On the Wolff-type Integral System with Negative Exponents

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Abstract: In this paper, we are concerned with the positive continuous entire solutions of the Wolff-type integral system

$$\begin{cases} u(x) = C_1(x)W_{\beta,\gamma}(v^{-q})(x), \\ v(x) = C_2(x)W_{\beta,\gamma}(u^{-p})(x), \end{cases}$$

where $n \geq 1$, $\min\{p,q\} > 0$, $\gamma > 1$, $\beta > 0$ and $\beta\gamma \neq n$. In addition, $C_i(x)$ $(i = 1,2)$ are some double bounded functions. If $\beta\gamma \in (0,n)$, the Serrin-type condition is critical for existence of the positive solutions for some double bounded functions $C_i(x)$ $(i = 1,2)$. Such an integral equation system is related to the study of the γ-Laplace system and k-Hessian system with negative exponents. Estimated by the integral of the Wolff type potential, we obtain the asymptotic rates and the integrability of positive solutions, and studied whether the radial solutions exist.

Key words: Wolff type potential; Serrin-type condition; γ-Laplace system; k-Hessian system; Asymptotic limit.

1 Introduction

In this paper, we are concerned with the Wolff-type integral system

$$\begin{cases} u(x) = C_1(x)W_{\beta,\gamma}(v^{-q})(x), \\ v(x) = C_2(x)W_{\beta,\gamma}(u^{-p})(x), \end{cases} \tag{1.1}$$

where $n \geq 1$, $\min\{p,q\} > 0$, $\gamma > 1$, $\beta > 0$ and $\beta\gamma \neq n$. In addition, $C_1(x)$ and $C_2(x)$ are double bounded functions, namely, there exist $C \geq c > 0$ such that

$$c \leq C_i(x) \leq C \ (i = 1,2).$$

If $n < \beta\gamma$,

$$\begin{cases} W_{\beta,\gamma}(v^{-q})(x) = \int_0^\infty (\frac{\int_{B_t^c(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t}, \\ W_{\beta,\gamma}(u^{-p})(x) = \int_0^\infty (\frac{\int_{B_t^c(x)} u^{-p}(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t}. \end{cases}$$

If $n > \beta\gamma$,

$$\begin{cases} W_{\beta,\gamma}(v^{-q})(x) = \int_0^{\infty} (\frac{\int_{B_t(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t}, \\ W_{\beta,\gamma}(u^{-p})(x) = \int_0^{\infty} (\frac{\int_{B_t(x)} u^{-p}(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t}, \end{cases}$$

where $B_t(x)$ is a ball of radius t centered at x, and $B_t^c(x)$ be the complement of $B_t(x)$.

The Wolff potential $W_{\beta,\gamma}(f)$ of a positive function $f \in L^1_{loc}(R^n)$ was introduced in [7]. According to the different forms of Wolff potentials, we respectively discuss the properties of the positive continuous entire solutions in two cases: $n < \beta\gamma$ and $n > \beta\gamma$.

First we consider the case $n < \beta\gamma$.
If $\gamma = 2$ and $\beta = \frac{\alpha}{2}$, (1.1) is reduced to

$$\begin{cases} u(x) = C_1(x) \int_{\mathbb{R}^n} \frac{v^{-q}(y)}{|x-y|^{n-\alpha}} dy, \\ v(x) = C_2(x) \int_{\mathbb{R}^n} \frac{u^{-p}(y)}{|x-y|^{n-\alpha}} dy. \end{cases} \tag{1.2}$$

If $c(x) \equiv 1$, (1.1) becomes

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^{-q})(x), \\ v(x) = W_{\beta,\gamma}(u^{-p})(x), \end{cases} \tag{1.3}$$

and (1.2) is reduced to

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} \frac{v^{-q}(y)}{|x-y|^{n-\alpha}} dy, \\ v(x) = \int_{\mathbb{R}^n} \frac{u^{-p}(y)}{|x-y|^{n-\alpha}} dy. \end{cases} \tag{1.4}$$

An analogous integral system is the one with positive exponent

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} \frac{v^{q}(y)}{|x-y|^{n-\alpha}} dy, \\ v(x) = \int_{\mathbb{R}^n} \frac{u^{p}(y)}{|x-y|^{n-\alpha}} dy. \end{cases} \tag{1.5}$$

It is the Euler-Lagrange equation satisfied by the extremal functions of the Hardy-Littlewood-Sobolev inequality. Lieb proved in [20] that (1.5) has a pair of positive solutions in $L^{p+1}(R^n) \times L^{q+1}(R^n)$ if and only if $\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\alpha}{n}$. Many qualitative properties (including the radial symmetry, the integrability, and the estimates of decay rates) can be found in [2], [3], [8], [18] and the references therein. Those corresponding properties of the Wolff-type integral system with positive exponent

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^{q})(x), \\ v(x) = W_{\beta,\gamma}(u^{p})(x), \end{cases}$$

can be seen in [1], [21] and [11] respectively. In 2016, Lei and Li proved in [17] that

$$\begin{cases} u(x) = C_1(x) W_{\beta,\gamma}(v^{q})(x), \\ v(x) = C_2(x) W_{\beta,\gamma}(u^{p})(x), \end{cases} \tag{1.6}$$

has positive solutions for some double bounded functions $C_1(x)$ and $C_2(x)$, if and only if the following Serrin-type condition holds: $pq > (\gamma - 1)^2$ and

$$\max\{\frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}, \frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}\} < \frac{n-\beta\gamma}{\gamma-1}.$$

These finding provide a theoretical basis for our study of the integrability and asymptotic behavior of positive solutions.

For the system with negative exponents, (1.4) is the Euler-Lagrange equation satisfied by the extremal functions of the Hardy-Littlewood-Sobolev inequality (cf. [6] and [22]). The existence, the integrability, and the estimates of decay rates of positive solutions can be found in [12]. The single equation with the negative exponent is associated with the higher-order equation with lower-dimension appearing in the conformal geometry (cf. [19, 27, 28]).

For system (1.1) with $\beta\gamma > n$, the following

$$\max\{\frac{p(\beta\gamma-n)}{\gamma-1}, \frac{q(\beta\gamma-n)}{\gamma-1}\} > \beta\gamma, \tag{1.7}$$

is called the Serrin-type condition.

Theorem 1.1. *Let (u, v) be a pair of positive continuous entire solution of (1.1). If Serrin-type condition (1.7) does hold, then*

$$\min\{\frac{p(\beta\gamma-n)}{\gamma-1}, \frac{q(\beta\gamma-n)}{\gamma-1}\} > \beta\gamma. \tag{1.8}$$

In addition, there exist positive constants c and $C > 0$ such that for large $|x|$,

$$c|x|^{\frac{\beta\gamma-n}{\gamma-1}} \leq u(x) \leq C|x|^{\frac{\beta\gamma-n}{\gamma-1}}, \tag{1.9}$$

$$c|x|^{\frac{\beta\gamma-n}{\gamma-1}} \leq v(x) \leq C|x|^{\frac{\beta\gamma-n}{\gamma-1}}, \tag{1.10}$$

and

$$(u^{-1}, v^{-1}) \in L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n), \quad \forall\ s > \frac{n(\gamma-1)}{\beta\gamma-n}. \tag{1.11}$$

Theorem 1.2. *Let (u, v) be a pair of positive continuous entire solution of (1.3) and (1.7) does hold, then*

$$\lim_{|x|\to\infty} u(x)|x|^{\frac{n-\beta\gamma}{\gamma-1}} = \frac{\gamma-1}{\beta\gamma-n}[\int_{\mathbb{R}^n} v^{-q}(y)dy]^{\frac{1}{\gamma-1}},$$

$$\lim_{|x|\to\infty} v(x)|x|^{\frac{n-\beta\gamma}{\gamma-1}} = \frac{\gamma-1}{\beta\gamma-n}[\int_{\mathbb{R}^n} u^{-p}(y)dy]^{\frac{1}{\gamma-1}}.$$

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Theorem 1.3. *If Serrin-type condition (1.7) does hold, then the radial functions*

$$u(x) = (1+|x|^2)^{\theta_1}, \quad v(x) = (1+|x|^2)^{\theta_2},$$

solve (1.1) for some double bounded functions $C_1(x), C_2(x)$, where $2\theta_1 = 2\theta_2 = \frac{\beta\gamma-n}{\gamma-1}$.

Next, we consider the case $n > \beta\gamma$.

We know that the Wolff potential is helpful to well understand the nonlinear PDEs and other nonlinear problems (cf. [9, 10, 14, 16, 23]). For example, $W_{1,\gamma}(f)$ and $W_{\frac{2k}{k+1},k+1}(f)$ can be used to estimate involving positive solution of the γ-Laplace system

$$\begin{cases} -div(|\nabla u|^{\gamma-2}\nabla u) = v^{-q}, \\ -div(|\nabla v|^{\gamma-2}\nabla v) = u^{-p}, \end{cases} \tag{1.12}$$

and the k-Hessian system

$$\begin{cases} F_k[u] = v^{-q}, \\ F_k[v] = u^{-p}, \end{cases} \tag{1.13}$$

where $F_k[u] = S_k(\lambda(D^2u))$, $\lambda(D^2u) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, with λ_i being eigenvalues of the Hessian matrix(D^2u), and $S_k(\cdot)$ is the $k-th$ symmetric function

$$S_k(\lambda) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

Special cases of (1.13) are $F_1[u] = \Delta u$ and $F_n[u] = det(D^2u)$.

Consider the positive solution of integral equation with negative exponents

$$u(x) = C(x)\int_0^\infty (\frac{\int_{B_t(x)} u^{-p}(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}}\frac{dt}{t}. \tag{1.14}$$

In 2015, Lei obtained the nonexistence of the radial solution for (1.14) and showed that the positive solutions of (1.14) neither decay nor increase uniformly with power functions (cf. [13]). Those results are different from the properties of the Wolff-type equations with positive exponents in [4], [5], [15], [24], [25] and [26]. For system (1.1), we have the same conclusion.

Theorem 1.4. *For any double bounded function $C_1(x), C_2(x)$, (1.1) has no radial solution as the form $u(x) = (1+|x|^2)^{\frac{\theta_3}{2}}$ and $v(x) = (1+|x|^2)^{\frac{\theta_4}{2}}$, where θ_3 and θ_4 are any real numbers.*

Theorem 1.5. *Assume that (1.1) has a pair of positive solutions (u, v). Then there do not exist $C_1, C_2 > 0$ such that $u(x) \le C_1(1+|x|^2)^{-\frac{\theta_5}{2}}$ and $v(x) \le C_2(1+|x|^2)^{-\frac{\theta_6}{2}}$, where $\theta_5 \ge 0$ and $\theta_6 \ge 0$.*

Theorem 1.6. *Assume that (1.1) has a pair of positive solutions (u, v). Then there do not exist $C_1, C_2 > 0$ such that $u(x) \ge C_1(1+|x|^2)^{\frac{\theta_7}{2}}$ and $v(x) \ge C_2(1+|x|^2)^{\frac{\theta_8}{2}}$, where $\theta_7 > 0$ and $\theta_8 > \beta\gamma/q$, or $\theta_7 > \beta\gamma/p$ and $\theta_8 > 0$.*

We have the following corollary naturally.

Corollary 1.1. *The conclusions in Theorems 1.1-1.6 are still true for (1.2).*

Corollary 1.2. *For (1.12) and (1.13), the conclusions in Theorems 1.4-1.6 are still true.*

2 Case of $n < \beta\gamma$

In this section, we prove of Theorems 1.1-1.3.

Proof of Theorem 1.1. Without loss of generality, we assume that $0 < p \le q$. Thus by the Serrin-Type condition (1.7), there holds

$$q > \frac{\beta\gamma(\gamma-1)}{\beta\gamma - n}. \tag{2.1}$$

For $|x| > R$ with some large $R > 0$, we have

$$u(x) \ge c \int_0^{\frac{|x|}{2}} \Big(\frac{\int_{B_1(0)} v^{-q}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \ge c|x|^{\frac{\beta\gamma-n}{\gamma-1}}, \tag{2.2}$$

$$v(x) \ge c \int_0^{\frac{|x|}{2}} \Big(\frac{\int_{B_1(0)} u^{-p}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \ge c|x|^{\frac{\beta\gamma-n}{\gamma-1}}. \tag{2.3}$$

Similarly, for large $|x|$,

$$u_1(x) := C_1(x) \int_0^{2|x|} \Big(\frac{\int_{B_t^c(x)\bigcap B_R(0)} v^{-q}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \le C|x|^{\frac{\beta\gamma-n}{\gamma-1}}.$$

By (2.1) and (2.3), we deduce

$$\begin{aligned} u_2(x) :=& C_1(x) \int_0^{2|x|} \Big(\frac{\int_{B_t^c(x)\setminus B_R(0)} v^{-q}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \le& C \int_0^{2|x|} \Big(\frac{\int_R^\infty r^{n-q\frac{\beta\gamma-n}{\gamma-1}} \frac{dr}{r}}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \le& C|x|^{\frac{\beta\gamma-n}{\gamma-1}}, \end{aligned}$$

and

$$\begin{aligned} u_3(x) :=& C_1(x) \int_{2|x|}^\infty \Big(\frac{\int_{B_t^c(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \le& C \int_{2|x|}^\infty \Big(\frac{\int_{t-|x|}^\infty r^{n-q\frac{\beta\gamma-n}{\gamma-1}} \frac{dr}{r}}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \le& C|x|^{\frac{\beta\gamma-q\frac{\beta\gamma-n}{\gamma-1}}{\gamma-1}} \\ \le& C|x|^{\frac{\beta\gamma-n}{\gamma-1}}. \end{aligned}$$

Therefore, we get

$$u(x) := u_1(x) + u_2(x) + u_3(x) \le C|x|^{\frac{\beta\gamma-n}{\gamma-1}}.$$

This result, together with (2.2), implies (1.9).

Now, (1.9) leads to

$$\begin{aligned} \infty >& v(x) \ge c \int_{|x|}^\infty \Big(\frac{\int_{B_t^c(x)} u^{-p}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \ge& c \int_{|x|}^\infty \Big(\frac{\int_{t-|x|}^\infty r^{n-p\frac{\beta\gamma}{\gamma-1}}}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \ge& c \int_{|x|}^\infty t^{\frac{\beta\gamma-p\frac{\beta\gamma-n}{\gamma-1}}{\gamma-1}} \frac{dt}{t}. \end{aligned}$$

This implies $\beta\gamma - p\frac{\beta\gamma-n}{\gamma-1} < 0$, i.e. $p > \frac{\beta\gamma(\gamma-1)}{\beta\gamma-n}$. Therefore, (1.8) does hold. By the same way of derivation of (1.9), using (1.8) we also get (1.10).

Finally, we prove (1.11). By (1.9), there exists $R > 0$ such that

$$u(x) \geq c|x|^{\frac{\beta\gamma-n}{\gamma-1}}, \quad |x| > R.$$

Therefore, for each $s > \frac{n(\gamma-1)}{\beta\gamma-n}$,

$$\begin{aligned}\int_{\mathbb{R}^n} u^{-s}(x)dx \leq & \int_{B_R(0)} u^{-s}(x)dx + \int_{B_R^c(0)} u^{-s}(x)dx \\ \leq & C + C\int_R^\infty r^{n-s\frac{\beta\gamma-n}{\gamma-1}}\frac{dr}{r} \\ < & \infty.\end{aligned}$$

Similarly, $v^{-1} \in L^s(\mathbb{R}^n)$. $\square$

Proof of Theorem 1.2. According to Theorem 1.1, we know that (1.8) holds. Therefore, by (1.11), we see

$$\int_{\mathbb{R}^n} u^{-p}(x)dx < \infty, \quad \int_{\mathbb{R}^n} v^{-q}(x)dx < \infty.$$

Step 1. We claim that

$$\liminf_{|x|\to\infty} |x|^{\frac{n-\beta\gamma}{\gamma-1}} u(x) \geq \frac{\gamma-1}{\beta\gamma-n}(\int_{\mathbb{R}^n} v^{-q}(x)dx)^{\frac{1}{\gamma-1}}.$$

In fact, for any given $R > 0$ and $\varepsilon \in (0,1)$, when $t \in (0, \varepsilon|x|)$, we have $B_R(0) \subset B_t^c(x)$ for large $|x|$. Therefore,

$$\begin{aligned}u(x) \geq & \int_0^{\varepsilon|x|} (\frac{\int_{B_t^c(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}}\frac{dt}{t} \\ \geq & \int_0^{\varepsilon|x|} t^{\frac{\beta\gamma-n}{\gamma-1}}\frac{dt}{t}(\int_{B_R(0)} v^{-q}(y)dy)^{\frac{1}{\gamma-1}} \\ = & \frac{\gamma-1}{\beta\gamma-n}(\varepsilon|x|)^{\frac{\beta\gamma-n}{\gamma-1}}(\int_{B_R(0)} v^{-q}(y)dy)^{\frac{1}{\gamma-1}}.\end{aligned}$$

Therefore,

$$\liminf_{|x|\to\infty} |x|^{\frac{n-\beta\gamma}{\gamma-1}} u(x) \geq \frac{\gamma-1}{\beta\gamma-n}\varepsilon^{\frac{\beta\gamma-n}{\gamma-1}}(\int_{B_R(0)} v^{-q}(y)dy)^{\frac{1}{\gamma-1}}.$$

Letting $\varepsilon \to 1$ and $R \to \infty$ in the inequality above yields our claim.

Step 2. We claim that

$$\limsup_{|x|\to\infty} |x|^{\frac{n-\beta\gamma}{\gamma-1}} u(x) \leq \frac{\gamma-1}{\beta\gamma-n}(\int_{\mathbb{R}^n} v^{-q}(x)dx)^{\frac{1}{\gamma-1}}.$$

For any given $\varepsilon > 1$, we have

$$\begin{aligned}u(x) = & \int_0^{\varepsilon|x|} (\frac{\int_{B_t^c(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}}\frac{dt}{t} + \int_{\varepsilon|x|}^\infty (\frac{\int_{B_t^c(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}}\frac{dt}{t} \\ := & u_1(x) + u_2(x).\end{aligned}$$

Clearly,

$$\begin{aligned}u_1(x) &= \int_0^{\varepsilon|x|} \left(\frac{\int_{B_t^c(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\leq \int_0^{\varepsilon|x|} t^{\frac{\beta\gamma-n}{\gamma-1}} \frac{dt}{t} \left(\int_{\mathbb{R}^n} v^{-q}(y)dy\right)^{\frac{1}{\gamma-1}} \\ &= \frac{\gamma-1}{\beta\gamma-n} (\varepsilon|x|)^{\frac{\beta\gamma-n}{\gamma-1}} \left(\int_{\mathbb{R}^n} v^{-q}(y)dy\right)^{\frac{1}{\gamma-1}}.\end{aligned}$$

Letting $|x| \to \infty$ and then letting $\varepsilon \to 1$, we get

$$\limsup_{|x|\to\infty} |x|^{\frac{n-\beta\gamma}{\gamma-1}} u_1(x) \leq \frac{\gamma-1}{\beta\gamma-n} \left(\int_{\mathbb{R}^n} v^{-q}(x)dx\right)^{\frac{1}{\gamma-1}}.$$

Next, we claim that

$$\limsup_{|x|\to\infty} |x|^{\frac{n-\beta\gamma}{\gamma-1}} u_2(x) = 0. \tag{2.4}$$

In fact, by Theorem 1.1 we get

$$\begin{aligned}|x|^{\frac{n-\beta\gamma}{\gamma-1}} u_2(x) &= |x|^{\frac{n-\beta\gamma}{\gamma-1}} \int_{\varepsilon|x|}^{\infty} \left(\frac{\int_{B_t^c(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\leq C|x|^{\frac{n-\beta\gamma}{\gamma-1}} \int_{\varepsilon|x|}^{\infty} \left(\frac{\int_{t-|x|}^{\infty} r^{n-q\frac{\beta\gamma-n}{\gamma-1}} \frac{dr}{r}}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\leq C|x|^{\frac{n-\beta\gamma}{\gamma-1}} \int_{\varepsilon|x|}^{\infty} t^{\frac{\beta\gamma-q\frac{\beta\gamma-n}{\gamma-1}}{\gamma-1}} \frac{dt}{t} \\ &\leq C|x|^{\frac{n-q\frac{\beta\gamma-n}{\gamma-1}}{\gamma-1}}.\end{aligned}$$

Letting $|x| \to \infty$ and using (1.8), we obtain (2.4), and hence the claim is proved.

Combining Step 1 and Step 2, we obtained the asymptotic estimation of positive solution $u(x)$. Similarly, the asymptotic estimation of $v(x)$ is also obtained. $\square$

Proof of Theorem 1.3. Set

$$u(x) = (1+|x|^2)^{\theta_1}, \quad v(x) = (1+|x|^2)^{\theta_2},$$

where $2\theta_1 = 2\theta_2 = \frac{\beta\gamma-n}{\gamma-1}$.

Clearly, for $|x| \leq R$ with some $R > 0$, $W_{\beta,\gamma}(v^{-q})$ and $W_{\beta,\gamma}(u^{-p})$ are bounded, and hence they are proportional to $u(x)$ and $v(x)$, respectively.

For $|x| > R$, similar to the proof of Theorem 1.1, there exist positive constants c_1, c_2, C_1, C_2 such that

$$c_1|x|^{\frac{n-\beta\gamma}{\gamma-1}} \leq W_{\beta,\gamma}(v^{-q}) \leq C_1|x|^{\frac{n-\beta\gamma}{\gamma-1}},$$

$$c_2|x|^{\frac{n-\beta\gamma}{\gamma-1}} \leq W_{\beta,\gamma}(u^{-p}) \leq C_2|x|^{\frac{n-\beta\gamma}{\gamma-1}}.$$

Thus, for large $|x|$, there holds

$$c_1 W_{\beta,\gamma}(v^{-q}) \leq u(x) \leq C_1 W_{\beta,\gamma}(v^{-q}),$$

$$c_2 W_{\beta,\gamma}(u^{-p}) \leq v(x) \leq C_2 W_{\beta,\gamma}(u^{-p}),$$

for all $x \in R^n$. Write

$$C_1(x) = u(x)[W_{\beta,\gamma}(v^{-q})]^{-1},$$
$$C_2(x) = v(x)[W_{\beta,\gamma}(u^{-p})]^{-1}.$$

Then $C_1(x)$ and $C_2(x)$ are double bounded, and system (1.1) has a pair of radial solutions (u, v). $\square$

3 Case of $n > \beta\gamma$

In this section, we prove of Theorems 1.4-1.6.

Proof of Theorem 1.4. Without loss of generality, if (u, v) solves (1.1) with

$$u(x) = (1 + |x|^2)^{\frac{\theta_3}{2}}, \ v(x) = (1 + |x|^2)^{\frac{\theta_4}{2}}, \tag{3.1}$$

for some double bounded $C_1(x)$ and $C_2(x)$. There holds

$$\begin{aligned} u(x) \geq & c \int_{2|x|}^{\infty} \left(\frac{\int_{B_t(x)\setminus B_1(0)} (1 + |y|^2)^{-\frac{q\theta_4}{2}} dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \geq & c \int_{2|x|}^{\infty} \left(\frac{\int_1^{t-|x|} r^{n-q\theta_4} \frac{dr}{r}}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t}. \end{aligned} \tag{3.2}$$

If $\theta_4 \leq \frac{\beta\gamma}{q}$, by (3.2),

$$u(x) \geq c \int_{2|x|}^{\infty} t^{\frac{\beta\gamma - q\theta_4}{\gamma-1}} \frac{dt}{t} = \infty.$$

It is impossible. Thus $\theta_4 > \frac{\beta\gamma}{q}$.

Suppose for large $R > 0$, there exists $C > 0$ such that

$$v(x) \geq C|x|^{\theta_4}, \tag{3.3}$$

with $\theta_4 > \frac{\beta\gamma}{q}$ as $|x| > R$.

If $t \in (0, \frac{|x|}{2})$ and $y \in B_t(x)$, it follows that $\frac{|x|}{2} < |y| < \frac{3|x|}{2}$. Therefore, by (3.3), we have

$$\begin{aligned} u_1(x) :=& C_1(x) \int_0^{\frac{|x|}{2}} \left(\frac{\int_{B_t(x)} v^{-q}(y) dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \leq & C|x|^{-\frac{q\theta_4}{\gamma-1}} \int_0^{\frac{|x|}{2}} t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} \\ \leq & C|x|^{\frac{\beta\gamma - q\theta_4}{\gamma-1}}, \end{aligned}$$

$$\begin{aligned} u_2(x) :=& C_1(x) \int_{\frac{|x|}{2}}^{\infty} \left(\frac{\int_{B_t(x)\bigcap B_R(0)} v^{-q}(y) dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \leq & C \int_{\frac{|x|}{2}}^{\infty} t^{\frac{\beta\gamma - n}{\gamma-1}} \frac{dt}{t} \\ \leq & C|x|^{\frac{\beta\gamma - n}{\gamma-1}}, \end{aligned}$$

and

$$u_3(x) := C_1(x)\int_{\frac{|x|}{2}}^{\infty}\Big(\frac{\int_{B_t(x)\setminus B_R(0)} v^{-q}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t}$$
$$\le c\int_{\frac{|x|}{2}}^{\infty}\Big(\frac{\int_R^{|x|+t} r^{n-q\theta_4}\frac{dr}{r}}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t}.$$

In view of $\theta_4 > \frac{\beta\gamma}{q}$, there holds

$$u_3(x) \le C\int_{\frac{|x|}{2}}^{\infty} t^{\frac{\beta\gamma-q\theta_4}{\gamma-1}}\frac{dt}{t} \le C|x|^{\frac{\beta\gamma-q\theta_4}{\gamma-1}}, \qquad n > q\theta_4,$$

$$u_3(x) \le C\int_{\frac{|x|}{2}}^{\infty}\Big(\frac{\ln t}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t} \le C|x|^{\frac{\beta\gamma-n+\delta}{\gamma-1}},\ n = q\theta_4,$$

$$u_3(x) \le C\int_{\frac{|x|}{2}}^{\infty} t^{\frac{\beta\gamma-n}{\gamma-1}}\frac{dt}{t} \le C|x|^{\frac{\beta\gamma-n}{\gamma-1}}, \qquad n < q\theta_4,$$

where $\delta > 0$ is sufficiently small. Clearly,

$$u(x) := u_1(x) + u_2(x) + u_3(x) \le C|x|^{\frac{\mu_1}{\gamma-1}}, \tag{3.4}$$

where $\mu_1 := \max\{\beta\gamma - q\theta_4, \beta\gamma - n + \delta\}$. We can see that $\mu_1 < 0$. It contradicts with $\theta_4 > \frac{\beta\gamma}{q} > 0$. $\square$

Lemma 3.1. *Let (1.1) has a pair of positive continuous entire solutions (u, v). Then there exists $C > 0$ such that for all $x \in B_R(0)$ with any $R > 0$,*
if $1 < \gamma < 2$,

$$u(x) \ge CR^{\frac{\beta\gamma-n}{\gamma-1}}\Big(\int_{B_R(0)} v^{-q}(y)dy\Big)^{\frac{1}{\gamma-1}},$$

$$v(x) \ge CR^{\frac{\beta\gamma-n}{\gamma-1}}\Big(\int_{B_R(0)} u^{-p}(y)dy\Big)^{\frac{1}{\gamma-1}},$$

if $\gamma \ge 2$,

$$u(x) \ge CR^{\frac{\beta\gamma}{\gamma-1}-n}\int_{B_R(0)} v^{-\frac{q}{\gamma-1}}(y)dy,$$

$$v(x) \ge CR^{\frac{\beta\gamma}{\gamma-1}-n}\int_{B_R(0)} u^{-\frac{p}{\gamma-1}}(y)dy.$$

Proof of Lemma 3.1. Step 1. If $1 < \gamma < 2$, by the *Hölder* inequality, for any $R > 0$, there holds

$$\int_0^R\int_{B_t(x)} v^{-q}(y)dydt \le \Big[\int_0^R\Big(\frac{\int_{B_t(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t}\Big]^{\gamma-1}\Big(\int_0^R t^{\frac{n-\beta\gamma+1}{2-\gamma}}\frac{dt}{t}\Big)^{2-\gamma}$$
$$\le CR^{n-\beta\gamma+1}\Big[\int_0^R\Big(\frac{\int_{B_t(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t}\Big]^{\gamma-1}.$$

Therefore, for all $x \in B_R(0)$, by exchanging the integral variants, we have

$$
\begin{aligned}
u(x) \geq & c\int_0^{2R}(\frac{\int_{B_t(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}}\frac{dt}{t} \\
\geq & cR^{-\frac{n-\beta\gamma+1}{\gamma-1}}[\int_0^{2R}\int_{B_t(x)} v^{-q}(y)dydt]^{\frac{1}{\gamma-1}} \\
\geq & cR^{-\frac{n-\beta\gamma+1}{\gamma-1}}[\int_{B_{2R}(0)} v^{-q}(y)\int_{|x-y|}^{2R} dtdy]^{\frac{1}{\gamma-1}} \\
\geq & cR^{-\frac{n-\beta\gamma}{\gamma-1}}(\int_{B_R(0)} v^{-q}(y)dy)^{\frac{1}{\gamma-1}}.
\end{aligned}
$$

This is the estimation of $u(x)$. Similarly, $v(x)$ also has the corresponding result.
Step 2. If $\gamma \geq 2$, for any $R > 0$, there holds

$$
\int_{B_t(x)} v^{-\frac{q}{\gamma-1}}(y)dy \leq (\int_{B_t(x)} v^{-q}(y)dy)^{\frac{1}{\gamma-1}} t^{\frac{n(\gamma-2)}{\gamma-1}}.
$$

Thus for all $x \in B_R(0)$, we get by exchanging the integral variants that

$$
\begin{aligned}
u(x) \geq & c\int_0^{2R}(\frac{\int_{B_t(x)} v^{-q}(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}}\frac{dt}{t} \\
\geq & c\int_0^{2R}\int_{B_t(x)} v^{-\frac{q}{\gamma-1}}(y)dyt^{\frac{\beta\gamma}{\gamma-1}-n}\frac{dt}{t} \\
\geq & c\int_{B_{2R}(0)} v^{-\frac{q}{\gamma-1}}(y)\int_{|x-y|}^{2R} t^{\frac{\beta\gamma}{\gamma-1}-n}\frac{dt}{t}dy \\
\geq & cR^{\frac{\beta\gamma}{\gamma-1}-n}\int_{B_R(0)} v^{-\frac{q}{\gamma-1}}(y)dy.
\end{aligned}
$$

This is the estimation of u. Similarly, v also has the corresponding result. □

Proof of Theorem 1.5. We prove Theorem 1.5 by contradiction. Without loss of generality, if there exists $C_2 > 0$ such that for $|x| > R$ with large $R > 0$, there holds

$$
v(x) \leq C_2|x|^{-\theta_6}. \tag{3.5}
$$

Step 1. When $\theta_6 = 0$, (3.5) means

$$
v(x) \leq C_2.
$$

Therefore

$$
u(x) \geq c\int_1^{\infty}(\frac{\int_{B_t(x)} C_2^{-q}dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}}\frac{dt}{t} \geq c\int_1^{\infty} t^{\frac{\beta\gamma}{\gamma-1}}\frac{dt}{t} = \infty.
$$

It is impossible.
Step 2. When $\theta_6 > 0$, (3.5) shows that for all $s > \frac{n}{\theta_6}$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} v^s(y)dy \leq & \int_{B_R(0)} v^s(y)dy + \int_{B_R^c(0)} v^s(y)dy \\
\leq & C + C\int_R^{\infty} r^{n-s\theta_6}\frac{dr}{r} \\
< & \infty.
\end{aligned} \tag{3.6}
$$

Step 2.1. If $1 < \gamma < 2$, by Lemma 3.1, for any $R > 0$, and some large $s > \frac{n}{\theta_6}$, we have

$$
\begin{aligned}
CR^n =& \int_{B_R(0)} v^t(y) v^{-t}(y) dy \\
\leq & C(\int_{B_R(0)} v^s(y)dy)^{t/s} (\int_{B_R(0)} v^{-q}(y)dy)^{t/q} R^{n(1-t/s-t/q)} \\
\leq & CR^{\frac{t(n-\beta\gamma)}{q}+n(1-\frac{t}{s}-\frac{t}{q})} u^{\frac{t(\gamma-1)}{q}}(0)(\int_{B_R(0)} v^s(y)dy)^{\frac{t}{s}},
\end{aligned}
\tag{3.7}
$$

where $t \in (0, \min\{s, q\})$. Clearly, $\frac{t(n-\beta\gamma)}{q} + n(1 - \frac{t}{s} - \frac{t}{q}) - n < 0$. Letting $R \to \infty$ in (3.7) and using (3.6), we can get a contradiction.

Step 2.2. If $\gamma \geq 2$, by Lemma 3.1, for any $R > 0$ some large $s > \frac{n}{\theta_6}$, by *Hölder* inequality, we have

$$
\begin{aligned}
CR^n =& \int_{B_R(0)} v^t(y) v^{-t}(y) dy \\
\leq & C(\int_{B_R(0)} v^{-\frac{q}{\gamma-1}}(y)dy)^{\frac{(\gamma-1)t}{q}} (\int_{B_R(0)} v^s(y)dy)^{\frac{t}{s}} R^{n(1-\frac{t}{s}-\frac{\gamma-1}{q}t)} \\
\leq & CR^{(n-\frac{\beta\gamma}{\gamma-1})\frac{t(\gamma-1)}{q}+n(1-\frac{t}{s}-\frac{\gamma-1}{q}t)} u^{\frac{t(\gamma-1)}{q}}(0)(\int_{B_R(0)} v^s(y)dy)^{\frac{t}{s}},
\end{aligned}
\tag{3.8}
$$

where $t \in (0, \min\{s, \frac{q}{\gamma-1}\})$. Similarly, $(n - \frac{\beta\gamma}{\gamma-1})\frac{t(\gamma-1)}{q} + n(1 - \frac{t}{s} - \frac{\gamma-1}{q}t) - n < 0$. Thus, letting $R \to \infty$ in (3.8), we can get a contradiction. $\square$

Proof of Theorem 1.6. Without loss of generality, we suppose that there exist $C_1, C_2 > 0$, there holds

$$
u(x) \geq C_1|x|^{\theta_7}, v(x) \geq C_2|x|^{\theta_8}, \tag{3.9}
$$

for $|x| > R$ with large $R > 0$, where $\theta_7 > 0$ and $\theta_8 > \beta\gamma/q$. Then, for all $k_1 > \frac{n}{\theta_8}$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} v^{-k_1}(y)dy \leq & \int_{B_R(0)} v^{-k_1}(y)dy + \int_{B_R^c(0)} v^{-k_1}(y)dy \\
\leq & C + C\int_R^\infty r^{n-k_1\theta_8}\frac{dr}{r} \\
< & \infty.
\end{aligned}
\tag{3.10}
$$

Similarly, for all $k_3 > \frac{n}{\theta_7}$,

$$
\int_{\mathbb{R}^n} u^{-k_3}(y)dy < \infty. \tag{3.11}
$$

By the Wolff-type inequality (cf. Corollary 2.1 in [21]), we have

$$
\|u\|_{k_2} \leq C\|W_{\beta,\gamma}(v^{-q})\|_{k_2} \leq C\|v^{-1}\|_{sq}^{\frac{q}{\gamma-1}}, \tag{3.12}
$$

where $k_2, s > 1$ satisfy

$$
\frac{\gamma-1}{k_2} = \frac{1}{s} - \frac{\beta\gamma}{n}.
$$

In view of $\theta_8 > \beta\gamma/q$, we can choose sk_2 suitably large such that $sq > n/\theta_8$. Therefore, by (3.10), from (3.12) it follows that

$$u \in L^{k_2}(\mathbb{R}^n). \tag{3.13}$$

By (3.11) and (3.13), for any $R > 0$ and some small $L > 0$, we get

$$R^n = C\int_{B_R(0)} u^L(y)u^{-L}(y)dy \le C(\int_{B_R(0)} u^{-k_3}(y)dy)^{\frac{L}{k_3}}(\int_{B_R(0)} u^{k_2}(y)dy)^{\frac{L}{k}2} < \infty.$$

Letting $R \to \infty$ in the result above, we get a contradiction. $\square$

4 Proof of Corollaries

In this section, we prove Corollaries 1.1 and 1.2.

Proof of Corollary 1.1..

If $n > \alpha$, by exchanging the integral variants we get

$$\begin{aligned}
&\frac{1}{\alpha - n}\int_{\mathbb{R}^n} \frac{v^{-q}(y)}{|x-y|^{n-\alpha}}dy \\
&= \int_{\mathbb{R}^n} v^{-q}(y)dy \int_{|x-y|}^{\infty} t^{\alpha-n}\frac{dt}{t} \\
&= \int_0^{\infty} \frac{\int_{B_t(x)} v^{-q}(y)dy}{t^{n-\alpha}}\frac{dt}{t}.
\end{aligned}$$

Similarly, if $n < \alpha$, we have

$$\begin{aligned}
&\frac{1}{\alpha - n}\int_{\mathbb{R}^n} \frac{v^{-q}(y)}{|x-y|^{n-\alpha}}dy \\
&= \int_{\mathbb{R}^n} v^{-q}(y)dy \int_0^{|x-y|} t^{\alpha-n}\frac{dt}{t} \\
&= \int_0^{\infty} \frac{\int_{B_t^c(x)} v^{-q}(y)dy}{t^{n-\alpha}}\frac{dt}{t}.
\end{aligned}$$

Similarly, by the same argument on $u(x)$, we get

$$\frac{1}{\alpha - n}\int_{\mathbb{R}^n} \frac{u^{-p}(y)}{|x-y|^{n-\alpha}}dy = \int_0^{\infty} \frac{\int_{B_t(x)} u^{-p}(y)dy}{t^{n-\alpha}}\frac{dt}{t}.$$

$$\frac{1}{\alpha - n}\int_{\mathbb{R}^n} \frac{u^{-p}(y)}{|x-y|^{n-\alpha}}dy = \int_0^{\infty} \frac{\int_{B_t^c(x)} u^{-p}(y)dy}{t^{n-\alpha}}\frac{dt}{t}.$$

This show that (1.1) is equivalent to (1.2) with $\gamma = 2$ and $\beta\gamma = \alpha$. $\square$

Proof of Corollary 1.2.

According to the results in [9], [10] and [23], if (u, v) is a pair of positive entire solutions of (1.12) or (1.13) in R^n, we can find $C_1, C_2 > 0$ such that

$$C_1^{-1}W_{\beta,\gamma}(v^{-q})(x) \le u(x) \le C_1 W_{\beta,\gamma}(v^{-q})(x) + C_1 \inf_{R^n} u, \quad x \in \mathbb{R}^n, \tag{4.1}$$

$$C_2^{-1}W_{\beta,\gamma}(u^{-p})(x) \le v(x) \le C_2 W_{\beta,\gamma}(u^{-p})(x) + C_2 \inf_{R^n} v, \quad x \in \mathbb{R}^n. \tag{4.2}$$

Step 1. We claim that the conclusions of Lemma 3.1 and Theorem 1.5 are true for (1.12) or (1.13).

Without loss of generality, if there exist $C_1, C_2 > 0$ such that $u(x) \le C_1(1 + |x|^2)^{-\theta_5/2}$ and $v(x) \le C_2(1+|x|^2)^{-\theta_6/2}$ with $\theta_5, \theta_6 \ge 0$, then

$$\inf_{\mathbb{R}^n} v \le 1, \tag{4.3}$$

$$\inf_{\mathbb{R}^n} u \le 1. \tag{4.4}$$

Clearly, (4.4) implies

$$W_{\beta,\gamma}(u^{-p})(x) \ge \int_2^3 (\frac{|B_t(x)|}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t} \ge c > 0. \tag{4.5}$$

Combining (4.3) with (4.5) yields

$$\inf_{\mathbb{R}^n} v \le C W_{\beta,\gamma}(u^{-p})(x).$$

Inserting this into (4.2) leads to

$$C_2^{-1}W_{\beta,\gamma}(u^{-p})(x) \le v(x) \le C_2 W_{\beta,\gamma}(u^{-p})(x), \quad x \in \mathbb{R}^n. \tag{4.6}$$

Set

$$C_2(x) = v(x)(W_{\beta,\gamma}(u^{-p})(x))^{-1}.$$

Then $C_2(x)$ are double bounded and

$$v(x) = C_2(x) W_{\beta,\gamma}(u^{-p})(x).$$

By the same proof of Theorem 1.5, we also deduce a contradiction.

Step 2. (i) If $\theta_3, \theta_4 \le 0$, we know that the conclusions of Theorem 1.4 are true for (1.12) or (1.13) by the same argument in Step 1.

(ii) If $\theta_3, \theta_4 > 0$, we deduce a contradiction in two cases.

In the case of $0 < \theta_4 \le \frac{\beta\gamma}{q}$, there holds

$$\begin{aligned} W_{\beta,\gamma}(v^{-q})(x) \ge & c\int_{2|x|}^{\infty} (\frac{\int_{B_t(x)\setminus B_1(0)} (1+|y|^2)^{-\frac{q\theta_4}{2}} dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \ge & c\int_{2|x|}^{\infty} (\frac{\int_1^{t-|x|} (r^{n-q\theta_4} \frac{dr}{r}}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ \ge & c\int_{2|x|}^{\infty} t^{\frac{\beta\gamma-q\theta_4}{\gamma-1}} \frac{dt}{t} \\ = & \infty. \end{aligned}$$

Therefore, by (4.1) we see $u(x) = \infty$. It is impossible.

In the case of $\theta_4 > \frac{\beta\gamma}{q}$, similar to the calculation of (3.4), we also have

$$W_{\beta,\gamma}(v^{-q})(x) \le C_1|x|^{\sigma}, \quad |x| > R,$$

where R is a suitably large positive constant, and

$$\sigma = \frac{1}{\gamma - 1} \max\{\beta\gamma - q\theta_4, \beta\gamma - n + \delta\},$$

with $\delta > 0$ is sufficiently small. Thus, $\sigma < 0$.

In addition,

$$\inf_{\mathbb{R}^n} u \leq u(0). \tag{4.7}$$

Therefore, by (4.1), we get

$$u(x) \leq C(|x|^\sigma + u(0)) \leq C, \quad |x| > R.$$

This is contradicts $u(x) \geq C_1(1 + |x|^2)^{\theta_3}$ with $\theta_3 > 0$.

(iii) When $\theta_3\theta_4 < 0$, without loss of generality, we assume $\theta_3 > 0$, $\theta_4 < 0$. By (4.1), we have

$$u(x) \leq CW_{\beta,\gamma}(v^{-q})(x) + \inf_{\mathbb{R}^n} u,$$

for $x \in R^n$. Thus,

$$W_{\beta,\gamma}(v^{-q})(x) \geq C^{-1}(u(x) - u(0)) \geq c(1 + |x|^2)^{\theta_3/2} \geq c,$$

as long as $|x| > R$ with some large $R > 0$. Combining with (4.7), we also get

$$\inf_{\mathbb{R}^n} u \leq CW_{\beta,\gamma}(v^{-q})(x).$$

Inserting this into (4.1) yields

$$C_1^{-1} W_{\beta,\gamma}(v^{-q})(x) \leq u(x) \leq C_1 W_{\beta,\gamma}(v^{-q})(x),$$

for $|x| > R$. Set

$$C_1(x) = u(x)(W_{\beta,\gamma}(v^{-p})(x))^{-1}.$$

Then $C_1(x)$ are double bounded and

$$u(x) = C_1(x) W_{\beta,\gamma}(v^{-q})(x).$$

By the same proof of Theorem 1.4, we also deduce a contradiction.

Step 3. The conclusions of Theorem 1.6 are still true for (1.12) or (1.13). The proof is same of (3) in Step 2. □

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