Liouville theorem and qualitative properties of solutions for an integral system

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Abstract: In this paper, we are concerned with an integral system

$$\begin{cases} u(x) = W_{\beta,\gamma}(u^{p-1}v)(x), \ u > 0 \text{ in } R^n, \\ v(x) = I_\alpha(u^p)(x), \ v > 0 \text{ in } R^n, \end{cases}$$

where $p > 0$, $0 < \alpha, \beta\gamma < n$, $\gamma > 1$. Base on the integrability of positive solutions, we obtain some Liouville theorems and the decay rates of positive solutions at infinity. In addition, we use the properties of the contraction map and the shrinking map to prove that u is Lipschitz continuous. In particular, the Serrin type condition is established, which plays an important role to classify the positive solutions.

Key words: integral equation; Liouville theorem; asymptotic behavior.

MSC2020: 45G15, 45M20

1 Introduction

In this paper, we study an integral system involving Wolff potential and Riesz potential:

$$\begin{cases} u(x) = W_{\beta,\gamma}(u^{p-1}v)(x), \ u > 0 \text{ in } R^n, \\ v(x) = I_\alpha(u^p)(x), \ v > 0 \text{ in } R^n, \end{cases} \tag{1.1}$$

where $p > 0$, $0 < \alpha, \beta\gamma < n$, $\gamma > 1$,

$$W_{\beta,\gamma}(u^{p-1}v)(x) = \int_0^\infty \left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t},$$

$$I_\alpha(u^p)(x) = \int_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy.$$

The Wolff potential $W_{\beta,\gamma}(f)$ of a positive function $f \in L^1_{loc}(R^n)$ was introduced in [8]. It is easy to verify that $W_{1,2}(\cdot)$ is the well-know Newton potential and $W_{\frac{\alpha}{2},2}(\cdot)$ is the Riesz potential.

As a special case of system (1.1), the following integral system was investigated extensively

$$\begin{cases} u(x) = \displaystyle\int_{R^n} \frac{u^{p-1}(y)v(y)}{|x-y|^{n-\alpha}} dy, \ \ u > 0 \text{ in } R^n, \\ v(x) = \displaystyle\int_{R^n} \frac{u^p(y)}{|x-y|^{n-\beta}} dy, \ \ v > 0 \text{ in } R^n, \end{cases} \tag{1.2}$$

and the corresponding nonlinear system is the fractional Choquard type equation

$$(-\Delta)^{\frac{\alpha}{2}} u = \left(\frac{1}{|x|^{n-\beta}} * u^p \right) u^{p-1} \quad \text{in } R^n, \tag{1.3}$$

which arises in the study of boson stars and other physical phenomena. Here $*$ represents convolution, $0 < \alpha < 2$ and $0 < \beta < n$.

There are lots of results on the qualitative properties of solutions for these equations. Le [17] proved (1.3) has no positive solution if $1 \le p < \frac{n+\beta}{n-\alpha}$ and classified all positive solutions to the equation in the critical case $p = \frac{n+\beta}{n-\alpha}$. Afterwards, he obtained that (1.3) has similar conclusions for positive $H^{\frac{\alpha}{2}}(R^n)$ solutions (see [18]). By the a direct method of moving planes which developed in [5], Ma, Shang and Zhang [25] proved that the symmetry and nonexistence of positive solutions in the critical and subcritical case respectively. Later, Dai, Qin and Huang et al. studied (1.3) with $\beta = n - 2\alpha$. They gave a regularity result for weak solutions in integral forms, and classified all positive solutions with $\alpha \in (0, \min\{2, \frac{n}{2}\})$ (cf. [6] and [7]).

In particular, when $\alpha = 2$, $\beta = n-4$, $p = 2$, Liu [23] classified all $L^{\frac{2n}{n-2}}(R^n)$ solutions for the equivalent integral system (1.2). When $\beta = \alpha \in (1, n)$ and $p = \frac{n+\alpha}{n-\alpha}$. Xu and Lei [28] and Lei [15] proved that all the positive solutions of equation (1.3) in $L^{\frac{n(p-1)}{\alpha}}(R^n)$ can be classified as $u(x) = v(x) = c(\frac{t}{t^2+|x-x^*|^2})^{\frac{n-\alpha}{2}}$, where c, t are positive constants and $x^* \in R^n$. In addition, Wang and Tian [27] obtained a classification result for weak solutions in $H^\alpha(R^n)$ to (1.3) when $\beta = \alpha \in (0, n/2)$ and $p = \frac{n+\alpha}{n-\alpha}$. Afterwards, [19] proved that some integrable solutions belong to $C^1(R^n)$.

When $\alpha = p = 2$, $n - \beta = \gamma$, (1.3) is reduced to the nonlinear static Hartree equation

$$-\Delta u = 2u(|x|^{-\gamma} * |u|^2), \ \ u > 0 \ \in R^n, \tag{1.4}$$

where $n \ge 3$ and $\gamma \in (0, n)$. This equation is helpful in understanding the blowing up or the global existence and scattering of the solutions of the dynamic Hartree equation. It also arises in the Hartree-Fock theory of the nonlinear Schödinger equations (see [20] and [21]). By studying the equivalent integral system, Lei [16] proved that (1.4) does not have any positive solution if $n + \gamma < 3\alpha$. In addition, he obtained the integrability result for the integrable solution and estimated some decay rates.

Recently, Lei also considered a general equation, that is the static Hartree-Poisson equation

$$-\Delta u = pu^{p-1}(|x|^{2-n} * u^p), \quad u > 0 \text{ in } R^n, \tag{1.5}$$

where $n \ge 3$ and $p \ge 1$. In [15], he gave three important exponents, namely the Serrin type $p_{se} = \frac{n}{n-2}$, the Sobolev type $p_{so} = \frac{n+2}{n-2}$, and the Joseph-Lundgren type $p_{jl}(n) = 1 + \frac{4}{n-4-2\sqrt{n-1}}$. Furthermore, some Liouville type theorems and the classification results on positive solutions of (1.5) were also established.

Recalling the work in [9], [10] and [26], we know that the Wolff potential is helpful to study the nonlinear PDEs. For example, $W_{1,p}(\omega)$ and $W_{\frac{2k}{k+1},k+1}(\omega)$ can be used to estimate the $\mathcal{A}$-superhamonic functions involving solutions of the p-Laplace equation

$$-div(|\nabla u|^{p-2}\nabla u) = \omega, \tag{1.6}$$

and the k-Hessian equation

$$F_k[-u] = \omega, \quad k = 1, 2, \cdots, n,$$

respectively. In 2009, Liu [23] discussed the following quasilinear partial differential equation on R^n

$$-\,div\mathcal{A}(x, \nabla u) = (|x|^{-\gamma} * |u|^2)u \tag{1.7}$$

with $\gamma = 2n - \frac{4(n-p)}{p}$, $1 < p < n$. In the special case $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$, $\mathcal{A}(x, \nabla u)$ is the usually p-Laplacian defined by $\Delta_p := div(|\nabla u|^{p-2}\nabla u)$. According to [9], if $\inf_{R^n} u = 0$, there exists $C > 0$ such that the positive solution u of (1.6) satisfies

$$\frac{1}{C} W_{1,p}(\omega)(x) \le u(x) \le C W_{1,p}(\omega)(x), \;\; x \in R^n.$$

Let $c(x) = u(x) W^{-1}_{1,p}(\omega)(x)$, then $1/C \le c(x) \le C$, (1.7) is reduced to

$$u(x) = c(x) W_{1,p}((|x|^{-\gamma} * |u|^2)u)(x), \;\; u > 0 \text{ in } R^n. \tag{1.8}$$

Recently, Liu and Li [22] considered more general equations:

$$\begin{cases} u(x) = c(x) W_{\beta,\gamma}(u^{p-1}v), & u > 0 \text{ in } R^n, \\ v(x) = |x|^{\alpha-n} * |u|^p, & v > 0 \text{ in } R^n. \end{cases} \tag{1.9}$$

Here $c(x)$ are double bounded function, i.e., there exist positive constants c and C such that $c \le c(x) \le C$. By a regularity lifting lemma which was established by Chen and Jin et al. (see [1]), they obtained the optimal integrability of integrable solutions, which is the key ingredients to study the regularity and decay rates at infinity of positive solutions.

Proposition 1.1. *([22]) Let $(u, v) \in L^{r_0}(R^n) \times L^{s_0}(R^n)$ be a pair of positive solutions of (1.1), where $r_0 = \frac{n(2p-\gamma)}{\alpha+\gamma\beta}$, $s_0 = \frac{n(2p-\gamma)}{p\gamma\beta-(p-\gamma)\alpha}$ and $p > \max\{\frac{n\gamma}{2n-\gamma\beta-\alpha}, \frac{n\gamma+(\gamma-1)\alpha-\gamma\beta}{2n-2\gamma\beta}\}$, $p \ge \gamma$, $1 < \gamma \le 2$, $p\gamma\beta - (p-\gamma)\alpha > 0$. Then for any $1 \le r, s < \infty$, $u \in L^r(R^n), v \in L^s(R^n)$ if and only if*

$$\frac{1}{r} \in \left(0, \frac{n-\gamma\beta}{n(\gamma-1)}\right),$$

and

(i) When $\frac{n-\gamma\beta}{n(\gamma-1)} > \frac{(2p-\gamma)n-(p-1)(\alpha+\gamma\beta)}{n(2p-\gamma)}$,

$$\frac{1}{s} \in \left(0, \frac{n-\alpha}{n}\right).$$

(ii) When $\frac{n-\gamma\beta}{n(\gamma-1)} < \frac{(2p-\gamma)n-(p-1)(\alpha+\gamma\beta)}{n(2p-\gamma)}$,

$$\frac{1}{s} \in \left(0, \min\left\{\frac{n-\alpha}{n}, \frac{p(n-\gamma\beta)-(\gamma-1)\alpha}{n(\gamma-1)}\right\}\right).$$

The right end values are optimal in the sense that if $\frac{1}{r}$ or $\frac{1}{s}$ exceed the right end values, then $\|u\|_r = \|v\|_s = \infty$.

Motivated by the work above, we continue to study some Liouville theorems and qualitative properties of positive solutions of (1.1) and (1.9). Consequently, we obtain the following main results.

Theorem 1.1. *If* $0 < p \le \frac{n\gamma+\alpha(\gamma-1)-\beta\gamma}{2(n-\beta\gamma)}$, *then the integral system (1.9) has no positive solution.*

On the contrary, under certain conditions, we can find positive solutions of (1.9) at different decay rates for some double bounded $c(x)$.

Theorem 1.2. *If* $p > \max\left\{\frac{n(\gamma-1)}{n-\beta\gamma}, \frac{\alpha(\gamma-1)}{n-\beta\gamma}+1, \frac{n\gamma+\alpha(\gamma-1)-\beta\gamma}{2(n-\beta\gamma)}\right\}$ *or* $p\beta\gamma-(p-\gamma)\alpha > 0$ *and* $p > \max\left\{\frac{n\gamma}{2n-\beta\gamma-\alpha}, \frac{n\gamma+(\gamma-1)\alpha-\beta\gamma}{2(n-\beta\gamma)}\right\}$, *then (1.9) has positive entire solution for some double bounded* $c(x)$.

Theorem 1.3. *Assume that* (u, v) *is a pair of positive solutions of (1.1) with* $p \ge 2$, $1 < \gamma \le 2$, $u \in L^{r_0}(R^n)$, *where* $r_0 = \frac{n(2p-\gamma)}{\alpha+\gamma\beta}$. *Then* u, v *must be radially symmetric and monotone decreasing about some point in* R^n.

For (1.8) with $c(x) \equiv 1$, the radial symmetry and monotonicity of positive solutions was established in [23]. Based on the results above, we study asymtotic behavior of positive solutions at infinity. Firstly, we give the ground state of positive solutions of (1.1).

Theorem 1.4. *Assume* $u \in L^r(R^n)$ *and* $v \in L^s(R^n)$ *solve (1.1), where* $\frac{1}{r} \in \left(0, \frac{n-\gamma\beta}{n(\gamma-1)}\right)$, $\frac{1}{s} \in \left(0, \min\left\{\frac{n-\alpha}{n}, \frac{p(n-\gamma\beta)-(\gamma-1)\alpha}{n(\gamma-1)}\right\}\right)$, *then* $u(x)$ *and* $v(x)$ *are bounded, and* $u(x), v(x) \to 0$ *as* $|x| \to \infty$.

Next, we derive the decay rates of positive solutions at infinity.

Theorem 1.5. *Assume that* (u, v) *is a pair of positive solutions of (1.1) with* $p \ge 2$, $1 < \gamma \le 2$, $u \in L^{r_0}(R^n)$, *where* $r_0 = \frac{n(2p-\gamma)}{\alpha+\gamma\beta}$. *If* $p > \frac{n\gamma}{2n-\beta\gamma-\alpha}$, $p\gamma\beta-(p-\gamma)\alpha > 0$, *then*

$$u(x) \simeq \frac{A_0}{|x|^{\frac{n-\beta\gamma}{\gamma-1}}}, \tag{1.10}$$

and

$$v(x) \simeq \begin{cases} \dfrac{A_1}{|x|^{n-\alpha}}, & p\dfrac{n-\beta\gamma}{\gamma-1} > n, \\ \dfrac{A_2 \ln|x|}{|x|^{n-\alpha}}, & p\dfrac{n-\beta\gamma}{\gamma-1} = n, \\ \dfrac{A_3}{|x|^{p\frac{n-\beta\gamma}{\gamma-1}-\alpha}}, & p\dfrac{n-\beta\gamma}{\gamma-1} < n, \end{cases}$$

where

$$A_0 = \frac{\gamma-1}{n-\beta\gamma}\int_{R^n} u^{p-1}(y)v(y)dy, \quad A_1 = \int_{R^n} u^p(y)dy,$$
$$A_2 = A_0^p|S^{n-1}|, \quad A_3 = A_0^p\int_{R^n}\frac{dz}{|z|^{p\frac{n-\beta\gamma}{\gamma-1}}|e-z|^{n-\alpha}}.$$

Here we define $f(x) \simeq \frac{A}{|x|^t}$ *at* ∞ *if* $\lim_{|x|\to\infty}|x|^t f(x) = A$.

According to Remark 3.3.5 in [3], Chen and Li give the second regularity lifting lemma which can be used to prove the Lipschitz continuity of positive solutions of integral systems involving the Riesz potential, the Bessel potential and the Wolff potential (cf. [11, 19, 24]).

Lemma 1.6. *(Regularity Lifting II.) Let $X = L^\infty(R^n)\times L^\infty(R^n)$ and $Y = C^{0,1}(R^n)\times C^{0,1}(R^n)$ with the norms*

$$\| (f,g) \|_X = \| f \|_\infty + \| g \|_\infty \quad \text{and} \quad \| (f,g) \|_Y = \| f \|_{0,1} + \| g \|_{0,1} .$$

Define their closed subset

$$X_1 = \{(f,g) \in X; \| f \|_\infty + \| g \|_\infty \leq C(\| u \|_\infty + \| v \|_\infty)\},$$

$$Y_1 = \{(f,g) \in Y; \| f \|_\infty + \| g \|_\infty \leq C(\| u \|_\infty + \| v \|_\infty)\}.$$

Assume
(i) T is a contracting map from $X_1 \to X$;
(ii) T is a shrinking map from $Y_1 \to Y$;
(iii) $(F,G) \in X_1 \cap Y_1$;
(iv) $T(\cdot,\cdot) + (F,G)$ is a map from $X_1 \cap Y_1$ to itself.
If $(u,v) \in X$ is a pair of solutions of the operator equation $(f,g) = T(f,g) + (F,G)$, then $(u,v) \in Y$.

Using this lemma, we can obtain the regularity of positive solutions of (1.1).

Theorem 1.7. *Under the same condition as in Theorem 1.4, then $u(x)$ and $v(x)$ are Lipschitz continuous.*

Finally, let us now recall several basic estimates for both the Riesz and Wolff potentials which we often invoke throughout this paper.

Lemma 1.8. *([24], Hardy-Littlewood-Sobolev inequality) Let*

$$f(x) = \int_{R^n} |x-y|^{\alpha-n} g(y)dy,$$

then for any $s > \frac{n}{n-\alpha}$, we have

$$\|f\|_s \leq C(n,s,\alpha)\|g\|_{\frac{ns}{n+\alpha s}}.$$

Lemma 1.9. *([24], Corollary 2.1.) Let $p, q > 1$, $\beta > 0$, $\gamma > 1$ and $\beta\gamma < n$, then there exists some positive constant C such that*

$$\|W_{\beta,\gamma}(f)\|_q \leq C\|f\|_p^{\frac{1}{\gamma-1}}, \quad f \in L^p(R^n),$$

where $\frac{1}{p} - \frac{\gamma-1}{q} = \frac{\beta\gamma}{n}$, $q > \gamma - 1$ and we denote $\|f\|_{L^q(R^n)}$ by $\|f\|_q$.

Remark 1.1. *From Theorem 1.1, we can see that if (1.9) has positive solution, then $p > \frac{n\gamma+\alpha(\gamma-1)-\beta\gamma}{2(n-\beta\gamma)}$, which implies $nr_0 > n(\gamma-1) + \beta\gamma r_0$. This result ensures Lemma 1.9 that can be use in (3.5).*

2 Serrin-type condition

In this section, we prove Theorem 1.1.

Theorem 2.1. *If $0 < p \leq \frac{n\gamma+\alpha(\gamma-1)-\beta\gamma}{2(n-\beta\gamma)}$, then (1.1) has no positive super-solution.*

Proof. Assume that (u,v) is a pair of positive super-solution of (1.1), we can get a contradiction.

In fact, for $|x|>R>1$ with $R>0$,

$$\begin{aligned} u(x) &\geq \int_{2|x|}^{\infty}\left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} \geq \int_{2|x|}^{\infty}\left(\frac{\int_{B_1(0)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} \\ &\geq \int_{2|x|}^{\infty} t^{\frac{\beta\gamma-n}{\gamma-1}}\frac{dt}{t} = \frac{c}{|x|^{a_0}}, \end{aligned} \tag{2.1}$$

where $a_0=\frac{n-\beta\gamma}{\gamma-1}$. By this estimate, we have

$$v(x) \geq c\int_{B_{\frac{|x|}{2}}(x)} \frac{|y|^{-pa_0}}{|x-y|^{n-\alpha}}dy \geq \frac{c}{|x|^{b_0}},$$

where $b_0=pa_0-\alpha$. This implies

$$u(x) \geq \int_{2|x|}^{\infty}\left(\frac{\int_{B_{t-|x|}(0)} |y|^{-(p-1)a_0-b_0}dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} \geq c\int_{2|x|}^{\infty} t^{\frac{\beta\gamma-(p-1)a_0-b_0}{\gamma-1}}\frac{dt}{t}.$$

When $\beta\gamma-(p-1)a_0-b_0\geq 0$, we have $u(x)=\infty$ for $|x|>R$, it is impossible. When $\beta\gamma-(p-1)a_0-b_0<0$, then $u(x)>\frac{c}{|x|^{a_1}}$, where $a_1=\frac{(p-1)a_0+b_0-\beta\gamma}{\gamma-1}$. Similarly, using this estimate, we have

$$v(x) \geq c\int_{B_{\frac{|x|}{2}}(x)} \frac{|y|^{-pa_1}}{|x-y|^{n-\alpha}}dy \geq \frac{c}{|x|^{b_1}},$$

where $b_1=pa_1-\alpha$. By induction, we obtain that for $|x|>R$,

$$u(x)\geq \frac{c}{|x|^{a_j}}, \quad v(x)\geq \frac{c}{|x|^{b_k}}.$$

Here $a_j=\frac{(p-1)a_{j-1}+b_{j-1}-\beta\gamma}{\gamma-1}$, $b_k=pa_k-\alpha$. Therefore,

$$\begin{aligned} a_j &= \frac{(p-1)a_{j-1}+pa_{j-1}-\alpha-\beta\gamma}{\gamma-1} = \frac{2p-1}{\gamma-1}a_{j-1}-\frac{\alpha+\beta\gamma}{\gamma-1} \\ &= \left(\frac{2p-1}{\gamma-1}\right)^2 a_{j-2}-\frac{\alpha+\beta\gamma}{\gamma-1}\left(1+\frac{2p-1}{\gamma-1}\right) \\ &= \cdots = \left(\frac{2p-1}{\gamma-1}\right)^j a_0-\frac{\alpha+\beta\gamma}{\gamma-1}\left[1+\frac{2p-1}{\gamma-1}+\cdots+\left(\frac{2p-1}{\gamma-1}\right)^{j-1}\right]. \end{aligned}$$

We claim that there exists j_0 such that $a_{j_0}<0$. This leads to

$$v(x)\geq c\int_{R^n\setminus B_R(0)} \frac{|y|^{-pa_{j_0}}}{|x-y|^{n-\alpha}}dy=\infty,$$

which contradicts with the fact that v is a positive solution. In fact, when $\frac{2p-1}{\gamma-1}=1$, then $a_j=a_0-\frac{\alpha+\beta\gamma}{\gamma-1}j$. Thus, we can find some large j_0 such that $a_{j_0}<0$. When $\frac{2p-1}{\gamma-1}\in(0,1)$, then

$$a_j=\left(\frac{2p-1}{\gamma-1}\right)^j\left[a_0-\frac{\alpha+\beta\gamma}{2p-\gamma}\right]+\frac{\alpha+\beta\gamma}{2p-\gamma}\to\frac{\alpha+\beta\gamma}{2p-\gamma}<0, \quad j\to\infty.$$

This implies $a_{j_0} < 0$ for some large j_0. When $\frac{2p-1}{\gamma-1} > 1$, by view of $0 < p < \frac{n\gamma+\alpha(\gamma-1)-\beta\gamma}{2(n-\beta\gamma)}$, then $a_0 < \frac{\alpha+\beta\gamma}{2p-\gamma}$. Thus, we can find sufficiently large j_0 such that $a_{j_0} < 0$. If $\frac{2p-1}{\gamma-1} \leq 0$, then $a_1 = \frac{2p-1}{\gamma-1}a_0 - \frac{\alpha+\beta\gamma}{\gamma-1} < 0$. In conclusion, we complete the proof of the claim.

Next, we prove that (1.1) has no positive super-solution if $p = \frac{n\gamma+\alpha(\gamma-1)-\beta\gamma}{2(n-\beta\gamma)}$. Otherwise, for any $x \in B_R(0)$ with $R > 0$, from (1.1), we have

$$\begin{aligned} u(x) &\geq \int_{2R}^{\infty} \left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \geq \int_{2R}^{\infty} \left(\frac{\int_{B_R(0)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\geq cR^{\frac{\beta\gamma-n}{\gamma-1}} \left(\int_{B_R(0)} u^{p-1}(y)v(y)dy \right)^{\frac{1}{\gamma-1}}. \end{aligned} \tag{2.2}$$

$$v(x) \geq \frac{c}{(|x|+R)^{n-\alpha}} \int_{B_R(0)} u^p(y)dy \geq cR^{\alpha-n} \int_{B_R(0)} u^p(y)dy. \tag{2.3}$$

Taking $p-1$ powers of (2.2) and multiplying (2.3), and then integrating on $B_R(0)$, we obtain

$$\begin{aligned} \int_{B_R(0)} u^{p-1}(y)v(y)dy &\geq cR^{n+\alpha-(2p-1)\frac{n-\beta\gamma}{\gamma-1}} \left(\int_{B_R(0)} u^{p-1}(y)v(y)dy \right)^{\frac{2p-1}{\gamma-1}} \\ &= c\left(\int_{B_R(0)} u^{p-1}(y)v(y)dy \right)^{\frac{2p-1}{\gamma-1}}. \end{aligned} \tag{2.4}$$

By view of $p = \frac{n\gamma+\alpha(\gamma-1)-\beta\gamma}{2(n-\beta\gamma)}$, then $n+\alpha-(2p-1)\frac{n-\beta\gamma}{\gamma-1} = 0$. Letting $R \to \infty$ in (2.4), it follows that $u^{p-1}v \in L^1(R^n)$.

Similar to (2.4), and integrating on $A_R := B_{2R}(0)\backslash B_R(0)$, we can get

$$\int_{A_R} u^{p-1}(y)v(y)dy \geq c\left(\int_{B_R(0)} u^{p-1}(y)v(y)dy \right)^{\frac{2p-1}{\gamma-1}}.$$

Letting $R \to \infty$ and noting $u^{p-1}v \in L^1(R^n)$, we see $\int_{R^n} u^{p-1}(y)v(y)dy = 0$, which contradicts with $u, v > 0$. □

Proof of Theorem 1.1. When $p \in \left(0, \frac{n\gamma+\alpha(\gamma-1)-\beta\gamma}{2(n-\beta\gamma)}\right]$, we claim that (1.9) has no positive solution. Assume $u(x), v(x)$ solves (1.9) for some double bounded $c(x)$, then

$$u(x) \geq c\int_0^{\infty} \left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}$$

and

$$v(x) \geq \int_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}dy.$$

Set

$$\tilde{u}(x) = c^{\frac{\gamma-1}{2p-\gamma}}u(x), \quad \tilde{v}(x) = c^{\frac{p(\gamma-1)}{2p-\gamma}}v(x).$$

Thus, $(\tilde{u}, \tilde{v})$ is a pair of super-solution to (1.1). This contradicts with Theorem 2.1. □

Proof of Theorem 1.2. Set

$$u(x) = \frac{1}{(1+|x|^2)^{\theta}}. \tag{2.5}$$

When $|x| \le 2R$ for some $R > 0$, $u(x)$ in (2.5) is proportional to $W_{\beta,\gamma}(u^{p-1}v)(x)$. Thus we only consider the case of $|x| > 2R$. Similar to the estimates of Theorem 2.2 and Theorem 2.3 in [13]. Inserting (2.5) into the right hand side of (1.9), we get

$$\begin{aligned}
&\int_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy = \int_{R^n} \frac{dy}{|x-y|^{n-\alpha}(1+|y|^2)^{p\theta}} \\
&= \left(\int_{B_R(0)} + \int_{B_{|x|/2}(x)} + \int_{B_{2|x|}(0)\setminus B_R(0)\setminus B_{|x|/2}(x)} + \int_{B^c_{2|x|}(0)} \right) \frac{dy}{|x-y|^{n-\alpha}(1+|y|^2)^{p\theta}} \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.6}$$

$$I_1 = \frac{c(x)}{(1+|x|^2)^{\frac{n-\alpha}{2}}} \int_{B_R(0)} \frac{dy}{(1+|y|^2)^{p\theta}} = \frac{c(x)}{(1+|x|^2)^{\frac{n-\alpha}{2}}}.$$

$$I_2 = \frac{c(x)}{(1+|x|^2)^{p\theta}} \int_{B_{|x|/2}(x)} \frac{dy}{|x-y|^{n-\alpha}} = \frac{c(x)}{(1+|x|^2)^{p\theta}} \int_0^{\frac{|x|}{2}} r^{\alpha} \frac{dr}{r} = \frac{c(x)}{(1+|x|^2)^{p\theta - \frac{\alpha}{2}}}.$$

$$0 \le I_3 \le \frac{c}{(1+|x|^2)^{\frac{n-\alpha}{2}}} \int_{B_{2|x|}(0)\setminus B_R(0)} \frac{dy}{|y|^{2p\theta}} = \frac{c}{(1+|x|^2)^{\frac{n-\alpha}{2}}} \int_R^{2|x|} r^{n-2p\theta} \frac{dr}{r}.$$

$$I_4 = c(x) \int_{B^c_{2|x|}(0)} \frac{dy}{|y|^{n-\alpha+2p\theta}} = c(x) \int_{2|x|}^{\infty} r^{\alpha - 2p\theta} \frac{dr}{r}.$$

If there holds

$$p > \max\left\{ \frac{n(\gamma-1)}{n-\beta\gamma}, \frac{\alpha(\gamma-1)}{n-\beta\gamma} + 1, \frac{n\gamma + \alpha(\gamma-1) - \beta\gamma}{2(n-\beta\gamma)} \right\}.$$

We take $2\theta = \frac{n-\beta\gamma}{\gamma-1}$, then $2p\theta > n$ and $2(p-1)\theta + n - \alpha > n$. Combining with I_1, I_2, I_3 and I_4, we have

$$\int_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy = \frac{c(x)}{(1+|x|^2)^{\frac{n-\alpha}{2}}} = c(x)v(x).$$

Similarly, we estimate $W_{\beta,\gamma}(u^{p-1}v)(x)$ yields

$$\begin{aligned}
W_{\beta,\gamma}(u^{p-1}v)(x) &= c(x) \left[\int_0^{\frac{|x|}{2}} + \int_{\frac{|x|}{2}}^{\infty} \right] \left(\frac{\int_{B_t(x)} (1+|y|^2)^{-(p-1)\theta - \frac{n-\alpha}{2}} dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\
&:= J_1 + J_2.
\end{aligned} \tag{2.7}$$

In view of $|y-x| < t < |x|/2$, then $|x|/2 \le |y| \le 3|x|/2$, we have

$$J_1 = c(x)(1+|x|^2)^{\frac{-(p-1)\theta - \frac{n-\alpha}{2}}{\gamma-1}} \int_0^{\frac{|x|}{2}} t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} = c(x)(1+|x|^2)^{\frac{\beta\gamma - 2(p-1)\theta - n + \alpha}{2(\gamma-1)}}.$$

Moreover, since $2(p-1)\theta+n-\alpha>n$, then

$$J_2=c(x)\int_{\frac{|x|}{2}}^{\infty}\left(\frac{\int_{B_t(x)\cap B_1(0)}(1+|y|^2)^{-(p-1)\theta-\frac{n-\alpha}{2}}dy+\int_{B_t(x)\backslash B_1(0)}(1+|y|^2)^{-(p-1)\theta-\frac{n-\alpha}{2}}dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t}$$
$$=c(x)\int_{\frac{|x|}{2}}^{\infty}t^{-\frac{n-\beta\gamma}{\gamma-1}}\frac{dt}{t}=c(x)(1+|x|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}.$$

Combining the estimates of J_1 and J_2, we get

$$W_{\beta,\gamma}(u^{p-1}v)(x)=c(x)(1+|x|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}=c(x)u(x).$$

Here $c(x)$ are unfixed double bounded functions. Thus, (1.9) has the radial entire solution as the form of (2.5) with $2\theta=\frac{n-\beta\gamma}{\gamma-1}$. It satisfies $u(x)\sim|x|^{-\frac{n-\beta\gamma}{\gamma-1}}$ at ∞ (i.e., u decays fast).

In addition, if another condition $p\beta\gamma-(p-\gamma)\alpha>0$ and

$$p>\max\left\{\frac{n\gamma}{2n-\beta\gamma-\alpha},\frac{n\gamma+(\gamma-1)\alpha-\beta\gamma}{2(n-\beta\gamma)}\right\}$$

hold, we take $2\theta=\frac{\alpha+\beta\gamma}{2p-\gamma}$, then $\alpha<2p\theta<n$ and $\beta\gamma<2(p-1)\theta+2p\theta-\alpha<n$. According to I_1, I_2, I_3 and I_4, we can see

$$\int_{R^n}\frac{u^p(y)}{|x-y|^{n-\alpha}}dy=\frac{c(x)}{(1+|x|^2)^{p\theta-\frac{\alpha}{2}}}=c(x)v(x).$$

On the other hand, $t>\frac{|x|}{2}$ and $y\in B_t(x)$ imply $|y|<3t$, thus

$$0\le J_2\le c\int_{\frac{|x|}{2}}^{\infty}\left(\frac{\int_{B_{3t}(0)}(1+|y|^2)^{-(p-1)\theta-p\theta+\alpha/2}dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t}$$
$$\le c\int_{\frac{|x|}{2}}^{\infty}\left(\frac{t^{n-2(p-1)\theta-2p\theta+\alpha}}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t}\le c(1+|x|^2)^{\frac{\beta\gamma-2(p-1)\theta-2p\theta+\alpha}{2(\gamma-1)}}.$$

Combining this with J_1, and then

$$W_{\beta,\gamma}(u^{p-1}v)(x)=c(x)(1+|x|^2)^{\frac{\beta\gamma-2(p-1)\theta-2p\theta+\alpha}{2(\gamma-1)}}=c(x)u(x).$$

This implies that (1.9) has the radial entire solution as the form of (2.5) with $2\theta=\frac{\alpha+\beta\gamma}{2p-\gamma}$. It satisfies $u(x)\sim|x|^{-\frac{\alpha+\beta\gamma}{2p-\gamma}}$ at ∞ (i.e., u decays slowly). $\square$

3 Symmetry of the solutions of (1.1)

In this section, we prove Theorem 1.3. We employ the method of moving planes introduced by Chen-Li in [2] and [3].

Proof of Theorem 1.3. Firstly, we introduce some notation. Let $\Sigma_\lambda=\{x=(x_1,x_2,\cdots,x_n)\mid x_1<\lambda\}$, $x^\lambda=(2\lambda-x_1,x_2,\cdots,x_n)$ be the reflection point of x about the plane $x_1=\lambda$. Write $u_\lambda(x)=u(x^\lambda)$ and $v_\lambda(x)=v(x^\lambda)$. Denote $D_t(x)$ the intersection of the ball $B_t(x)$ with its mirror image $B_t(x^\lambda)$, and $\Omega_t(x)=B_t(x)\backslash D_t(x)$.

Step 1. We claim that $v \in L^{s_0}(R^n)$, where $s_0 = \frac{n(2p-\gamma)}{p\gamma\beta-(p-\gamma)\alpha}$.

Applying the Hardy-Littlewood-Sobolev inequatility, from the second equation of (1.1), we get

$$\| v \|_{s_0} \leq C \| u^p \|_{\frac{ns_0}{n+\alpha s_0}} = C \| u \|^p_{\frac{nps_0}{n+\alpha s_0}} = C \| u \|^p_{r_0} < \infty.$$

Since $u \in L^{r_0}(R^n)$ and $\frac{nps_0}{n+\alpha s_0} = r_0$, here $s_0 = \frac{n(2p-\gamma)}{p\gamma\beta-(p-\gamma)\alpha}$.

Step 2. We show that for λ sufficiently negative,

$$u_\lambda(x) \geq u(x), \quad v_\lambda(x) \geq v(x) \quad \text{for all } x \in \Sigma_\lambda. \tag{3.1}$$

To show (3.1), we will prove that $\Sigma^u_\lambda := \{x \in \Sigma_\lambda \mid u(x) > u_\lambda(x)\}$ and $\Sigma^v_\lambda := \{x \in \Sigma_\lambda \mid v(x) > v_\lambda(x)\}$ must have measure zero for λ sufficiently negative.

For $x \in \Sigma^u_\lambda$, by the the mean value theorem and the Hölder inequality, we have

$$\begin{aligned}
&0 < u(x) - u_\lambda(x) \\
&= \int_0^\infty \left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} - \int_0^\infty \left(\frac{\int_{B_t(x)} u_\lambda^{p-1}(y)v_\lambda(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\
&= \int_0^\infty \left[\left(\frac{\int_{\Omega_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}} + A_t(x)\right)^{\frac{1}{\gamma-1}} + \left(\frac{\int_{\Omega_t(x)} u_\lambda^{p-1}(y)v_\lambda(y)dy}{t^{n-\beta\gamma}} + A_t(x)\right)^{\frac{1}{\gamma-1}}\right] \frac{dt}{t} \\
&= \frac{1}{\gamma-1}\int_0^\infty \left[\left(\frac{\xi_t(x)}{t^{n-\beta\gamma}}\right)^{\frac{2-\gamma}{\gamma-1}} \frac{\int_{\Omega_t(x)} u^{p-1}(y)v(y) - u_\lambda^{p-1}(y)v_\lambda(y)dy}{t^{n-\beta\gamma}}\right] \frac{dt}{t} \\
&= \frac{1}{\gamma-1}\int_0^\infty \left[\left(\frac{\xi_t(x)}{t^{n-\beta\gamma}}\right)^{\frac{2-\gamma}{\gamma-1}} \frac{\int_{\Omega_t(x)} u^{p-1}(y)(v-v_\lambda)(y)dy}{t^{n-\beta\gamma}}\right] \frac{dt}{t} \\
&\quad + \int_0^\infty \left[\left(\frac{\xi_t(x)}{t^{n-\beta\gamma}}\right)^{\frac{2-\gamma}{\gamma-1}} \frac{\int_{\Omega_t(x)} (u^{p-1} - u_\lambda^{p-1})(y)v_\lambda(y)dy}{t^{n-\beta\gamma}}\right] \frac{dt}{t} \\
&\leq C[u(x)+u_\lambda(x)]^{2-\gamma} \cdot \left\{\int_0^\infty \left(\frac{\int_{B_t(x)} u^{p-1}(v-v_\lambda)^+(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t}\right\}^{\gamma-1} \\
&\quad + C[u(x)+u_\lambda(x)]^{2-\gamma} \cdot \left\{\int_0^\infty \left(\frac{\int_{B_t(x)} u^{p-2}v_\lambda(u-u_\lambda)(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t}\right\}^{\gamma-1} \\
&\leq C[u(x)+u_\lambda(x)]^{2-\gamma} \cdot \{[W_{\beta,\gamma}(u^{p-1}(v-v_\lambda)^+)(x)]^{\gamma-1} + [W_{\beta,\gamma}(u^{p-2}v_\lambda(u-u_\lambda))(x)]^{\gamma-1}\} \\
&\leq Cu^{2-\gamma}(x) \cdot \{[W_{\beta,\gamma}(u^{p-1}(v-v_\lambda)^+)(x)]^{\gamma-1} + [W_{\beta,\gamma}(u^{p-2}v_\lambda(u-u_\lambda))(x)]^{\gamma-1}\},
\end{aligned} \tag{3.2}$$

where

$$A_t(x) = \frac{1}{t^{n-\beta\gamma}}\int_{D_t(x)} u^{p-1}(y)v(y)dy = \frac{1}{t^{n-\beta\gamma}}\int_{D_t(x)} u_\lambda^{p-1}(y)v_\lambda(y)dy.$$

Since $\xi_t(x)$ is value between

$$\int_{B_t(x)} u^{p-1}(y)v(y)dy \quad \text{and} \quad \int_{B_t(x)} u_\lambda^{p-1}(y)v_\lambda(y)dy,$$

so that there holds

$$\xi_t(x) \leq \int_{B_t(x)} u^{p-1}(y)v(y) + u_\lambda^{p-1}(y)v_\lambda(y)dy.$$

Similar to the calculation of Lemma 2.1 in [4], we also have

$$v(x) - v_\lambda(x) = \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) (u^p - u_\lambda^p)(y)dy. \tag{3.3}$$

For $x \in \Sigma_\lambda^v$, we can deduce

$$0 < v(x) - v_\lambda(x) \leq C \int_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\alpha}} [u^{p-1}(u-u_\lambda)^+](y)dy. \tag{3.4}$$

Applying the fact of Lemma 1.8, 1.9 and the Hölder inequality to (3.2) and (3.4), we obtain

$$\begin{aligned}
\| u - u_\lambda \|_{L^{r_0}(\Sigma_\lambda^u)} &\leq C \| u \|_{L^{r_0}(\Sigma_\lambda^u)} \| W_{\beta,\gamma}(u^{p-1}(v-v_\lambda)^+) \|_{L^{r_0}(\Sigma_\lambda^u)}^{\gamma-1} \\
&\quad + \| u \|_{L^{r_0}(\Sigma_\lambda^u)} \| W_{\beta,\gamma}(u^{p-2}v_\lambda(u-u_\lambda)) \|_{L^{r_0}(\Sigma_\lambda^u)}^{\gamma-1} \\
&\leq C \| u \|_{L^{r_0}(\Sigma_\lambda^u)} \| u^{p-1}(v-v_\lambda)^+ \|_{L^{\frac{nr_0}{n(\gamma-1)+\beta\gamma r_0}}(\Sigma_\lambda^u)} \\
&\quad + \| u \|_{L^{r_0}(\Sigma_\lambda^u)} \| u^{p-2}v_\lambda(u-u_\lambda) \|_{L^{\frac{nr_0}{n(\gamma-1)+\beta\gamma r_0}}(\Sigma_\lambda^u)} \\
&\leq C \| u \|_{L^{r_0}(\Sigma_\lambda^u)} \| u \|_{L^{r_0}(\Sigma_\lambda^u)}^{p-1} \| v - v_\lambda \|_{L^{s_0}(\Sigma_\lambda^v)} \\
&\quad + \| u \|_{L^{r_0}(\Sigma_\lambda^u)} \| u \|_{L^{r_0}(\Sigma_\lambda^u)}^{p-2} \| v \|_{L^{s_0}(\Sigma_\lambda^c)} \| u - u_\lambda \|_{L^{r_0}(\Sigma_\lambda^u)} .
\end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}
\| v - v_\lambda \|_{L^{s_0}(\Sigma_\lambda^v)} &\leq C \| u^{p-1}(u-u_\lambda) \|_{L^{\frac{ns_0}{n+\alpha s_0}}(\Sigma_\lambda^u)} \\
&\leq C \| u \|_{L^{r_0}(\Sigma_\lambda^u)}^{p-1} \| u - u_\lambda \|_{L^{r_0}(\Sigma_\lambda^u)} .
\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), it follows that

$$\begin{aligned}
\| u - u_\lambda \|_{L^{r_0}(\Sigma_\lambda^u)} &\leq C \| u \|_{L^{r_0}(\Sigma_\lambda^u)} \| u \|_{L^{r_0}(\Sigma_\lambda^u)}^{p-1} \| u \|_{L^{r_0}(\Sigma_\lambda^u)}^{p-1} \| u - u_\lambda \|_{L^{r_0}(\Sigma_\lambda^u)} \\
&\quad + C \| u \|_{L^{r_0}(\Sigma_\lambda^u)} \| u \|_{L^{r_0}(\Sigma_\lambda^u)}^{p-2} \| v \|_{L^{s_0}(\Sigma_\lambda^c)} \| u - u_\lambda \|_{L^{r_0}(\Sigma_\lambda^u)} \\
&\leq C \left(\| u \|_{L^{r_0}(\Sigma_\lambda^u)}^{2p-1} + \| u \|_{L^{r_0}(\Sigma_\lambda^u)}^{p-1} \| v \|_{L^{s_0}(\Sigma_\lambda^c)} \right) \| u - u_\lambda \|_{L^{r_0}(\Sigma_\lambda^u)} .
\end{aligned}$$

By the integrability condition $u \in L^{r_0}(R^n)$ and $v \in L^{s_0}(R^n)$, for sufficiently negative λ, we arrive at

$$\| u - u_\lambda \|_{L^{r_0}(\Sigma_\lambda^u)} \leq \frac{1}{2} \| u - u_\lambda \|_{L^{r_0}(\Sigma_\lambda^u)} . \tag{3.7}$$

Therefore, the measure of Σ_λ^u must be zero. From (3.6), we also deduce that Σ_λ^v has measure zero.

Step 3. We move the plane $x_1 = \lambda$ to the right as long as (3.1) holds. Define

$$\lambda_0 = \sup\{\mu \mid (3.1) \text{ holds for any } \lambda \leq \mu\}.$$

Next, we show

$$u_{\lambda_0}(x) \equiv u(x), \quad v_{\lambda_0}(x) \equiv v(x), \quad \forall\, x \in \Sigma_{\lambda_0}. \tag{3.8}$$

Otherwise, we can prove that the plane can be moved further to the right. Similar to the above discussion, for $\lambda \in [\lambda_0, \lambda_0+\varepsilon)$, we choose ε sufficiently small, so that (3.7) holds. Therefore, Σ_λ^u and Σ_λ^v must be measure zero, i.e., $u_\lambda(x) \geq u(x)$ and $v_\lambda(x) \geq v(x)$ on Σ_λ for all $\lambda \in [\lambda_0, \lambda_0+\varepsilon)$. This contradicts with the definition of λ_0.

If the plane stops at $x_1 = \lambda_0$ for some $\lambda_0 < 0$, then $u(x)$ and $v(x)$ must be radially symmetric and decreasing about the plane $x_1 = \lambda_0$. Otherwise, we can move the plane all the way to $x_1 = 0$. Since the x_1 direction can be chosen arbitrarily, we deduce that $u(x)$ and $v(x)$ must be radially symmetric and decreasing about some point x^*. $\square$

4 Decay rates of solutions

In this section, we estimate the decay rates of u and v. The idea of proof comes from references [12] and [14]. Firstly, we gave a lemma that will use later.

According to Theorem 1.3, we know that u, v are radially symmetric and monotone decreasing about x^*. We write

$$u(x) = \tilde{u}(r), \quad v(x) = \tilde{v}(r), \quad r = |x - x^*|.$$

Therefore, similar to Proposition 2.2 in [12], we have the following estimate.

Lemma 4.1. *There exists $C > 0$ such that for any $r > 0$,*

$$\tilde{u}(R) \leq CR^{-n/r}, \quad \tilde{v}(R) \leq CR^{-n/s},$$

where r, s satisfy

$$\frac{1}{r} \in \left(0, \frac{n-\gamma\beta}{n(\gamma-1)}\right), \quad \frac{1}{s} \in \left(0, \min\left\{\frac{n-\alpha}{n}, \frac{p(n-\gamma\beta)-(\gamma-1)\alpha}{n(\gamma-1)}\right\}\right). \tag{4.1}$$

Proof of Theorem 1.5. Step 1. We claim that $\int_{R^n} u^{p-1}(x)v(x)dx < \infty$.

Applying the Hardy-Littlewood-Sobolev inequality, we have

$$\left|\int_{R^n} u^{p-1}(x)v(x)dx\right| = \left|\int_{R^n}\int_{R^n} \frac{u^{p-1}(x)u^p(y)}{|x-y|^{n-\alpha}}dydx\right| \leq C \parallel u \parallel_r^{p-1} \parallel u \parallel_s^p,$$

where

$$\frac{p-1}{r} + \frac{p}{s} = \frac{n+\alpha}{n}.$$

We take $r = s = \frac{n(2p-1)}{n+\alpha}$, then $\frac{1}{r} \in (0, \frac{n-\beta\gamma}{n(\gamma-1)})$ from $p > \frac{n\gamma+\alpha(\gamma-1)-\beta\gamma}{2(n-\beta\gamma)}$. According to the integral integrable interval of u in Proposition 1.1, we can see that $u \in L^r(R^n)$. Therefore, we complete the proof of the claim.

Step 2. We prove that $p > \frac{\alpha(\gamma-1)}{n-\beta\gamma} + 1$.

For large $|x| > R > 1$, we have

$$v(x) \geq \int_{B_R(0)} \frac{u^p(y)}{|x-y|^{n-\alpha}}dy \geq \frac{c}{|x|^{n-\alpha}}.$$

Combining with (2.1) and the fact of $u^{p-1}v \in L^1(R^n)$, then

$$+\infty > \int_{R^n} u^{p-1}(x)v(x)dx \geq \int_{B_R^c(0)} \frac{c}{|x|^{(p-1)\frac{n-\beta\gamma}{\gamma-1}+n-\alpha}}dx = \int_R^{+\infty} r^{\alpha-(p-1)\frac{n-\beta\gamma}{\gamma-1}} \frac{dr}{r},$$

which implies $\alpha - (p-1)\frac{n-\beta\gamma}{\gamma-1} < 0$. This leads to $p > \frac{\alpha(\gamma-1)}{n-\beta\gamma} + 1$.

Step 3. There holds

$$\varliminf_{|x|\to\infty} |x|^{\frac{n-\beta\gamma}{\gamma-1}} u(x) \geq \frac{\gamma-1}{n-\beta\gamma}\left(\int_{R^n} u^{p-1}(y)v(y)dy\right)^{\frac{1}{\gamma-1}}. \tag{4.2}$$

For any given $\lambda > 1$, then $B_{(\lambda-1)|x|}(0) \subset B_t(x)$ when $t > \lambda|x|$. Thus, from (1.1), we have

$$\begin{aligned}
u(x) &\geq \int_{\lambda|x|}^{\infty}\left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} \\
&\geq \int_{\lambda|x|}^{\infty} t^{-\frac{n-\beta\gamma}{\gamma-1}}\frac{dt}{t}\left(\int_{B_{(\lambda-1)|x|}(0)} u^{p-1}(y)v(y)dy\right)^{\frac{1}{\gamma-1}} \\
&= \frac{\gamma-1}{n-\beta\gamma}(\lambda|x|)^{-\frac{n-\beta\gamma}{\gamma-1}}\left(\int_{B_{(\lambda-1)|x|}(0)} u^{p-1}(y)v(y)dy\right)^{\frac{1}{\gamma-1}}.
\end{aligned}$$

Letting $|x| \to \infty$, then

$$\varliminf_{|x|\to\infty} |x|^{\frac{n-\beta\gamma}{\gamma-1}} u(x) \geq \frac{\gamma-1}{n-\beta\gamma}(\lambda)^{-\frac{n-\beta\gamma}{\gamma-1}}\left(\int_{R^n} u^{p-1}(y)v(y)dy\right)^{\frac{1}{\gamma-1}}.$$

Letting $\lambda \to 1$ in the inequality above, we arrive at (4.2).

Step 4. There holds

$$\varlimsup_{|x|\to\infty} |x|^{\frac{n-\beta\gamma}{\gamma-1}} u(x) \leq \frac{\gamma-1}{n-\beta\gamma}\left(\int_{R^n} u^{p-1}(y)v(y)dy\right)^{\frac{1}{\gamma-1}}. \tag{4.3}$$

For any given $\lambda > 1$, we write

$$\begin{aligned}
u(x) &= W_{\beta,\gamma}(u^{p-1}v)(x) \\
&= \int_0^{\frac{1}{\lambda}|x|}\left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} + \int_{\frac{1}{\lambda}|x|}^{\infty}\left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} \\
&:= H_1 + H_2.
\end{aligned}$$

When $y \in B_t(x) \subset B_{\frac{1}{\lambda}|x|}(x)$, then $(1-\frac{1}{\lambda})|x| \leq |y| \leq (1+\frac{1}{\lambda})|x|$. According to Theorem 1.3, for large $|x|$, we have

$$u(y) = \tilde{u}(|y-x^*|) \leq \tilde{u}\left(\frac{(1-\frac{1}{\lambda})|x|}{2}\right), \quad v(y) = \tilde{v}(|y-x^*|) \leq \tilde{v}\left(\frac{(1-\frac{1}{\lambda})|x|}{2}\right).$$

Using the result of Lemma 4.1, it follows that

$$\begin{aligned}
|x|^{\frac{n-\beta\gamma}{\gamma-1}} H_1 &= |x|^{\frac{n-\beta\gamma}{\gamma-1}}\int_0^{\frac{1}{\lambda}|x|}\left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} \\
&\leq |x|^{\frac{n-\beta\gamma}{\gamma-1}}\int_0^{\frac{1}{\lambda}|x|}\left(\frac{\int_{B_t(x)} \tilde{u}^{p-1}((1-\frac{1}{\lambda})|x|/2)(y)\tilde{v}((1-\frac{1}{\lambda})|x|/2)(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} \qquad (4.4) \\
&\leq C|x|^{\frac{n-\beta\gamma}{\gamma-1}-\frac{n}{\gamma-1}(\frac{p-1}{r}+\frac{1}{s})}\int_0^{\frac{1}{\lambda}|x|} t^{\frac{\beta\gamma}{\gamma-1}}\frac{dt}{t} \\
&\leq C|x|^{\frac{n}{\gamma-1}(1-\frac{p-1}{r}-\frac{1}{s})}.
\end{aligned}$$

According to (4.1), we take $\frac{1}{r}=\frac{n-\gamma\beta-\varepsilon}{n(\gamma-1)}$. If $\frac{p(n-\gamma\beta)-(\gamma-1)\alpha}{n(\gamma-1)}\leq\frac{n-\alpha}{n}$, we choose $\frac{1}{s}=\frac{p(n-\gamma\beta)-(\gamma-1)\alpha-\varepsilon}{n(\gamma-1)}$ with sufficiently small $\varepsilon>0$. Since $p>\frac{n\gamma+\alpha(\gamma-1)-\beta\gamma}{2(n-\beta\gamma)}$, then $\frac{n}{\gamma-1}(1-\frac{p-1}{r}-\frac{1}{s})<0$. If $\frac{p(n-\gamma\beta)-(\gamma-1)\alpha}{n(\gamma-1)}>\frac{n-\alpha}{n}$, we choose $\frac{1}{s}=\frac{n-\alpha-\varepsilon}{n}$. By virtue of $p>\frac{\alpha(\gamma-1)}{n-\gamma\beta}+1$, then we also have $\frac{n}{\gamma-1}(1-\frac{p-1}{r}-\frac{1}{s})<0$. Letting $|x|\to\infty$ in (4.4), we obtain

$$\lim_{|x|\to\infty}|x|^{\frac{n-\beta\gamma}{\gamma-1}}H_1=0. \tag{4.5}$$

On the other hand,

$$\begin{aligned}
H_2&=\int_{\frac{1}{\lambda}|x|}^{\infty}\left(\frac{\int_{B_t(x)}u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t}\\
&\leq\left(\int_{R^n}u^{p-1}(y)v(y)dy\right)^{\frac{1}{\gamma-1}}\int_{\frac{1}{\lambda}|x|}^{\infty}t^{-\frac{n-\beta\gamma}{\gamma-1}}\frac{dt}{t}\\
&=\frac{\gamma-1}{n-\beta\gamma}\left(\frac{1}{\lambda}|x|\right)^{-\frac{n-\beta\gamma}{\gamma-1}}\left(\int_{R^n}u^{p-1}(y)v(y)dy\right)^{\frac{1}{\gamma-1}}.
\end{aligned}$$

Letting $|x|\to\infty$, then

$$\lim_{|x|\to\infty}|x|^{\frac{n-\beta\gamma}{\gamma-1}}H_2\leq\frac{\gamma-1}{n-\beta\gamma}\lambda^{\frac{n-\beta\gamma}{\gamma-1}}\left(\int_{R^n}u^{p-1}(y)v(y)dy\right)^{\frac{1}{\gamma-1}}.$$

Letting $\lambda\to1$ and combining with (4.5), we complete the proof of (4.3).

Finally, from (4.2) and (4.3), we can get (1.10).

For the decay rate of $v(x)$, according to Proposition 1.1 in [12] (see also Theorem 1.5 in [14]), we can come to the conclusion immediately, i.e.,

$$v(x)\simeq\begin{cases}\dfrac{A_1}{|x|^{n-\alpha}}, & p\dfrac{n-\beta\gamma}{\gamma-1}>n,\\ \dfrac{A_2\ln|x|}{|x|^{n-\alpha}}, & p\dfrac{n-\beta\gamma}{\gamma-1}=n,\\ \dfrac{A_3}{|x|^{p\frac{n-\beta\gamma}{\gamma-1}-\alpha}}, & p\dfrac{n-\beta\gamma}{\gamma-1}<n.\end{cases}$$

Here we only need replace $\lim_{|x|\to\infty}u(x)|x|^{n-\alpha}=B_0$ with $B_0=\int_{R^n}v^q(y)dy$ by $\lim_{|x|\to\infty}u(x)|x|^{\frac{n-\beta\gamma}{\gamma-1}}=A_0$ with $A_0=\int_{R^n}u^{p-1}(y)v(y)dy$. □

5 Lipschitz continuity

In this section, we prove Theorem 1.4 and Theorem 1.7. Obviously, we can immediately deduce them by Theorem 5.1, 5.2 and Corollary 5.1.

Theorem 5.1. *Assume $u\in L^r(R^n)$ and $v\in L^s(R^n)$ solve (1.1), where $\frac{1}{r}\in\left(0,\frac{n-\gamma\beta}{n(\gamma-1)}\right)$, $\frac{1}{s}\in\left(0,\min\left\{\frac{n-\alpha}{n},\frac{p(n-\gamma\beta)-(\gamma-1)\alpha}{n(\gamma-1)}\right\}\right)$, then $u,v\in L^\infty(R^n)$.*

Proof. Let $d<1$ be a positive constant, from (1.1), we have

$$\begin{aligned}u(x)=&\int_0^d\left(\frac{\int_{B_t(x)}u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t}+\int_d^\infty\left(\frac{\int_{B_t(x)}u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t}\\ :=&I_1+I_2.\end{aligned}$$

Applying the Hölder inequality and the fact that $(u,v)\in L^r(R^n)\times L^s(R^n)$ with $\frac{1}{r}\in\left(0,\frac{n-\gamma\beta}{n(\gamma-1)}\right)$, $\frac{1}{s}\in\left(0,\min\left\{\frac{n-\alpha}{n},\frac{p(n-\gamma\beta)-(\gamma-1)\alpha}{n(\gamma-1)}\right\}\right)$, there holds

$$\begin{aligned}I_1&\le C\parallel u\parallel_r^{\frac{p-1}{\gamma-1}}\parallel v\parallel_s^{\frac{1}{\gamma-1}}\int_0^d t^{\frac{n}{\gamma-1}(1-\frac{p-1}{r}-\frac{1}{s})+\frac{\gamma\beta-n}{\gamma-1}}\frac{dt}{t}\\ &\le C\parallel u\parallel_r^{\frac{p-1}{\gamma-1}}\parallel v\parallel_s^{\frac{1}{\gamma-1}}\int_0^d t^{\frac{1}{\gamma-1}[\beta\gamma-n(\frac{p-1}{r}+\frac{1}{s})]}\frac{dt}{t}\le C,\end{aligned}\tag{5.1}$$

here we can choose r,s sufficiently large such that $\beta\gamma>n(\frac{p-1}{r}+\frac{1}{s})$.

On the other hand, Assume $|x-y|=\delta<1$, then

$$\begin{aligned}I_2&\le C\int_d^\infty\left(\frac{\int_{B_{t+\delta}(y)}u^{p-1}(z)v(z)dz}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t}\\ &\le C\int_d^\infty\left(\frac{\int_{B_{t+\delta}(y)}u^{p-1}(z)v(z)dz}{(t+\delta)^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\left(\frac{t+\delta}{t}\right)^{\frac{n-\beta\gamma}{\gamma-1}+1}\frac{d(t+\delta)}{t+\delta}\\ &\le C\int_{d+\delta}^\infty\left(\frac{\int_{B_t(y)}u^{p-1}(z)v(z)dz}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t}(1+\delta)^{\frac{n-\beta\gamma}{\gamma-1}+1}\le Cu(y).\end{aligned}\tag{5.2}$$

Therefore,

$$u(x)\le C+Cu(y),\quad |x-y|=\delta<1.\tag{5.3}$$

Since $u\in L^r(R^n)$ for any large $r>\frac{n(\gamma-1)}{n-\gamma\beta}$, integrating both side of (5.3) with respect to y on $B_\delta(x)$, we obtain

$$u^r(x)\le C+\frac{C}{|B_\delta(x)|}\int_{B_\delta(x)}u^r(y)dy\le C.$$

Similarly, exchanging the integral variants, we get

$$v(x)=(n-\alpha)\int_{R^n}u^p(y)dy\int_{|x-y|}^\infty t^{\alpha-n}\frac{dt}{t}=(n-\alpha)\int_0^\infty\frac{\int_{B_t(x)}u^p(y)dy}{t^{n-\alpha}}\frac{dt}{t}.$$

Thus, $v(x)\in L^\infty(R^n)$. □

Corollary 5.1. *Under the same condition as in Theorem 5.1, then $u(x),v(x)\to 0$ when $|x|\to\infty$.*

Proof. Take d sufficiently small in (5.1) such that $I_1\le C\varepsilon$. Combining this with (5.2), we have

$$u^r(x)\le C\varepsilon+\frac{C}{|B_\delta(x)|}\int_{B_\delta(x)}u^r(y)dy.$$

Using the fact that $u\in L^r(R^n)$, letting $\varepsilon\to 0$ and $|x|\to\infty$, thus $u(x)\to 0$. Similarly, we have $v(x)\to 0$. □

Theorem 5.2. *Under the assumption of Theorem 5.1, $u(x)$ and $v(x)$ are Lipschitz continuous.*

Proof. Let $X = L^\infty(R^n) \times L^\infty(R^n)$ and $Y = C^{0,1}(R^n) \times C^{0,1}(R^n)$ with the norms

$$\| (f,g) \|_X = \| f \|_\infty + \| g \|_\infty \quad \text{and} \quad \| (f,g) \|_Y = \| f \|_{0,1} + \| g \|_{0,1} .$$

Define their closed subset

$$X_1 = \{(f,g) \in X; \| f \|_\infty + \| g \|_\infty \leq C(\| u \|_\infty + \| v \|_\infty)\},$$

$$Y_1 = \{(f,g) \in Y; \| f \|_\infty + \| g \|_\infty \leq C(\| u \|_\infty + \| v \|_\infty)\}.$$

Set

$$T_1(f,g) = \int_0^d \left(\frac{\int_{B_t(x)} f^{p-1}(y)g(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t},$$

$$T_2(f) = \int_{B_d(x)} \frac{f^p(y)}{|x-y|^{n-\alpha}} dy,$$

$$F(x) = \int_d^\infty \left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t},$$

$$G(x) = \int_{R^n \setminus B_d(x)} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy,$$

where d is a small number to be determined later. Write $T(f,g) = (T_1(f,g), T_2(f))$, obviously, (u,v) solves

$$(f,g) = T(f,g) + (F,G). \tag{5.4}$$

Step 1. T is a contracting map from X_1 to X.

In fact, for two functions $(f_1,g_1),(f_1,g_1) \in X_1$, we deduce that

$$\begin{aligned}
&\| T_1(f_1,g_1) - T_1(f_2,g_2) \|_\infty \\
\leq\ & C \int_0^d \frac{\int_{B_t(x)} |f_1^{p-1}g_1(y) - f_2^{p-1}g_2(y)|dy}{t^{n-\beta\gamma}} \left(\frac{\int_{B_t(x)} f_1^{p-1}g_1(y) + f_2^{p-1}g_2(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t} \\
\leq\ & C \int_0^d \frac{\int_{B_t(x)} |g_1(f_1^{p-1} - f_2^{p-1})| + |f_2^{p-1}(g_1 - g_2)|dy}{t^{n-\beta\gamma}} \left(\frac{\int_{B_t(x)} f_1^{p-1}g_1(y) + f_2^{p-1}g_2(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t} \\
\leq\ & C(\| u \|_\infty + \| v \|_\infty)^{\frac{p-1}{\gamma-1}} (\| f_1 - f_2 \|_\infty + \| g_1 - g_2 \|_\infty) \int_0^d t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} \\
\leq\ & Cd^{\frac{\beta\gamma}{\gamma-1}} (\| u \|_\infty + \| v \|_\infty)^{\frac{p-1}{\gamma-1}} (\| f_1 - f_2 \|_\infty + \| g_1 - g_2 \|_\infty),
\end{aligned} \tag{5.5}$$

here the third inequality is derived by Mean Value Theorem and the definition of X_1.

Similarly, we obtain

$$\| T_2(f_1) - T_2(f_2) \|_\infty \leq Cd^\alpha (\| u \|_\infty + \| v \|_\infty)^{p-1} \| f_1 - f_2 \|_\infty .$$

Choose d sufficiently small such that

$$Cd^{\frac{\beta\gamma}{\gamma-1}} (\| u \|_\infty + \| v \|_\infty)^{\frac{p-1}{\gamma-1}} \leq \frac{1}{2}, \qquad Cd^\alpha (\| u \|_\infty + \| v \|_\infty)^{p-1} \leq \frac{1}{2}.$$

Therefore, T is a contracting map.
Step 2. T is a shrinking map from Y_1 to Y.
In fact, for $(f,g) \in Y_1$ and for any $x_1, x_2 \in R^n$, we have

$$
\begin{aligned}
&|T_1(f,g)(x_1) - T_1(f,g)(x_2)| \\
\leq\ & C \int_0^d \frac{\int_{B_t(0)} |f^{p-1}g(x_1+y) - f^{p-1}g(x_2+y)|dy}{t^{n-\beta\gamma}} \left(\frac{\int_{B_t(x_1)} f^{p-1}g(y)dy + \int_{B_t(x_2)} f^{p-1}g(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t}.
\end{aligned}
$$

According to Mean Value Theorem and the fact that $(f,g) \in Y_1$, it follows that

$$
\begin{aligned}
&|f^{p-1}(x_1+y)g(x_1+y) - f^{p-1}(x_2+y)g(x_2+y)| \\
&\leq |g(x_1+y)(f^{p-1}(x_1+y) - f^{p-1}(x_2+y))| + |f^{p-1}(x_2+y)(g(x_1+y) - g(x_2+y))| \\
&\leq C(\| u \|_\infty + \| v \|_\infty)^{p-1}(\| f \|_{0,1} + \| g \|_{0,1})|x_1 - x_2|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|T_1(f,g)(x_1) - T_1(f,g)(x_2)| \\
&\leq C(\| u \|_\infty + \| v \|_\infty)^{\frac{p-1}{\gamma-1}}(\| f \|_{0,1} + \| g \|_{0,1})|x_1 - x_2| \int_0^d t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} \\
&\leq Cd^{\frac{\beta\gamma}{\gamma-1}}(\| u \|_\infty + \| v \|_\infty)^{\frac{p-1}{\gamma-1}}(\| f \|_{0,1} + \| g \|_{0,1})|x_1 - x_2|.
\end{aligned}
\tag{5.6}
$$

Choosing d sufficiently small such that

$$
\frac{|T_1(f,g)(x_1) - T_1(f,g)(x_2)|}{|x_1 - x_2|} \leq \frac{1}{2}(\| f \|_{0,1} + \| g \|_{0,1}).
$$

Similarly, we obtain

$$
\frac{|T_2(f)(x_1) - T_2(f)(x_2)|}{|x_1 - x_2|} \leq Cd^\alpha(\| u \|_\infty + \| v \|_\infty)^{p-1} \| f \|_{0,1} \ \leq \frac{1}{2} \| f \|_{0,1}.
$$

Therefore, T is a shrinking map.
Step 3. $(F,G) \in X_1 \cap Y_1$.
According to Theorem 5.1 and (1.1), we have $u, v \in L^\infty(R^n)$ and $F \leq u$, $G \leq v$, thus $(F,G) \in X_1$. Next, we prove $(F,G) \in Y_1$.
Write

$$
F(x) = \left(\int_d^1 + \int_1^\infty \right) \left(\frac{\int_{B_t(x)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t} := F_1(x) + F_2(x).
$$

Let $|x_1 - x_2| := \delta < 1/3$, for any $x_1, x_2 \in R^n$, without loss of generality, we assume $F_1(x_1) \leq F_1(x_2)$ and $F_2(x_1) \leq F_2(x_2)$. In view of $u, v \in L^\infty(R^n)$, we get

$$
\begin{aligned}
&F_1(x_2) - F_1(x_1) \\
&\leq C \int_d^1 \frac{\int_{B_t(x_2)\setminus B_t(x_1)} u^{p-1}v(y)dy}{t^{n-\beta\gamma}} \left(\frac{\int_{B_t(x_1)} u^{p-1}v(y)dy + \int_{B_t(x_2)} u^{p-1}v(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t} \\
&\leq C \int_d^1 \left(\frac{t^{n-1}\delta}{t^{n-\beta\gamma}} \right) \left(\frac{t^n}{t^{n-\beta\gamma}} \right)^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t} \\
&\leq Ct^{\frac{\beta\gamma}{\gamma-1}-1}\Big|_d^1 \delta = C(d)\delta.
\end{aligned}
$$

Here the second inequality is derived by the fact that when $|x_1 - x_2| = \delta$,

$$\text{the volume of } (B_t(x_2)\backslash B_t(x_1)) \cup (B_t(x_1)\backslash B_t(x_2)) \leq Ct^{n-1}\delta.$$

On the other hand, similar to (5.2), we also get

$$\begin{aligned} F_2(x_2) &\leq C\int_1^\infty \left(\frac{\int_{B_{t+\delta}(x_1)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\leq C\int_{1+\delta}^\infty \left(\frac{\int_{B_t(x_1)} u^{p-1}(y)v(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t}(1+\delta)^{\frac{n-\beta\gamma}{\gamma-1}+1} \\ &\leq C(1+\delta)^{\frac{n-\beta\gamma}{\gamma-1}+1} F_2(x_1). \end{aligned}$$

Therefore,

$$F_2(x_2) - F_2(x_1) \leq C\delta.$$

Combining the estimates of $F_1(x)$ and $F_2(x)$, we know that $F(x_2) - F(x_1) \leq C\delta$.

Similarly, by exchanging the integral variants,

$$G(x) = (n-\alpha)\int_d^\infty \frac{\int_{B_t(x)} u^p(y)dy}{t^{n-\alpha}}\frac{dt}{t},$$

we also prove that $G(x)$ is Lipschitz continuous. Thus, $(F, G) \in Y_1$.

Step 4. $T(\cdot,\cdot) + (F, G)$ is a map from $X_1 \cap Y_1$ to itself.

Similar to (5.5) and (5.6), we also have

$$\begin{aligned} \| T(f,g) \|_\infty &= \| (T_1(f,g), T_2(f)) \|_\infty = \| T_1(f,g) \|_\infty + \| T_2(f) \|_\infty \\ &\leq C(\| u \|_\infty + \| v \|_\infty)^{\frac{p}{\gamma-1}}(d^{\frac{\beta\gamma}{\gamma-1}} + d^\alpha), \end{aligned} \tag{5.7}$$

and

$$\| T(f,g) \|_{0,1} = \| T_1(f,g) \|_{0,1} + \| T_2(f) \|_{0,1} \leq C.$$

Moreover, (5.7) implies $\| T(f,g) \|_\infty \leq C(\| u \|_\infty + \| v \|_\infty)$ as long as d is chosen suitably small. Thus,

$$\| T(f,g) + (F,G) \|_\infty = \| T(f,g) \|_\infty + \| (F,G) \|_\infty \leq C(\| u \|_\infty + \| v \|_\infty).$$

Step 4 is verified.

Since (u, v) solves (5.4), according to Lemma 1.6, then $u, v \in C^{0,1}(R^n)$. □

Acknowledgements. This research was supported by NNSF (11871278) of China.

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