Numerical Solution of Volterra-Fredholm Integral Equations with Hosoya Polynomials

Merve Zeynep Geçmen[1,*] and Ercan Çelik[2]

[1]Distance Education Center,Ağrı İbrahim Çeçen University, Ağrı,
Email: mzgecmen@agri.edu.tr

[2]Department of Mathematics, Atatürk University, Erzurum,
Email: ercelik@atauni.edu.tr

Abstract

In this study, Volterra-Fredholm integral equation is solved by Hosoya Polynomials. The solutions obtained with these methods were compared on the figure and table. And error analysis was done. Matlab programming language has been used to obtain conclusitions, tables and error analysis within a certain algorithm.

Keywords— Volterra-Fredholm Integral Equations, Hosoya Polynomial, Path Graphs

1 Introduction

The integral equations are the base for many mathematical models of diverse fact in engineering, chemistry, mathematics and other disciplines of science. Numerous problems of applied mathematics and engineering are relating to integral equations [1]. Integral equations are defined as equations which the unknown function is under the integral sign [14]. There are three types of integral equation. These are Fredholm, Volterra and Volterra-Fredholm integral equations. While both integral boundaries can be constant, equations that are infinite on one or both are called Fredholm integral equations. This integral equation is a practicable method to solve this kind of equations such as the Adomian decomposition method [15]. In recent years, there has been a expanding attention in the Volterra integral equations ascending invarious disciplines of engineering and applied mathematics. This equation has been studied in many fields of study such as Banach space, Haar functions problems, spectral methods, numerical computational problems and computer science problems [16]. The Volterra-Fredholm integral equations are associated with many scientific fields of study and have made an important contribution to the solution of many methods in applied mathematics, engineering and physics [2]. This integral equations are a combination of Volterra and Fredholm integral equations. Many methods have been used for solving these equations with sufficient accuracy and efficiency [3, 5].

2 Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations come in sight in two forms, namely

$$v(x) = w(x) = \lambda_1 \int_0^a L_1(x,t)v(t)dt + \lambda_2 \int_c^d L_2(x,t)v(t)dt \tag{1}$$

or,

$$v(x) = w(x) = \lambda \int_0^a \int_c^d L(r,t)v(t)dtdr \tag{2}$$

Here (1) implicates Volterra and Fredholm integrals separately, however (2) implicates both of the integrals [6, 7].

3 Hosoya Polynomials

Distance-based index was the Wiener index presented in 1947 by H. Wiener. Later, Haruo Hosoya proved the Wiener polynomial in chemistry [13]. The Wiener Polynomial is named The Hosoya Polynomial. Its generalize from the Wiener number. The Hosoya Polynomial is obtained from path graphs of certain pairs of graphs [8]. Many studies have been done from the combination of the Hosoya Polynomial and graphs. For a related graph with the Hosoya polynomial is describe as,

$$H(P,\gamma) = \sum_{l \geq 0} d(P,l)\gamma^k \tag{3}$$

where $d(P,l)$ is the distance between vertex pairs in the path graph. The relationship between the Hosoya polynomial and the Wiener index is [8, 9],

$$W(P) = H'(P,1) \tag{4}$$

where $H'(P,\gamma)$ is the first derivative of $H(P,\gamma)$ [10]. Sum of the path graph vertices m with $1, 2, ..., m$ are multipled γ parameter. Then Hosoya values are calculated based on m vertex values. For m integer values we represent path as ρ_m, then Hosoya polynomial of path compute as:

$$\begin{aligned}
&H(\rho_1,\lambda) = \sum_{l \geq 0} d(\rho_1,l)\gamma^l = 1 \\
&H(\rho_2,\gamma) = \sum_{l \geq 0} d(\rho_2,l)\gamma^l = \gamma + 2 \\
&H(\rho_3,\gamma) = \sum_{l \geq 0} d(\rho_3,l)\gamma^l = \gamma^2 + 2\gamma + 3 \\
&\vdots \\
&H(\rho_m,\gamma) = m + (m-1)\gamma + (m-2)\gamma^2 + ... \\
&+ (m-(m-2))\gamma^{m-2} + (m-(m-1))\gamma^{m-1}
\end{aligned}$$

For more information about the Hosoya polynomial one can refer [11, 12]. A function $w(x) \in L_2[0;\ 1]$ is expanded as,

$$w(x) = \sum_{i=1}^{n} z_i H(\rho_i, x) = Z^T H_\rho(x), \tag{5}$$

where Z and $H_\rho(x)$ are $mx1$ matrices shown as,

$$Z = [z_1, z_2, z_3, ..., z_m]^T \tag{6}$$

and

$$H_\rho(x) = [H(\rho_1, x), H(\rho_2, x), ..., H(\rho_m, x)]^T \tag{7}$$

4 Hosoya Polynomial Method

Consider the Volterra-Fredholm integral equation,

$$v(x) = w(x) + \int_0^x L(x,t)v(t)dt + \int_0^1 L(x,t)v(t)dt, 0 \leq x, t \leq 1 \tag{8}$$

to solve (8) , the method is as follows:

1. First we taking $v(x)$ as defined in Equation (5). This equation is,

$$v(x) = Z^T H_\rho(x) \tag{9}$$

2. Then using place of (9) in (8), we get,

$$Z^T H_\rho(x) = w(x) + \int_0^1 L(x,t) \left[Z^T H_\rho(t)\right] dt + \int_0^x L(x,t) \left[Z^T H_\rho(t)\right] dt \tag{10}$$

3. Substituting the collocation point $x_j = \frac{j-0.5}{m}, j = 1, 2, \cdots, m$ in Equation (10). Then we obtain,

$$Z^T H_\rho(x_j) = w(x_j) + Z^T \left[\int_0^1 L(x_j,t) H_p(t)dt + \int_0^x L(x_j,t) H_\rho(t)dt\right] \tag{11}$$

$$Z^T (H_\rho(x_j) - Y) = w$$

where

$$Y = \int_0^1 L(x_j,t) H_\rho(t)dt + \int_0^x L(x_j,t) H_\rho(t)dt$$

4. Finally, we get the results of unknown Hosoya values,

$$Z^T L = w$$

where

$$L = H_\rho(x_j) - Y$$

resolving this system of equations, we obtain coefficients Z and then use in place of these coefficients in (9), we get the necessary solution of (8).

5 Numerical Examples

5.1 Example

Consider Volterra-Fredholm integral equations,

$$v(x) = w(x) + \int_0^x (x-t)u(t)dt + \int_0^2 (xt)u(t)dt \tag{12}$$

$$w(x) = 2\cos(x) - x\cos(2) - 2\sin(2) + x - 1$$

which has the exact solution $v(x) = cos(x)$. First we substitute $v(x) = Z^T H_\rho(x)$ in (12). We get,

$$Z^T H_\rho(x) = w(x) + \int_0^x (x-t)\left[Z^T H_\rho(t)\right] dt + \int_0^2 (xt)\left[Z^T H_\rho(t)\right] dt \tag{13}$$

Therefore for $m = 3$

$$\begin{aligned} & Z_1[H_1(x) - \left(\int_0^x (x-t)H_1(t)dt + \int_0^2 (xt)H_1(t)dt\right)] \\ & + Z_2[H_2(x) - \left(\int_0^x (x-t)H_2(t)dt + \int_0^2 (xt)H_2(t)dt\right)] \\ & + Z_3[H_3(x) - \left(\int_0^x (x-t)H_3(t)dt + \int_0^2 (xt)H_3(t)dt\right)] = w(x) \end{aligned} \tag{14}$$

Next, we substitute the Hosoya polynomials as,

$$\begin{aligned} & Z_1[1 - \left(\int_0^x (x-t)dt + \int_0^2 (xt)dt\right)] \\ & + Z_2[(x+2) - \left(\int_0^x (x-t)(t+2)dt + \int_0^2 (xt)(t+2)dt\right)] \\ & + Z_3[(x^2+2x+3) - \\ & \left(\int_0^x (x-t)(t^2+2t+3)dt + \int_0^2 (xt)(t^2+2t+3)dt\right)] \\ & = w(x) \end{aligned} \tag{15}$$

Next,

$$
\begin{aligned}
&Z_1[1-\frac{x^2}{2}-x]\\
&+Z_2[2-\frac{x^3}{6}-x^2-\frac{17x}{3}]\\
&+Z_3[3-\frac{x^2}{2}-\frac{40x}{3}-\frac{x^4}{12}-\frac{x^3}{3}]=w(x)
\end{aligned}
\tag{16}
$$

if it is calculated as $x_j=\frac{j-0.5}{m}$ and putting in place of the collocation points x_1, x_2, x_3, we obtain the system of three equations with three unknowns as,

$$
\begin{aligned}
&Z_1[1-\frac{{x_1}^2}{2}-x_1]+Z_2[2-\frac{{x_1}^3}{6}-{x_1}^2-\frac{17x_1}{3}]\\
&+Z_3[3-\frac{{x_1}^2}{2}-\frac{40x_1}{3}-\frac{{x_1}^4}{12}-\frac{{x_1}^3}{3}]=w(x_1)\\
\\
&Z_1[1-\frac{{x_2}^2}{2}-x_2]+Z_2[2-\frac{{x_2}^3}{6}-{x_2}^2-\frac{17x_2}{3}]\\
&+Z_3[3-\frac{{x_2}^2}{2}-\frac{40x_2}{3}-\frac{{x_2}^4}{12}-\frac{{x_2}^3}{3}]=w(x_2)\\
\\
&Z_1[1-\frac{{x_3}^2}{2}-x_3]+Z_2[2-\frac{{x_3}^3}{6}-{x_3}^2-\frac{17x_3}{3}]\\
&+Z_3[3-\frac{{x_3}^2}{2}-\frac{40x_3}{3}-\frac{{x_3}^4}{12}-\frac{{x_3}^3}{3}]=w(x_3)
\end{aligned}
\tag{17}
$$

Resolving these systems we obtain the three unknown Hosoya values,

$$Z_1=0.5012,\ Z_2=0.8672,\ Z_3=\text{-}0.4101$$

Replacing with these coefficients in the approximation, We obtain

$$v(x)=Z_1[H_1(x)]+Z_2[H_2(x)+Z_3[H_3(x)]$$

If in (17) is written instead of the x_1, x_2, x_3 values, approximate solutions are obtained,

$$
\begin{aligned}
v_1(x)&=Z_1[H_1(x_1)]+Z_2[H_2(x_1)+Z_3[H_3(x_1)]\\
v_2(x)&=Z_1[H_1(x_2)]+Z_2[H_2(x_2)+Z_3[H_3(x_2)]\\
v_3(x)&=Z_1[H_1(x_3)]+Z_2[H_2(x_3)+Z_3[H_3(x_3)]
\end{aligned}
\tag{18}
$$

We get the approximate values,

$$v_1=1.00183,\ v_2=0.926377,\ v_3=0.759796$$

Maximun Error analyzed for $m=3$,

$$
\begin{aligned}
E_{\max}&=\sqrt{\sum_{i=1}^{m}\left(v_e(x_i)-v_a(x_i)\right)^2}=\\
&\sqrt{(\cos(x_1)-v_1)^2+(\cos(x_2)-v_2)^2+(\cos(x_3)-v_3)^2}=\ 0.1013
\end{aligned}
\tag{19}
$$

Table 1: Conclution of Hosoya Polynomial Method, for $m = 3$

x	**Hosoya Polynomial Method**	**Exact Solution**
0.1667	10.018	0.9861
0.5	0.9263	0.8775
0.8333	0.7597	0.6724

and for $m = 3, 8, 10$ are shown in the Table $1, 2, 3$.

Figure 1: Example 5.1 for $m = 3$

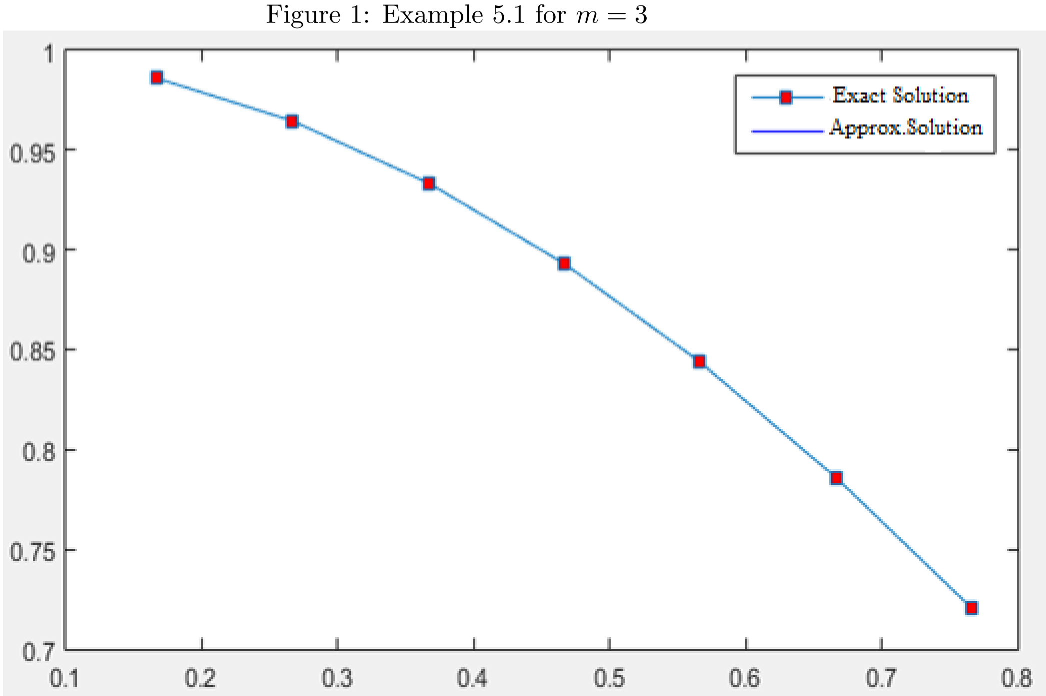


Table 2: Conclution of Hosoya Polynomial for $m = 8$

x	**Hosoya Polynomial Method**	**Exact Solution**
0.0625	0.9980	0.9980
0.1875	0.9824	0.9824
0.3125	0.9515	0.9515
0.4375	0.9058	0.9058
0.5625	0.8459	0.8459
0.6875	0.7728	0.7728
0.8125	0.6877	0.6877
0.9375	0.5918	0.5918

Figure 2: Example 5.1 for $m = 8$

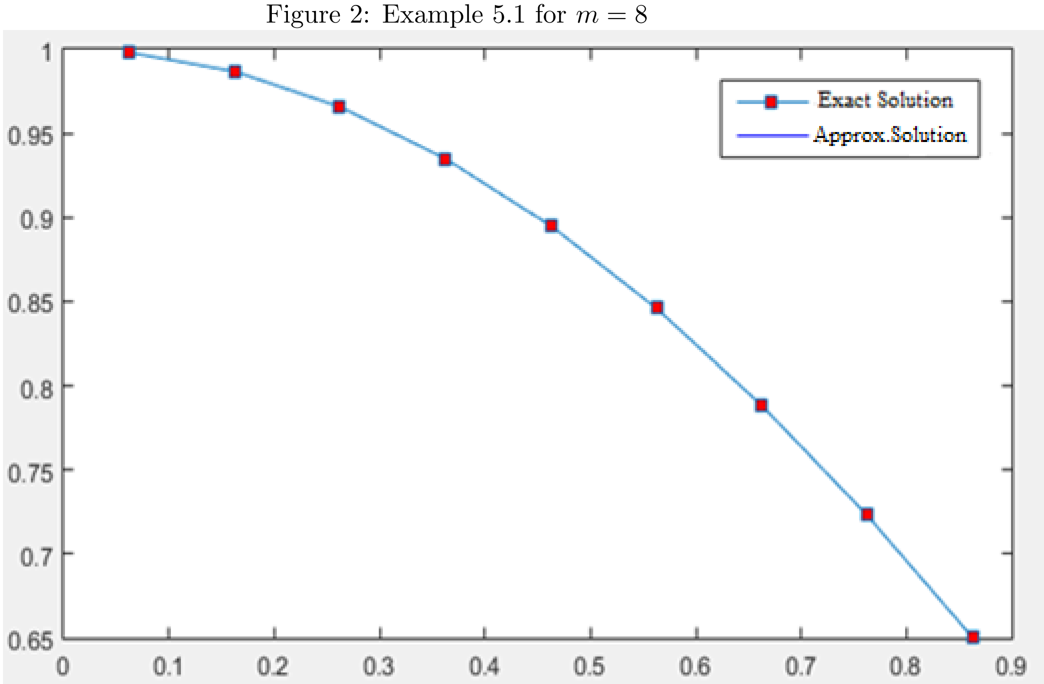
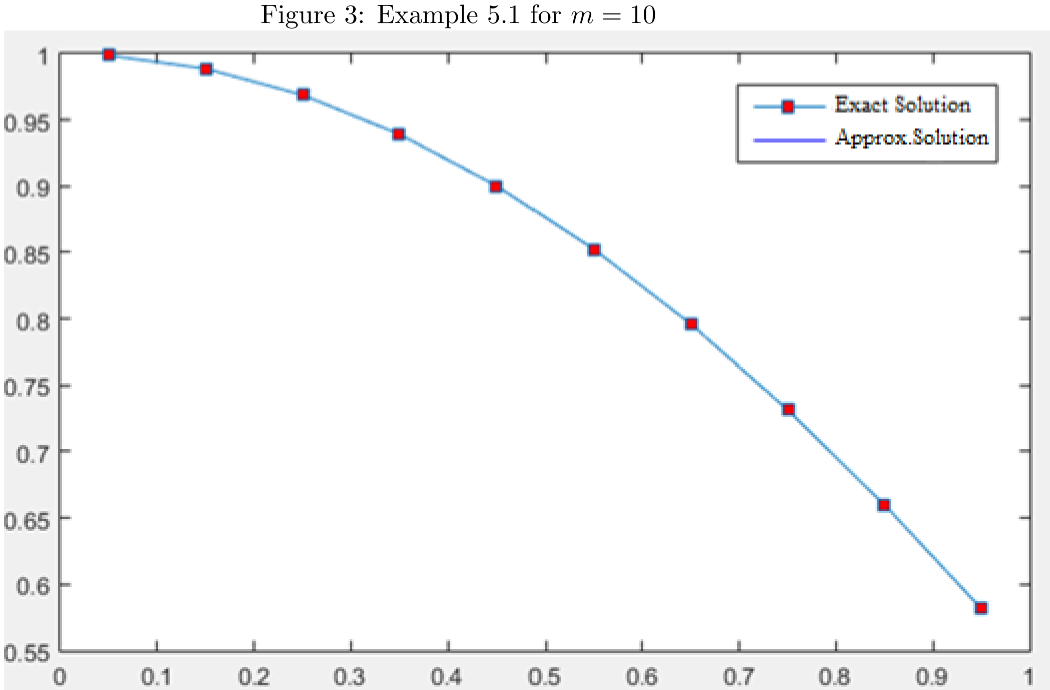


Table 3: . Conclution of Hosoya Polynomial Method for $m = 10$

x	**Hosoya Polynomial Method**	**Exact Solution**
0.050	0.99875	0.99875
0.150	0.98877	0.98877
0.250	0.96891	0.96891
0.350	0.93937	0.93937
0.450	0.90044	0.90044
0.550	0.85252	0.85252
0.650	0.79608	0.79608
0.750	0.73168	0.73168
0.850	0.65998	0.65998
0.950	0.58168	0.58168

Figure 3: Example 5.1 for $m = 10$



5.2 Example

Consider Volterra-Fredholm integral equations,

$$v(x) = w(x) + \int_0^x (x^2 - t)u(t)dt + \int_0^1 (xt + x)u(t)dt$$
$$w(x) = e^x + e^x(x-1) - xe - x^2(e^x - 1) + 1 \tag{20}$$

which has the exact solution $v(x) = e^x$. Applying the proposed method to solve Equation (20) for $m = 3, 8, 10$. We obtain the approximate solution $v(x)$ as shown in Table $4, 5, 6$ and Figure $4, 5, 6$. Error analysis is shown in **Table 7**.

Table 4: Conclution of Hosoya Polynomial Method for $m = 3$

x	Hosoya Polynomial Method	Exact Solution
0.1667	1.1803	1.1813
0.5	1.6461	1.6487
0.8333	2.2962	2.3009

Figure 4: Example 5.1 for $m = 3$

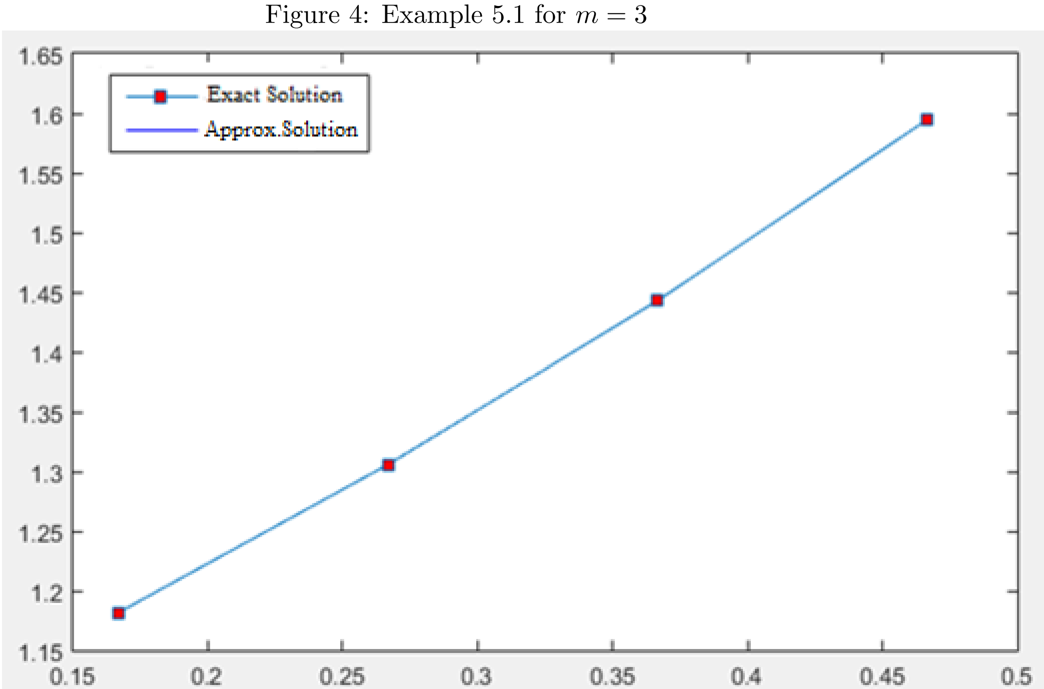


Table 5: Conclution of Hosoya Polynomial Method for $m = 8$

x	**Hosoya Polynomial Method**	**Exact Solution**
0.0625	1.06449	1.06449
0.1875	1.20623	1.17644
0.3125	1.36684	1.30017
0.4375	1.54883	1.43691
0.5625	1.75505	1.58803
0.6875	1.98874	1.75505
0.8125	2.25353	1.93963
0.9375	2.55359	2.14362

Figure 5: Example 5.1 for $m = 8$

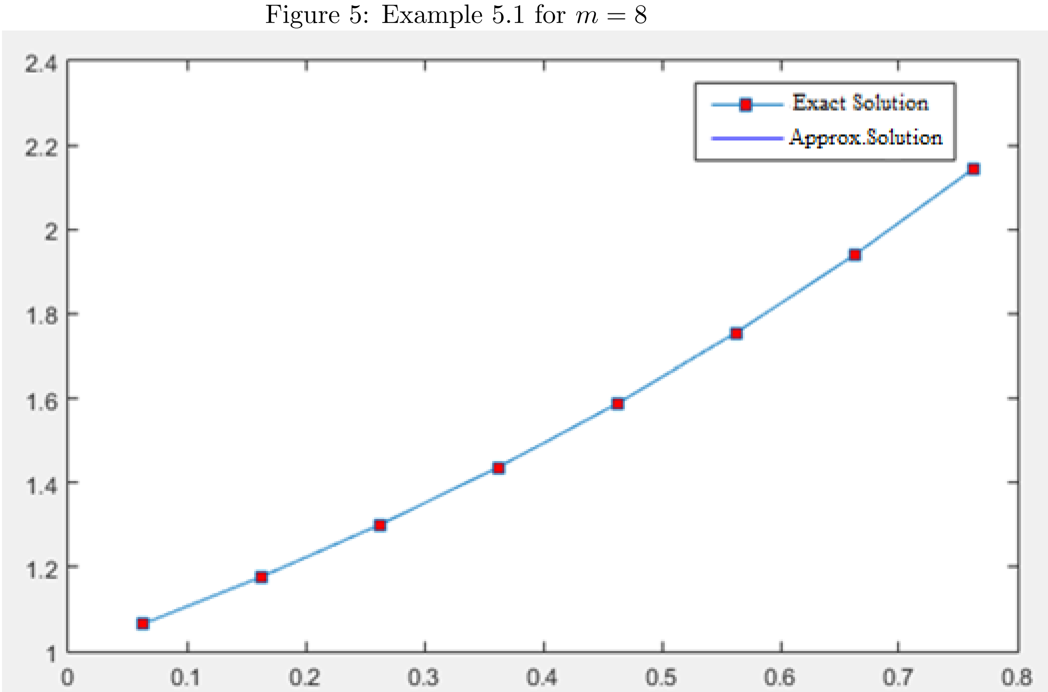


Table 6: Conclution of Hosoya Polynomial Method for $m = 10$

x	**Hosoya Polynomial Method**	**Exact Solution**
0.050	1.05127	1.05127
0.150	1.16183	1.16183
0.250	1.28403	1.28402
0.350	1.41907	1.41906
0.450	1.56831	1.56831
0.550	1.73325	1.73325
0.650	1.91554	1.91554
0.750	2.11700	2.11700
0.850	2.33965	2.33964
0.950	2.58571	2.58570

Figure 6: Example 5.1 for $m = 10$

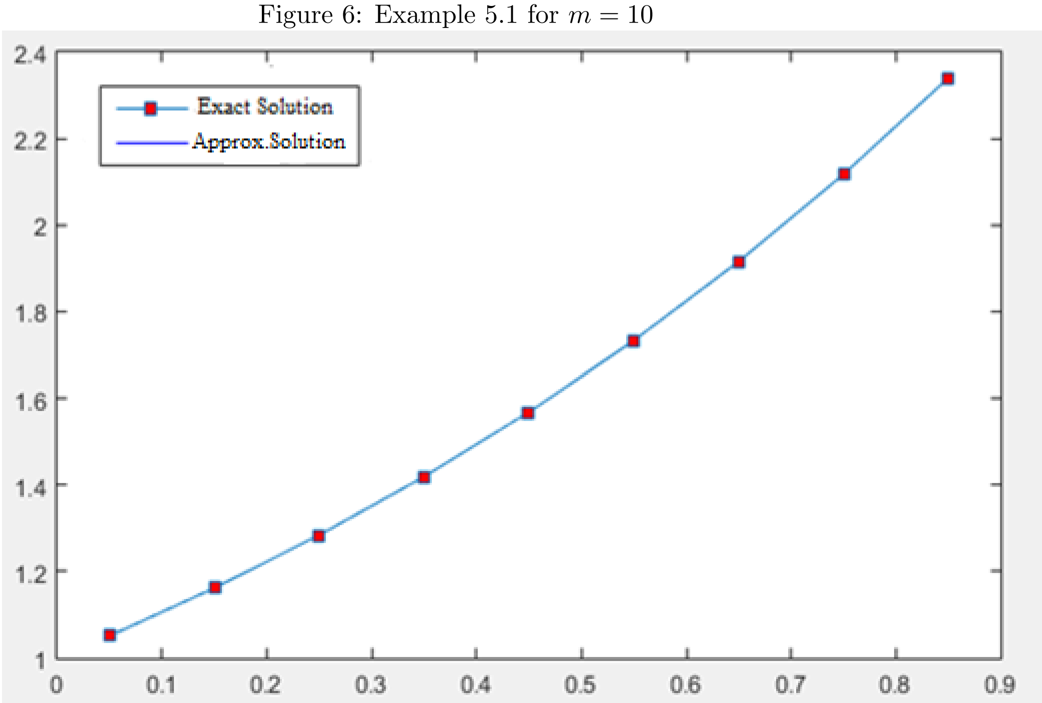


Table 7: Maximum error analysis of Hosoya Polynomial Method, for $m = 3, 8, 10$

m	Example 5.1	Example 5.2
3	2.4074e-15	0.0055
8	1.6431e-12	1.6160e-08
10	1.1172e-11	2.1883e-11

6 Conclusion

In this article, the solution of Volterra-Fredholm integral equations with Hosoya method is examined. The method was applied to two test problems in the matlab program created with a specific algorithm. Hosoya method is calculated for$m = 3, m = 8, m = 10$ values. By determining the x ranking points according to the m values, the maximum error analysis was made and the solutions acquired by the method were compared with the exact solutions. The approximate solution obtained in each test problem, numerical solution, maximum error rate and graphs are shown with tables and figures. When the obtained results are examined, it is seen that the Hosoya method is an effective method for solving the Volterra-Fredholm integral equation.

References

[1] M. A. Abdou, A. A. Soliman and M. A. Abdel–Aty, On a discussion of Volterra Fredholm integral equation with discontinuous kernel, Journal of the Egyptian Mathematical Society, (2020),1-10.

[2] K. Maleknejad, J. Rashidinia and T. Eftekhari, Numerical solution of three-dimensional Volterra–Fredholm integral equations of the first and second kinds based on Bernstein's approximation, Applied Mathematics and Computation Volume 339,(2018), 272-285.

[3] Micula, S., (2015), An iterative numerical method for Fredholm-Volterra integral equations of the second kind, Applied Mathematics and Computation, 270(C), pp.935-942.

[4] Mirzaee, F., Hoseini, A.A., (2013), Numerical solution of nonlinear Volterra-Fredholm integral equations using hybrid of Block-pulse functions and Taylor series, Alexandria Engineering Journal, 52(3), pp.551-555.

[5] Shali, J.A., Akbarfam, A.A., Ebadi, J.G., (2012), Approximate solutions of nonlinear Volterra Fredholm integral equations, Int. J. Nonlin. Sci. 14(4), pp.425-433.

[6] K. Maleknejad and M. Hadizadeh, A New computational method for Volterra-Fredholm integral equations, Comput. Math. Appl., 37 (1999) 1–8.

[7] A.M. Wazwaz, A reliable treatment for mixed Volterra-Fredholm integral equations, Appl. Math. Comput., 127 (2002), 405–414.

[8] D. Stevanovic, Hosoya polynomial of composite graphs, Department of Mathematics, Faculty of Philosophy, Cirila i Metodija 2, Niˇs 18000, Yugoslavia.

[9] E. V. Konstantinova, M. V. Diudea, The Wiener polynomial derivatives and other topological indices in chemical research, Croat. Chem. Acta 73 (2000) 383-403.

[10] G. Cash, S. Klavžar, M. Petkovšek, Three methods for calculation of the hyper–Wiener index of molecular graphs, J.Chem. Inf. Comput. Sci. 43 (2002) 571-576.

[11] I. Gutman, Hosoya polynomial and the distance of the total graph of a tree, Publ. Elektrotehn. Fak. (Beograd) Ser. Mat.,10 (1999) 53-58.

[12] H. S. Ramane, K. P. Narayankar, S. S. Shirkol, A. B. Ganagi, Terminal Wiener index of line graphs, MATCH Commun.Math. Comput. Chem., 69 (2013) 775-782.

[13] N.Tratnika, P. Z. Pletersek, Relationship Between the Hosoya Polynomial and the Edge- Hosoya Polynomial of Trees, MATCH Commun. Math. Comput. Chem. 78 (2017) 181-187.

[14] S. Upadhyay and K. N. Rai, Integral equation, https://www.researchgate.net/publication/283829675, November, 2015.

[15] Fredholm Integral Equation https://en.wikipedia.org/wiki/Fredholmintegralequation

[16] J. A.Zarnan, International Journal of Computer Science and Mobile Computing, Vol.5 Issue.7, July- 2016, pg. 509-514.

[17] H. S. Ramane, S. C. Shiralashetti, R. A. Mundewadi,R. Jummannaver,Numerical Solution of Fredholm Integral Equations Using Hosoya Polynomial of Path Graphs, American Journal of Numerical Analysis, 2017, Vol. 5, No. 1, 11-15.