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Global strong solutions and large time behavior of 2D tropical climate model with zero thermal diffusion

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Abstract

In this article, we study the global well-posedness and large-time behaviors of solutions to the two-dimensional tropical climate system with zero thermal diffusion for a small initial data in the whole space. The main approaches include high and low frequency decomposition method and exploiting the structure of system (1) to obtain the estimates of thermal dissipation. We utilize the time decay properties of the kernels to a linear differential equation to obtain the decay rates of solutions of the low frequency part and the decay property of exponential operator for the high frequency part. The key ingredient here is the explicit large-time decay rate of solutions.

KEYWORDS:
Tropical climate model, Global well-posedness, Large-time behavior

1 | INTRODUCTION

The tropical climate models can be used to predict the weather in hydrodynamics and also have a wide range of applications in many fields such as molecular biology, agricultural production, and so on[1,2]. By performing a Galerkin truncation up to the hydrostatic Boussinesq equations, the system without any dissipation terms was first derived by Frierson-Majda-Pauluis for large-scale dynamics of precipitation fronts in the tropical atmosphere[3]. For abbreviation, a general tropical climate model with both dissipation and damping terms is given as follows,

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u + \alpha u + \mathrm{div}(v\otimes v) + \nabla P = 0, \\ \partial_t v + u\cdot\nabla v - \nu\Delta v + \beta v + v\cdot\nabla u + \nabla\theta = 0, \\ \partial_t \theta + u\cdot\nabla\theta - \eta\Delta\theta + \mathrm{div} v = 0, \\ \mathrm{div} u = 0, \\ (u, v, \theta)(t, x)|_{t=0} = (u_0, v_0, \theta_0)(x), \end{cases}$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ denote the barotropic mode and the first baroclinic mode of velocity field, P denotes the pressure, and θ represents the temperature. Here μ, ν, η, α and β are non-negative parameters.

There are considerable mathematical studies on the tropical climate model recently from various authors. In the 2D case, Li-Titi[4] had proved the global well-posedness of strong solutions for the tropical climate model with $\mu = \nu = 1$ and $\alpha = \beta = \eta = 0$. They obtained the higher regularity of u and the gradient estimate of (u, v, θ) by constructing a new unknown $\omega = v - \nabla(-\Delta)^{-1}\theta$ to bypass the obstacle caused by the absence of thermal diffusion. In this article some background on tropical models was introduced and more relevant background can be found in[5,6,7,8]. Wan[9] studied a tropical climate model with $\mu = \eta = 0$ and $\nu, \alpha, \beta > 0$. The author obtained the global regularity of classical solutions with small initial data in $H^s(\mathbb{R}^2)$, $s > 2$. And Ma

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and Wan[10] proved the global well-posedness of global strong solutions for the system with $\nu = \eta = \alpha = 0$ and $\mu > 0, \beta > 0$, under the hypotheses of small initial data in $H^s(\mathbb{R}^2)$, $s > 3$. Ye and Zhu[11] studied the existence of global strong solutions for the tropical climate model with temperature-dependent diffusion on the barotropic mode ($\nu = \eta = 0$ and $\alpha, \beta \geq 0$) under small initial data $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$, $s > 1$. Zhou[12] established the global regularity for the 2D tropical climate model with generalized nonlocal dissipation of the barotropic mode ($\nu = 1$ and $\alpha = \beta = 0$). We also mention the 3D case. Wang-Zhang-Pan[13] considered the global existence of unique strong solutions for the system with $\mu = \nu = \eta = 1$ and $\alpha = \beta = 0$ under the assumption of small initial data. Zhu[14] obtained the global well-posedness of strong solutions for the system with fractional diffusion on barotropic mode ($\nu = \eta = \alpha = \beta = 0$) under initial data in $H^3(\mathbb{R}^3)$. The issue of global regularity of the generalized tropical climate system has been also investigated in a number of articles[15,16,17].

In this paper, we consider the cauchy problem of a two-dimensional tropical climate model with a damping term and zero thermal diffusion, namely

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = -\mathrm{div}(v \otimes v), \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\ \partial_t v + u \cdot \nabla v + v \cdot \nabla u + v = -\nabla \theta, \\ \partial_t \theta + u \cdot \nabla \theta = -\mathrm{div} v, \\ \mathrm{div} u = 0, \end{cases} \tag{1}$$

with the initial data

$$(u, v, \theta)(t,x)|_{t=0} = (u_0, v_0, \theta_0)(x), \quad x \in \mathbb{R}^2. \tag{2}$$

We emphasize that in this paper the equation for θ contains no dissipation or any damping term and then powerful classical tools such as the Fourier splitting methods designed for the systems with full dissipation no longer apply here. Without the dissipation term $-\Delta v$ and add a damping term v, it is well known that the local and global well-posedness of solutions to system (1) under the hypotheses of small initial data studied by Ma and Wan[10]. They used the spectral analysis technique and constructed a new energy estimate of u in Besov space to achieve the goal. To our knowledge, many models with damping terms such as damped Navier-Stokes equations[18], compressible Euler equations[19],[20] and MHD equations[21] have been studied. Wang[20] studied the large-time behavior of the smooth solutions of three-dimensional compressible Euler equations of isentropic gases with damping. Wu et al.[21] proved the global well-posedness of the incompressible MHD system with a velocity damping term when the initial data is close to an equilibrium state $(0, y)$. They also obtained the explicit large-time decay rates of the solutions and their approach exploits the time decay properties of the kernel functions to a linear differential equation. Lai and Wu et al.[22] had investigated the large-time behavior of solutions to a special Boussinesq system with partial dissipation $\partial_{22} u$ and a damping term θ under small initial data in $H^2(\mathbb{R}^2)$. The tool commonly used in dealing with such problems is the bootstrapping argument. We also refer to related works[23,24,25,26].

Our effort here is mainly related to the global well-posedness and large-time behaviors of the solutions of system (1). The main methods to establish the global existence and uniqueness of smooth solutions and obtain the explicit large-time decay rates for various sobolev norms, i.e. $\|\nabla^i (u, v, \theta)\|_{L^\infty}, i = 0, 1$, of the solutions are by the high and low frequency decomposition method to overcome the difficulties caused by nonlinear terms F_1 and F_2 (definition in Section 2). And we exploit the structure of system (1) coming from the coupled terms and damping term to overcome this difficulty caused by the absence of thermal dissipation. We utilize the time decay properties of the kernel functions to a linear differential equation to obtain the decay rates of solutions of the low frequency part and the decay property of exponential operator to get the decay rates of solutions of the high frequency part. More details can be found in Section 3.

Before precisely stating the main result, we introduce the functional settings. In this paper, a generic constant C means a positive constant independent of time. The operator $\langle \nabla \rangle$ is defined by $\langle \nabla \rangle := (I - \Delta)^{\frac{1}{2}}$. The operator $\langle t \rangle$ denotes $\langle t \rangle := 1 + t$. For any constant $N \geq 3$, we define working space X with its norm given by

$$\begin{aligned} \|(u, v, \theta)\|_X = & \|\langle \nabla \rangle^N (u, v, \theta)\|_{L^2} + \langle t \rangle \|v_t\|_{L^2} + \sum_{j=0}^{1} \langle t \rangle^{\frac{j+2}{2}} \|\nabla^j v\|_{L^2} + \\ & \langle t \rangle^{\frac{3}{2}} \|(u_t, \theta_t)\|_{L^2} + \langle t \rangle^{\frac{3}{2}} \|\nabla \Delta (u, \theta)\|_{L^2} + \langle t \rangle^{\frac{3}{2}} \|\Delta v\|_{L^2} + \\ & \sum_{i=0}^{2} \langle t \rangle^{\frac{i+1}{2}} \|\nabla^i (u, \theta)\|_{L^2} + \langle t \rangle^{\frac{3}{2}} \|\mathrm{div} v_t\|_{L^2}, \end{aligned} \tag{3}$$

and space X_0,

$$||(u_0, v_0, \theta_0)||_{X_0} = \|\langle \nabla \rangle^N (u_0, v_0, \theta_0)\|_{L^2} + \|(u_0, u_1, \theta_0, \theta_1)\|_{L^1}, \tag{4}$$

where $\|f\|_{L^p} = \|f\|_{L^p_{xy}}$ for $1 \le p \le \infty$.

Theorem 1. Let $N \ge 3$ be a constant. Assume the initial data $(u_0, v_0, \theta_0) \in X_0$ with $\nabla \cdot u_0 = 0$. Then, there exists a small constant $\delta > 0$ such that if (u_0, v_0, θ_0) satisfies

$$||(u_0, v_0, \theta_0)||_{X_0} \le \delta, \tag{5}$$

then the cauchy problem (1) admits a unique global solution (u, v, θ) satisfying

$$(u, v, \theta) \in X. \tag{6}$$

Moreover, the following time decay estimates hold

$$\begin{aligned} &\|(u, \theta)\|_{L^\infty} \le C\langle t\rangle^{-1}, \\ &\|(\nabla u, \nabla\theta)\|_{L^\infty} \le C\langle t\rangle^{-\frac{11}{8}}, \\ &\|v\|_{L^\infty} \le C\langle t\rangle^{-\frac{11}{8}}, \\ &\|\nabla v\|_{L^\infty} \le C\langle t\rangle^{-\frac{9}{8}}. \end{aligned} \tag{7}$$

Remark 1. The proof of Theorem 1.1 relies on the bootstrapping argument (see, e.g.[27]). More precisely, we verify that

$$U(t) \le C(U(0) + Q(U(t))), \quad t > 0. \tag{8}$$

where $Q(m) = m^\beta$ for $m < 1$ and $\beta > 1$, we denote $U(t) := \|(u, v, \theta)\|_X$ and accordingly $U(0)$ as $||(u, v, \theta)||_{X_0}$. Once (8) is confirmed, a bootstrapping argument applied to (8) would lead to the desired global existence and stability of Theorem 1.1. Our main goals are to check (8).

The rest of the paper mainly includes the expressions and estimates of solutions of the linearized system in Section 2 and several estimates for the proof of Theorem 1.1 in Section 3.

2 | PRELIMINARY

This section is divided into three parts. First, we derive the linearized equation and then obtain integral formulation (19) with explicit representation and the time decay estimates of the kernels of the linear equation. In addition, several technical lemmas are given here.

2.1 | Reformulation of system

Firstly, (1) can be rewritten as the following equivalent system

$$\begin{cases} \partial_{tt}u + \partial_t u - \Delta u = \partial_t \pi_1 + \pi_1 + \partial_t \Delta u, \\ \partial_{tt}v + \partial_t v - \nabla \mathrm{div} v = -\partial_t(u \cdot \nabla v) - \partial_t(v \cdot \nabla u) - \nabla(u \cdot \nabla\theta), \\ \partial_{tt}\theta + \partial_t\theta - \Delta\theta = \mathrm{div}(u \cdot \nabla v) + \mathrm{div}(v \cdot \nabla u) - \partial_t(u \cdot \nabla\theta) - u \cdot \nabla\theta, \\ \mathrm{div} u = 0, \\ (u, v, \theta)|_{t=0} = (u_0, v_0, \theta_0), \\ (u_t, v_t, \theta_t)|_{t=0} = (u_1, v_1, \theta_1), \end{cases} \tag{9}$$

where π_1 is expressed by

$$\begin{aligned} \pi_1 :=& -u \cdot \nabla u - \mathrm{div}(v \otimes v) - \nabla P \\ =& -\nabla(-\Delta)^{-1}(\nabla \cdot (u \cdot \nabla u) + \nabla \cdot (\nabla \cdot (v \otimes v))) \\ & - u \cdot \nabla u - \mathrm{div}(v \otimes v). \end{aligned} \tag{10}$$

From (9), we obtain

$$\begin{cases} \partial_{tt}u + \partial_t u - \Delta u = F_1, \\ \partial_{tt}\theta + \partial_t\theta - \Delta\theta = F_2, \\ (u,\theta)|_{t=0} = (u_0,\theta_0), \\ (u_t,\theta_t)|_{t=0} = (u_1,\theta_1), \end{cases} \tag{11}$$

with

$$\begin{cases} F_1 := \partial_t\pi_1 + \pi_1 + \partial_t\Delta u, \\ F_2 := \operatorname{div}(u\cdot\nabla v) + \operatorname{div}(v\cdot\nabla u) - \partial_t(u\cdot\nabla\theta) - u\cdot\nabla\theta. \end{cases} \tag{12}$$

It can be easily found that u and θ have the same structure of the linear part in (11), which plays a crucial role in ensuring the global existence of small solutions.

2.2 | Representation of solution of linearization problem

For simplicity, we consider the linear system as follows,

$$\begin{cases} \partial_{tt}\phi + \partial_t\phi - \Delta\phi = F, \\ \phi(0,x) = \phi_0, \quad \phi_t(0,x) = \phi_1, \end{cases} \tag{13}$$

where F is given by (11)-(12).

Taking the Fourier transform of above formula yields

$$\begin{cases} \partial_{tt}\hat\phi + \partial_t\hat\phi + |\xi|^2\hat\phi = \hat F, \\ \hat\phi(0,x) = \hat\phi_0, \quad \hat\phi_t(0,x) = \hat\phi_1, \end{cases} \tag{14}$$

where the Fourier transform $\hat\phi$ is defined as

$$\hat\phi(t,\xi) := \int_{\mathbb{R}^2} e^{-ix\cdot\xi}\phi(t,x)dx. \tag{15}$$

Then we solve (14) by a simple ODE theory to obtain

$$\lambda_1 = -\frac12 + \sqrt{\frac14 - |\xi|^2}, \quad \lambda_2 = -\frac12 - \sqrt{\frac14 - |\xi|^2}, \tag{16}$$

And by Duhamel's principle, it follows

$$\hat\phi(t,\xi) = \hat\omega_0(\xi)\hat\phi_0(\xi) + \hat\omega_1(\xi)(\frac12\hat\phi_0(\xi) + \hat\phi_1(\xi)) + \int_0^t \hat\omega_1(t-s,\xi)\hat F(s,\xi)ds, \tag{17}$$

with

$$\begin{cases} \hat\omega_0(t,\xi) := \dfrac12(e^{\lambda_1 t} + e^{\lambda_2 t}), \\ \hat\omega_1(t,\xi) := \dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}. \end{cases} \tag{18}$$

Thus, we can get the integral representation of solution of (13):

$$\phi(t,x) = \omega_0(x)\phi_0(x) + \omega_1(x)(\frac12\phi_0(x) + \phi_1(x)) + \int_0^t \omega_1(t-s,x)F(s,x)ds. \tag{19}$$

In the following, we present the decay estimate on $\hat\omega_i(t,\xi)$.

Lemma 1. Let $\hat\omega_i(t,\xi)$, $i = 0,1$, be defined in (16) and (18). Then for $1 \le p \le +\infty$ and $\alpha \ge 0$, it holds

$$\||\xi|^\alpha\hat\omega_i\|_{L^p_{(|\xi|\le\frac12)}} \le C\langle t\rangle^{-\frac1p-\frac\alpha2}. \tag{20}$$

Proof. According to the expressions of ω_0 and ω_1, we have for $|\xi| \le \frac12$ that

$$\hat\omega_0(t,\xi) = \frac12(e^{(-\frac12+\sqrt{\frac14-|\xi|^2})t} + e^{(-\frac12-\sqrt{\frac14-|\xi|^2})t}), \tag{21}$$

and

$$\hat{\omega}_1(t,\xi) = \frac{1}{2\sqrt{\frac{1}{4}-|\xi|^2}}\left(e^{(-\frac{1}{2}+\sqrt{\frac{1}{4}-|\xi|^2})t} - e^{(-\frac{1}{2}-\sqrt{\frac{1}{4}-|\xi|^2})t}\right). \tag{22}$$

Since it holds

$$\frac{-1-\sqrt{1-4|\xi|^2}}{2} \le \frac{-1+\sqrt{1-4|\xi|^2}}{2} = \frac{-2\xi^2}{1+\sqrt{1-4|\xi|^2}} \le -\xi^2, \tag{23}$$

we obtain

$$0 \le \hat{\omega}_i(t,\xi) \le Ce^{-|\xi|^2 t}. \tag{24}$$

For $1 \le p < +\infty$ and $\alpha \ge 0$, multiplying the above equation by $|\xi|^\alpha$ and integrating over $\{\xi \in \mathbb{R} \quad | \quad |\xi| \le \frac{1}{2}\}$, we deduce

$$\||\xi|^\alpha \hat{\omega}_i\|^p_{L^p_{(|\xi|\le\frac{1}{2})}} \le C\int_{|\xi|\le\frac{1}{2}} |\xi|^{p\alpha}e^{-p|\xi|^2 t}d\xi. \tag{25}$$

Set $y_i = \xi_i t^{\frac{1}{2}}$, i.e. $t^{-\frac{1}{2}}dy_i = d\xi_i$ so that the right hand of (25) changes into

$$\int_{|y|\le\frac{1}{2}t^{\frac{1}{2}}} t^{-\frac{p}{2}\alpha}|y|^{p\alpha}e^{-p|y|^2}t^{-1}dy \le C\langle t\rangle^{-1-\frac{p}{2}\alpha}. \tag{26}$$

And for $p = +\infty$, we have

$$||\xi|^\alpha \hat{\omega}_i| \le |\xi|^\alpha e^{-|\xi|^2 t} = |\xi|^\alpha t^{\frac{\alpha}{2}}e^{-|\xi|^2 t}\cdot t^{-\frac{\alpha}{2}} \le C\langle t\rangle^{-\frac{\alpha}{2}}, \tag{27}$$

where $\langle t\rangle = Ct$ for $t \gg 0$. The proof of Lemma 1 is completed. □

Lemma 2. Let $\hat{\omega}(t,\xi)$ denote a Fourier multiplier operator with $\|\hat{\omega}(t,\xi)\|_{L^i_{\xi_1\xi_2}} < \infty$, $i = 1,2$. Then for any Schwartz function f, there holds

$$\|\omega(t,\partial_x,\partial_y)f\|_{L^2} \le C\|\hat{\omega}\|_{L^2_{\xi_1\xi_2}}\|f\|_{L^1}, \tag{28}$$

$$\|\omega(t,\partial_x,\partial_y)f\|_{L^2} \le C\|\hat{\omega}\|_{L^\infty_{\xi_1\xi_2}}\|f\|_{L^2}. \tag{29}$$

Proof. The above two inequalities can be derived by $\|f\|_{L^\infty} \le C\|\hat{f}\|_{L^1}$ and Plancherel formula. Indeed, we obtain

$$\begin{aligned}\|\omega(t,\partial_x,\partial_y)f\|_{L^2} &= \|\hat{\omega}(t,\xi_1,\xi_2)\hat{f}\|_{L^2_{\xi_1\xi_2}}\\ &\le C\|\hat{\omega}\|_{L^2_{\xi_1\xi_2}}\|\hat{f}\|_{L^\infty_{\xi_1\xi_2}}\\ &\le C\|\hat{\omega}\|_{L^2_{\xi_1\xi_2}}\|f\|_{L^1},\end{aligned}$$

and

$$\begin{aligned}\|\omega(t,\partial_x,\partial_y)f\|_{L^2} &= \|\hat{\omega}(t,\xi_1,\xi_2)\hat{f}\|_{L^2_{\xi_1\xi_2}}\\ &\le C\|\hat{\omega}\|_{L^\infty_{\xi_1\xi_2}}\|f\|_{L^2}.\end{aligned}$$

□

2.3 | Auxiliary lemmas

Lemma 3. For any $\alpha, \beta \ge 0$, $\alpha + \beta > 1$, it holds

$$\int_0^t \langle t-\tau\rangle^{-\alpha}\langle\tau\rangle^{-\beta}d\tau \le C\begin{cases}\langle t\rangle^{-\alpha-\beta+1}, & if \quad \beta < 1, \quad \alpha < 1,\\ \langle t\rangle^{-\alpha}+\langle t\rangle^{-\beta}, & if \quad \beta > 1 \quad or \quad \alpha > 1.\end{cases} \tag{30}$$

The proof is shown in Appendix A.1. And a similar result can be found in[28].

Lemma 4. For any $s > 0$, $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$ with $p_1, p_2, q_1, q_2 \in (1,\infty]$, $r \in (1,\infty)$, the commutator estimate holds:

$$\|[\langle\nabla\rangle^s, u\cdot\nabla]v\|_r \le C(\|\langle\nabla\rangle^s u\|_{p_1}\|\nabla v\|_{p_2} + \|\nabla u\|_{q_1}\|\langle\nabla\rangle^s v\|_{q_2}), \tag{31}$$

which can be seen in[29].

Lemma 5. For a constant $C > 0$, it holds

$$\|f\|_{L^\infty_{xy}} \le C\|f\|^{\frac{1}{4}}_{L^2_{xy}}\|\nabla f\|^{\frac{1}{2}}_{L^2_{xy}}\|\Delta f\|^{\frac{1}{4}}_{L^2_{xy}}. \tag{32}$$

Proof. By

$$\|f\|_{L^\infty_x} \le C\|f\|^{\frac{1}{2}}_{L^2_x}\|\partial_x f\|^{\frac{1}{2}}_{L^2_x},$$

we infer

$$\begin{aligned}\|f\|_{L^\infty_{xy}} &\le C\|\sup_y f\|^{\frac{1}{2}}_{L^2_x}\|\sup_y \partial_x f\|^{\frac{1}{2}}_{L^2_x}\\ &\le C\|f\|^{\frac{1}{4}}_{L^2_{xy}}\|\partial_y f\|^{\frac{1}{4}}_{L^2_{xy}}\|\partial_x f\|^{\frac{1}{4}}_{L^2_{xy}}\|\partial_{xy} f\|^{\frac{1}{4}}_{L^2_{xy}}\\ &\le C\|f\|^{\frac{1}{4}}_{L^2_{xy}}\|\nabla f\|^{\frac{1}{2}}_{L^2_{xy}}\|\Delta f\|^{\frac{1}{4}}_{L^2_{xy}}.\end{aligned}$$

□

3 | PROOF OF THE MAIN THEOREM

In this section, we establish the global existence and uniqueness of smooth solutions and obtain the explicit large-time decay rates for various sobolev norms of the solutions, i.e. $\|\nabla^i(u,v,\theta)\|_{L^\infty}, i = 0,1$, and $\|\nabla^j(u,v,\theta)\|_{L^2}, j = 0,1,2$. The main techniques include the high and low frequency decomposition method and exploiting the structure of system (1) to overcome this difficulty caused by the absence of thermal dissipation. Here we utilize the estimates of $\hat{\omega}_i(t,\xi)$ to obtain the decay rates of solutions of the low frequency part and the decay property of exponential operator to get the decay rates of solutions of the high frequency part.

For the sake of clarity, we divide this section into several parts to obtain the relevant estimates of (u,v,θ). Recall the definition of (3), the derivation of the estimate of $\|\langle\nabla\rangle^N(u,v,\theta)\|_{L^2}$ is given first.

3.1 | Estimates of u

Lemma 6. For any $t > 0$, there holds

$$\|\langle\nabla\rangle^N(u,v,\theta)\|_{L^2} \le C(U(t) + Q(U(t))). \tag{33}$$

Proof. Firstly, applying $\langle\nabla\rangle^N$ to (1), we obtain

$$\begin{cases}\partial_t\langle\nabla\rangle^N u + \langle\nabla\rangle^N(u\cdot\nabla u) - \langle\nabla\rangle^N\Delta u + \langle\nabla\rangle^N\nabla P = -\langle\nabla\rangle^N\nabla\cdot(v\otimes v),\\ \partial_t\langle\nabla\rangle^N v + \langle\nabla\rangle^N(u\cdot\nabla v) + \langle\nabla\rangle^N v + \langle\nabla\rangle^N(v\cdot\nabla u) = -\langle\nabla\rangle^N\nabla\theta,\\ \partial_t\langle\nabla\rangle^N\theta + \langle\nabla\rangle^N(u\cdot\nabla\theta) = -\langle\nabla\rangle^N \mathrm{div} v,\\ \langle\nabla\rangle^N\mathrm{div} u = 0,\end{cases} \tag{34}$$

then taking the inner product with $\langle\nabla\rangle^N u$, $\langle\nabla\rangle^N v$, $\langle\nabla\rangle^N\theta$, respectively, yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\langle\nabla\rangle^N u\|^2_{L^2} + \|\langle\nabla\rangle^N\nabla u\|^2_{L^2} &= -\int\langle\nabla\rangle^N(\nabla\cdot(v\otimes v))\cdot\langle\nabla\rangle^N u dx\\ &\quad-\int\langle\nabla\rangle^N(u\cdot\nabla u)\cdot\langle\nabla\rangle^N u dx - \int\langle\nabla\rangle^N\nabla P\cdot\langle\nabla\rangle^N u dx,\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\langle\nabla\rangle^N v\|^2_{L^2} + \|\langle\nabla\rangle^N v\|^2_{L^2} &= -\int\langle\nabla\rangle^N\nabla\theta\cdot\langle\nabla\rangle^N v dx\\ &\quad-\int\langle\nabla\rangle^N(u\cdot\nabla v)\cdot\langle\nabla\rangle^N v dx - \int\langle\nabla\rangle^N(v\cdot\nabla u)\cdot\langle\nabla\rangle^N v dx,\end{aligned} \tag{36}$$

$$\frac{1}{2}\frac{d}{dt}\|\langle\nabla\rangle^N\theta\|^2_{L^2} = -\int\langle\nabla\rangle^N(u\cdot\nabla\theta)\langle\nabla\rangle^N\theta dx - \int\langle\nabla\rangle^N\mathrm{div} v\langle\nabla\rangle^N\theta dx. \tag{37}$$

Applying the standard commutator notation

$$\langle\nabla\rangle^N(u\cdot\nabla v) = [\langle\nabla\rangle^N, u\cdot\nabla]v + u\cdot\nabla\langle\nabla\rangle^N v, \tag{38}$$

after integrating by parts and invoking $\nabla\cdot u=0$, we infer

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\langle\nabla\rangle^N(u,v,\theta)\|_{L^2}^2+\|\langle\nabla\rangle^N\nabla u\|_{L^2}^2+\|\langle\nabla\rangle^N v\|_{L^2}^2\\
=&-\int\langle\nabla\rangle^N\nabla\cdot(v\otimes v)\cdot\langle\nabla\rangle^N u dx-\int[\langle\nabla\rangle^N,u\cdot\nabla]u\cdot\langle\nabla\rangle^N u dx\\
&-\int[\langle\nabla\rangle^N,u\cdot\nabla]v\cdot\langle\nabla\rangle^N v dx-\int[\langle\nabla\rangle^N,v\cdot\nabla]u\cdot\langle\nabla\rangle^N v dx\\
&-\int v\cdot\nabla\langle\nabla\rangle^N u\cdot\langle\nabla\rangle^N v dx-\int[\langle\nabla\rangle^N,u\cdot\nabla]\theta\langle\nabla\rangle^N\theta dx\\
=&(1)+(2)+(3)+(4)+(5)+(6),
\end{aligned}\tag{39}$$

where one has used the fact

$$\int\langle\nabla\rangle^N\nabla\theta\cdot\langle\nabla\rangle^N v dx=-\int\langle\nabla\rangle^N \mathrm{div} v\langle\nabla\rangle^N\theta dx.\tag{40}$$

Then by Hölder inequality, Young inequality, and Lemma 4, we deduce

$$\begin{aligned}
(1)+(5)\leq&C\|v\|_{L^\infty}\|\langle\nabla\rangle^N\nabla u\|_{L^2}\|\langle\nabla\rangle^N v\|_{L^2}\\
\leq&C\|v\|_{L^\infty}^2\|\langle\nabla\rangle^N v\|_{L^2}^2+\frac{1}{2}\|\langle\nabla\rangle^N\nabla u\|_{L^2}^2,
\end{aligned}$$

and

$$(2)\leq\|[\langle\nabla\rangle^N,u\cdot\nabla]u\|_{L^2}\|\langle\nabla\rangle^N u\|_{L^2}\leq C\|\nabla u\|_{L^\infty}\|\langle\nabla\rangle^N u\|_{L^2}^2,$$

similarly, there holds

$$\begin{aligned}
(3)\leq&\|[\langle\nabla\rangle^N,u\cdot\nabla]v\|_{L^2}\|\langle\nabla\rangle^N v\|_{L^2}\\
\leq&C\|\nabla v\|_{L^\infty}\|\langle\nabla\rangle^N u\|_{L^2}\|\langle\nabla\rangle^N v\|_{L^2}+C\|\nabla u\|_{L^\infty}\|\langle\nabla\rangle^N v\|_{L^2}^2\\
\leq&C(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty})(\|\langle\nabla\rangle^N u\|_{L^2}^2+\|\langle\nabla\rangle^N v^2\|_{L^2}),
\end{aligned}$$

we also get

$$\begin{aligned}
(4)\leq&\|[\langle\nabla\rangle^N,v\cdot\nabla]u\|_{L^2}\|\langle\nabla\rangle^N v\|_{L^2}\\
\leq&C\|\nabla v\|_{L^\infty}\|\langle\nabla\rangle^N u\|_{L^2}\|\langle\nabla\rangle^N v\|_{L^2}+C\|\nabla u\|_{L^\infty}\|\langle\nabla\rangle^N v\|_{L^2}^2\\
\leq&C(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty})(\|\langle\nabla\rangle^N u\|_{L^2}^2+\|\langle\nabla\rangle^N v^2\|_{L^2}),
\end{aligned}$$

and

$$\begin{aligned}
(6)\leq&\|[\langle\nabla\rangle^N,u\cdot\nabla]\theta\|_{L^2}\|\langle\nabla\rangle^N\theta\|_{L^2}\\
\leq&C(\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})(\|\langle\nabla\rangle^N u\|_{L^2}^2+\|\langle\nabla\rangle^N\theta\|_{L^2}^2).
\end{aligned}$$

Substituting the above estimates of (1)-(6) into (39), we have

$$\begin{aligned}
&\frac{d}{dt}(\|\langle\nabla\rangle^N(u,v,\theta)\|_{L^2}^2+\|\langle\nabla\rangle^N\nabla u\|_{L^2}^2+\|\langle\nabla\rangle^N v\|_{L^2}^2\\
\leq&C\|\langle\nabla\rangle^N(u,v,\theta)\|_{L^2}^2(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty}+\|\nabla\theta\|_{L^\infty}+\|v\|_{L^\infty}^2).
\end{aligned}\tag{41}$$

It follows from (41) that

$$\begin{aligned}
\|\langle\nabla\rangle^N(u,v,\theta)\|_{L^2}\leq&\|\langle\nabla\rangle^N(u_0,v_0,\theta_0)\|_{L^2}+C\int_0^t(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty}+\|\nabla\theta\|_{L^\infty}\\
&+\|v\|_{L^\infty}^2)\|\langle\nabla\rangle^N(u,v,\theta)\|_{L^2}ds\\
\leq&\|\langle\nabla\rangle^N(u_0,v_0,\theta_0)\|_{L^2}+C\int_0^t\langle s\rangle^{-\frac{9}{8}}ds(U(t)^2+U(t)^3)\\
\leq&U(0)+C(U(t)^2+U(t)^3),
\end{aligned}\tag{42}$$

Therefore, we obtain

$$\|\langle\nabla\rangle^N(u,v,\theta)\|_{L^2}\leq C(U(0)+Q(U(t))).\tag{43}$$

□

Lemma 7. For any $t>0$, there holds

$$\||\nabla|^{-1}(\partial_t\pi_1+\pi_1)\|_{L^1}\leq C\langle t\rangle^{-\frac{3}{2}}U(t)^2.\tag{44}$$

Proof. The expression of π_1 directly gives

$$
\begin{aligned}
\||\nabla|^{-1}\pi_1\|_{L^2} \leq & \||\nabla|^{-1}\nabla(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u)\|_{L^2}+\||\nabla|^{-1}(\nabla\cdot(v\otimes v))\|_{L^2}\\
&+\||\nabla|^{-1}\nabla(-\Delta)^{-1}\nabla\cdot(\nabla\cdot(v\otimes v))\|_{L^2}+\||\nabla|^{-1}(u\cdot\nabla u)\|_{L^2}\\
\leq & C(\||\nabla|^{-1}\nabla\cdot(u\otimes u)\|_{L^2}+\||\nabla|^{-1}\nabla\cdot(v\otimes v)\|_{L^2})\\
\leq & C(\|u_iu_j\|_{L^2}+\|v_iv_j\|_{L^2})\\
\leq & C(\|u\|_{L^4}^2+\|v\|_{L^4}^2)\\
\leq & C(\|u\|_{L^2}\|\nabla u\|_{L^2}+\|v\|_{L^2}\|\nabla v\|_{L^2})\\
\leq & C\langle t\rangle^{-\frac{3}{2}}U(t)^2.
\end{aligned}
\tag{45}
$$

Similarly, one has

$$
\begin{aligned}
\||\nabla|^{-1}\partial_t\pi_1\|_{L^2} \leq & \||\nabla|^{-1}\partial_t\nabla(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u)\|_{L^2}+\||\nabla|^{-1}\partial_t(\nabla\cdot(v\otimes v))\|_{L^2}\\
&+\||\nabla|^{-1}\partial_t\nabla(-\Delta)^{-1}\nabla\cdot(\nabla\cdot(v\otimes v))\|_{L^2}+\||\nabla|^{-1}\partial_t(u\cdot\nabla u)\|_{L^2}\\
\leq & C(\||\nabla|^{-1}\nabla\cdot\partial_t(u\otimes u)\|_{L^2}+\||\nabla|^{-1}\nabla\cdot\partial_t(v\otimes v)\|_{L^2})\\
\leq & C(\|\partial_t(u_iu_j)\|_{L^2}+\|\partial_t(v_iv_j)\|_{L^2})\\
\leq & C(\|u\|_{L^\infty}\|\partial_t u\|_{L^2}+\|v\|_{L^\infty}\|\partial_t v\|_{L^2})\\
\leq & C\langle t\rangle^{-\frac{19}{8}}U(t)^2.
\end{aligned}
\tag{46}
$$

Thus, it holds

$$\||\nabla|^{-1}(\partial_t\pi_1+\pi_1)\|_{L^1}\leq C\langle t\rangle^{-\frac{3}{2}}U(t)^2. \tag{47}$$

□

Lemma 8. For any $t>0$, there holds

$$\|u\|_{L^2}\leq C\langle t\rangle^{-\frac{1}{2}}(U(0)+U(t)^2). \tag{48}$$

Proof. Firstly, we will split $\|u\|_{L^2}$ into the following two parts

$$\|\hat{u}\|_{L^2}=\|\hat{u}\|_{L^2_{(|\xi|\leq\frac{1}{2})}}+\|\hat{u}\|_{L^2_{(|\xi|>\frac{1}{2})}}. \tag{49}$$

For case $|\xi|\leq\frac{1}{2}$, by integral representation of solutions (19), there holds

$$\|u\|_{L^2}\leq\|\omega_0u_0\|_{L^2}+\|\omega_1(\frac{1}{2}u_0+u_1)\|_{L^2}+\int_0^t\|\omega_1(t-\tau)F_1\|_{L^2}d\tau, \tag{50}$$

then it follows by Lemma $1-2$ that

$$\|\hat{\omega}_0\hat{u}_0\|_{L^2}\leq\|\hat{\omega}_0\|_{L^2}\|u_0\|_{L^1}\leq C\langle t\rangle^{-\frac{1}{2}}U(0), \tag{51}$$

and

$$\|\hat{\omega}_1(\frac{1}{2}\hat{u}_0+\hat{u}_1)\|_{L^2}\leq\|\hat{\omega}_1\|_{L^2}\|(\frac{1}{2}u_0+u_1)\|_{L^1}\leq C\langle t\rangle^{-\frac{1}{2}}U(0). \tag{52}$$

Moreover, we obtain

$$
\begin{aligned}
&\int_0^t \|\omega_1(t-\tau)F_1\|_{L^2} d\tau \\
\leq &\int_0^t \|\omega_1(t-\tau)(F_1-\partial_t\Delta u)\|_{L^2} d\tau + \int_0^t \|\omega_1(t-\tau)\partial_t\Delta u\|_{L^2} d\tau \\
\leq &C\int_0^t \||\xi|\widehat{\omega}_1(t-\tau)\|_{L^\infty} \||\nabla|^{-1}(F_1-\partial_t\Delta u)\|_{L^2} d\tau \\
&+ C\int_0^t \|\widehat{\Delta\omega}_1(t-\tau)\|_{L^\infty}\|\partial_t u\|_{L^2} d\tau \\
\leq &C\int_0^t \langle t-\tau\rangle^{-\frac{1}{2}} \cdot \langle\tau\rangle^{-\frac{3}{2}} d\tau U(t)^2 \\
&+ C\int_0^t \langle t-\tau\rangle^{-1} \cdot \langle\tau\rangle^{-\frac{3}{2}} d\tau (U(0)+U(t)^2)) \\
\leq &C\langle t\rangle^{-\frac{1}{2}}(U(0)+U(t)^2),
\end{aligned}
\tag{53}
$$

where by (54) derived from the expression of F_1 and Lemma 7, one has used

$$
\||\nabla|^{-1}(F_1-\partial_t\Delta u)\|_{L^2} = \||\nabla|^{-1}(\partial_t\pi_1+\pi_1)\|_{L^2} \leq C\langle t\rangle^{-\frac{3}{2}}U(t)^2. \tag{54}
$$

Hence, we have

$$
\|u\|_{L^2_{(|\xi|\leq\frac{1}{2})}} \leq C\langle t\rangle^{-\frac{1}{2}}(U(0)+U(t)^2). \tag{55}
$$

For case $|\xi| > \frac{1}{2}$, taking the Fourier transform of the $(1)_1$ yields

$$
\partial_t\hat{u} + |\xi|^2\hat{u} = -\widehat{u\cdot\nabla u} - \mathrm{div}\widehat{(v\otimes v)} - \widehat{\nabla P} = G(\xi,t), \tag{56}
$$

multiplying (56) by the integrating factor $e^{|\xi|^2 t}$ and integrating in time gives

$$
\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0 + \int_0^t e^{-|\xi|^2(t-\tau)}G(\xi,\tau)d\tau. \tag{57}
$$

By Plancherel theorem and Hölder inequality and space X, we get

$$
\begin{aligned}
\|\hat{u}\|_{L^2} \le& \|e^{-|\xi|^2 t}\hat{u}_0\|_{L^2} + \int_0^t \|e^{-|\xi|^2(t-\tau)}G\|_{L^2} d\tau \\
\le& e^{-\frac{1}{4}t}\|\hat{u}_0\|_{L^2} + \int_0^t e^{-\frac{1}{4}(t-\tau)}\|G\|_{L^2} d\tau \\
\le& e^{-\frac{1}{4}t}\|u_0\|_{L^2} + \int_0^t e^{-\frac{1}{4}(t-\tau)}(\|u\cdot\nabla u\|_{L^2} + \|\nabla\cdot(v\otimes v)\|_{L^2}) d\tau \\
\le& e^{-\frac{1}{4}t}\|u_0\|_{L^2} + \int_0^t e^{-\frac{1}{4}(t-\tau)}(\|u\|_{L^\infty}\|\nabla u\|_{L^2} + \|v\|_{L^\infty}\|\nabla v\|_{L^2}) d\tau \\
\le& e^{-\frac{1}{4}t}\|u_0\|_{L^2} + C\int_0^t e^{-\frac{1}{4}(t-\tau)}(\langle\tau\rangle^{-1}\cdot\langle\tau\rangle^{-1} \\
&+ \langle\tau\rangle^{-\frac{11}{8}}\cdot\langle\tau\rangle^{-\frac{3}{2}}) d\tau U(t)^2 \\
\le& e^{-\frac{1}{4}t}\|u_0\|_{L^2} + C\int_0^t e^{-\frac{1}{4}(t-\tau)}\cdot\langle\tau\rangle^{-2} d\tau U(t)^2 \\
\le& C\langle t\rangle^{-2}(U(0) + U(t)^2),
\end{aligned}
\tag{58}
$$

which gives rise to

$$
\|\hat{u}\|_{L^2_{(|\xi|>\frac{1}{2})}} \le C\langle t\rangle^{-2}(U(0) + U(t)^2). \tag{59}
$$

The combination of (55) and (59) yields

$$
\langle t\rangle^{\frac{1}{2}}\|u\|_{L^2} \le C(U(0) + U(t)^2). \tag{60}
$$

□

Lemma 9. For any $t > 0$, there holds

$$
\|\nabla u\|_{L^2} \le C\langle t\rangle^{-1}(U(0) + U(t)^2). \tag{61}
$$

Proof. Similarly, we split $\|\nabla u\|_{L^2}$ into the following two parts:

$$
\|\widehat{\nabla u}\|_{L^2} = \|\widehat{\nabla u}\|_{L^2_{(|\xi|\le\frac{1}{2})}} + \|\widehat{\nabla u}\|_{L^2_{(|\xi|>\frac{1}{2})}}. \tag{62}
$$

For case $|\xi| \le \frac{1}{2}$, by integral representation of solutions (19), there holds

$$
\|\nabla u\|_{L^2} \le \|\nabla\omega_0 u_0\|_{L^2} + \|\nabla\omega_1(\frac{1}{2}u_0 + u_1)\|_{L^2} + \int_0^t \|\nabla\omega_1(t-\tau)F_1\|_{L^2} d\tau, \tag{63}
$$

the right-hand side of (63) can be controlled as follows. We deduce by Lemma $1-2$ that

$$
\|\widehat{\nabla\omega_0}\widehat{u}_0\|_{L^2} \le \||\xi|\hat{\omega}_0\|_{L^2}\|u_0\|_{L^1} \le C\langle t\rangle^{-1}U(0), \tag{64}
$$

and

$$
\begin{aligned}
\|\widehat{\nabla\omega_1}(\frac{1}{2}\hat{u}_0 + \hat{u}_1)\|_{L^2} \le& \||\xi|\hat{\omega}_1\|_{L^2}\|(\frac{1}{2}u_0 + u_1)\|_{L^1} \\
\le& C\langle t\rangle^{-1}U(0).
\end{aligned}
\tag{65}
$$

Moreover, one has

$$
\begin{aligned}
&\int_0^t \|\nabla\omega_1(t-\tau)F_1\|_{L^2}d\tau \\
\le&\int_0^t \|\nabla\omega_1(t-\tau)(F_1-\partial_t\Delta u)\|_{L^2}d\tau+\int_0^t \|\nabla\omega_1(t-\tau)\partial_t\Delta u\|_{L^2}d\tau \\
\le& C\int_0^t \||\xi|^2\widehat{\omega}_1(t-\tau)\|_{L^\infty}\||\nabla|^{-1}(F_1-\partial_t\Delta u)\|_{L^2}d\tau \\
&+C\int_0^t \||\xi|^3\hat{\omega}_1(t-\tau)\|_{L^\infty}\|\partial_t u\|_{L^2}d\tau \\
\le& C\int_0^t \langle t-\tau\rangle^{-1}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau U(t)^2 \\
&+C\int_0^t \langle t-\tau\rangle^{-\frac{3}{2}}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau(U(0)+U(t)^2) \\
\le& C\langle t\rangle^{-1}(U(0)+U(t)^2).
\end{aligned}
\tag{66}
$$

Combining the above estimates together, we obtain

$$\|\nabla u\|_{L^2_{(|\xi|\le\frac{1}{2})}}\le C\langle t\rangle^{-1}(U(0)+U(t)^2). \tag{67}$$

For case $|\xi|>\frac{1}{2}$, similar to $\|\hat{u}\|_{L^2}$, we derive from (56), (57), Plancherel theorem, Hölder inequality, and space X that

$$
\begin{aligned}
\||\xi|\hat{u}\|_{L^2}&\le\|e^{-|\xi|^2t}|\xi|\hat{u}_0\|_{L^2}+\int_0^t \|e^{-|\xi|^2(t-\tau)}|\xi|G\|_{L^2}d\tau \\
&\le e^{-\frac{1}{8}t}\|e^{-\frac{1}{2}|\xi|^2t}|\xi|\hat{u}_0\|_{L^2}+\int_0^t e^{-\frac{1}{8}(t-\tau)}\|e^{-\frac{1}{2}|\xi|^2(t-\tau)}|\xi|G\|_{L^2}d\tau \\
&\le Ce^{-\frac{1}{8}t}\|u_0\|_{L^2}+C\int_0^t e^{-\frac{1}{8}(t-\tau)}(\|u\cdot\nabla u\|_{L^2}+\|\nabla\cdot(v\otimes v)\|_{L^2})d\tau \\
&\le Ce^{-\frac{1}{8}t}\|u_0\|_{L^2}+C\int_0^t e^{-\frac{1}{8}(t-\tau)}(\|u\|_{L^\infty}\|\nabla u\|_{L^2}+\|v\|_{L^\infty}\|\nabla v\|_{L^2})d\tau \\
&\le Ce^{-\frac{1}{8}t}\|u_0\|_{L^2}+C\int_0^t e^{-\frac{1}{8}(t-\tau)}\cdot\langle\tau\rangle^{-2}d\tau U(t)^2 \\
&\le C\langle t\rangle^{-2}(U(0)+U(t)^2),
\end{aligned}
\tag{68}
$$

hence

$$\|\widehat{\nabla u}\|_{L^2_{(|\xi|>\frac{1}{2})}}\le C\langle t\rangle^{-2}(U(0)+U(t)^2). \tag{69}$$

The combination of (67) and (69) yields

$$\langle t\rangle\|\nabla u\|_{L^2}\le C(U(0)+U(t)^2). \tag{70}$$

□

Lemma 10. For any $t>0$, there holds

$$\|\Delta u\|_{L^2}\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{71}$$

Proof. Similarly, we split $\|\Delta u\|_{L^2}$ into the following two parts:

$$\|\widehat{\Delta u}\|_{L^2} = \|\widehat{\Delta u}\|_{L^2_{(|\xi|\le\frac{1}{2})}} + \|\widehat{\Delta u}\|_{L^2_{(|\xi|>\frac{1}{2})}}. \tag{72}$$

For case $|\xi| \le \frac{1}{2}$, we obtain the estimate of $\|\Delta u\|_{L^2}$ as

$$\|\Delta u\|_{L^2} \le \|\Delta\omega_0 u_0\|_{L^2} + \|\Delta\omega_1(\frac{1}{2}u_0 + u_1)\|_{L^2} + \int_0^t \|\Delta\omega_1(t-\tau)F_1\|_{L^2} d\tau, \tag{73}$$

then it follows by Lemma $1-2$ that

$$\|\widehat{\Delta\omega_0}\hat{u}_0\|_{L^2} \le \||\xi|^2\hat{\omega}_0\|_{L^2}\|u_0\|_{L^1} \le C\langle t\rangle^{-\frac{3}{2}}U(0), \tag{74}$$

and

$$\begin{aligned}\|\widehat{\Delta\omega_1}(\frac{1}{2}\hat{u}_0 + \hat{u}_1)\|_{L^2} &\le \||\xi|^2\hat{\omega}_1\|_{L^2}\|(\frac{1}{2}u_0 + u_1)\|_{L^1}\\ &\le C\langle t\rangle^{-\frac{3}{2}}U(0).\end{aligned} \tag{75}$$

Moreover, one has

$$\begin{aligned}&\int_0^t \|\Delta\omega_1(t-\tau)F_1\|_{L^2} d\tau\\ \le&\int_0^t \|\Delta\omega_1(t-\tau)(F_1-\partial_t\Delta u)\|_{L^2}d\tau + \int_0^t \|\Delta\omega_1(t-\tau)\partial_t\Delta u\|_{L^2}d\tau\\ \le& C\int_0^t \||\xi|^3\hat{\omega}_1(t-\tau)\|_{L^\infty}\||\nabla|^{-1}(F_1-\partial_t\Delta u)\|_{L^2}d\tau\\ &+C\int_0^t \||\xi|^4\hat{\omega}_1(t-\tau)\|_{L^\infty}\|\partial_t u\|_{L^2}d\tau\\ \le& C\int_0^t \langle t-\tau\rangle^{-\frac{3}{2}}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau U(t)^2\\ &+C\int_0^t \langle t-\tau\rangle^{-2}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau(U(0)+U(t)^2)\\ \le& C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2).\end{aligned} \tag{76}$$

Hence, we have

$$\|\widehat{\Delta u}\|_{L^2_{(|\xi|\le\frac{1}{2})}} \le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{77}$$

For case $|\xi| > \frac{1}{2}$, similar to $\|\nabla u\|_{L^2}$, we derive from (56), (57), Plancherel theorem, Hölder inequality, and space X that

$$
\begin{aligned}
\||\xi|^2\hat{u}\|_{L^2} &\le \|e^{-|\xi|^2 t}|\xi|^2\hat{u}_0\|_{L^2} + \int_0^t \|e^{-|\xi|^2(t-\tau)}|\xi|^2 G\|_{L^2} d\tau \\
&\le e^{-\frac{1}{8}t}\|e^{-\frac{1}{2}|\xi|^2 t}|\xi|^2\hat{u}_0\|_{L^2} + \int_0^t e^{-\frac{1}{8}(t-\tau)}\|e^{-\frac{1}{2}|\xi|^2(t-\tau)}|\xi|^2 G\|_{L^2} d\tau \\
&\le Ce^{-\frac{1}{8}t}\|u_0\|_{L^2} + C\int_0^t e^{-\frac{1}{8}(t-\tau)}(\|u\cdot\nabla u\|_{L^2} + \|\nabla\cdot(v\otimes v)\|_{L^2})d\tau \\
&\le Ce^{-\frac{1}{8}t}\|u_0\|_{L^2} + C\int_0^t e^{-\frac{1}{8}(t-\tau)}\cdot\langle\tau\rangle^{-2}d\tau U(t)^2 \\
&\le C\langle t\rangle^{-2}(U(0)+U(t)^2),
\end{aligned} \tag{78}
$$

from the above inequality, we deduce

$$
\|\widehat{\Delta u}\|_{L^2_{(|\xi|>\frac{1}{2})}} \le C\langle t\rangle^{-2}(U(0)+U(t)^2). \tag{79}
$$

The combination of (77) and (79) gives

$$
\langle t\rangle^{\frac{3}{2}}\|\Delta u\|_{L^2} \le C(U(0)+U(t)^2). \tag{80}
$$

□

Lemma 11. For any $t > 0$, there holds

$$
\|\nabla\Delta u\|_{L^2} \le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{81}
$$

Proof. Firstly, we split $\|\nabla\Delta u\|_{L^2}$ into the following two parts:

$$
\|\widehat{\nabla\Delta u}\|_{L^2} = \|\widehat{\nabla\Delta u}\|_{L^2_{(|\xi|\le\frac{1}{2})}} + \|\widehat{\nabla\Delta u}\|_{L^2_{(|\xi|>\frac{1}{2})}}. \tag{82}
$$

For case $|\xi| \le \frac{1}{2}$, we obtain the estimate of $\|\nabla\Delta u\|_{L^2}$ as

$$
\|\nabla\Delta u\|_{L^2} \le \|\nabla\Delta\omega_0 u_0\|_{L^2} + \|\nabla\Delta\omega_1(\frac{1}{2}u_0+u_1)\|_{L^2} + \int_0^t \|\nabla\Delta\omega_1(t-\tau)F_1\|_{L^2}d\tau, \tag{83}
$$

then we derive by Lemma $1-2$ that

$$
\|\widehat{\nabla\Delta\omega_0}\hat{u}_0\|_{L^2} \le \||\xi|^3\hat{\omega}_0\|_{L^2}\|u_0\|_{L^1} \le C\langle t\rangle^{-2}U(0), \tag{84}
$$

and

$$
\begin{aligned}
\|\widehat{\nabla\Delta\omega_1}(\frac{1}{2}\hat{u}_0+\hat{u}_1)\|_{L^2} &\le \||\xi|^3\hat{\omega}_1\|_{L^2}\|(\frac{1}{2}u_0+u_1)\|_{L^1} \\
&\le C\langle t\rangle^{-2}U(0).
\end{aligned} \tag{85}
$$

Moreover, there holds

$$
\begin{aligned}
&\int_0^t \|\nabla\Delta\omega_1(t-\tau)F_1\|_{L^2}d\tau\\
\le&\int_0^t \|\nabla\Delta\omega_1(t-\tau)(F_1-\partial_t\Delta u)\|_{L^2}d\tau+\int_0^t \|\nabla\Delta\omega_1(t-\tau)\partial_t\Delta u\|_{L^2}d\tau\\
\le&C\int_0^t \||\xi|^4\widehat{\omega}_1(t-\tau)\|_{L^\infty}\||\nabla|^{-1}(F_1-\partial_t\Delta u)\|_{L^2}d\tau\\
&+C\int_0^t \||\xi|^5\widehat{\omega}_1(t-\tau)\|_{L^\infty}\|\partial_t u\|_{L^2}d\tau\\
\le&C\int_0^t \langle t-\tau\rangle^{-2}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau U(t)^2\\
&+C\int_0^t \langle t-\tau\rangle^{-\frac{5}{2}}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau(U(0)+U(t)^2)\\
\le&C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2).
\end{aligned}
\tag{86}
$$

Combining the above estimates together, we have

$$
\|\widehat{\nabla\Delta u}\|_{L^2_{(|\xi|\le\frac{1}{2})}}\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{87}
$$

For case $|\xi|>\frac{1}{2}$, similar to $\|\nabla u\|_{L^2}$, we derive from (56), (57), Plancherel theorem, Hölder inequality, and space X that

$$
\begin{aligned}
\||\xi|^3\hat{u}\|_{L^2}\le&\|e^{-|\xi|^2t}|\xi|^3\hat{u}_0\|_{L^2}+\int_0^t \|e^{-|\xi|^2(t-\tau)}|\xi|^3G\|_{L^2}d\tau\\
\le&e^{-\frac{1}{8}t}\|e^{-\frac{1}{2}|\xi|^2t}|\xi|^3\hat{u}_0\|_{L^2}+\int_0^t e^{-\frac{1}{8}(t-\tau)}\|e^{-\frac{1}{2}|\xi|^2(t-\tau)}|\xi|^3G\|_{L^2}d\tau\\
\le&Ce^{-\frac{1}{8}t}\|u_0\|_{L^2}+C\int_0^t e^{-\frac{1}{8}(t-\tau)}(\|u\cdot\nabla u\|_{L^2}+\|\nabla\cdot(v\otimes v)\|_{L^2})d\tau\\
\le&Ce^{-\frac{1}{8}t}\|u_0\|_{L^2}+C\int_0^t e^{-\frac{1}{8}(t-\tau)}\cdot\langle\tau\rangle^{-2}d\tau U(t)^2\\
\le&C\langle t\rangle^{-2}(U(0)+U(t)^2),
\end{aligned}
\tag{88}
$$

which gives rise to

$$
\|\widehat{\nabla\Delta u}\|_{L^2_{(|\xi|>\frac{1}{2})}}\le C\langle t\rangle^{-2}(U(0)+U(t)^2). \tag{89}
$$

Combining (87) and (89), we obtain

$$
\langle t\rangle^{\frac{3}{2}}\|\nabla\Delta u\|_{L^2}\le C(U(0)+U(t)^2). \tag{90}
$$

□

Lemma 12. For any $t>0$, there holds

$$
\|\partial_t u\|_{L^2}\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{91}
$$

Proof. We deduce from $(1)_1$ that

$$\begin{aligned}\|\partial_t u\|_{L^2} &\le \|u\cdot\nabla u\|_{L^2} + \|\Delta u\|_{L^2} + \|\nabla\cdot(v\otimes v)\|_{L^2} + \|\nabla P\|_{L^2}\\ &\le C(\|u\|_{L^\infty}\|\nabla u\|_{L^2} + \|\Delta u\|_{L^2} + \|v\|_{L^\infty}\|\nabla v\|_{L^2})\\ &\le C(\langle t\rangle^{-1}\cdot\langle t\rangle^{-1} + \langle t\rangle^{-\frac{3}{2}} + \langle t\rangle^{-\frac{11}{8}}\cdot\langle t\rangle^{-\frac{3}{2}})U(t)^2\\ &\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2).\end{aligned} \tag{92}$$

□

Lemma 13. For any $t>0$, there holds

$$\begin{aligned}\|u\|_{L^\infty} &\le C\langle t\rangle^{-1}U(t),\\ \|\nabla u\|_{L^\infty} &\le C\langle t\rangle^{-\frac{11}{8}}U(t).\end{aligned} \tag{93}$$

Proof. By lemma 5 and interpolation inequality, we infer

$$\begin{aligned}\|u\|_{L^\infty} &\le C\|u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\\ &\le C\langle t\rangle^{-\frac{1}{4}}\cdot\langle t\rangle^{-\frac{3}{4}}U(t)\\ &\le C\langle t\rangle^{-1}U(t),\end{aligned} \tag{94}$$

and

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta u\|_{L^2}^{\frac{1}{4}}\\ &\le C\langle t\rangle^{-\frac{1}{4}}\cdot\langle t\rangle^{-\frac{3}{4}}\cdot\langle t\rangle^{\frac{3}{8}}U(t)\\ &\le C\langle t\rangle^{-\frac{11}{8}}U(t).\end{aligned} \tag{95}$$

□

3.2 | Estimates of v

In this part, we establish some estimates on v. Importantly, we take advantage of the damping term of $(1)_2$. It provides the decay property of exponential operator to get the decay rates for various sobolev norms of solutions.

Lemma 14. For any $t>0$, it holds

$$\|v\|_{L^2} \le C\langle t\rangle^{-1}(U(0)+U(t)^2). \tag{96}$$

Proof. By $(1)_2$,

$$\partial_t v + u\cdot\nabla v + v\cdot\nabla u + v = -\nabla\theta, \tag{97}$$

multiplying (97) by v and integrating over $\mathbb{R}^2$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \|v\|_{L^2}^2\\ =&-\int\nabla\theta\cdot v\,dx - \int v\cdot\nabla u\cdot v\,dx\\ \le& C\|\nabla\theta\|_{L^2}\|v\|_{L^2} + C\|v\|_{L^\infty}\|\nabla u\|_{L^2}\|v\|_{L^2}\\ \le& C(\|\nabla\theta\|_{L^2}^2 + \|v\|_{L^\infty}^2\|\nabla u\|_{L^2}^2) + \frac{1}{2}\|v\|_{L^2}^2,\end{aligned} \tag{98}$$

then we derive from multiplying (98) by the integrating factor e^t and integrating in time that

$$\begin{aligned}\|v\|_{L^2}^2 &\le e^{-t}\|v_0\|_{L^2}^2 + C\int_0^t e^{-(t-\tau)}(\|\nabla\theta\|_{L^2}^2 + \|v\|_{L^\infty}^2\|\nabla u\|_{L^2}^2)d\tau\\ &\le e^{-t}\|v_0\|_{L^2}^2 + C\int_0^t e^{-(t-\tau)}(\langle\tau\rangle^{-2} + \langle\tau\rangle^{-\frac{11}{4}}\cdot\langle\tau\rangle^{-2})d\tau(U(0)^2+U(t)^4)\\ &\le C\langle t\rangle^{-2}(U(0)^2+U(t)^4).\end{aligned} \tag{99}$$

Therefore, it follows by (99) that

$$\langle t\rangle\|v\|_{L^2}\le C(U(0)+U(t)^2). \tag{100}$$

□

Lemma 15. For any $t>0$, it holds

$$\|\nabla v\|_{L^2}\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{101}$$

Proof. Similar to the proof of $\|v\|_{L^2}$, there holds

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v\|_{L^2}^2+\|\nabla v\|_{L^2}^2\\
=&-\int\nabla\theta\cdot\Delta v dx+\int u\cdot\nabla v\cdot\Delta v dx+\int v\cdot\nabla u\cdot\Delta v dx\\
\le& C(\|\Delta\theta\|_{L^2}\|\nabla v\|_{L^2}+\|\nabla v\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}+\|v\|_{L^\infty}\|\Delta u\|_{L^2}\|\nabla v\|_{L^2})\\
\le& C(\|\Delta\theta\|_{L^2}^2+\|\nabla v\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+\|v\|_{L^\infty}^2\|\Delta u\|_{L^2}^2)+\frac{1}{2}\|\nabla v\|_{L^2}^2,
\end{aligned} \tag{102}$$

then applying Gronwall's inequality gives

$$\begin{aligned}
\|\nabla v\|_{L^2}^2\le& e^{-t}\|\nabla v_0\|_{L^2}^2+C\int_0^t e^{-(t-\tau)}(\|\Delta\theta\|_{L^2}^2+\|\nabla v\|_{L^\infty}^2\|\nabla u\|_{L^2}^2\\
&+\|v\|_{L^\infty}^2\|\Delta u\|_{L^2}^2)d\tau\\
\le& e^{-t}\|\nabla v_0\|_{L^2}^2+C\int_0^t e^{-(t-\tau)}(\langle\tau\rangle^{-3}+\langle\tau\rangle^{-2}\cdot\langle\tau\rangle^{-2}\\
&+\langle\tau\rangle^{-\frac{11}{4}}\cdot\langle\tau\rangle^{-3})d\tau(U(0)^2+U(t)^4)\\
\le& C\langle t\rangle^{-3}(U(0)^2+U(t)^4).
\end{aligned} \tag{103}$$

Hence, we have

$$\langle t\rangle^{\frac{3}{2}}\|\nabla v\|_{L^2}\le C(U(0)+U(t)^2). \tag{104}$$

□

Lemma 16. For any $t>0$, it holds

$$\|\Delta v\|_{L^2}\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{105}$$

Proof. First, by Holder's inequality and a standard commutator estimate (see,e.g.,[30]), there holds

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta v\|_{L^2}^2+\|\Delta v\|_{L^2}^2\\
=&-\int[\Delta,u\cdot\nabla]v\cdot\Delta v dx-\int[\Delta,v\cdot\nabla]u\cdot\Delta v dx\\
&-\int v\cdot\nabla\Delta u\cdot\Delta v dx-\int\Delta\nabla\theta\cdot\Delta v dx\\
\le& C(\|[\Delta,u\cdot\nabla]v\|_{L^2}\|\Delta v\|_{L^2}+\|[\Delta,v\cdot\nabla]u\|_{L^2}\|\Delta v\|_{L^2}\\
&+\|v\cdot\nabla\Delta u\|_{L^2}\|\Delta v\|_{L^2}+\|\Delta\nabla\theta\|_{L^2}\|\Delta v\|_{L^2}\\
\le& C(\|\Delta u\|_{L^2}^2\|\nabla v\|_{L^\infty}^2+\|\nabla u\|_{L^\infty}^2\|\Delta v\|_{L^2}^2\\
&+\|\nabla\Delta u\|_{L^2}^2\|v\|_{L^\infty}^2+\|\Delta\nabla\theta\|_{L^2}^2)+\frac{1}{2}\|\Delta v\|_{L^2}^2,
\end{aligned} \tag{106}$$

then we deduce from applying the Gronwall's inequality that

$$\begin{aligned}
\|\Delta v\|_{L^2}^2 \le & e^{-t}\|\Delta v_0\|_{L^2}^2 + C\int_0^t e^{-(t-\tau)}(\|\Delta\nabla\theta\|_{L^2}^2 + \|\nabla v\|_{L^\infty}^2\|\Delta u\|_{L^2}^2 \\
& + \|\Delta v\|_{L^2}^2\|\nabla u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2\|\nabla\Delta u\|_{L^2}^2)d\tau \\
\le & e^{-t}\|\Delta v_0\|_{L^2}^2 + C\int_0^t e^{-(t-\tau)}(\langle\tau\rangle^{-3} + \langle\tau\rangle^{-5} \\
& + \langle\tau\rangle^{-3}\cdot\langle\tau\rangle^{-\frac{11}{4}})d\tau(U(0)^2 + U(t)^4) \\
\le & C\langle t\rangle^{-3}(U(0)^2 + U(t)^4),
\end{aligned} \tag{107}$$

Hence, we have

$$\langle t\rangle^{\frac{3}{2}}\|\Delta v\|_{L^2} \le C(U(0) + U(t)^2). \tag{108}$$

□

Lemma 17. For any $t > 0$, it holds

$$\|\partial_t v\|_{L^2} \le C\langle t\rangle^{-1}(U(0) + U(t)^2). \tag{109}$$

Proof. . Using $(1)_2$, we have the following estimate:

$$\begin{aligned}
\|\partial_t v\|_{L^2} & \le \|v\|_{L^2} + \|u\cdot\nabla v\|_{L^2} + \|v\cdot\nabla u\|_{L^2} + \|\nabla\theta\|_{L^2} \\
& \le \|v\|_{L^2} + \|u\|_{L^2}\|\nabla v\|_{L^\infty} + \|v\|_{L^2}\|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^2} \\
& \le C(\langle t\rangle^{-1} + \langle t\rangle^{-\frac{3}{2}} + \langle t\rangle^{-\frac{19}{8}})(U(0) + U(t)^2) \\
& \le C\langle t\rangle^{-1}(U(0) + U(t)^2).
\end{aligned} \tag{110}$$

□

Lemma 18. For any $t > 0$, it holds

$$\|\mathrm{div}\partial_t v\|_{L^2} \le C\langle t\rangle^{-\frac{3}{2}}(U(0) + U(t)^2). \tag{111}$$

Proof. Similarly, by $(1)_2$, we obtain

$$\begin{aligned}
\|\mathrm{div}\partial_t v\|_{L^2} \le & \|\nabla\cdot v\|_{L^2} + \|\nabla\cdot(u\cdot\nabla v)\|_{L^2} + \|\nabla\cdot(v\cdot\nabla u)\|_{L^2} + \|\Delta\theta\|_{L^2} \\
\le & \|\nabla v\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla v\|_{L^2} + \|u\|_{L^\infty}\|\Delta v\|_{L^2} \\
& + \|v\|_{L^\infty}\|\Delta u\|_{L^2} + \|\Delta\theta\|_{L^2} \\
\le & C(\langle t\rangle^{-\frac{3}{2}} + \langle t\rangle^{-\frac{23}{8}} + \langle t\rangle^{-\frac{5}{2}})(U(0) + U(t)^2) \\
\le & C\langle t\rangle^{-\frac{3}{2}}(U(0) + U(t)^2).
\end{aligned} \tag{112}$$

□

Lemma 19. For any $t > 0$, it holds

$$\begin{aligned}
\|v\|_{L^\infty} & \le C\langle t\rangle^{-\frac{11}{8}}U(t), \\
\|\nabla v\|_{L^\infty} & \le C\langle t\rangle^{-\frac{9}{8}}U(t).
\end{aligned} \tag{113}$$

Proof. We derive from Lemma 5 that

$$\begin{aligned}
\|v\|_{L^\infty} & \le C\|v\|_{L^2}^{\frac{1}{4}}\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\Delta v\|_{L^2}^{\frac{1}{4}} \\
& \le C\langle t\rangle^{-\frac{1}{4}}\cdot\langle t\rangle^{-\frac{3}{4}}\cdot\langle t\rangle^{-\frac{3}{8}}U(t) \\
& \le C\langle t\rangle^{-\frac{11}{8}}U(t),
\end{aligned} \tag{114}$$

and

$$
\begin{aligned}
\|\nabla v\|_{L^\infty} &\le C\|\nabla v\|_{L^2}^{\frac14}\|\Delta v\|_{L^2}^{\frac12}\|\nabla\Delta v\|_{L^2}^{\frac14}\\
&\le C\|\nabla v\|_{L^2}^{\frac14}\|\Delta v\|_{L^2}^{\frac34-\frac{1}{4(N-2)}}\|\langle\nabla\rangle^N v\|_{L^2}^{\frac{1}{4(N-2)}}\\
&\le C\langle t\rangle^{-\frac32(1-\frac{1}{4(N-2)})}U(t)\\
&\le C\langle t\rangle^{-\frac98}U(t),
\end{aligned}
\tag{115}
$$

for $N \ge 3$. □

3.3 | Estimates of θ

To overcome the difficulty by the absence of thermal dissipation, we exploit the structure of system (1) coming from the coupled terms and damping term. By taking div to the equation of v and then we obtain the thermal dissipation $-\Delta\theta$.

Lemma 20. For any $t > 0$, it holds

$$
\|F_2\|_{L^1} \le \langle t\rangle^{-\frac32}U(t)^2. \tag{116}
$$

Proof. Owing to the expression of F_2, we infer

$$
\begin{aligned}
\|F_2\|_{L^1} \le&\|\mathrm{div}(u\cdot\nabla v)\|_{L^1} + \|\mathrm{div}(v\cdot\nabla u)\|_{L^1} + \|\partial_t(u\cdot\nabla\theta)\|_{L^1} + \|u\cdot\nabla\theta\|_{L^1}\\
\le&\|u\|_{L^2}\|\nabla\theta\|_{L^2} + \|u_t\|_{L^2}\|\nabla\theta\|_{L^2} + \|u\|_{L^2}\|\nabla\theta_t\|_{L^2}\\
&+ \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \|u\|_{L^2}\|\Delta v\|_{L^2} + \|v\|_{L^2}\|\Delta u\|_{L^2}\\
\le&C(\langle t\rangle^{-\frac12}\cdot\langle t\rangle^{-1} + \langle t\rangle^{-\frac32}\cdot\langle t\rangle^{-1} + \langle t\rangle^{-\frac12}\cdot\langle t\rangle^{-\frac32})U(t)^2\\
\le&C\langle t\rangle^{-\frac32}U(t)^2.
\end{aligned}
\tag{117}
$$

□

Lemma 21. For any $t > 0$, it holds

$$
\|\theta\|_{L^2} \le C\langle t\rangle^{-\frac12}(U(0) + U(t)^2). \tag{118}
$$

Proof. For case $|\xi| \le \frac12$, by integral representation of solutions (19), we get

$$
\|\theta\|_{L^2} \le \|\omega_0\theta_0\|_{L^2} + \|\omega_1(\frac12\theta_0 + \theta_1)\|_{L^2} + \int_0^t \|\omega_1(t-\tau)F_2\|_{L^2}d\tau. \tag{119}
$$

then we derive from Lemma 1 and Lemma 2 that

$$
\|\omega_0\theta_0\|_{L^2} \le \|\hat\omega_0\|_{L^2}\|\theta_0\|_{L^1} \le C\langle t\rangle^{-\frac12}U(0), \tag{120}
$$

and

$$
\|\omega_1(\frac12\theta_0 + \theta_1)\|_{L^2} \le C\langle t\rangle^{-\frac12}\|(\frac12\theta_0 + \theta_1)\|_{L^1} \le C\langle t\rangle^{-\frac12}U(0). \tag{121}
$$

Moreover, there holds

$$
\begin{aligned}
&\int_0^t \|\omega_1(t-\tau)F_2\|_{L^2}d\tau\\
\le&\int_0^t \|\hat\omega_1(t-\tau)\|_{L^2}\|F_2\|_{L^1}d\tau\\
\le&C\int_0^t \langle t-\tau\rangle^{-\frac12}\cdot\langle\tau\rangle^{-\frac32}d\tau U(t)^2\\
\le&C\langle t\rangle^{-\frac12}U(t)^2.
\end{aligned}
\tag{122}
$$

Inserting the above estimstes of (120)-(122) into (119), we obtain

$$\|\hat{\theta}\|_{L^2_{(|\xi|\le\frac{1}{2})}} \le C\langle t\rangle^{-\frac{1}{2}}(U(0)+U(t)^2). \tag{123}$$

For case $|\xi| > \frac{1}{2}$, combining $(1)_2$ with $(1)_3$, and taking div to the equation of v to obtain the thermal diffusion $-\Delta\theta$, and then adding it to the equation of θ, we deduce the following equation:

$$\partial_t\theta - \Delta\theta = -u\cdot\nabla\theta + \nabla\cdot(\partial_t v) + \nabla\cdot(u\cdot\nabla v) + \nabla\cdot(v\cdot\nabla u) = H. \tag{124}$$

Then taking the Fourier transform to (124) yields

$$\partial_t\hat{\theta} + |\xi|^2\hat{\theta} = \hat{H}, \tag{125}$$

multiplying (125) by the integrating factor $e^{|\xi|^2 t}$ and integrating in time, we have

$$\hat{\theta} = e^{-|\xi|^2 t}\hat{\theta}_0 + \int_0^t e^{-|\xi|^2(t-\tau)}\hat{H}\,d\tau. \tag{126}$$

Next, we deduce from Plancherel theorem and Hölder inequality and space X that

$$\begin{aligned}
\|\hat{\theta}\|_{L^2} \le& \|e^{-|\xi|^2 t}\hat{\theta}_0\|_{L^2} + \int_0^t \|e^{-|\xi|^2(t-\tau)}\hat{H}\|_{L^2}d\tau \\
\le& e^{-\frac{1}{4}t}\|\hat{\theta}_0\|_{L^2} + \int_0^t e^{-\frac{1}{4}(t-\tau)}\|\hat{H}\|_{L^2}d\tau \\
\le& e^{-\frac{1}{4}t}\|\theta_0\|_{L^2} + \int_0^t e^{-\frac{1}{4}(t-\tau)}(\|u\cdot\nabla\theta\|_{L^2} + \|\nabla\cdot\partial_t v\|_{L^2} \\
&+ \|\nabla\cdot(u\cdot\nabla v)\|_{L^2} + \|\nabla\cdot(v\cdot\nabla u)\|_{L^2})d\tau \\
\le& e^{-\frac{1}{4}t}\|\theta_0\|_{L^2} + \int_0^t e^{-\frac{1}{4}(t-\tau)}(\|u\|_{L^\infty}\|\nabla\theta\|_{L^2} + \|\nabla\cdot\partial_t v\|_{L^2} \\
&+ \|\nabla u\|_{L^\infty}\|\nabla v\|_{L^2} + \|u\|_{L^\infty}\|\Delta v\|_{L^2} + \|v\|_{L^\infty}\|\Delta u\|_{L^2})d\tau \\
\le& e^{-\frac{1}{4}t}\|\theta_0\|_{L^2} + C\int_0^t e^{-\frac{1}{4}(t-\tau)}(\langle\tau\rangle^{-1}\cdot\langle\tau\rangle^{-1} + \langle\tau\rangle^{-\frac{3}{2}} \\
&+ \langle\tau\rangle^{-\frac{11}{8}}\cdot\langle\tau\rangle^{-\frac{3}{2}} + \langle\tau\rangle^{-1}\cdot\langle\tau\rangle^{-\frac{3}{2}})d\tau(U(0)+U(t)^2) \\
\le& e^{-\frac{1}{4}t}\|\theta_0\|_{L^2} + C\int_0^t e^{-\frac{1}{4}(t-\tau)}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau(U(0)+U(t)^2) \\
\le& C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2),
\end{aligned} \tag{127}$$

hence, we get

$$\|\hat{\theta}\|_{L^2_{(|\xi|>\frac{1}{2})}} \le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{128}$$

The combination of (123) and (128) gives

$$\langle t\rangle^{\frac{1}{2}}\|\theta\|_{L^2} \le C(U(0)+U(t)^2). \tag{129}$$

□

Lemma 22. For any $t > 0$, it holds

$$\|\nabla\theta\|_{L^2} \le C\langle t\rangle^{-1}(U(0)+U(t)^2). \tag{130}$$

Proof. For case $|\xi| \leq \frac{1}{2}$, it follows by (19) that

$$\|\nabla\theta\|_{L^2} \leq \|\nabla\omega_0\theta_0\|_{L^2} + \|\nabla\omega_1(\frac{1}{2}\theta_0 + \theta_1)\|_{L^2} + \int_0^t \|\nabla\omega_1(t-\tau)F_2\|_{L^2}d\tau. \tag{131}$$

Similarly, we get

$$\|\nabla\omega_0\theta_0\|_{L^2} \leq \||\xi|\hat{\omega}_0\|_{L^2}\|\theta_0\|_{L^1} \leq C\langle t\rangle^{-1}U(0), \tag{132}$$

and

$$\|\nabla\omega_1(\frac{1}{2}\theta_0 + \theta_1)\|_{L^2} \leq C\langle t\rangle^{-1}U(0). \tag{133}$$

Moreover, there holds

$$\begin{aligned}
&\int_0^t \|\nabla\omega_1(t-\tau)F_2\|_{L^2}d\tau \\
\leq &\int_0^t \||\xi|\hat{\omega}_1(t-\tau)\|_{L^2}\|F_2\|_{L^1}d\tau \\
\leq &C\int_0^t \langle t-\tau\rangle^{-1}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau U(t)^2 \\
\leq &C\langle t\rangle^{-1}U(t)^2,
\end{aligned} \tag{134}$$

thus combining the above estimates together, we obtain

$$\|\widehat{\nabla\theta}\|_{L^2_{(|\xi|\leq\frac{1}{2})}} \leq C\langle t\rangle^{-1}(U(0) + U(t)^2). \tag{135}$$

For case $|\xi| > \frac{1}{2}$, similar to $\|\theta\|_{L^2}$, one has

$$\hat{\theta} = e^{-|\xi|^2t}\hat{\theta}_0 + \int_0^t e^{-|\xi|^2(t-\tau)}\hat{H}d\tau. \tag{136}$$

By Plancherel theorem and Hölder inequality and space X, we infer

$$\begin{aligned}
\||\xi|\hat{\theta}\|_{L^2} \leq &\|e^{-|\xi|^2t}|\xi|\hat{\theta}_0\|_{L^2} + \int_0^t \|e^{-|\xi|^2(t-\tau)}|\xi|\hat{H}\|_{L^2}d\tau \\
\leq &e^{-\frac{1}{8}t}\|e^{-\frac{1}{2}|\xi|^2t}|\xi|\hat{\theta}_0\|_{L^2} + \int_0^t e^{-\frac{1}{8}(t-\tau)}\|e^{-\frac{1}{2}|\xi|^2(t-\tau)}|\xi|\hat{H}\|_{L^2}d\tau \\
\leq &e^{-\frac{1}{8}t}\|\theta_0\|_{L^2} + \int_0^t e^{-\frac{1}{8}(t-\tau)}(\|u\cdot\nabla\theta\|_{L^2} + \|\nabla\cdot\partial_t v\|_{L^2} \\
&+ \|\nabla\cdot(u\cdot\nabla v)\|_{L^2} + \|\nabla\cdot(v\cdot\nabla u)\|_{L^2})d\tau \\
\leq &e^{-\frac{1}{8}t}\|\theta_0\|_{L^2} + \int_0^t e^{-\frac{1}{8}(t-\tau)}(\|u\|_{L^\infty}\|\nabla\theta\|_{L^2} + \|\nabla\cdot\partial_t v\|_{L^2} \\
&+ \|\nabla u\|_{L^\infty}\|\nabla v\|_{L^2} + \|u\|_{L^\infty}\|\Delta v\|_{L^2} + \|v\|_{L^\infty}\|\Delta u\|_{L^2})d\tau \\
\leq &e^{-\frac{1}{8}t}\|\theta_0\|_{L^2} + C\int_0^t e^{-\frac{1}{8}(t-\tau)}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau(U(0) + U(t)^2) \\
\leq &C\langle t\rangle^{-\frac{3}{2}}(U(0) + U(t)^2),
\end{aligned} \tag{137}$$

which gives rise to

$$\|\widehat{\nabla\theta}\|_{L^2_{(|\xi|>\frac{1}{2})}} \leq C\langle t\rangle^{-\frac{3}{2}}(U(0) + U(t)^2). \tag{138}$$

The combination of (135) and (138) yields

$$\langle t\rangle\|\nabla\theta\|_{L^2}\leq C(U(0)+U(t)^2). \tag{139}$$

□

Lemma 23. For any $t>0$, it holds

$$\|\Delta\theta\|_{L^2}\leq C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{140}$$

Proof. For case $|\xi|\leq\frac{1}{2}$, we derive from (19) that

$$\|\Delta\theta\|_{L^2}\leq\|\Delta\omega_0\theta_0\|_{L^2}+\|\Delta\omega_1(\frac{1}{2}\theta_0+\theta_1)\|_{L^2}+\int_0^t\|\Delta\omega_1(t-\tau)F_2\|_{L^2}d\tau, \tag{141}$$

then there hold

$$\|\Delta\omega_0\theta_0\|_{L^2}\leq\||\xi|^2\hat{\omega}_0\|_{L^2}\|\theta_0\|_{L^1}\leq C\langle t\rangle^{-\frac{3}{2}}U(0), \tag{142}$$

and

$$\|\Delta\omega_1(\frac{1}{2}\theta_0+\theta_1)\|_{L^2}\leq C\langle t\rangle^{-\frac{3}{2}}U(0). \tag{143}$$

Furthermore, one has

$$\begin{aligned}
&\int_0^t\|\Delta\omega_1(t-\tau)F_2\|_{L^2}d\tau\\
\leq&\int_0^t\||\xi|^2\hat{\omega}_1\|_{L^2}\|F_2\|_{L^1}d\tau\\
\leq&C\int_0^t\langle t-\tau\rangle^{-\frac{3}{2}}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau U(t)^2\\
\leq&C\langle t\rangle^{-\frac{3}{2}}U(t)^2.
\end{aligned} \tag{144}$$

Hence, we have

$$\|\widehat{\Delta\theta}\|_{L^2_{(|\xi|\leq\frac{1}{2})}}\leq C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{145}$$

For case $|\xi|>\frac{1}{2}$, similar to $\|\nabla\theta\|_{L^2}$, we derive from (136), Plancherel theorem, Hölder inequality, and space X that

$$\begin{aligned}
\||\xi|^2\hat{\theta}\|_{L^2}\leq&\|e^{-|\xi|^2t}|\xi|^2\hat{\theta}_0\|_{L^2}+\int_0^t\|e^{-|\xi|^2(t-\tau)}|\xi|^2\hat{H}\|_{L^2}d\tau\\
\leq&e^{-\frac{1}{8}t}\|e^{-\frac{1}{2}|\xi|^2t}|\xi|^2\hat{\theta}_0\|_{L^2}+\int_0^t e^{-\frac{1}{8}(t-\tau)}\|e^{-\frac{1}{2}|\xi|^2(t-\tau)}|\xi|^2\hat{H}\|_{L^2}d\tau\\
\leq&Ce^{-\frac{1}{8}t}\|\theta_0\|_{L^2}+C\int_0^t e^{-\frac{1}{8}(t-\tau)}(\|u\cdot\nabla\theta\|_{L^2}+\|\nabla\cdot\partial_t v\|_{L^2}\\
&+\|\nabla\cdot(u\cdot\nabla v)\|_{L^2}+\|\nabla\cdot(v\cdot\nabla u)\|_{L^2})d\tau\\
\leq&Ce^{-\frac{1}{8}t}\|\theta_0\|_{L^2}+C\int_0^t e^{-\frac{1}{8}(t-\tau)}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau(U(0)+U(t)^2)\\
\leq&C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2),
\end{aligned} \tag{146}$$

which gives rise to

$$\|\widehat{\Delta\theta}\|_{L^2_{(|\xi|>\frac{1}{2})}}\leq C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{147}$$

Thus, we derive from the combination of (145) and (147)

$$\langle t\rangle^{\frac{3}{2}}\|\Delta\theta\|_{L^2}\le C(U(0)+U(t)^2). \tag{148}$$

□

Lemma 24. For any $t>0$, it holds

$$\|\nabla\Delta\theta\|_{L^2}\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{149}$$

Proof. For case $|\xi|\le\frac{1}{2}$, we obtain the estimate of $\|\nabla\Delta\theta\|_{L^2}$ by (19):

$$\|\nabla\Delta\theta\|_{L^2}\le\|\nabla\Delta\omega_0\theta_0\|_{L^2}+\|\nabla\Delta\omega_1(\frac{1}{2}\theta_0+\theta_1)\|_{L^2}+\int_0^t\|\nabla\Delta\omega_1(t-\tau)F_2\|_{L^2}d\tau, \tag{150}$$

the right-hand side of (150) can be controlled as follows. We deduce by Lemma $1-2$ that

$$\|\nabla\Delta\omega_0\theta_0\|_{L^2}\le\||\xi|^3\hat{\omega}_0\|_{L^2}\|\theta_0\|_{L^1}\le C\langle t\rangle^{-2}U(0), \tag{151}$$

and

$$\|\nabla\Delta\omega_1(\frac{1}{2}\theta_0+\theta_1)\|_{L^2}\le C\langle t\rangle^{-2}U(0). \tag{152}$$

Furthermore, one has

$$\begin{aligned}
&\int_0^t\|\nabla\Delta\omega_1(t-\tau)F_2\|_{L^2}d\tau\\
\le&\int_0^t\||\xi|^3\hat{\omega}_1\|_{L^2}\|F_2\|_{L^1}d\tau\\
\le&C\int_0^t\langle t-\tau\rangle^{-2}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau U(t)^2\\
\le&C\langle t\rangle^{-\frac{3}{2}}U(t)^2,
\end{aligned} \tag{153}$$

Combining the above estimates together, we have

$$\|\widehat{\nabla\Delta\theta}\|_{L^2_{(|\xi|\le\frac{1}{2})}}\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{154}$$

For case $|\xi|>\frac{1}{2}$, similar to $\|\nabla\theta\|_{L^2}$, we derive from (136), Plancherel theorem, Hölder inequality, and space X that

$$\begin{aligned}
\||\xi|^3\hat{\theta}\|_{L^2}\le&\|e^{-|\xi|^2t}|\xi|^3\hat{\theta}_0\|_{L^2}+\int_0^t\|e^{-|\xi|^3(t-\tau)}|\xi|^3\hat{H}\|_{L^2}d\tau\\
\le&e^{-\frac{1}{8}t}\|e^{-\frac{1}{2}|\xi|^3t}|\xi|^2\hat{\theta}_0\|_{L^2}+\int_0^t e^{-\frac{1}{8}(t-\tau)}\|e^{-\frac{1}{2}|\xi|^2(t-\tau)}|\xi|^3\hat{H}\|_{L^2}d\tau\\
\le&Ce^{-\frac{1}{8}t}\|\theta_0\|_{L^2}+C\int_0^t e^{-\frac{1}{8}(t-\tau)}(\|u\cdot\nabla\theta\|_{L^2}+\|\nabla\cdot\partial_t v\|_{L^2}\\
&+\|\nabla\cdot(u\cdot\nabla v)\|_{L^2}+\|\nabla\cdot(v\cdot\nabla u)\|_{L^2})d\tau\\
\le&Ce^{-\frac{1}{8}t}\|\theta_0\|_{L^2}+C\int_0^t e^{-\frac{1}{8}(t-\tau)}\cdot\langle\tau\rangle^{-\frac{3}{2}}d\tau(U(0)+U(t)^2)\\
\le&C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2),
\end{aligned} \tag{155}$$

which gives rise to

$$\|\widehat{\nabla\Delta\theta}\|_{L^2_{(|\xi|>\frac{1}{2})}}\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{156}$$

The combination of (154) and (156) yields

$$\langle t\rangle^{\frac{3}{2}}\|\nabla\Delta\theta\|_{L^2}\le C(U(0)+U(t)^2). \tag{157}$$

□

Lemma 25. For any $t>0$, it holds

$$\|\partial_t\theta\|_{L^2}\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{158}$$

Proof. Due to $(1)_3$, one has

$$\begin{aligned}\|\partial_t\theta\|_{L^2}&\le\|(\nabla\cdot v)\|_{L^2}+\|u\cdot\nabla\theta\|_{L^2}\\&\le\|\nabla v\|_{L^2}+\|u\|_{L^\infty}\|\nabla\theta\|_{L^2}\\&\le C(\langle t\rangle^{-\frac{3}{2}}+\langle t\rangle^{-1}\cdot\langle t\rangle^{-1})(U(0)+U(t)^2)\\&\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2).\end{aligned} \tag{159}$$

□

Lemma 26. For any $t>0$, it holds

$$\|\nabla\theta_t\|_{L^2}\le C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2). \tag{160}$$

Proof. We derive from $(1)_3$ that

$$\begin{aligned}\|\nabla\theta_t\|_{L^2}\le&\|\nabla(\nabla\cdot v)\|_{L^2}+\|\nabla(u\cdot\nabla\theta)\|_{L^2}\\\le&\|\Delta v\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}+\|u\|_{L^\infty}\|\Delta\theta\|_{L^2}\\\le&C(\langle t\rangle^{-\frac{3}{2}}+\langle t\rangle^{-\frac{11}{8}}\cdot\langle t\rangle^{-1}+\langle t\rangle^{-1}\cdot\langle t\rangle^{-\frac{3}{2}})(U(0)+U(t)^2)\\\le&C\langle t\rangle^{-\frac{3}{2}}(U(0)+U(t)^2).\end{aligned} \tag{161}$$

□

Lemma 27. For any $t>0$, it holds

$$\|\theta\|_{L^\infty}\le C\langle t\rangle^{-1}U(t), \tag{162}$$

$$\|\nabla\theta\|_{L^\infty}\le C\langle t\rangle^{-\frac{11}{8}}U(t). \tag{163}$$

Proof. From Lemma 5, we deduce

$$\begin{aligned}\|\theta\|_{L^\infty}&\le C\|\theta\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}^{\frac{1}{2}}\\&\le C\langle t\rangle^{-\frac{1}{4}}\cdot\langle t\rangle^{-\frac{3}{4}}U(t)\\&\le C\langle t\rangle^{-1}U(t),\end{aligned} \tag{164}$$

and

$$\begin{aligned}\|\nabla\theta\|_{L^\infty}&\le C\|\nabla\theta\|_{L^2}^{\frac{1}{4}}\|\Delta\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta\theta\|_{L^2}^{\frac{1}{4}}\\&\le C\langle t\rangle^{-\frac{1}{4}}\cdot\langle t\rangle^{-\frac{3}{4}}\cdot\langle t\rangle^{\frac{3}{8}}U(t)\\&\le C\langle t\rangle^{-\frac{11}{8}}U(t).\end{aligned} \tag{165}$$

□

Remark 2. Combining Section 3.1, 3.2, and 3.3, for any $t>0$, there holds

$$U(t)\le C(U(0)+U(t)^2+U(t)^3). \tag{166}$$

An application of the bootstrapping argument would yield the global uniform bound that would guarantee the global existence and the stability of the solution. One example of this process can be found in[23].

To apply a bootstrapping argument, we take $U(0)\le\mu$, μ is a suitable small constant, e.g., $\mu=\frac{1}{16C^2}$. The argument starts with the ansatz that

$$U(t)\le h=\frac{1}{4C}, \tag{167}$$

then

$$\begin{aligned} U(t) &\le CU(0) + \frac{1}{4}U(t) + \frac{1}{16C}U(t) \\ &\le CU(0) + \frac{1}{2}U(t), \end{aligned} \tag{168}$$

and we get

$$U(t) \le 2CU(0) \le \frac{1}{8C} = \frac{1}{2}h. \tag{169}$$

The bootstrapping argument then concludes that, for all $t > 0$,

$$U(t) \le 2C\mu. \tag{170}$$

This completes the proof of Theorem 1.1.

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$\square$

APPENDIX

A .1

proof of lemma 2.3. Firstly, for any $\alpha, \beta \ge 0$ and $\alpha + \beta > 1$, we deduce

$$\begin{aligned} \int_0^t \langle t-\tau\rangle^{-\alpha}\langle\tau\rangle^{-\beta}d\tau &= \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\alpha}\langle\tau\rangle^{-\beta}d\tau + \int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\alpha}\langle\tau\rangle^{-\beta}d\tau \\ &:= \Gamma_1 + \Gamma_2, \end{aligned} \tag{A1}$$

then for case $\alpha, \beta \ne 1$, there holds

$$\begin{aligned} \Gamma_1 &\le \int_0^{\frac{t}{2}} (\frac{t}{2}+1)^{-\alpha}(\tau+1)^{-\beta}d\tau \\ &\le (\frac{t}{2}+1)^{-\alpha}\int_0^{\frac{t}{2}}(\tau+1)^{-\beta}d\tau \\ &\le (\frac{t}{2}+1)^{-\alpha}\frac{1}{1-\beta}(\tau+1)^{1-\beta}|_0^{\frac{t}{2}} \\ &\le \frac{1}{1-\beta}(\frac{t}{2}+1)^{1-\alpha-\beta} - \frac{1}{1-\beta}(\frac{t}{2}+1)^{-\alpha}, \end{aligned} \tag{A2}$$

which gives rise to

$$\Gamma_1 \le C\begin{cases} \langle t\rangle^{-\alpha-\beta+1}, & if \quad \beta < 1, \\ \langle t\rangle^{-\alpha}, & if \quad \beta > 1. \end{cases} \tag{A3}$$

And we obtain

$$
\begin{aligned}
\Gamma_2 &\le \int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\alpha}(\frac{t}{2}+1)^{-\beta}d\tau \\
&\le (\frac{t}{2}+1)^{-\beta}\int_{\frac{t}{2}}^{t}(t-\tau+1)^{-\alpha}d\tau \\
&\le (\frac{t}{2}+1)^{-\beta}\int_{0}^{\frac{t}{2}}(y+1)^{-\alpha}dy(y=t-\tau) \\
&\le (\frac{t}{2}+1)^{-\beta}\frac{1}{1-\alpha}(y+1)^{1-\alpha}|_0^{\frac{t}{2}} \\
&\le (\frac{t}{2}+1)^{-\beta}[\frac{1}{1-\alpha}(\frac{t}{2}+1)^{1-\alpha}-\frac{1}{1-\alpha}],
\end{aligned}
\tag{A4}
$$

hence, we have

$$
\Gamma_2 \le C\begin{cases}\langle t\rangle^{-\alpha-\beta+1}, & if \quad \alpha<1,\\ \langle t\rangle^{-\beta}, & if \quad \alpha>1.\end{cases}
\tag{A5}
$$

For $\alpha = 1$ and $\beta < 1$, there holds

$$
\begin{aligned}
\Gamma_1 &\le \int_{0}^{\frac{t}{2}}(\frac{t}{2}+1)^{-1}(\tau+1)^{-\beta}d\tau \\
&\le (\frac{t}{2}+1)^{-\beta+\varepsilon}\int_{0}^{\frac{t}{2}}(\frac{t}{2}+1)^{\beta-\varepsilon-1}(\tau+1)^{-\beta}d\tau \\
&\le (\frac{t}{2}+1)^{-\beta+\varepsilon}\int_{0}^{\frac{t}{2}}(\tau+1)^{-1-\varepsilon}d\tau \\
&\le C\langle t\rangle^{-\beta+\varepsilon},
\end{aligned}
\tag{A6}
$$

and

$$
\begin{aligned}
\Gamma_2 &\le \int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-1}(\frac{t}{2}+1)^{-\beta}d\tau \\
&\le (\frac{t}{2}+1)^{-\beta}\int_{0}^{\frac{t}{2}}(y+1)^{-1}dy(y=t-\tau) \\
&\le (\frac{t}{2}+1)^{-\beta+\varepsilon}\int_{0}^{\frac{t}{2}}(\frac{t}{2}+1)^{-\varepsilon}(y+1)^{-1}dy \\
&\le (\frac{t}{2}+1)^{-\beta+\varepsilon}\int_{0}^{\frac{t}{2}}(y+1)^{-1-\varepsilon}d\tau \\
&\le C\langle t\rangle^{-\beta+\varepsilon}.
\end{aligned}
\tag{A7}
$$

For $\alpha = 1$ and $\beta > 1$, we derive

$$\begin{aligned}\Gamma_1 &\leq \int_0^{\frac{t}{2}} (\frac{t}{2}+1)^{-1}(\tau+1)^{-\beta} d\tau \\ &\leq (\frac{t}{2}+1)^{-1} \int_0^{\frac{t}{2}} (\tau+1)^{-\beta} d\tau \\ &\leq C\langle t\rangle^{-1},\end{aligned} \tag{A8}$$

and

$$\begin{aligned}\Gamma_2 &= \int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-1}(\frac{t}{2}+1)^{-\beta} d\tau \\ &\leq (\frac{t}{2}+1)^{-\beta} \int_0^{\frac{t}{2}} (y+1)^{-1} dy (y = t-\tau) \\ &\leq (\frac{t}{2}+1)^{-\beta+\varepsilon} \int_0^{\frac{t}{2}} (\frac{t}{2}+1)^{-\varepsilon}(y+1)^{-1} dy \\ &\leq C\langle t\rangle^{-\beta+\varepsilon}.\end{aligned} \tag{A9}$$

Similarly, we can obtain $\Gamma_1 \leq C\langle t\rangle^{-\alpha+\varepsilon}$ and $\Gamma_2 \leq C\langle t\rangle^{-1}$ for $\beta = 1$ and $\alpha > 1$. And one have $\Gamma_1 \leq C\langle t\rangle^{-\alpha+\varepsilon}$ and $\Gamma_2 \leq C\langle t\rangle^{-\alpha+\varepsilon}$ for $\beta = 1$ and $\alpha < 1$. Also, there holds $\Gamma_1 \leq C\langle t\rangle^{-1+\varepsilon}$ and $\Gamma_2 \leq C\langle t\rangle^{-1+\varepsilon}$ for $\beta = 1$ and $\alpha = 1$.

Combining all of the above, we have

$$\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\alpha}\langle\tau\rangle^{-\beta} d\tau \leq C \begin{cases} \langle t\rangle^{-\alpha-\beta+1}, & if \quad \beta < 1, \\ \langle t\rangle^{-\alpha}, & if \quad \beta > 1, \\ \langle t\rangle^{-\alpha+\varepsilon}, & if \quad \beta = 1, \end{cases} \tag{A10}$$

and

$$\int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\alpha}\langle\tau\rangle^{-\beta} d\tau \leq C \begin{cases} \langle t\rangle^{-\alpha-\beta+1}, & if \quad \alpha < 1, \\ \langle t\rangle^{-\beta}, & if \quad \alpha > 1, \\ \langle t\rangle^{-\beta+\varepsilon}, & if \quad \alpha = 1. \end{cases} \tag{A11}$$

In addition, it holds

$$\int_0^{t} e^{-\gamma(t-\tau)}\langle\tau\rangle^{-\beta} d\tau \leq C \int_0^{t} \langle t-\tau\rangle^{-\alpha}\langle\tau\rangle^{-\beta} d\tau, \tag{A12}$$

due to $e^{-\gamma(t-\tau)} \cdot \langle t-\tau\rangle^{\alpha} \leq C$ for $\gamma > 0$. □

References

1. Z. Ye. Global regularity of 2D tropical climate model with zero thermal diffusion. *ZAMM.Z.Angew.Math.Mech.*. 2020;100(7).

2. B. Dong, J. Wu, Z. Ye. 2D tropical climate model with fractional dissipation and without thermal diffusion. *Commun.Math.Sci.* 2020;18(1):259–292.

3. D. Frierson, A. Majda, O. Pauluis. Large scale dynamics of precipitation fronts in the tropical atmosphere: a novel relaxation limit. *Commun.Math.Sci.* 2004;2:591–626.

4. J. Li, E.S. Titi. Global well-posedness of strong solutions to a tropical climate model. *Discrete.Contin.Dyn.Syst.* 2016;36(8):4495–4916.

5. J. Li, E.S. Titi. A tropical atmosphere model with moisture: global well-posedness and relaxation limit. *Nonlinearity.* 2016;29(9):2674–2714.

6. A. Gill. Some simple solutions for heat-induced tropical circulation. *Quart.J.Roy.Meteor.Soc.* 1980;106:447–462.

7. A. Majda, J. Biello. The nonlinear interaction of barotropic and equatorial baroclinic Rossby waves. *J.Atmos.Sci.* 2003;60:1809–1821.

8. T. Matsuno. Quasi-geostrophic motions in the equatorial area. *J.Meteor.Soc.Japan.* 1966;44:25–43.

9. R. Wan. Global small solutions to a tropical climate model without thermal diffusion. *J.Math.Phys.* 2016;57:021507.

10. C. Ma, R. Wan. Spectral analysis and global well-posedness for a viscous tropical climate model with only a damp term. *Nonlinear.Anal.Real.World.Appl.* 2018;39:554–567.

11. X. Ye, M. Zhu. Global strong solutions of the tropical climate model with temperature-dependent diffusion on the barotropic mode. *Applied Mathematics Letters.* 2019;89:8–14.

12. M. Zhou. On well-posedness and blow-up criterion for the 2D tropical climate model. *Bull.Korean.Math.Soc.* 2020;57:891–907.

13. Y. Wang, S. Zhang, N. Pan. Regularity and global existence on the 3D tropical climate model. *Bull.Malays.Math.Sci.Soc.* 2020;43:641–650.

14. M. Zhu. Global regularity for the tropical climate model with fractional diffusion on barotropic mode. *Appl.Math.Lett.* 2018;81:99–104.

15. Z. Ye. Global regularity for a class of 2D tropical climate model. *J.Math.Anal.Appl.* 2017;446:307–321.

16. B. Dong, W. Wang, J. Wu, H. Zhang. Global regularity results for the climate model with fractional dissipation. *Discrete.Contin.Dyn.Syst.Ser.B.* 2019;24:211–229.

17. C. Ma, Z. Jiang, R. Wan. Local well-posedness for the tropical climate model with fractional velocity diffusion. *Kinet.Relat.Models.* 2016;9:551–570.

18. A.A. Ilyin, A. Miranville, E.S. Titi. Small viscosity sharp estimates for the global attractor of the 2-D damped-driven Navier-Stokes equations. *Commun.Math.Sci.* 2004;2:403–426.

19. T. Sideris, B. Thomases, D. Wang. Long time behavior of solutions to the 3D compressible Euler equations with damping. *Comm.Partial Differential Equations.* 2003;28:795–816.

20. D. Wang. Large-time behavior of solutions to the multi-dimensional Euler equations with damping. 2003;:919–928.

21. J. Wu, Y. Wu, X. Xu. Global small solution to the 2D MHD system with a velocity damping term. *J.Math.Anal.* 2015;47:2630–2656.

22. S. Lai, J. Wu, X. Xu, J. Zhang, Y. Zhang. Optimal decay estimates for 2D boussinesq equations with partial dissipation. *Journal of Nonlinear Science.* 2021;31(16).

23. C. Li, J. Wu, X. Xu. Smoothing and stabilization effects of magnetic field on electrically conducting fluids. *Journal of Differential Equations.* 2021;276:368–403.

24. L. Tao, J. Wu, K. Zhao, X. Zhang. Stability near hydrostatic equilibrium to the 2D boussinesq equations without thermal diffusion. *Arch.Rational.Mech.Anal.* 2020;237:585–630.

25. J. Wu, Y. Zhu. Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium. *Advances in Mathematics.* 2021;377:107466.

26. J. Wu, Q. Zhang. Stability and optimal decay for a system of 3D anisotropic Boussinesq equations. *Nonlinearity.* 2021;34:5456–5484.

27. T. Tao. Nonlinear Dispersive Equations, Local and Global Analysis. 2006;.

28. R. Wan. Optimal decay estimate of strong solutions for the 3D incompressible oldroyd-B model without damping. *Pacific Journal of Mathematics.* 2019;301(2):667–701.

29. P. D'Ancona. A Short Proof of Commutator Estimates. *J.Fourier.Anal.Appl.* 2019;25:1134–1146.

30. T. Kato, G. Ponce. Commutator estimates and the euler and navier-stokes equations. *Communications on Pure and Applied Mathematics.* 1988;41:891–907.

31. J. Wu, Y. Wu. Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion. *Adv.Math.* 2017;310:759–888.

32. L. Bisconti. A regularity criterion for a 2D tropical climate model with fractional dissipation. *Monatshefte für Mathematik*. 2021;194:719–736.

33. Gala. S, Ragusa , M.A. , Z. Zhang. A regularity criterion in terms of pressure for the 3D viscous MHD equations. *Bull.Malays.Math.Sci.Soc.* 2017;40(4):1677–1690.

34. Gala. S, Ragusa , M.A. . A logarithmic regularity criterion for the two-dimensional MHD equations. *J.Math.Anal.Appl.* 2016;444:1752–1758.

35. Gala. S, Ragusa , M.A. . A new regularity criterion for the Navier-Stokes equations in terms of the two components of the velocity. *Electron.J.Qual.Theory.Differ.Equ.* 2016;26:1–9.