Ground state solutions of Pohožaev type for Kirchhoff type problems with general convolution nonlinearity and variable potential

Qiongfen[1,2] Zhang[*], Hai Xie [1,2], Yi-rong Jiang[1,2]

[1] College of Science, Guilin University of Technology, Guilin, Guangxi 541004, PR China

[2] Guangxi Colleges and Universities Key Laboratory of Applied Statistics, Guilin, Guangxi 541004, PR China

Abstract: This paper is devoted to dealing with the following nonlinear Kirchhoff type problem with general convolution nonlinearity and variable potential:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u + V(x)u = (I_\alpha * F(u))f(u), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $a > 0$, $b \geq 0$ are constants, $V \in C^1(\mathbb{R}^3, [0, +\infty))$, $f \in C(\mathbb{R}, \mathbb{R})$, $F(t) = \int_0^t f(s)ds$, $I_\alpha : \mathbb{R}^3 \to \mathbb{R}$ is the Riesz potential, $\alpha \in (0, 3)$. By applying some new analytical tricks introduced by [X.H. Tang, S.T. Chen, Adv. Nonlinear Anal. 9 (2020) 413-437], the existence results of ground state solutions of Pohožaev type for the above Kirchhoff type problem are obtained under some mild assumptions on V and the general "Berestycki-Lions assumptions" on the nonlinearity f. Our results generalize and improve the ones in [P. Chen, X.C. Liu, J. Math. Anal. Appl. 473 (2019) 587-608.] and other related results in the literature.

Keywords: Berestycki-Lions assumptions; Ground state solutions; Convolution nonlinearity; Variable potential; Kirchhoff type problem

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1. Introduction

In this paper, the following nonlinear Kirchhoff type problem with general convolution nonlinearity and

[*]Corresponding author: qfzhangcsu@163.com, qfzhang@glut.edu.cn (Q. F. Zhang); http://orcid.org/0000-0002-7037-1961.

variable potential is considered:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u + V(x)u = (I_\alpha * F(u))f(u), \quad \text{in } \ \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $a > 0$, $b \geq 0$ are constants, $\alpha \in (0,3)$, the Riesz potential $I_\alpha : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{3-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha \pi^{\frac{3}{2}}|x|^{3-\alpha}}, \quad x \in \mathbb{R}^3\backslash\{0\},$$

$F(t) = \int_0^t f(s)ds$, $f : \mathbb{R} \to \mathbb{R}$ and $V : \mathbb{R}^3 \to \mathbb{R}$. We assume that the nonlinearity f, F and the potential V satisfy the following basic conditions:

(V1) $V \in C(\mathbb{R}^3, [0, +\infty))$ and $V_\infty := \lim_{|x|\to\infty} V(x) \geq (\neq) V(x)$, $\forall\ x \in \mathbb{R}^3$;

(S1) $f \in C(\mathbb{R}, \mathbb{R})$ and there is a constant $c_0 > 0$ such that

$$|f(t)t| \leq c_0(|t|^{1+\alpha/3} + |t|^{3+\alpha}), \quad \forall\ t \in \mathbb{R};$$

(S2) $\lim_{|t|\to 0} \frac{F(t)}{|t|^{1+\alpha/3}} = 0$ and $\lim_{|t|\to\infty} \frac{F(t)}{|t|^{3+\alpha}} = 0$;

(S3) there exists $t_0 > 0$ such that $F(t_0) \neq 0$.

When $b = 0$ and $a = 1$, then problem (1.1) becomes to the following form with variable potential:

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * F(u))f(u), \quad \text{in } \ \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.2}$$

which is called generalized Choquard equation. In [1], Moroz and Van Schaftingen introduced (S1)-(S3) to investigate ground state solutions for problem (1.2). It is known that (S1)-(S3) were regarded as the Berestycki-Lions type conditions and were almost necessary and sufficient conditions for studying Choquard equations.

When the potential $V(x)$ is always equal to 1, $f(u) = u$ and $\alpha = 2$, problem (1.2) reduces to the well known Choquard-Pekar equation, which was introduced by Pekar [2] in 1954 for describing the quantum mechanics of a polaron at rest. It is also known as the Schrödinger-Newton equation or the stationary Hartree equation. Now, let's recall some works on Choquard equations. In [3], Lieb obtained some existence and uniqueness results for a nonlinear Choquard equation; Clapp and Salazar [4] studied positive and sign changing solutions for a kind of nonlinear Choquard equation in an exterior domain of $\mathbb{R}^N$ and obtained some existence results under some symmetry assumptions on the exterior domain and the potential; Chen, Tang and Wei [5] investigated Nehari-type ground state solutions for a kind of Choquard equation with doubly critical exponents; Tang, Wei and Chen [6] obtained existence results of Nehari-type ground state solutions for a kind of Choquard equation with local nonlinear perturbation and lower critical exponent; In [7], Chen and Tang established existence results of ground state solutions for a general Choquard equation; Li, Li and Tang [8] considered ground state solutions for a class of Choquard equations with potential vanishing at infinity which is Hardy-Littlewood-Sobolev upper critical growth; In [9], Deng, Jin and Shuai considered positive ground state solutions for Choquard equations; There are also semiclassical state solutions [10], bound state solutions [11], nontrivial solutions [12] for other kinds of Choquard equations. However, there is few literature which only uses (S1)-(S3) to deal with Choquard equations. As far as we known, only Tang and Chen [13] used (S1)-(S3) to investigate a kind of singularly perturbed Choquard equations by introducing some new techniques and obtained some nice results, which generalize and improve many works in the literature.

When $b \neq 0$, problem (1.1) is Kirchhoff type problem with variable potential and general convolution nonlinearity. Now, let us review some results about (1.1). The Kirchhoff type problems appear in real world with an interesting physical background. Indeed, if we let $\alpha \to 0$ in (1.1), then problem (1.1) reduces to the following Kirchhoff type problem

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u = g(u), \quad x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.3}$$

where $a, b > 0$ are constants, $g = Ff$ is local nonlinearity, $V \in C(\mathbb{R}^3, \mathbb{R})$. Such a problem is seen as being nonlocal since problem (1.3) is no longer a pointwise identity due to the presence of $\int_{\mathbb{R}^3} |\nabla u|^2 dx$. Problem (1.3) is related to the stationary analogue of the following Kirchhoff equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.4}$$

where ρ is the mass density, h is the area of cross-section, ρ_0 is the initial tension, L is the length of the string and E is the Young modulus of the material. When Kirchhoff [14] investigated the free vibrations of elastic strings, he extend the classical D'Alembert's wave equation to this situation, and then put forward problem (1.4), which considers the changes in length of the string caused by transverse vibrations. Please see [15, 16] and reference therein for more details of mathematical and physical background.

Kirchhoff equation (1.3) has received more and more attention from mathematical community after Lions [17] brought forward an abstract functional analysis framework for studying it. Via the variational method, there have been many important results on the existence and multiplicity of solutions for problem (1.3), when the nonlinearity g is autonomous or nonautonomous and satisfies different kinds of conditions, see for example [18] -[36] and the references therein. A classical way to deal with (1.3) is to use the mountain-pass theorem. To this end, one usually assumes that the potential $V(x)$ is periodic or is radial or $V(x) \equiv 1$, while the nonlinearity $g(t)$ is subcritical and satisfies the following condition:

(G1) $g(t)/|t|^3$ is increasing for $t \in \mathbb{R} \backslash \{0\}$;

or satisfies the following classical Ambrosetti-Rabinowitz type condition

(AR) there exists $\mu > 4$ such that $0 \leq \mu G(t) \leq g(t)t$, $\forall\ t \in \mathbb{R}$, where $G(t) = \int_0^t g(s)ds$.

Under (G1) or (AR), one can easily verify the Mountain Pass geometry and the boundedness of Palais-Smale sequences (PS-sequences for short) for the energy functional. The existence result of ground state solutions for problem (1.3) was first proved by He and Zou [37]. When the nonlinearity in problem (1.3) is a special form with respect to u, for example, problem (1.3) reduces to the following form

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V(x)u=|u|^{p-2}u, & x\in\mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.5}$$

Li and Ye [30] first established the existence of positive ground state solutions when $2<p\leq 4$ and $V(x)\equiv 1$ by applying a minimizing argument on a new manifold

$$\widetilde{M}=\{u\in H^1(\mathbb{R}^3):\langle \Phi_0'(u),u\rangle+P_0(u)=0\},$$

where $\Phi_0(u)$ and $P_0(u)$ are the energy functional and the Pohožaev equality for problem (1.5), respectively. The idea used in [30] comes from [38], in which a kind of nonlinear Schrödinger-Poisson system was investigated. Subsequently, Guo [39] generalized the results of [30] to problem (1.3). Later, Chen and Tang [19], Tang and Chen [35] improved the above results to problem (1.3) under (V1), some standard growth assumptions on g, and the following additional conditions:

(V2)' $V\in C^1(\mathbb{R}^3,\mathbb{R})$ and there is $\theta'\in(0,1)$ such that

$$\nabla V(x)\cdot x\leq\frac{\theta' a}{2|x|^2},\quad \text{a.e. } x\in\mathbb{R}^3\backslash\{0\};$$

(G2) $g\in C(\mathbb{R},\mathbb{R})$ and $\frac{g(t)t+6G(t)}{t|t|}$ is nondecreasing on $(-\infty,0)\cup(0,+\infty)$;

(G3) $g\in C^1(\mathbb{R}^+,\mathbb{R})$ and $\frac{g(t)}{t}$ is increasing on $(0,+\infty)$.

Compared with problem (1.2) and problem (1.3), it feels more difficult to study problem (1.1) since it contains two nonlocal terms. Chen and Liu [40] studied the following Kirchhoff type equation

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V(x)u=(I_\alpha*|u|^p)|u|^{p-2}u, & x\in\mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.6}$$

they obtained a ground state solution for all $p\in(1+\alpha/3,3+\alpha)$ under some assumptions on the potential $V(x)$. In [41], Lü investigated (1.6) with $p\in(2,3+\alpha)$ and $V(x)=1+\mu h_0(x)$, where $h_0(x)\geq 0$ is a steep potential well function and $\mu>0$ is a parameter. By applying the concentration compactness principle and the Nehari manifold, some existence results of ground state solutions for problem (1.6) were obtained by Lü when μ is sufficiently large. We must point out that the results obtained in [40] fulfill the gap of $p\in(1+\alpha/3,2]$ and improve the result obtained in [41]. Very recently, by applying variational methods and some new analytical skills, Chen, Zhang and Tang [42] obtained some results for problem (1.1) under (V1), (V2), (S2) and the following conditions:

(S2)' $\lim_{|t|\to\infty}\frac{F(t)}{|t|^{1-\alpha}}=\infty$;

(S3)' the function $\frac{f(t)t+(3+\alpha)F(t)}{t|t|^{-\alpha}}$ is nondecreasing on $(-\infty,0)\cup(0,+\infty)$.

(V3)' $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and there is $\theta \in [0,1)$ such that for every $x \in \mathbb{R}^3 \backslash \{0\}$, $t \mapsto 4V(tx) + \nabla V(tx) \cdot tx + \frac{\theta a}{2t^2|x|^2}$ is nonincreasing on $(0, +\infty)$.

We must point out that when $f(u) = |u|^{p-2}u$, the results obtained in [42] also fill the gap with $p \in (1 + \alpha/3, 2]$ and cover many results in the literature, such as the results in [30, 35, 39] when $\alpha \to 0$. Besides, the results obtained in [42] are more general than those in [40] since the nonlinearity F is more general than that of [40]. However, there seems to be no results for (1.1) when the nonlinearity F is more general which only satisfies (S1)-(S3). Our main purpose is to deal with this case. For some $s \in (2, 2^*)$ and any $\varepsilon > 0$, by Hardy-Littlewood-Sobolev inequality, (S1) and (S2), one has

$$\begin{aligned} \int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)dx &= \frac{\Gamma(\frac{3-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha \pi^{3/2}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{F(u(x))F(u(y))}{|x-y|^{3-\alpha}}dxdy \\ &\leq C_0 \|F(u)\|^2_{6/(3+\alpha)} \\ &\leq \varepsilon(\|u\|_2^{2(3+\alpha)/3} + \|u\|_{2^*}^{2(3+\alpha)}) + C_\varepsilon \|u\|_s^{s(3+\alpha)/3}, \quad \forall\, u \in H^1(\mathbb{R}^3), \end{aligned} \tag{1.7}$$

where C_0 ia a positive constant and C_ε is a positive constant which depends on ε. Under (V1), (S1), (S2) and using (1.7), it is standard to check that the energy functional for problem (1.1) defined by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + V(x)u^2]dx - \frac{1}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2, \quad \forall\, u \in H^1(\mathbb{R}^3) \tag{1.8}$$

is of C^1 and the critical points of (1.8) correspond to the weak solutions of problem (1.1).

When the potential $V(x)$ is a constant V_∞, problem (1.1) becomes to the following so called "limiting problem" which is autonomous:

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V_\infty u = (I_\alpha * F(u))f(u), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.9}$$

whose energy functional is defined as follows:

$$\Phi^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + V_\infty u^2]dx - \frac{1}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2, \quad \forall\, u \in H^1(\mathbb{R}^3). \tag{1.10}$$

The Pohožaev type identity corresponds to problem (1.9) is defined as follows:

$$P^\infty(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{3V_\infty}{2}\|u\|_2^2 - \frac{3+\alpha}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)dx + \frac{b}{2}\|\nabla u\|_2^4 = 0, \quad \forall\, u \in H^1(\mathbb{R}^3). \tag{1.11}$$

The following set is related to $P^\infty(u)$ which is defined as

$$M^\infty := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : P^\infty(u) = 0\}.$$

In view of [39], if u is a solution of problem (1.9), then it must satisfy (1.11). Hence, M^∞ is a natural constraint for Φ^∞. In most of the previous literature, the obtained least energy solution u_0 of problem (1.9) satisfies $\Phi^\infty(u_0) \geq \inf_{M^\infty} \Phi^\infty$. There is a natural question: can one find a solution $\tilde{u} \in M^\infty$ such that

$$\Phi^\infty(\tilde{u}) = \inf_{M^\infty} \Phi^\infty. \tag{1.12}$$

Motivated mainly by [13, 19, 40, 42], we will use a more direct method to prove some existence results on ground state solutions for problem (1.1). A weak solution of problem (1.1) is called a ground state solution if it has minimal "energy" Φ among all nontrivial weak solutions. Moreover, the ground state solution obtained in this paper also minimizes the functional Φ on Pohožaev manifold associated with problem (1.1), under (V1), (S1)-(S3) and the following growth condition on V:

(V2) $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and there exists $\theta \in [0, 1)$ such that $t \mapsto \frac{3V(tx)+\nabla V(tx)\cdot(tx)}{t^\alpha} + \frac{\theta a}{4t^{2+\alpha}|x|^2}$ is nonincreasing on $(0, +\infty)$ for every $x \in \mathbb{R}^3\backslash\{0\}$.

To state our results, similar to (1.11), we define the Pohožaev functional on $H^1(\mathbb{R}^3)$ as follows:

$$P(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[3V(x) + \nabla V(x) \cdot x]u^2dx - \frac{3+\alpha}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx + \frac{b}{2}\|\nabla u\|_2^4 = 0. \quad (1.13)$$

Similarly, we set

$$M := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : P(u) = 0\}.$$

Then one knows that every nontrivial solution of problem (1.1) is contained in M. Now, we can state our first main result.

Theorem 1.1. *Suppose that* (V1)-(V2) *and* (S1)-(S3) *hold. Then problem* (1.1) *admits a solution* $\tilde{u} \in H^1(\mathbb{R}^3)$ *such that* $\Phi(\tilde{u}) = \inf_M \Phi = \inf_{u\in\Upsilon} \max_{t>0} \Phi(u_t) > 0$*, where*

$$\Upsilon := \left\{u \in H^1(\mathbb{R}^3) : 0 < \int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx\right\} \text{ and } u_t := u_t(x) = u(t^{-1}x).$$

Since problem (1.9) is autonomous form of problem (1.1), the following corollary follows from Theorem 1.1 obviously.

Corollary 1.2. *Suppose that* (S1)-(S3) *hold. Then problem* (1.9) *admits a solution* $\tilde{u} \in H^1(\mathbb{R}^3)$ *such that* $\Phi^\infty(\tilde{u}) = \inf_{M^\infty} \Phi^\infty = \inf_{u\in\Upsilon} \max_{t>0} \Phi^\infty(u_t) > 0$.

For the autonomous problem (1.9), we can easily show that its least energy solution corresponds to the obtained solution $\tilde{u}$ in the above Corollary 1.2 under the Pohožaev type identity (1.11). Specially, we estabilsh the following result.

Theorem 1.3. *Suppose that* (S1)-(S3) *hold. Then problem* (1.9) *admits a solution* $\tilde{u} \in H^1(\mathbb{R}^3)$ *such that* $\Phi^\infty(\tilde{u}) = \inf_{M^\infty} \Phi^\infty = \inf\{\Phi^\infty(u) : u \in H^1(\mathbb{R}^3)\backslash\{0\}$ *is a soltuion of problem* (1.9)$\}$.

Remark 1.4. *As pointed out in [13], from Theorem 1.1, we know that the least energy value* $m := \inf_M \Phi$ *possesses a minimax characterization* $m := \inf_{u\in\Upsilon} \max_{t>0} \Phi(u_t)$*, which seems much simpler than the usual characterization related to the Mountain Pass level.*

In order to obtain the existence of the least energy solutions for problem (1.1), besides the conditions (V1) and (S1)-(S3), we also need the following decay assumption on ∇V which is weaker than (V2):

(V3) $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and there is $\theta' \in [0,1)$ and $\tilde{R} \geq 0$ such that

$$\nabla V(x) \cdot x \leq \begin{cases} \theta' \alpha V(x), & |x| \geq \tilde{R}; \\ \frac{a}{2|x|^2}, & 0 < |x| \leq \tilde{R}. \end{cases}$$

Now, we can state the following theorem.

Theorem 1.5. *Suppose that* (V1), (V3) *and* (S1)-(S3) *hold. Then problem* (1.1) *admits a solution* $\tilde{u} \in H^1(\mathbb{R}^3)$ *such that* $\Phi(\tilde{u}) = \inf_\Lambda \Phi$, *where*

$$\Lambda := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \Phi'(u) = 0\}.$$

Remark 1.6. *In fact, there are many functions which satisfy* (V1)-(V3). *For example,* $V(x) = A_1 - \frac{A_2}{1+|x|^3}$ *satisfies* (V1) *and* (V3) *for* $0 < (3+\alpha)A_2 < \alpha A_1$ *and* $V(x) = A_3 - A_4 e^{-|x|^\alpha}$ *for* $A_3 > A_4 > 0$ *satisfies* (V1) *and* (V2). *The results obtained in this paper generalize and improve some previous results on problem* (1.1) *in the literature, which seem also new for the "limiting problem" of problem* (1.1), *that is* $V(x) \equiv V_\infty$. *Specially speaking, Theorem 1.1 and Theorem 1.3 complete the gap of* $p \in (1+\alpha/3, 2]$ *when* $f(u) = |u|^{p-2}u$.

Remark 1.7. *Let* $\alpha \to 0$, *the results obtained in this paper also cover many results in the literature, for example, the ones in* [19, 21, 22, 30, 35, 39, 40], *in which the nonlinearity is a special form of* $(I_\alpha * F(u))f(u)$. *Besides, the nonlinearity* F *in this paper only need to satisfy the Berestycki-Lion type conditions* (S1)-(S3), *which seem more simpler.*

To prove Theorem 1.1, motivated by [13], firstly, choosing a sequence $\{u_n\}$ of Φ on M such that

$$P(u_n) = 0, \quad \Phi(u_n) \to m := \inf_M \Phi. \tag{1.14}$$

Secondly, proving that $u_n \rightharpoonup \tilde{u} \in H^1(\mathbb{R}^3)\backslash\{0\}$ by "the least energy squeeze approach" and the concentration-compactness argument, and then checking that $\tilde{u} \in M$ and $\Phi(\tilde{u}) = \inf_M \Phi$. Finally, checking that $\tilde{u}$ is a critical point of Φ. We would like to point out that it is very difficult to prove the solution $\tilde{u} \in M$ and $\Phi(\tilde{u}) = \inf_M \Phi$ because of lack of adequate information on $\Phi'(u_n)$ and global compactness. To deal with this difficulty, we construct an important inequality which is related to $\Phi(u)$, $\Phi(u_t)$ and $P(u)$. We point out that this inequality is very important in the sequent arguments. Compared with most of the existence of results obtained in the existing literature, we do not need to compare the critical level of problem (1.1) with the one of the "limiting problem" (1.9). Besides, we do not need to construct the following strict inequality:

$$\max_{t\in[0,1]} \Phi(x_0(t)) < \inf\{\Phi^\infty(u) : u \in H^1(\mathbb{R}^3)\backslash\{0\} \text{ is a solution of problem (1.9)}\} \tag{1.15}$$

for some path $x_0 \in C([0,1], H^1(\mathbb{R}^3))$. It is easy to see that $x_0(t) > 0$ under (V1). It is also known that (1.15) is often proved under (V1), (S1)-(S3) and other additional assumptions on f, such as $tf(t) \geq 0$ and f is odd. However, we do not need the strict inequality (1.15) in our proofs in this paper. Our approach could be used for studying problems where the ground state solutions or paths of the problems at infinity are indefinite.

To prove Theorem 1.5, we borrow the idea from Jeanjean-Tanaka [43], that is, a bounded (PS)-sequence for Φ is obtained by using an approximation procedure, but not starting directly from an arbitrary (PS)-sequence. Specially speaking, for $\lambda \in [1/2, 1]$, a family of functionals $\Phi_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}[a|\nabla u|^2 + V(x)u^2]dx - \frac{\lambda}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 \tag{1.16}$$

are considered. These functionals have a Mountain Pass geometry, whose corresponding Mountain Pass levels are denoted by c_λ. Corresponding to (1.16), we also define

$$\Phi_\lambda^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3}[a|\nabla u|^2 + V_\infty u^2]dx - \frac{\lambda}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2, \quad \forall\, u \in H^1(\mathbb{R}^3). \tag{1.17}$$

From Corollary 1.2, there exists a minimizer u_λ^∞ of Φ_λ^∞ on M_λ^∞ for every $\lambda \in [1/2, 1]$, where the set M_λ^∞ is defined as follows

$$M_\lambda^\infty := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : P_\lambda^\infty(u) = 0\} \tag{1.18}$$

and

$$P_\lambda^\infty(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{3V_\infty}{2}\|u\|_2^2 - \frac{(3+\alpha)\lambda}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx + \frac{b}{2}\|\nabla u\|_2^4, \quad \forall\, u \in H^1(\mathbb{R}^3). \tag{1.19}$$

Let

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^3}[a|\nabla u|^2 + V(x)u^2]dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2, \quad B(u) = \frac{1}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx, \quad \forall\, u \in H^1(\mathbb{R}^3).$$

Then $\Phi_\lambda(u) = A(u) - \lambda B(u)$. Because $B(u)$ is indefinite, we can't use the classical monotonicity trick. Moreover, since the minimizer u_λ^∞ is not positive definite, it is also more difficult to prove the following key inequality

$$c_\lambda < m_\lambda^\infty := \inf_{u\in M_\lambda^\infty} \Phi_\lambda^\infty(u)\ (= \Phi_\lambda^\infty(u_\lambda^\infty)), \quad \lambda \in [1/2, 1]. \tag{1.20}$$

By the excellent work of Jeanjean-Tolan [44], for a.e. $\lambda \in [1/2, 1]$, the functional Φ_λ still possesses a bounded (PS)-sequence $\{u_n(\lambda)\} \subset H^1(\mathbb{R}^3)$ at level c_λ. Different from the classical way to obtain (1.20) in the existing literature, following the strategy in [13], by means of u_1^∞ and the following key inequality established in Lemma 2.2

$$\Phi(u) \geq \Phi(u_t) + \frac{1-t^{3+\alpha}}{3+\alpha}P(u) + \frac{a(1-\theta)g(t)}{2(3+\alpha)}\|\nabla u\|_2^2 + \frac{bi(t)}{4(3+\alpha)}\|\nabla u\|_2^4, \quad \forall\, u \in H^1(\mathbb{R}^3),\ t > 0, \tag{1.21}$$

we can find $\bar\lambda \in [1/2, 1]$ such that

$$c_\lambda < m_\lambda^\infty, \quad \lambda \in (\bar\lambda, 1]. \tag{1.22}$$

In our arguments, we do not need any information on sign of u_1^∞. By the idea of a precise decomposition of bounded (PS)-sequence in [43] and (1.22), one can obtain a nontrivial critical point u_λ of Φ_λ with $c_\lambda = \Phi(u_\lambda)$

for almost every $\lambda \in (\bar{\lambda}, 1]$. Finally, we prove that problem (1.1) has a least energy solution under (V1), (V3) and (S1)-(S3) with the help of Pohožaev identity.

Now, we give out the following notations which are used in the whole paper:

♣ $L^s(\mathbb{R}^3)(1 \le s \le +\infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{1/s}$;

♣ For any $x \in \mathbb{R}^3$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$;

♣ $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)dx, \quad \|u\| = (u, v)^{1/2}, \ \forall \ u, v \in H^1(\mathbb{R}^3);$$

♣ For any $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ and $t > 0$, $u_t := u_t(x) = u(x/t)$;

♣ $C_1, C_2, C_3, \cdots$ are positive constants possibly different in different space.

The rest of the paper is organized as follows. In Section 2, some preliminaries are given and the proofs of Theorem 1.1 and Theorem 1.3 are given too. In Section 3, we study the existence of a least energy solution for problem (1.1) and Theorem 1.5 will be proved.

2. Ground state solutions

In this section, we give the proofs of Theorem 1.1 and Theorem 1.3. In order to do this, some useful lemmas are given. When the potential $V(x)$ is a constant, for example $V(x) \equiv V_\infty$, since it satisfies (V1) and (V2) too, all conclusions on Φ also hold true for Φ^∞. When studying problem (1.9), for convenience, we always assume that $V_\infty > 0$. First, in view of [13], we can verify Lemma 2.1 by a simple calculation.

Lemma 2.1. *For all $t \in [0, 1) \cup (1, +\infty)$, the following inequalities hold:*

$$g(t) := 2 + \alpha - (3 + \alpha)t + t^{3+\alpha} > g(1) = 0, \tag{2.1}$$

$$h(t) := \alpha - (3 + \alpha)t^3 + 3t^{3+\alpha} > h(1) = 0, \tag{2.2}$$

$$i(t) := 1 + \alpha - (3 + \alpha)t^2 + 2t^{3+\alpha} > i(1) = 0. \tag{2.3}$$

Moreover, under (V2), *for all $t \ge 0$ and $x \in \mathbb{R}^3\backslash\{0\}$, the following inequality is true*:

$$(\alpha + 3t^{3+\alpha})V(x) + (t^{3+\alpha} - 1)\nabla V(x) \cdot x - (3 + \alpha)t^3 V(tx) \ge -\frac{a\theta[2 + \alpha - (3 + \alpha)t + t^{3+\alpha}]}{4|x|^2}. \tag{2.4}$$

Lemma 2.2. *Suppose that* (V1), (V2) *and* (S2) *hold. Then for all $t > 0$ and $u \in H^1(\mathbb{R}^3)$, the following inequality holds:*

$$\Phi(u) \ge \Phi(u_t) + \frac{1 - t^{3+\alpha}}{3 + \alpha} P(u) + \frac{a(1 - \theta)g(t)}{2(3 + \alpha)} \|\nabla u\|_2^2 + \frac{bi(t)}{4(3 + \alpha)} \|\nabla u\|_2^4. \tag{2.5}$$

Proof. From Hardy inequality, one has

$$\|\nabla u\|_2^2 \ge \frac{1}{4} \int_{\mathbb{R}^3} \frac{u^2}{|x|^2} dx, \quad \forall \ u \in H^1(\mathbb{R}^3). \tag{2.6}$$

From the definition of u_t and (1.8), we have

$$\Phi(u_t) = \frac{at}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{t^3}{2}\int_{\mathbb{R}^3}V(tx)u^2dx - \frac{t^{3+\alpha}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx + \frac{bt^2}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2. \quad (2.7)$$

Hence, from (1.8), (1.13), (2.1), (2.3), (2.4), (2.6) and (2.7), we have

$$\begin{aligned}
\Phi(u) - \Phi(u_t) &= \frac{(1-t)a}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[V(x) - t^3V(tx)]u^2dx \\
&\quad -\frac{1-t^{3+\alpha}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx + \frac{(1-t^2)b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 \\
&= \frac{1-t^{3+\alpha}}{3+\alpha}\left\{\frac{a}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[3V(x) + \nabla V(x)\cdot x]u^2dx\right. \\
&\quad \left.-\frac{3+\alpha}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx + \frac{b}{2}\|\nabla u\|_2^4\right\} \\
&\quad +\frac{a[2+\alpha-(3+\alpha)t+t^{3+\alpha}]}{2(3+\alpha)}\|\nabla u\|_2^2 + \frac{b[1+\alpha-(3+\alpha)t^2+2t^{3+\alpha}]}{4(3+\alpha)}\|\nabla u\|_2^4 \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^3}\left\{\left[\frac{\alpha+3t^{3+\alpha}}{3+\alpha}V(x) - t^3V(tx)\right] - \frac{1-t^{3+\alpha}}{3+\alpha}\nabla V(x)\cdot x\right\}u^2dx \\
&\geq \frac{1-t^{3+\alpha}}{3+\alpha}P(u) + \frac{a(1-\theta)g(t)}{2(3+\alpha)}\|\nabla u\|_2^2 + \frac{bi(t)}{4(3+\alpha)}\|\nabla u\|_2^4. \quad (2.8)
\end{aligned}$$

It follows from (2.8) that (2.5) holds. □

The following two corollaries are obtained by Lemma 2.2.

Corollary 2.3. *Suppose that* (S1) *and* (S2) *hold. Then*

$$\Phi^\infty(u) = \Phi^\infty(u_t) + \frac{1-t^{3+\alpha}}{3+\alpha}P^\infty(u) + \frac{bi(t)}{4(3+\alpha)}\|\nabla u\|_2^4 + \frac{ag(t)\|\nabla u\|_2^2 + V_\infty h(t)\|u\|_2^2}{2(3+\alpha)}, \quad \forall\, u \in H^1(\mathbb{R}^3). \quad (2.9)$$

Corollary 2.4. *Suppose that* (V1), (V2), (S1) *and* (S2) *hold. Then*

$$\Phi(u) = \max_{t>0}\Phi(u_t), \quad \forall\, u \in M. \quad (2.10)$$

Lemma 2.5. *Suppose that* (V1) *and* (V2) *hold. Then there exist two positive constants* z_1 *and* z_2 *such that*

$$z_1\|u\|^2 \leq a\|\nabla u\|_2^2 + \int_{\mathbb{R}^3}[3V(x) + \nabla V(x)\cdot x]u^2dx \leq z_2\|u\|^2, \quad \forall\, u \in H^1(\mathbb{R}^3). \quad (2.11)$$

Proof. The proof of Lemma 2.5 is similar to that of [13], for the readers' convenience, we give the details here. Letting $t = 0$ in (2.4), we have

$$\alpha V(x) - \nabla V(x)\cdot x \geq -\frac{a(2+\alpha)\theta}{4|x|^2}, \quad \forall\, x \in \mathbb{R}^3\backslash\{0\}. \quad (2.12)$$

By (V1) and (2.12), we have

$$\nabla V(x)\cdot x \leq \frac{a(2+\alpha)\theta}{4|x|^2} + \alpha V_\infty, \quad \forall\, x \in \mathbb{R}^3\backslash\{0\}. \quad (2.13)$$

Similarly, letting $t \to \infty$ in (2.4), then, by (V1), we obtain

$$-3V_\infty - \frac{a\theta}{4|x|^2} \le -3V(x) - \frac{a\theta}{4|x|^2} \le \nabla V(x) \cdot x, \quad \forall\, x \in \mathbb{R}^3 \backslash \{0\}. \tag{2.14}$$

From (V1), (2.13) and (2.14), there is a positive constant C_1 such that

$$|\nabla V(x) \cdot x| \le C_1, \quad \forall\, x \in \mathbb{R}^3. \tag{2.15}$$

By (2.4), for any $t > 0$ and $x \in \mathbb{R}^3$, we have

$$3V(x) + \nabla V(x) \cdot x \ge -\frac{a\theta}{4|x|^2} + (3+\alpha)t^{-\alpha}V(tx) - \left[\alpha V(x) - \nabla V(x) \cdot x + \frac{a(2+\alpha)\theta}{4|x|^2}\right] t^{-(3+\alpha)}. \tag{2.16}$$

From (V1), there exists a positive constant R such that

$$\frac{V_\infty}{2} \le V(x) \ \text{ for all } \ |x| \ge R \tag{2.17}$$

and

$$\left[C_1 + \alpha V_\infty + \frac{a(2+\alpha)\theta}{4}\right] \le \frac{(3+\alpha)V_\infty R^3}{4}. \tag{2.18}$$

From (V1), (2.15), (2.16) and (2.18), one has

$$\begin{aligned} 3V(x) + \nabla V(x) \cdot x &\ge -\frac{a\theta}{4|x|^2} + (3+\alpha)R^{-\alpha}V(Rx) - R^{-3-\alpha}\left[\alpha V(x) - \nabla V(x) \cdot x + \frac{a(2+\alpha)\theta}{4|x|^2}\right] \\ &\ge -\frac{a\theta}{4|x|^2} + \frac{(3+\alpha)R^{-\alpha}V_\infty}{4}, \quad \forall\, |x| \ge 1. \end{aligned} \tag{2.19}$$

By Sobolev inequality and Hölder inequality, we have

$$\int_{|x|<1} u^2 dx \le \omega_3^{(2^*-2)/2} \left(\int_{|x|<1} u^{2^*} dx\right)^{2/2^*} \le S^{-1}\omega_3^{2/3}\|\nabla u\|_2^2, \tag{2.20}$$

where ω_3 denotes the volume of the unit ball of $\mathbb{R}^3$. Hence, from (2.6), (2.13), (2.14), (2.19) and (2.20), one obtains

$$\begin{aligned} &\int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x]u^2 dx + a\int_{\mathbb{R}^3} |\nabla u|^2 dx \\ \le\ & (3+\alpha)V_\infty\|u\|_2^2 + a[1 + (2+\alpha)\theta]\|\nabla u\|_2^2 \\ \le\ & [(3+\alpha)V_\infty + a + a\theta(2+\alpha)]\|u\|^2 := z_2\|u\|^2, \quad \text{for all } u \in H^1(\mathbb{R}^3) \end{aligned} \tag{2.21}$$

and

$$
\begin{aligned}
& \int_{\mathbb{R}^3}[3V(x)+\nabla V(x)\cdot x]u^2dx + a\int_{\mathbb{R}^3}|\nabla u|^2dx \\
= \quad & \int_{|x|<1}[3V(x)+\nabla V(x)\cdot x]u^2dx + \int_{|x|\geq 1}[3V(x)+\nabla V(x)\cdot x]u^2dx + a\int_{\mathbb{R}^3}|\nabla u|^2dx \\
\geq \quad & -\frac{a\theta}{4}\int_{\mathbb{R}^3}\frac{u^2}{|x|^2}dx + \frac{(3+\alpha)R^{-\alpha}V_\infty}{4}\int_{|x|\geq 1}u^2dx + a\|\nabla u\|_2^2 \\
\geq \quad & \frac{(3+\alpha)R^{-\alpha}V_\infty}{4}\int_{|x|\geq 1}u^2dx + a(1-\theta)\|\nabla u\|_2^2 \\
\geq \quad & \frac{(3+\alpha)R^{-\alpha}V_\infty}{4}\int_{|x|\geq 1}u^2dx + \frac{a(1-\theta)S}{2\omega_3^{2/3}}\int_{|x|<1}u^2dx + \frac{a(1-\theta)}{2}\|\nabla u\|_2^2 \\
\geq \quad & \min\left\{\frac{a(1-\theta)S}{2\omega_3^{2/3}}, \frac{(3+\alpha)R^{-\alpha}V_\infty}{4}\right\}\|u\|_2^2 + \frac{a(1-\theta)}{2}\|\nabla u\|_2^2 \\
\geq \quad & \min\left\{\frac{a(1-\theta)S}{2\omega_3^{2/3}}, \frac{(3+\alpha)R^{-\alpha}V_\infty}{4}, \frac{a(1-\theta)}{2}\right\}\|u\|^2 := z_1\|u\|^2, \quad \text{for all } u\in H^1(\mathbb{R}^3). \qquad (2.22)
\end{aligned}
$$

It follows from (2.21) and (2.22) that (2.11) holds. □

Lemma 2.6. *Suppose that* (V1), (V2) *and* (S1)-(S3) *hold. Then* $\Upsilon\neq\emptyset$ *and*

$$\{u\in H^1(\mathbb{R}^3)\backslash\{0\} : P^\infty(u)\leq 0 \text{ or } P(u)\leq 0\}\subset\Upsilon. \qquad (2.23)$$

Proof. From the proof of Claim 1 in Proposition 2.1 of [1], (S3) implies that $\Upsilon\neq\emptyset$. In the following, we have two cases to consider:

1). $u\in H^1(\mathbb{R}^3)\backslash\{0\}$ with $P^\infty(u)\leq 0$, then (1.11) implies that $u\in\Upsilon$.

2). $u\in H^1(\mathbb{R}^3)\backslash\{0\}$ with $P(u)\leq 0$, then from (1.13), (2.6) and (2.14), we have

$$
\begin{aligned}
& \frac{3+\alpha}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx \\
= \quad & -P(u) + \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}[3V(x)+\nabla V(x)\cdot x]u^2dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 \\
\geq \quad & \frac{a}{2}\|\nabla u\|_2^2 - \frac{a\theta}{8}\int_{\mathbb{R}^3}\frac{u^2}{|x|^2}dx + \frac{b}{2}\|\nabla u\|_2^4 \\
\geq \quad & \frac{a(1-\theta)}{2}\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 > 0. \qquad (2.24)
\end{aligned}
$$

(2.24) implies that $u\in\Upsilon$. □

Lemma 2.7. *Suppose that* (V1), (V2) *and* (S1)-(S3) *hold. Then for any* $u\in\Upsilon$*, there is a unique* $t_u>0$ *such that* $u_{t_u}\in M$.

Proof. Let $u \in \Upsilon$ be fixed and define a function $\xi(t) := \Phi(u_t)$ on $(0, \infty)$. From (1.13) and (2.7), we have

$$\begin{aligned}
\xi'(t) = 0 \quad \Leftrightarrow \quad & \frac{at}{2}\|\nabla u\|_2^2 + \frac{bt^2}{2}\|\nabla u\|_2^4 - \frac{(3+\alpha)t^{3+\alpha}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx \\
& + \frac{t^3}{2}\int_{\mathbb{R}^3}[3V(tx) + \nabla V(tx)\cdot(tx)]u^2dx = 0 \\
\Leftrightarrow \quad & P(u_t) = 0 \Leftrightarrow u_t \in M. \qquad (2.25)
\end{aligned}$$

From the definition of Υ and using (S1), (V1) and (2.7), we have $\lim_{t\to 0}\xi(t) = 0$, $\xi(t) < 0$ for t large enough and $\xi(t) > 0$ for $t > 0$ small enough. Hence, $\max_{t\in(0,\infty)}\xi(t)$ can be attained at some $t_u > 0$ such that $u_{t_u} \in M$ and $\xi'(t_u) = 0$.

Now, we prove that t_u is unique for any $u \in \Upsilon$. In fact, for any given $u \in \Upsilon$, let $t_1, t_2 > 0$ be such that $u_{t_1}, u_{t_2} \in M$. Then $P(u_{t_1}) = P(u_{t_2}) = 0$. Together with (2.5), one has

$$\begin{aligned}
\Phi(u_{t_1}) \quad \geq \quad & \Phi(u_{t_2}) + \frac{t_1^{3+\alpha} - t_2^{3+\alpha}}{(3+\alpha)t_1^{3+\alpha}}P(u_{t_1}) + \frac{bi(t_2/t_1)}{4(3+\alpha)}\|\nabla u_{t_1}\|_2^4 + \frac{a(1-\theta)g(t_2/t_1)}{2(3+\alpha)}\|\nabla u_{t_1}\|_2^2 \\
= \quad & \Phi(u_{t_2}) + \frac{bt_1^2 i(t_2/t_1)}{4(3+\alpha)}\|\nabla u\|_2^4 + \frac{a(1-\theta)t_1 g(t_2/t_1)}{2(3+\alpha)}\|\nabla u\|_2^2 \qquad (2.26)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(u_{t_2}) \quad \geq \quad & \Phi(u_{t_1}) + \frac{t_2^{3+\alpha} - t_1^{3+\alpha}}{(3+\alpha)t_2^{3+\alpha}}P(u_{t_2}) + \frac{bi(t_1/t_2)}{4(3+\alpha)}\|\nabla u_{t_2}\|_2^4 + \frac{a(1-\theta)g(t_1/t_2)}{2(3+\alpha)}\|\nabla u_{t_2}\|_2^2 \\
= \quad & \Phi(u_{t_1}) + \frac{bt_2^2 i(t_1/t_2)}{4(3+\alpha)}\|\nabla u\|_2^4 + \frac{a(1-\theta)t_2 g(t_1/t_2)}{2(3+\alpha)}\|\nabla u\|_2^2. \qquad (2.27)
\end{aligned}$$

It follows from (2.1), (2.3), (2.26) and (2.27) that $t_1 = t_2$. Hence, for any $u \in \Upsilon$, $t_u > 0$ is unique. $\square$

Corollary 2.8. *Suppose that* (S1)-(S3) *hold. Then, there is a unique $t_u > 0$ for any $u \in \Upsilon$, such that $u_{t_u} \in M^\infty$.*

It follows from Corollary 2.4, Lemma 2.6, Lemma 2.7 and Corollary 2.8 that $M \neq \emptyset$, $M^\infty \neq \emptyset$. Besides, we have the following lemma.

Lemma 2.9. *Suppose that* (V1), (V2) *and* (S1)-(S3) *hold. Then*

$$\inf_{u\in M}\Phi(u) := m = \inf_{u\in\Lambda}\max_{t>0}\Phi(u_t).$$

By a standard argument as that in [45], the following Brezis-Lieb type lemma is easy to be obtained.

Lemma 2.10. *Suppose that* (V1) *and* (S2) *hold. If $u_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^3)$, then*

$$\Phi(u_n) = \Phi(\tilde{u}) + \Phi(u_n - \tilde{u}) + \frac{b}{2}\|\nabla\tilde{u}\|_2^2\|\nabla(u_n - \tilde{u})\|_2^2 + o(1)$$

and

$$P(u_n) = P(\tilde{u}) + P(u_n - \tilde{u}) + b\|\nabla\tilde{u}\|_2^2\|\nabla(u_n - \tilde{u})\|_2^2 + o(1).$$

Lemma 2.11. *Suppose that* (V1), (V2) *and* (S1)-(S3) *hold. Then*
(i) *there is a positive constant* ρ *such that* $\|u\| \geq \rho, \ \forall\, u \in M$;
(ii) $m = \inf_{u\in M} \Phi(u) > 0$.

Proof. (i). From (1.7), (1.13), (2.11), Sobolev embedding theorem and $P(u) = 0$ for any $u \in M$, we have

$$\begin{aligned} \frac{z_1}{2}\|u\|^2 &\leq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 + \frac{1}{2}\int_{\mathbb{R}^3}[3V(x) + \nabla V(x)\cdot x]u^2 dx \\ &= \frac{3+\alpha}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx \\ &\leq \|u\|^{2(3+\alpha)/3} + C_2\|u\|^{2(3+\alpha)}. \end{aligned} \tag{2.28}$$

It follows from (2.28) that

$$\|u\| \geq \rho := \min\left\{1, \left[\frac{z_1}{2(1+C_2)}\right]^{3/2\alpha}\right\}, \quad \forall\, u \in M. \tag{2.29}$$

(ii). Letting $\{u_n\} \subset M$ with $\Phi(u_n) \to m$. Now, we consider two possible cases.
Case 1). $\inf_{n\in\mathbb{N}} \|\nabla u_n\|_2 := \sigma_0 > 0$. From (2.1), (2.3) and (2.5) with $t \to 0$, one has

$$\begin{aligned} o(1) + m = \Phi(u_n) &\geq \frac{a(1-\theta)(2+\alpha)}{2(3+\alpha)}\|\nabla u_n\|_2^2 + \frac{b(1+\alpha)}{4(3+\alpha)}\|\nabla u\|_2^4 \\ &\geq \frac{a(1-\theta)(2+\alpha)}{2(3+\alpha)}\sigma_0^2 + \frac{b(1+\alpha)}{4(3+\alpha)}\sigma_0^4. \end{aligned} \tag{2.30}$$

Case 2). $\inf_{n\in\mathbb{N}} \|\nabla u_n\|_2 = 0$. After passing to a subsequence and using (2.29), we have

$$\|u_n\|_2 \geq \frac{1}{2}\rho, \quad \|\nabla u_n\|_2 \to 0. \tag{2.31}$$

From Sobolev inequality and (1.7), it yields that

$$\begin{aligned} \int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx &\leq C_3\left(\|u\|_2^{2(3+\alpha)/3} + \|u\|_{2^*}^{2(3+\alpha)}\right) \\ &\leq C_3\left(\|u\|_2^{2(3+\alpha)/3} + S^{-(3+\alpha)}\|\nabla u\|_2^{2(3+\alpha)}\right), \quad \forall\, u \in H^1(\mathbb{R}^3). \end{aligned} \tag{2.32}$$

From (2.17), we get

$$\int_{|tx|\geq R} V(tx)u^2 dx \geq \frac{V_\infty}{2}\int_{|tx|\geq R} u^2 dx, \quad \forall\, u \in H^1(\mathbb{R}^3),\ t > 0. \tag{2.33}$$

It follows from Hölder inequality and Sobolev inequality that

$$\begin{aligned} \int_{|tx|\leq R} u^2 dx &\leq \left(\frac{\omega_3 R^3}{t^3}\right)^{(2^*-2)/2^*}\left(\int_{|tx|\leq R} u^{2^*} dx\right)^{2/2^*} \\ &\leq \omega_3^{2/3} R^2 S^{-1} t^{-2}\|\nabla u\|_2^2, \quad \forall\, u \in H^1(\mathbb{R}^3),\ t > 0. \end{aligned} \tag{2.34}$$

Set

$$\delta_0 = \min\{V_\infty, a\omega_3^{-2/3}R^{-2}S\} \tag{2.35}$$

and

$$t_n = \left(\frac{\delta_0}{3C_3}\right)^{1/\alpha} \|u_n\|_2^{-2/3}. \tag{2.36}$$

From (2.31), one has that $\{t_n\}$ is bounded. Hence, from (2.7), (2.10), (2.31), (2.32), (2.33), (2.34), (2.35) and (2.36), it yields that

$$\begin{aligned}
o(1) + m &= \Phi(u_n) \geq \Phi((u_n)_{t_n}) \\
&= \frac{at_n}{2}\|\nabla u_n\|_2^2 + \frac{t_n^3}{2}\int_{\mathbb{R}^3} V(t_n x)u_n^2 dx + \frac{bt_n^2}{4}\|\nabla u_n\|_2^4 - \frac{t_n^{3+\alpha}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u_n))F(u_n)dx \\
&\geq \frac{aSt_n^3}{2R^2\omega_3^{2/3}}\int_{|t_n x|\leq R} u_n^2 dx + \frac{V_\infty t_n^3}{4}\int_{|t_n x|\geq R} u_n^2 dx + \frac{bt_n^2}{4}\|\nabla u_n\|_2^4 \\
&\quad -\frac{C_3}{2}t_n^{3+\alpha}\|u_n\|_2^{2(3+\alpha)/3} - \frac{C_3 t_n^{3+\alpha}}{2S^{3+\alpha}}\|\nabla u_n\|_2^{2(3+\alpha)} \\
&\geq \frac{\delta_0}{4}t_n^3\|u_n\|_2^2 - \frac{C_3}{2}t_n^{3+\alpha}\|u_n\|_2^{2(3+\alpha)/3} + o(1) \\
&= \frac{1}{4}t_n^3\|u_n\|_2^2\left(\delta_0 - 2C_3 t_n^\alpha\|u_n\|_2^{2\alpha/3}\right) + o(1) \\
&= \frac{\delta_0}{12}\left(\frac{\delta_0}{3C_3}\right)^{3/\alpha} + o(1).
\end{aligned} \tag{2.37}$$

It follows from Case 1) and Case 2) that $m = \inf_{u\in M}\Phi(u) > 0$. □

Lemma 2.12. *Suppose that* (V1), (V2) *and* (S1)-(S3) *hold. Then* $m \leq m^\infty$.

Proof. Arguing indirectly, assume that $m > m^\infty$. Let $\epsilon := m - m^\infty$. Then there exists u_ϵ^∞ such that

$$u_\epsilon^\infty \in M^\infty \quad \text{and } m^\infty + \frac{\epsilon}{2} > \Phi^\infty(u_\epsilon^\infty). \tag{2.38}$$

According to Lemma 2.6 and Lemma 2.7, there exists $t_\epsilon > 0$ such that $(u_\epsilon^\infty)_{t_\epsilon} \in M$. Hence, from (V1), (V2), (1.8), (1.10), (2.9) and (2.38), we have

$$m^\infty + \frac{\epsilon}{2} > \Phi^\infty(u_\epsilon^\infty) \geq \Phi^\infty((u_\epsilon^\infty)_{t_\epsilon}) \geq \Phi(u_\epsilon^\infty) \geq m.$$

This is a contradiction, which shows that the conclusion of Lemma 2.12 is true. □

Lemma 2.13. *Suppose that* (V1), (V2) *and* (S1)-(S3) *hold. Then* m *is achieved.*

Proof. From Lemma 2.11, one has $m > 0$. Let $\{u_n\} \subset M$ be such that $\Phi(u_n) \to m$. Since $P(u_n) = 0$, from (2.5) with $t \to 0$, we have

$$m + o(1) = \Phi(u_n) \geq \frac{a(1-\theta)(2+\alpha)}{2(3+\alpha)}\|\nabla u_n\|_2^2 + \frac{b(1+\alpha)}{4(3+\alpha)}\|\nabla u_n\|_2^4. \tag{2.39}$$

It follows from (2.39) that $\{\|\nabla u_n\|_2\}$ is bounded. In the following, we will prove that $\{\|u_n\|_2\}$ is also bounded. Arguing indirectly, assume that $\|u_n\|_2 \to \infty$. From Sobolev inequality and (1.7) , we have

$$\begin{aligned}
\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)dx &\leq \frac{\delta_0}{4}\left(\frac{\delta_0}{24m}\right)^{\alpha/3} \|u_n\|_2^{2(3+\alpha)/3} + C_4\|u_n\|_{2^*}^{2(3+\alpha)} \\
&\leq \frac{\delta_0}{4}\left(\frac{\delta_0}{24m}\right)^{\alpha/3} \|u_n\|_2^{2(3+\alpha)/3} + C_4 S^{-(3+\alpha)}\|\nabla u\|_2^{2(3+\alpha)}, \quad \forall\, u \in H^1(\mathbb{R}^3), \qquad (2.40)
\end{aligned}$$

where δ_0 is given in (2.35). Let

$$\check{t}_n = \left(\frac{24m}{\delta_0}\right)^{1/3} \|u_n\|_2^{-2/3}. \tag{2.41}$$

Then it is easy to see that $\check{t}_n \to 0$ as $n \to \infty$. From (2.7), (2.10), (2.33), (2.34), (2.35), (2.40) and (2.41), we get

$$\begin{aligned}
o(1) + m &= \Phi(u_n) \geq \Phi((u_n)_{\check{t}_n}) \\
&= \frac{a\check{t}_n}{2}\|\nabla u_n\|_2^2 + \frac{\check{t}_n^3}{2}\int_{\mathbb{R}^3} V(\check{t}_n x)u_n^2 dx + \frac{b\check{t}_n^2}{4}\|\nabla u_n\|_2^4 - \frac{\check{t}_n^{3+\alpha}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u_n))F(u_n)dx \\
&\geq \frac{aS\check{t}_n^3}{2R^2\omega_3^{2/3}}\int_{|\check{t}_n x|<R} u_n^2 dx + \frac{V_\infty \check{t}_n^3}{4}\int_{|\check{t}_n x|\geq R} u_n^2 dx + \frac{b\check{t}_n^2}{4}\|\nabla u_n\|_2^4 \\
&\quad -\frac{\delta_0}{8}\left(\frac{\delta_0}{24m}\right)^{\alpha/3} \check{t}_n^{3+\alpha}\|u_n\|_2^{2(3+\alpha)/3} - \frac{C_4\check{t}_n^{3+\alpha}}{2S^{3+\alpha}}\|\nabla u_n\|_2^{2(3+\alpha)} \\
&\geq \frac{\delta_0}{4}\check{t}_n^3\|u_n\|_2^2 - \frac{\delta_0}{8}\left(\frac{\delta_0}{24m}\right)^{\alpha/3} \check{t}_n^3\|u_n\|_2^{2(3+\alpha)/3} + o(1) \\
&= \frac{\delta_0}{4}\check{t}_n^3\|u_n\|_2^2\left[1 - \frac{1}{2}\left(\frac{\delta_0}{24m}\right)^{\alpha/3} \check{t}_n^\alpha \|u_n\|_2^{2\alpha/3}\right] + o(1) \\
&= 3m + o(1), \qquad (2.42)
\end{aligned}$$

a contradiction, which implies the boundedness of $\{\|u_n\|_2\}$. Therefore, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Going to a subsequence, one has $u_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^3)$. Then $u_n \to \tilde{u}$ in $L^s_{Loc}(\mathbb{R}^3)$ for $s \in [2, 2^*)$ and $u_n \to \tilde{u}$ a.e. in $\mathbb{R}^3$. We have two possible cases: i) $\tilde{u} = 0$ and ii) $\tilde{u} \neq 0$.

Case i). $\tilde{u} = 0$, i.e. $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Then $u_n \to 0$ in $L^s_{Loc}(\mathbb{R}^3)$ for $s \in [2, 2^*)$ and $u_n \to 0$ a.e. in $\mathbb{R}^3$. From (V1) and (V2), we can show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}[V_\infty - V(x)]u_n^2 dx = \lim_{n\to\infty}\int_{\mathbb{R}^3}\nabla V(x)\cdot x u_n^2 dx = 0. \tag{2.43}$$

By (1.8), (1.10), (1.11), (1.13) and (2.43), we have

$$\Phi^\infty(u_n) \to m, \quad P^\infty(u_n) \to 0. \tag{2.44}$$

By Lemma 2.11(i), (1.11) and (2.44), we have

$$\begin{aligned}\min\{a, 3V_\infty\}\rho^2 &\leq \min\{a, 3V_\infty\}\|u_n\|^2 + b\|\nabla u_n\|_2^4 \\ &\leq a\|\nabla u_n\|_2^2 + b\|\nabla u_n\|_2^4 + 3V_\infty\|u_n\|_2^2 \\ &= (3+\alpha)\int_{\mathbb{R}^3}(I_\alpha * F(u_n))F(u_n)dx + o(1). \end{aligned} \tag{2.45}$$

By (1.7), (2.45) and Lion's concentration compactness principle [46, Lemma 2.1], one can show that there are $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that $\int_{B_1(y_n)} |u_n|^2 dx > \delta$. Let $\check{u}_n(x) = u_n(x + y_n)$. Then one has $\|\check{u}_n\| = \|u_n\|$ and

$$P^\infty(\check{u}_n) = o(1), \quad \Phi^\infty(\check{u}_n) \to m, \quad \int_{B_1(0)} |\check{u}_n|^2 dx > \delta. \tag{2.46}$$

Hence, there exists $\check{u} \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that, passing to a subsequence,

$$\begin{cases} \check{u}_n \rightharpoonup \check{u}, & \text{in } H^1(\mathbb{R}^3); \\ \check{u}_n \to \check{u}, & \text{in } L^s_{Loc}(\mathbb{R}^3), \quad \forall\, s \in [1, 2^*); \\ \check{u}_n \to \check{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{2.47}$$

Let $w_n = \check{u}_n - \check{u}$. Then from Lemma 2.10 and (2.47), we have

$$\Phi^\infty(\check{u}_n) = \Phi^\infty(\check{u}) + \Phi^\infty(w_n) + \frac{b}{2}\|\nabla \check{u}\|_2^2 \|w_n\|_2^2 + o(1) \tag{2.48}$$

and

$$P^\infty(\check{u}_n) = P^\infty(\check{u}) + P^\infty(w_n) + b\|\nabla \check{u}\|_2^2 \|w_n\|_2^2 + o(1). \tag{2.49}$$

Let

$$\Psi^\infty(u) := \Phi^\infty(u) - \frac{1}{3+\alpha}P^\infty(u) = \frac{\alpha V_\infty \|u\|_2^2 + a(2+\alpha)\|\nabla u\|_2^2}{2(3+\alpha)} + \frac{b(1+\alpha)\|\nabla u\|_2^4}{4(3+\alpha)}. \tag{2.50}$$

By (1.10), (1.11), (2.46), (2.48) and (2.49), we have

$$P^\infty(w_n) = -P^\infty(\check{u}) + o(1), \quad \Psi^\infty(w_n) = m - \Psi^\infty(\check{u}) + o(1). \tag{2.51}$$

If there is a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then after passing to this subsequence, one has

$$P^\infty(\check{u}) = 0, \quad \Phi^\infty(\check{u}) = m. \tag{2.52}$$

Next, assume that $w_n \neq 0$. Now, we prove that $P^\infty(\check{u}) \leq 0$. Else, if $P^\infty(\check{u}) > 0$, then, for large n, it follows from (2.51) that $P^\infty(w_n) < 0$. It follows from Lemma 2.6 and Corollary 2.8 that there is $t_n > 0$ such that $(w_n)_{t_n} \in M^\infty$ for large n. Thanks to (1.10), (1.11), (2.9), (2.50) and (2.51), it yields that

$$\begin{aligned} o(1) + m - \Psi^\infty(\check{u}) = \Psi^\infty(w_n) &= \Phi^\infty(w_n) - \frac{1}{3+\alpha}P^\infty(w_n) \\ &\geq \Phi^\infty((w_n)_{t_n}) - \frac{t_n^3}{3+\alpha}P^\infty(w_n) \\ &\geq m^\infty - \frac{t_n^3}{3+\alpha}P^\infty(w_n) \geq m^\infty, \end{aligned} \tag{2.53}$$

which implies that $P^\infty(\check{u}) \le 0$ due to $m \le m^\infty$ and $\Psi^\infty(\check{u}) > 0$. Since $\check{u} \ne 0$ and $P^\infty(\check{u}) \le 0$, it follows from Lemma 2.6 and Corollary 2.8 that there exists $\check{t} > 0$ such that $\check{u}_{\check{t}} \in M^\infty$. By (1.10), (1.11), (2.9), (2.44), (2.46), (2.50) and the weak semicontinuity of norm, we obtain

$$\begin{aligned} m &= \lim_{n\to\infty} \left[\Phi^\infty(\check{u}_n) - \frac{1}{3+\alpha} P^\infty(\check{u}_n)\right] \\ &= \lim_{n\to\infty} \Psi^\infty(\check{u}_n) \ge \Psi^\infty(\check{u}) \\ &= \Phi^\infty(\check{u}) - \frac{1}{3+\alpha} P^\infty(\check{u}) \\ &\ge \Phi^\infty(\check{u}_{\check{t}}) - \frac{\check{t}^3}{3+\alpha} P^\infty(\check{u}) \\ &\ge m^\infty - \frac{\check{t}^3}{3+\alpha} P^\infty(\check{u}) \\ &\ge m - \frac{\check{t}^3}{3+\alpha} P^\infty(\check{u}) \ge m. \end{aligned} \tag{2.54}$$

From (2.54), we know that (2.52) also holds. By Lemma 2.6 and Lemma 2.7, there exists $\bar{t} > 0$ such that $\check{u}_{\bar{t}} \in M$, moreover, from (V1), (V2), (1.8), (1.10), (2.52) and Corollary 2.3, we have

$$m \le \Phi(\check{u}_{\bar{t}}) \le \Phi^\infty(\check{u}_{\bar{t}}) \le \Phi^\infty(\check{u}) = m,$$

which implies that m is achieved at $\check{u}_{\bar{t}} \in M$.

Case ii). $\tilde{u} \ne 0$. Let $v_n = u_n - \tilde{u}$. Then from Lemma 2.10, we have

$$\Phi(u_n) = \Phi(\tilde{u}) + \Phi(v_n) + \frac{b}{2}\|\nabla\tilde{u}\|_2^2\|\nabla v_n\|_2^2 + o(1) \tag{2.55}$$

and

$$P(u_n) = P(\tilde{u}) + P(v_n) + b\|\nabla\tilde{u}\|_2^2\|\nabla v_n\|_2^2 + o(1). \tag{2.56}$$

Set

$$\Psi(u) = \frac{a(2+\alpha)}{2(3+\alpha)}\|\nabla u\|_2^2 + \frac{1}{2(3+\alpha)}\int_{\mathbb{R}^3}[\alpha V(x) - \nabla V(x)\cdot x]u^2 dx + \frac{b(1+\alpha)}{4(3+\alpha)}\|\nabla u\|_2^4. \tag{2.57}$$

From (2.6) and (2.12), we have

$$\begin{aligned} &\int_{\mathbb{R}^3}[\alpha V(x) - \nabla V(x)\cdot x]u^2 dx + a(2+\alpha)\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{b(1+\alpha)}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 \\ \ge\ & -\frac{a\theta(2+\alpha)}{4}\int_{\mathbb{R}^3}\frac{u^2}{|x|^2}dx + a(2+\alpha)\|\nabla u\|_2^2 + \frac{b(1+\alpha)}{2}\|\nabla u\|_2^4 \\ \ge\ & a(1-\theta)(2+\alpha)\|\nabla u\|_2^2 + \frac{b(1+\alpha)}{2}\|\nabla u\|_2^4, \quad \forall\, u \in H^1(\mathbb{R}^3). \end{aligned} \tag{2.58}$$

By (1.8), (1.13), (2.55), (2.56), (2.57), $P(u_n) = 0$ and $\Phi(u_n) \to m$, we have

$$P(v_n) = -P(\tilde{u}) + o(1), \quad \Psi(v_n) = m - \Psi(\tilde{u}) + o(1). \tag{2.59}$$

If there is a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} = 0$, then after passing to this subsequence, one obtains

$$P(\tilde{u}) = 0, \quad \Phi(\tilde{u}) = m, \tag{2.60}$$

which shows that the conclusion of Lemma 2.13 is true. Next, suppose that $v_n \neq 0$. Now, we prove that $P(\tilde{u}) \leq 0$. Else, if $P(\tilde{u}) > 0$, then from (2.59), we know that that $P(v_n) < 0$ for large n. From Lemma 2.6 and Lemma 2.7, for large n, there is $t_n > 0$ such that $(v_n)_{t_n} \in M$. By (1.8), (1.13), (2.5), (2.57) and (2.59), one gets

$$\begin{aligned} o(1) + m - \Psi(\tilde{u}) = \Psi(v_n) &= \Phi(v_n) - \frac{1}{3+\alpha} P(v_n) \\ &\geq \Phi((v_n)_{t_n}) - \frac{t_n^3}{3+\alpha} P(v_n) \\ &\geq m - \frac{t_n^3}{3+\alpha} P(v_n) \geq m, \end{aligned} \tag{2.61}$$

which implies that $P(\tilde{u}) \leq 0$ due to $\Psi(\tilde{u}) > 0$. Since $\tilde{u} \neq 0$ and $P(\tilde{u}) \leq 0$, it follows from Lemma 2.6 and Lemma 2.7 that there is $\tilde{t} > 0$ such that $\tilde{u}_{\tilde{t}} \in M$. By (1.8), (1.13), (2.5), (2.57), (2.58) and the weak semicontinuity of norm, we obtain

$$\begin{aligned} m &= \lim_{n\to\infty} \left[\Phi(u_n) - \frac{1}{3+\alpha} P(u_n) \right] \\ &= \lim_{n\to\infty} \Psi(u_n) \geq \Psi(\tilde{u}) \\ &= \Phi(\tilde{u}) - \frac{1}{3+\alpha} P(\tilde{u}) \\ &\geq \Phi(\tilde{u}_{\tilde{t}}) - \frac{\tilde{t}^3}{3+\alpha} P(\tilde{u}) \\ &\geq m - \frac{\tilde{t}^3}{3+\alpha} P(\tilde{u}) \geq m, \end{aligned}$$

which implies that (2.60) also holds. □

Lemma 2.14. *Suppose that* (V1), (V2) *and* (F1)-(F3) *hold. If* $\tilde{u} \in M$ *and* $\Phi(\tilde{u}) = m$*, Then* $\tilde{u}$ *is critical point of* Φ*.*

Proof. Following the ideas of [13, Lemma 2.14] and [42, Lemma 2.15], we can use the intermediary theorem and deformation lemma to show this lemma. Suppose that $\Phi'(\tilde{u}) \neq 0$. Then, there are $\delta > 0$ and $\sigma > 0$ such that

$$\|u - \tilde{u}\| \leq 3\delta \Rightarrow \|\Phi'(u)\| \geq \sigma.$$

It follows from [35, equation (2.47)] that $\lim_{t\to 1} \|\tilde{u}_t - \tilde{u}\| = 0$. Hence, there is $\delta_1 > 0$ such that

$$|t - 1| < \delta_1 \Rightarrow \|\tilde{u}_t - \tilde{u}\| < \delta.$$

From (2.25), there exist $T_1 \in (0, 1)$ and $T_2 \in (1, \infty)$ such that

$$P(\tilde{u}_{T_1}) > 0, \quad P(\tilde{u}_{T_2}) < 0.$$

The rest of the proof is similar to the proofs of [13, Lemma 2.14] and [42, Lemma 2.15]. In fact, the desired conclusion can be obtained by using

$$\begin{aligned}\Phi(\tilde{u}_t) &\leq \Phi(\tilde{u}) - \frac{a(1-\theta)g(t)}{2(3+\alpha)}\|\nabla\tilde{u}\|_2^2 - \frac{bi(t)}{4(3+\alpha)}\|\nabla\tilde{u}\|_2^4 \\ &= m - \frac{a(1-\theta)g(t)}{2(3+\alpha)}\|\nabla\tilde{u}\|_2^2 - \frac{bi(t)}{4(3+\alpha)}\|\nabla\tilde{u}\|_2^4, \quad \forall\, t > 0 \end{aligned} \tag{2.62}$$

and

$$\varepsilon := \min\left\{\frac{a(1-\theta)g(T_1)}{6(3+\alpha)}\|\nabla\tilde{u}\|_2^2 + \frac{bi(T_1)}{12(3+\alpha)}\|\nabla\tilde{u}\|_2^4, \frac{a(1-\theta)g(T_2)}{6(3+\alpha)}\|\nabla\tilde{u}\|_2^2 + \frac{bi(T_2)}{12(3+\alpha)}\|\nabla\tilde{u}\|_2^4, 1, \frac{\sigma\delta}{8}\right\}$$

instead of [13, (2.56) and ε], respectively. □

Proof of Theorem 1.1. According to Lemma 2.9, Lemma 2.13 and Lemma 2.14, there is $\tilde{u} \in M$ such that

$$\Phi(\tilde{u}) = m = \inf_{u\in\Upsilon}\max_{t>0}\Phi(u_t) > 0, \quad \Phi'(\tilde{u}) = 0$$

which implies that $\tilde{u}$ is a nontrivial solution of problem (1.1). □

Proof of Theorem 1.3. Set $\bar{m}^\infty := \inf_{u\in\Lambda^\infty}\Phi^\infty(u)$, where $\Lambda^\infty := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : (\Phi^\infty)'(u) = 0\}$. From Corollary 1.2, we know that there is $\tilde{u} \in M^\infty$ such that $\Phi^\infty(\tilde{u}) = m^\infty$ and $(\Phi^\infty)'(\tilde{u}) = 0$, which implies that $\bar{m}^\infty \leq m^\infty$ and $\Lambda^\infty \neq \emptyset$. Besides, if $v \in \Lambda^\infty$, then, by (1.11) (i.e. [1, Theorem 3]), we have $v \in M^\infty$. Hence, for all $v \in \Lambda^\infty$, we have $\Phi^\infty(v) \geq m^\infty$, which implies that $\bar{m}^\infty \geq m^\infty$. Hence, $\bar{m}^\infty = m^\infty$. The proof is completed. □

3. The least energy solutions

We will use the following proposition to prove Theorem 1.5.

Proposition 3.1. [44] *Let X be a Banach space and let $J \subset \mathbb{R}^+$ be en interval, and*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall\, \lambda \in J,$$

be a family of continuously differential functional on X such that
(i) *either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\| \to \infty$;*
(ii) *B maps every bounded set of X into a set of $\mathbb{R}$ bounded below;*
(iii) *there are two points v_1 and v_2 in X such that*

$$\tilde{c}_\lambda := \inf_{x\in\Gamma}\max_{t\in[0,1]} I_\lambda(x(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\},$$

where

$$\Gamma = \{x \in C([0,1], X) : x(0) = v_1, x(1) = v_2\}.$$

Then, for almost every $\lambda \in J$, there exists a sequence $\{u_n(\lambda)\}$ such that

(1) $\{u_n(\lambda)\}$ *is bounded in* X*;*

(2) $I_\lambda(u_n(\lambda)) \to \tilde{c}_\lambda$*;*

(3) $I'_\lambda(u_n(\lambda)) \to 0$ *in* X^**, where* X^* *is the dual of* X*.*

Moreover, $\tilde{c}_\lambda$ *is nonincreasing and left continuous on* $\lambda \in [1/2, 1]$*.*

Similar to [1, 39], we have the following lemma.

Lemma 3.2. *Suppose that* (V1), (S1) *and* (S2) *hold. Let* $u \in H^1(\mathbb{R}^3)$ *be a critical point of* Φ_λ*, then one has the following Pohožaev type identity:*

$$P_\lambda(u) := \frac{a}{2}\|\nabla u\|_2^2 - \frac{(3+\alpha)\lambda}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx + \frac{1}{2}\int_{\mathbb{R}^3}[3V(x) + \nabla V(x)\cdot x]u^2dx + \frac{b}{2}\|\nabla u\|_2^4 = 0. \quad (3.1)$$

The following lemma can be obtained easily from Corollary 2.3.

Lemma 3.3. *Suppose that* (S1) *and* (S2) *hold. Then for all* $t > 0$, $\lambda \geq 0$ *and* $u \in H^1(\mathbb{R}^3)$*, the following equality holds*

$$\Phi_\lambda^\infty(u) := \Phi_\lambda^\infty(u_t) + \frac{1-t^3}{3+\alpha}P_\lambda^\infty(u) + \frac{ag(t)\|\nabla u\|_2^2 + V_\infty h(t)\|u\|_2^2}{2(3+\alpha)} + \frac{bi(t)}{4(3+\alpha)}\|\nabla u\|_2^4. \quad (3.2)$$

By Corollary 1.2, we know that $\Phi_1^\infty = \Phi^\infty$ has a minimizer $u_1^\infty \neq 0$ on $M_1^\infty = M^\infty$, that is

$$u_1^\infty \in M_1^\infty, \quad m_1^\infty = \Phi_1^\infty(u_1^\infty) \text{ and } (\Phi_1^\infty)'(u_1^\infty) = 0. \quad (3.3)$$

Since problem (1.9) is autonomous, from (V1), there are $\tilde{x} \in \mathbb{R}^3$ and $\tilde{r} > 0$ such that for almost every $|x - \tilde{x}| \leq \tilde{r}$, one has

$$V_\infty - V(x) > 0, \quad |u_1^\infty(x)| > 0. \quad (3.4)$$

From (V1), one has $V_{\max} := \max_{x\in\mathbb{R}^3} V(x) \in (0, \infty)$. Set

$$\Phi_\lambda^*(u) = \frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2 + V_{\max}u^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 - \frac{\lambda}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)dx. \quad (3.5)$$

By (2.7) and (3.3), there is $T > 0$ such that

$$\Phi_{1/2}^*((u_1^\infty)_t) < 0 \text{ for all } t \geq T. \quad (3.6)$$

Lemma 3.4. [13] *Suppose that* (V1) *and* (S1)-(S3) *hold. Then*

(i) $\Phi_\lambda((u_1^\infty)_T) < 0$ *for all* $\lambda \in [0.5, 1]$*;*

(ii) *there exists a positive constant* k_0 *independent of* λ *such that for all* $\lambda \in [0.5, 1]$*,*

$$c_\lambda := \inf_{y\in\tilde{\Gamma}}\max_{t\in[0,1]}\Phi_\lambda(y(t)) \geq k_0 > \max\{\Phi_\lambda(0), \Phi_\lambda((u_1^\infty)_T)\},$$

where

$$\widetilde{\Gamma} = \{y \in C([0,1], H^1(\mathbb{R}^3)) : y(0) = 0, y(1) = (u_1^\infty)_T\};$$

(iii) c_λ *is bounded for* $\lambda \in [0.5, 1]$*;*

(iv) m_λ^∞ *is non-increasing on* $\lambda \in [0.5, 1]$*;*

(v) $\limsup_{\lambda \to \lambda_0} c_\lambda \leq c_{\lambda_0}$ *for* $\lambda_0 \in [0.5, 1]$.

Lemma 3.5. *Suppose that* (V1) *and* (S1)-(S3) *hold. Then there is* $\widetilde{\lambda} \in [0.5, 1]$ *such that* $c_\lambda < m_\lambda^\infty$ *for* $\lambda \in (\widetilde{\lambda}, 1]$.

Proof. It is easy to see that $\Phi_\lambda((u_1^\infty)_t)$ is continuous for $t \in (0, +\infty)$. Hence, for any $\lambda \in [0.5, 1]$, we can choose $t_\lambda \in (0, T)$ such that $\Phi_\lambda((u_1^\infty)_{t_\lambda}) = \max_{t \in [0,T]} \Phi_\lambda((u_1^\infty)_t)$. Set

$$y_0(t) = \begin{cases} (u_1^\infty)_{(tT)}, & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases} \tag{3.7}$$

Then $y_0 \in \widetilde{\Gamma}$ defined by Lemma 3.4(ii). Moreover

$$\Phi_\lambda((u_1^\infty)_{t_\lambda}) = \max_{t \in [0,1]} \Phi_\lambda(y_0(t)) \geq c_\lambda. \tag{3.8}$$

It follows from $P^\infty(u_1^\infty) = 0$ that $0 < \int_{\mathbb{R}^3} (I_\alpha * F(u_1^\infty)) F(u_1^\infty) dx$. Set

$$\zeta_0 := \min\{1/4, 3\tilde{r}/8(1 + |\tilde{x}|)\}. \tag{3.9}$$

Hence, by (3.4) and (3.9), one obtains

$$|x - \tilde{x}| \leq \frac{\tilde{r}}{2} \text{ and } s \in [1 - \zeta_0, 1 + \zeta_0] \Rightarrow |sx - \tilde{x}| \leq \bar{r}. \tag{3.10}$$

Set

$$\begin{aligned} \widetilde{\lambda} \quad := \quad & \max\left\{ \frac{1}{2}, 1 - \frac{(1-\zeta_0)^3 \min_{s \in [1-\zeta_0, 1+\zeta_0]} \int_{\mathbb{R}^3} [V_\infty - V(sx)] |u_1^\infty|^2 dx}{T^{3+\alpha} \int_{\mathbb{R}^3} (I_\alpha * F(u_1^\infty)) F(u_1^\infty) dx}, \right. \\ & 1 - \frac{a \min\{g(1-\zeta_0), g(1+\zeta_0)\} \|\nabla u_1^\infty\|_2^2 + V_\infty \min\{h(1-\zeta_0), h(1+\zeta_0)\} \|u_1^\infty\|_2^2}{(3+\alpha) T^{3+\alpha} \int_{\mathbb{R}^3} (I_\alpha * F(u_1^\infty)) F(u_1^\infty) dx} \\ & \left. - \frac{b \min\{i(1-\zeta_0), i(1+\zeta_0)\} \|\nabla u_1^\infty\|_2^4}{2(3+\alpha) T^{3+\alpha} \int_{\mathbb{R}^3} (I_\alpha * F(u_1^\infty)) F(u_1^\infty) dx} \right\}. \end{aligned} \tag{3.11}$$

By (2.1), (2.2), (2.3), (3.4) and (3.11), we have $1/2 \leq \widetilde{\lambda} < 1$. We have the following two cases to distinguish:

Case i): $t_\lambda \in [1 - \zeta_0, 1 + \zeta_0]$. By (1.16), (1.17), (3.2)-(3.8), (3.10), (3.11) and Lemma 3.4(iv), one obtains

$$\begin{aligned} m_\lambda^\infty \quad & \geq \quad m_1^\infty = \Phi_1^\infty(u_1^\infty) \geq \Phi_1^\infty((u_1^\infty)_{t_\lambda}) \\ & = \quad \Phi_\lambda^\infty((u_1^\infty)_{t_\lambda}) + \frac{t_\lambda^3}{2} \int_{\mathbb{R}^3} [V_\infty - V(t_\lambda x)] |u_1^\infty|_2^2 dx - \frac{(1-\lambda) t_\lambda^{3+\alpha}}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u_1^\infty)) F(u_1^\infty) dx \\ & \geq \quad c_\lambda + \frac{(1-\zeta_0)^3}{2} \int_{\mathbb{R}^3} [V_\infty - V(t_\lambda x)] |u_1^\infty|_2^2 dx - \frac{(1-\lambda) T^{3+\alpha}}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u_1^\infty)) F(u_1^\infty) dx \\ & > \quad c_\lambda, \quad \forall \lambda \in (\widetilde{\lambda}, 1]. \end{aligned} \tag{3.12}$$

Case ii): $t_\lambda \in (0, 1-\zeta_0) \cup (1+\zeta_0, T]$. By (1.16), (1.17), (2.1)-(2.3), (3.2), (3.3), (3.8), (3.11) and Lemma 3.4(iv), one has

$$\begin{aligned}
m_\lambda^\infty &\geq m_1^\infty = \Phi_1^\infty(u_1^\infty) = \Phi_1^\infty((u_1^\infty)_{t_\lambda}) + \frac{ag(t_\lambda)\|\nabla u_1^\infty\|_2^2 + V_\infty h(t_\lambda)\|u_1^\infty\|_2^2}{2(3+\alpha)} + \frac{bi(t_\lambda)}{4(3+\alpha)}\|\nabla u_1^\infty\|_2^4 \\
&= \Phi_\lambda^\infty((u_1^\infty)_{t_\lambda}) + \frac{t_\lambda^3}{2}\int_{\mathbb{R}^3}[V_\infty - V(t_\lambda x)]|u_1^\infty|_2^2 dx - \frac{(1-\lambda)t_\lambda^{3+\alpha}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u_1^\infty))F(u_1^\infty)dx \\
&\quad + \frac{ag(t_\lambda)\|\nabla u_1^\infty\|_2^2 + V_\infty h(t_\lambda)\|u_1^\infty\|_2^2}{2(3+\alpha)} + \frac{bi(t_\lambda)}{4(3+\alpha)}\|\nabla u_1^\infty\|_2^4 \\
&\geq c_\lambda - \frac{(1-\lambda)T^{3+\alpha}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u_1^\infty))F(u_1^\infty)dx + \frac{b\min\{i(1-\zeta_0), i(1+\zeta_0)\}}{4(3+\alpha)}\|\nabla u_1^\infty\|_2^4 \\
&\quad + \frac{a\min\{g(1-\zeta_0), g(1+\zeta_0)\}\|\nabla u_1^\infty\|_2^2 + V_\infty \min\{h(1-\zeta_0), h(1+\zeta_0)\}\|u_1^\infty\|_2^2}{2(3+\alpha)} \\
&> c_\lambda, \quad \forall\, \lambda \in (\widetilde{\lambda}, 1].
\end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we can see that $c_\lambda < m_\lambda^\infty$ for $\lambda \in (\widetilde{\lambda}, 1]$ in both cases. □

Lemma 3.6.[47] *Suppose that* (V1) *and* (S1)-(S3) *hold. For* $\lambda \in [1/2, 1]$*, let* $\{u_n\}$ *be a bounded* (PS)-*sequence for* Φ_λ*. Then there are a subsequence of* $\{u_n\}$*, still denoted by* $\{u_n\}$*, an integer* $l \in \mathbb{N} \cup \{0\}$*, a sequence* $\{y_n^k\}$ *and* $w^k \in H^1(\mathbb{R}^3)$ *for* $1 \leq k \leq l$*, such that*
(i) $u_n \rightharpoonup u_0$ *with* $\Phi_\lambda'(u_0) = 0$*;*
(ii) $w^k \neq 0$ *and* $(\Phi_\lambda^\infty)'(w^k) = 0$ *for* $1 \leq k \leq l$*;*
(iii) $\|u_n - u_0 - \sum_{k=1}^l w^k(\cdot + y_n^k)\| \to 0$*;*
(iv) $\Phi_\lambda(u_n) \to \Phi_\lambda(u_0) + \sum_{i=1}^l \Phi_\lambda^\infty(w^i)$.
We agree that the above holds without w^k if $l = 0$.

Lemma 3.7. *Suppose that* (V1) *and* (V3) *hold. Then there exists* $z_3 > 0$ *such that*

$$a(2+\alpha)\|\nabla u\|_2^2 + \int_{\mathbb{R}^3}[\alpha V(x) - \nabla V(x)\cdot x]u^2 dx \geq z_3\|u\|^2, \quad \forall\, u \in H^1(\mathbb{R}^3). \tag{3.14}$$

Proof. By (V1), (V3) and (2.6), we obtain

$$\begin{aligned}
& a(2+\alpha)\|\nabla u\|_2^2 + \int_{\mathbb{R}^3}[\alpha V(x) - \nabla V(x)\cdot x]u^2 dx \\
=\ & a(2+\alpha)\|\nabla u\|_2^2 - \frac{a}{2}\int_{\mathbb{R}^3}\frac{u^2}{|x|^2}dx + \int_{\mathbb{R}^3}\left[\alpha V(x) - \nabla V(x)\cdot x + \frac{a}{2|x|^2}\right]u^2 dx \\
\geq\ & a\alpha\|\nabla u\|_2^2 + \alpha(1-\theta')\int_{\mathbb{R}^3}V(x)u^2 dx \\
\geq\ & z_3\|u\|^2,
\end{aligned}$$

where $z_3 > 0$ is a positive constant. □

Lemma 3.8. *Suppose that* (V1), (V3) *and* (S1)-(S3) *hold. Then for almost every* $\lambda \in (\widetilde{\lambda}, 1]$, *there is* $u_\lambda \in H^1(\mathbb{R}^3)\backslash\{0\}$ *such that*

$$\Phi_\lambda(u_\lambda) = c_\lambda, \quad \Phi'_\lambda(u_\lambda) = 0. \tag{3.15}$$

Proof. Let $X = H^1(\mathbb{R}^3)$, $J = (\widetilde{\lambda}, 1]$ and $\Phi_\lambda = I_\lambda$, then it follows from Lemma 3.4, (V1) and (S1)-(S3) that $\Phi_\lambda(u)$ satisfies all the assumptions of Proposition 3.1. Hence, for almost every $\lambda \in (\widetilde{\lambda}, 1]$, there is a bounded sequence $\{u_n(\lambda)\} \subset H^1(\mathbb{R}^3)$ denoted by $\{u_n\}$ such that

$$\Phi_\lambda(u_n) \to c_\lambda, \quad \Phi'_\lambda(u_n) \to 0. \tag{3.16}$$

From Lemma 3.2 and Lemma 3.6, there exist a subsequence of $\{u_n\}$, for simplicity, still denoted by $\{u_n\}$, $u_\lambda \in H^1(\mathbb{R}^3)$, an integer $l \in \mathbb{N}\backslash\{0\}$, and $w^1, \dots, w^l \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that

$$u_n \rightharpoonup u_\lambda \ \text{ in } H^1(\mathbb{R}^3), \quad \Phi'_\lambda(u_\lambda) = 0. \tag{3.17}$$

$$(\Phi^\infty_\lambda)'(w^k) = 0, \quad \Phi^\infty_\lambda(w^k) \geq m^\infty_\lambda, \quad 1 \leq k \leq l \tag{3.18}$$

and

$$c_\lambda = \Phi_\lambda(u_\lambda) + \sum_{i=1}^{l} \Phi^\infty_\lambda(w^i). \tag{3.19}$$

Since $\Phi'_\lambda(u_\lambda) = 0$, from Lemma 3.2, we have

$$\begin{aligned} P_\lambda(u_\lambda) \quad := \quad & \frac{1}{2}\|\nabla u_\lambda\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [3V(x) + \nabla V(x)\cdot x]u_\lambda^2 dx \\ & - \frac{(3+\alpha)\lambda}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u_\lambda))F(u_\lambda)dx + \frac{b}{2}\|\nabla u_\lambda\|_2^4 = 0. \end{aligned} \tag{3.20}$$

It follows from (3.18), (3.19) and $\|u_n\| \not\to 0$ that if $u_\lambda = 0$, then $l \geq 1$ and

$$c_\lambda = \Phi_\lambda(u_\lambda) + \sum_{i=1}^{l} \Phi^\infty_\lambda(w^i) \geq m^\infty_\lambda,$$

which is a contradiction due to Lemma 3.5. Hence, $u_\lambda \neq 0$. Thanks to (1.16), (3.14) and (3.20), it yields that

$$\begin{aligned} \Phi_\lambda(u_\lambda) \quad &= \quad \Phi_\lambda(u_\lambda) - \frac{1}{3+\alpha}P_\lambda(u_\lambda) \\ &= \quad \frac{a(2+\alpha)}{2(3+\alpha)}\|\nabla u_\lambda\|_2^2 + \frac{b(1+\alpha)}{4(3+\alpha)}\|\nabla u_\lambda\|_2^4 + \frac{1}{2(3+\alpha)}\int_{\mathbb{R}^3} [\alpha V(x) - \nabla V(x)\cdot x]u_\lambda^2 dx \\ &\geq \quad \frac{z_3}{2(3+\alpha)}\|u_\lambda\|^2 + \frac{b(1+\alpha)}{4(3+\alpha)}\|\nabla u_\lambda\|_2^4 \\ &\geq \quad \frac{z_3}{2(3+\alpha)}\|u_\lambda\|^2 > 0. \end{aligned} \tag{3.21}$$

On the one hand, it follows from (3.19) and (3.21) that

$$c_\lambda = \Phi_\lambda(u_\lambda) + \sum_{i=1}^{l} \Phi^\infty_\lambda(w^i) \geq l m^\infty_\lambda. \tag{3.22}$$

On the other hand, it follows from Lemma 3.5 that

$$c_\lambda < m_\lambda^\infty \quad \text{for} \quad \lambda \in (\widetilde{\lambda}, 1]. \tag{3.23}$$

Hence, from (3.22) and (3.23), we can know that $l = 0$ and $\Phi_\lambda(u_\lambda) = c_\lambda$. The proof is completed. □

Lemma 3.9. *Suppose that* (V1) *and* (S1)-(S3) *hold. Then there is* $\tilde{u} \in H^1(\mathbb{R}^3)\backslash\{0\}$ *such that*

$$0 < \Phi(\tilde{u}) \leq c_1, \quad \Phi'(\tilde{u}) = 0. \tag{3.24}$$

Proof. It follows from Lemma 3.4(iii) and Lemma 3.8 that there are two sequences $\{\lambda_n\} \subset (\bar{\lambda}, 1]$ and $\{u_{\lambda_n}\} \subset H^1(\mathbb{R}^3)\backslash\{0\}$, denoted by $\{u_n\}$, such that

$$\Phi_{\lambda_n}(u_n) = c_{\lambda_n}, \quad \Phi'_{\lambda_n}(u_n) = 0, \quad \lambda_n \to 1, \quad c_{\lambda_n} \to c^*. \tag{3.25}$$

By Lemma 3.2 and (3.25), we have $P_{\lambda_n}(u_n) = 0$. By Lemma 3.4(iii), (1.16), (3.14), (3.20) and (3.25), we obtain

$$\begin{aligned}
C_4 \quad &\geq \quad c_{\lambda_n} = \Phi_{\lambda_n}(u_n) - \frac{1}{3+\alpha} P_{\lambda_n}(u_n) \\
&= \quad \frac{a(2+\alpha)}{2(3+\alpha)}\|\nabla u_n\|_2^2 + \frac{b(1+\alpha)}{4(3+\alpha)}\|\nabla u_n\|_2^4 + \frac{1}{2(3+\alpha)}\int_{\mathbb{R}^3}[\alpha V(x) - \nabla V(x)\cdot x]u_n^2 dx \\
&\geq \quad \frac{z_3}{2(3+\alpha)}\|u_n\|^2 + \frac{b(1+\alpha)}{4(3+\alpha)}\|\nabla u_n\|_2^4 \\
&\geq \quad \frac{z_3}{2(3+\alpha)}\|u_n\|^2,
\end{aligned}$$

which implies that $\{\|u_n\|\}$ is bounded in $H^1(\mathbb{R}^3)$. From Lemma 3.4(v), we have $\lim_{n\to\infty} c_{\lambda_n} = c^* \leq c_1$. Thus, by (1.16) and (3.25), we have

$$\Phi(u_n) \to c^*, \quad \Phi'(u_n) \to 0,$$

which implies that $\{u_n\}$ satisfies (3.16) with $c_\lambda = c^*$. From the proof of Lemma 3.8, one can prove that $\tilde{u} \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that (3.24) is true. The proof is completed. □

Proof of Theorem 1.5. Let $\check{m} := \inf_{u\in\Lambda} \Phi(u)$. Therefore, from Lemma 3.9, we have $\check{m} \leq c_1$ and $\Lambda \neq \emptyset$. For any $u \in \Lambda$, it follows from Lemma 3.2 that $P(u) = P_1(u) = 0$. Therefore, by (3.21), we have $\Phi(u) = \Phi_1(u) > 0$ for all $u \in \Lambda$, and so $\check{m} > 0$. Let $\{u_n\} \subset \Lambda$ be such that

$$\Phi(u_n) \to \check{m}, \quad \Phi'(u_n) = 0.$$

From Lemma 3.5, we have $\check{m} \leq c_1 < m_1^\infty$. By a similar argument as that in Lemma 3.8, one can show that there is $\tilde{u} \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that

$$\Phi(\tilde{u}) = \check{m}, \quad \Phi'(\tilde{u}) = 0,$$

which implies that $\tilde{u}$ is a least energy solution of (1.1). □

Declarations

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The authors declare that they have no competing interests.

Availability of data and material

No data and material were used to support to the work.

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Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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