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Nonnegative solutions to the reaction–diffusion equations for prey-predator models with the dormancy of predators†

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Abstract

The time-global unique solvability on the reaction–diffusion equations for prey-predator models and dormancy on predators is established. The crucial step is to construct time-local nonnegative classical solutions by using a new approximation associated with time-evolution operators. Although the system does not equip usual comparison principles, a priori bounds are derived, so solutions are extended time-globally. Via observations to the corresponding ordinary differential equations, invariant regions and asymptotic behaviors of solutions are also investigated.

KEYWORDS:
reaction diffusion equation, prey-predator model, time-evolution operator, invariant region

1 | INTRODUCTION

Some systems of reaction–diffusion equations have attracted much interest as a prototype model for pattern formation[6]. In particular, the Turing instability is mainly observed by numerical researchers. In this paper, we deal with the following reaction diffusion equations in the whole space $\mathbb{R}^n$ for $n \in \mathbb{N}$:

$$\text{(LV)} \qquad \begin{cases} \partial_t u = \delta\Delta u + r\,(1 - u/k)\,u - \gamma uv/(u+h), \\ \partial_t v = d\,\Delta v + \mu(u)uv/(u+h) + \alpha w - \theta v - \iota v - \beta v^2, \\ \partial_t w = \nu(u)uv/(u+h) + \theta v - \alpha w - \tilde{\iota} w. \end{cases}$$

This is a system of Lotka-Volterra type equations with diffusions. More precisely, this is a prey-predator model with dormancy of predators[4,5,6]. Here, $u = u(x,t)$, $v = v(x,t)$, and $w = w(x,t)$, define as the density of prey, the density of active predator, and the density of dormant predator, respectively, stand for the unknown scalar nonnegative functions at $x \in \mathbb{R}^n$ and $t > 0$. To avoid the effects from boundaries, the Cauchy problem is considered, in what follows. We have denoted the nonnegative constants by

δ	the diffusion coefficient of prey,	h	the constant of foraging efficiency and handling time,
d	the diffusion coefficient of active predator,	α	the rate of awakening,
r	the growth rate of prey,	β	the mortality rate by competitions of active predators
k	the capacity of prey,	$\tilde{\iota}$	the mortality rate of dormant predator,
γ	the mortality rate of prey,	ι	the mortality rate of active predator.
θ	the rate of sleeping.		

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Also, $\mu(u)$ and $\nu(u)$ are smooth positive functions of u denoting growth rates of active and dormant predators, respectively. In some research, e.g. μ is given as a sigmoid function $\mu(u) := \gamma(1+\tanh(\xi(u-\eta)))/2 \in (0,\gamma)$ with some constants ξ and η; $\nu(u) := \gamma - \mu(u)$[4]. In addition, we have used the notations of differentiation; $\partial_t := \partial/\partial t$ and $\Delta := \sum_{i=1}^n \partial_i^2$, where $\partial_i := \partial/\partial x_i$ for $i = 1, \dots, n$.

By change of variables and constants, we can replace $\delta = 1$, $k = 1$, $r = 1$ and $\beta = 1$. For the simplicity of notations, we put $m := \theta + \iota$, $\rho := \alpha + \tilde{\iota}$, in addition, assume that μ and ν are positive constants independent of u. So, we consider the initial value problem:

$$
\text{(P)} \qquad \begin{cases} \partial_t u = \Delta u + (1-u)u - \gamma uv/(u+h) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t v = d\Delta v + \mu uv/(u+h) + \alpha w - (m+v)v & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t w = \nu uv/(u+h) + \theta v - \rho w & \text{in } \mathbb{R}^n \times (0,\infty), \\ \big(u, v, w\big)\Big|_{t=0} = \big(u_0, v_0, w_0\big) & \text{in } \mathbb{R}^n. \end{cases}
$$

The bifurcation between stability and instability of stationary solutions to (LV) was concerned with some specific parameters, associated with numerical investigation[5]. Furthermore, a numerical study of Turing instability on (LV) was done[4]. Besides, in this paper, we focus on the mathematical theory for the existence of time-global nonnegative unique classical solutions to (P), and the invariant region which includes the trivial solution $\big(0,0,0\big)$.

This paper is organized as follows. In Section 2, we will present the main results of this paper, and in the next Section we will define function spaces, and recall some properties of the heat semigroup and time-evolution operators. Section 4 will be devoted to the proof of the time-local existence of nonnegative unique classical solutions with nonnegative initial data. We will discuss the time-global solvability in Section 5, deriving a priori estimates of solutions and their derivatives. In Section 6, some invariant regions and asymptotic behaviors of solutions to (P) will be argued.

2 | MAIN RESULTS

For the definition of function spaces BUC and BUC^1, see Section 3.

Theorem 1. Let $n \in \mathbb{N}$, d, $h > 0$, and let m, θ, ρ, α, γ, μ, $\nu \geq 0$. If u_0, $v_0 \in BUC(\mathbb{R}^n)$ and $w_0 \in BUC^1(\mathbb{R}^n)$ are nonnegative, then there exists a triplet $\big(u, v, w\big)$ of nonnegative time-global unique classical solutions to (P).

Remark 1. (i) We can find at most five stationary constant states (solutions independent of x and t), including the trivial solution $\big(0,0,0\big)$ and $\big(1,0,0\big)$. The trivial solution $\big(0,0,0\big)$ is always instable. Besides, the stabilities of non-trivial constant states depend on parameters; see Remark 4.
(ii) Even if μ and ν are positive smooth functions of u, the same time-global solvability can be proved. In here, we may relax the condition $\gamma = \mu + \nu$, mathematically.
(iii) When the initial data belong to L^∞, we can get the same assertion, although there is a lack of continuity of solutions in t at $t = 0$.

We will explain the strategy of the proof of Theorem 1, briefly. Using the heat semigroups, (P) is written as the forms of integral equations:

$$
u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \left[(1-u)\,u - \frac{\gamma uv}{u+h}\right](s)\,ds, \tag{1}
$$

$$
v(t) = e^{dt\Delta} v_0 + \int_0^t e^{d(t-s)\Delta} \left[\frac{\mu uv}{u+h} + \alpha w - (m+v)v\right](s)\,ds, \tag{2}
$$

$$
w(t) = e^{-\rho t} w_0 + \int_0^t e^{-\rho(t-s)} \left[\frac{\nu uv}{u+h} + \theta v\right](s)\,ds. \tag{3}
$$

Although we can show the uniqueness and regularity of solutions by these forms, the nonnegativity of solutions are not ensured, as long as we use the standard successive approximation. In fact, the following standard iteration scheme is often employed:

$$\widehat{u}_{\ell+1}(t) := e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left[\left(1-\widehat{u}_\ell\right)\widehat{u}_\ell - \frac{\gamma\widehat{u}_\ell\widehat{v}_\ell}{\widehat{u}_\ell + h}\right](s)\,ds.$$

See, for example, the book of Smoller[7]. However, it is hard to show the positivity of $\widehat{u}_\ell$ for $\ell \geq 2$. Thus, we have to look for the other integral forms for proving the existence of nonnegative solutions. To do so, as the same spirit[3], we will construct a triplet of the solutions (u, v, w) as the limits of the following successive approximations:

$$u_{\ell+1}(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left[\left(1-u_{\ell+1}\right)u_\ell - \frac{\gamma u_{\ell+1}v_\ell}{u_\ell + h}\right]ds, \tag{4}$$

$$v_{\ell+1}(t) = e^{dt\Delta}v_0 + \int_0^t e^{d(t-s)\Delta}\left[\frac{\mu u_\ell v_\ell}{u_\ell + h} + \alpha w_\ell - (m+v_\ell)v_{\ell+1}\right]ds, \tag{5}$$

$$w_{\ell+1}(t) = e^{-\rho t}w_0 + \int_0^t e^{-\rho(t-s)}\left[\frac{\nu u_\ell v_\ell}{u_\ell + h} + \theta v_\ell\right]ds \tag{6}$$

for $\ell \in \mathbb{N}$. Here and hereafter, we often omit the notation (s) in the integrand. These are corresponding to the abstract equations:

$$\partial_t u_{\ell+1} = \Delta u_{\ell+1} - \left(u_\ell + \frac{\gamma v_\ell}{u_\ell + h}\right)u_{\ell+1} + u_\ell, \qquad u_{\ell+1}|_{t=0} = u_0, \tag{7}$$

$$\partial_t v_{\ell+1} = d\Delta v_{\ell+1} - (m+v_\ell)v_{\ell+1} + \frac{\mu u_\ell v_\ell}{u_\ell + h} + \alpha w_\ell, \quad v_{\ell+1}|_{t=0} = v_0, \tag{8}$$

$$\partial_t w_{\ell+1} = -\rho w_{\ell+1} + \frac{\nu u_\ell v_\ell}{u_\ell + h} + \theta v_\ell, \qquad w_{\ell+1}|_{t=0} = w_0. \tag{9}$$

Our idea is to involve the coefficients of negative terms into the generators. Instead of the formal expression (4) − (6), we rewrite them using the time-evolution operators:

$$u_{\ell+1}(t) = U_\ell(t,0)u_0 + \int_0^t U_\ell(t,s)\left[u_\ell\right]ds, \tag{10}$$

$$v_{\ell+1}(t) = V_\ell(t,0)v_0 + \int_0^t V_\ell(t,s)\left[\frac{\mu u_\ell v_\ell}{u_\ell + h} + \alpha w_\ell\right]ds, \tag{11}$$

$$w_{\ell+1}(t) = e^{-\rho t}w_0 + \int_0^t e^{-\rho(t-s)}\left[\frac{\nu u_\ell v_\ell}{u_\ell + h} + \theta v_\ell\right]ds \tag{12}$$

for $\ell \in \mathbb{N}$, starting at

$$u_1(t) := e^{t\Delta}u_0, \quad v_1(t) := e^{t(d\Delta - m)}v_0 \quad \text{and} \quad w_1(t) := e^{-\rho t}w_0. \tag{13}$$

Here, $\{U_\ell(t,s)\}$ and $\{V_\ell(t,s)\}$ are time-evolution operators associated with $A_\ell := \Delta - u_\ell - \gamma v_\ell/(u_\ell + h)$ and $B_\ell := d\Delta - m - v_\ell$ for regarding u_ℓ, v_ℓ and w_ℓ as given nonnegative functions, respectively. The definition and estimates of time-evolution operators are given in Section 2. These approximations enable us to show nonnegativities of (u_ℓ, v_ℓ, w_ℓ) for each $\ell \in \mathbb{N}$, as well as its limit (u, v, w). We will derive the estimates $\|u_\ell, v_\ell, w_\ell\|_\infty$ by (10) − (12), inductively, in the fixed point arguments. Besides, for estimates $\|\partial_i u_\ell, \partial_i v_\ell, \partial_i w_\ell\|_\infty$, (4) − (6) are used. Once we get uniform bounds of u_ℓ, v_ℓ, w_ℓ, $\partial_i u_\ell$, $\partial_i v_\ell$ and $\partial_i w_\ell$, we can easily see that the limit (u, v, w) becomes a classical solution.

On the other hand, it is rather standard to extend the obtained solutions time-globally, deriving a priori estimates of solutions. The key idea is to apply the maximum principle to the classical solutions. We can also investigate asymptotic behaviors of solutions, more precisely. Via analysis of solutions to the system of corresponding ordinary differential equations, we obtain invariant regions as follows.

Theorem 2. Let

$$\overline{v} := \mu/(1+h) + \alpha(\nu+\theta+\theta h)/(\rho+\rho h) - m,$$
$$\overline{w} := (\nu+\theta+\theta h)\overline{v}/(\rho+\rho h)$$

be a stationary solution to the second and third equations of (P) with $u \equiv 1$.
(i) If $\overline{v} \le 0$, and if $u_0 \not\equiv 0$, then $\big(u, v, w\big) \to \big(1, 0, 0\big)$ as $t \to \infty$. Besides, if $u_0 \equiv 0$, then the solution tends to the trivial solution $\big(0, 0, 0\big)$ as $t \to \infty$.
(ii) If $\overline{v} > 0$, then for $0 < \varepsilon \ll 1$, there exists a $T_\varepsilon \ge 0$ such that

$$\big(u, v, w\big) \in R_\varepsilon := [0, 1+\varepsilon) \times [0, \overline{v}+\varepsilon) \times [0, \overline{w}+\varepsilon)$$

for $x \in \mathbb{R}^n$ and $t \ge T_\varepsilon$. Moreover, if $\big(u_0, v_0, w_0\big) \in R := [0,1] \times [0, \overline{v}] \times [0, \overline{w}]$, then $\big(u, v, w\big) \in R$ for $t > 0$.
(iii) Let $\underline{u} := (1-h)/2 + \sqrt{(1+h)^2 - 4\gamma\overline{v}}/2$, and let

$$\underline{v} := \mu\underline{u}/(\underline{u}+h) + \alpha\nu\underline{u}/(\rho\underline{u}+\rho h) + \alpha\theta/\rho - m,$$
$$\underline{w} := \nu\underline{u}\,\underline{v}/(\rho\underline{u}+\rho h) + \theta\underline{v}/\rho$$

be a stationary solution to the second and third equations of (P) with $u \equiv \underline{u}$. Assume that $\overline{v} > 0$, $\underline{u} > 0$, $\underline{v} > 0$ and $\underline{w} > 0$. If u, v, $w \ge c_\star$ for $x \in \mathbb{R}^n$ at $t = t_\star \ge 0$ with some $c_\star > 0$, then for $0 < \varepsilon \ll 1$, there exists a $T'_\varepsilon \ge t_\star$ such that

$$\big(u, v, w\big) \in R'_\varepsilon := (\underline{u}-\varepsilon, 1+\varepsilon) \times (\underline{v}-\varepsilon, \overline{v}+\varepsilon) \times (\underline{w}-\varepsilon, \overline{w}+\varepsilon)$$

for $x \in \mathbb{R}^n$ and $t \ge T'_\varepsilon$. Moreover, if $\big(u_0, v_0, w_0\big) \in R_\natural := [\underline{u}, 1] \times [\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}]$, then $\big(u, v, w\big) \in R_\natural$ for $t > 0$.

The sets R and $R_\natural$ are invariant regions. The reader may find another (narrower) invariant regions for each individual parameter. Theorem 2 implies that an absorbing set always exists in R or $R_\natural$.

Our conjecture is that we can also obtain similar results in several domains with suitable boundary conditions. Throughout this paper, we denote positive constants by C the value of which may differ from one occasion to another.

3 | SEMIGROUPS AND TIME-EVOLUTION OPERATORS

In this section, we recall the definitions of function spaces and properties of the heat semigroup as well as time-evolution operators.

Let $n \in \mathbb{N}$, $1 \le p < \infty$, and let $L^p := L^p(\mathbb{R}^n)$ be the space of all p-th integrable functions in $\mathbb{R}^n$ with the norm $\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}$. We often omit the notation of the domain $(\mathbb{R}^n)$, if no confusion occurs. We do not distinguish scalar-valued functions and vector, as well as function spaces. Let L^∞ be the space of all bounded functions with the norm $\|f\| := \|f\|_\infty := \operatorname{ess.sup}_{x\in\mathbb{R}^n}|f(x)|$; BUC as the space of all bounded uniformly continuous functions. For $k \in \mathbb{N}$, let $W^{k,\infty}$ be a set of all bounded functions whose k-th derivatives are also bounded. Furthermore, we define

$$BUC^k := \left\{ f \in W^{k,\infty};\ \partial_i^j f \in BUC \quad \text{for} \quad 1 \le i \le n,\ 0 \le j \le k \right\}.$$

In the whole space $\mathbb{R}^n$, for $\vartheta_0 \in L^\infty(\mathbb{R}^n)$, the heat equation

$$\text{(H)} \qquad \begin{cases} \partial_t \vartheta = \Delta\vartheta & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ \vartheta|_{t=0} = \vartheta_0 & \text{in} \quad \mathbb{R}^n \end{cases}$$

admits a time-global unique smooth solution

$$\begin{aligned} \vartheta := \vartheta(t) := \vartheta(x,t) &:= (e^{t\Delta}\vartheta_0)(x) := (H_t * \vartheta_0)(x) \\ &:= \int_{\mathbb{R}^n} (4\pi t)^{-n/2} \exp(-|x-y|^2/4t)\vartheta_0(y)dy \end{aligned}$$

in $C_w((0,\infty); L^\infty(\mathbb{R}^n))$, that is, $\vartheta \in C([\tau,\infty); L^\infty(\mathbb{R}^n))$ for any $\tau > 0$. Here, $H_t := H_t(x) := (4\pi t)^{-n/2}\exp(-|x|^2/4t)$ is the heat kernel. Since $\|H_t\|_1 = 1$ for $t > 0$, by Young's inequality we have $\|\vartheta(t)\| \le \|H_t\|_1\|\vartheta_0\| = \|\vartheta_0\|$ for $t > 0$. In particular, if $\vartheta_0(x) \ge 0$ for $x \in \mathbb{R}^n$, then $\vartheta(x,t) \ge 0$ holds true for $x \in \mathbb{R}^n$ and $t > 0$; so-called the maximum principle. Furthermore, if additionally $\vartheta_0 \in BUC(\mathbb{R}^n)$ and $\vartheta_0 \not\equiv 0$, then $\vartheta(x,t) > 0$ for $x \in \mathbb{R}^n$ and $t > 0$ by the strong maximum principle. For $\vartheta_0 \in L^\infty(\mathbb{R}^n)$, there is a lack of the continuity of solutions to (H) in time at $t = 0$, in general. Note that $e^{t\Delta}\vartheta_0 \to \vartheta_0$ in L^∞ as $t \to 0$, if and only if $\vartheta_0 \in BUC(\mathbb{R}^n)$. The reader may find its proof in e.g.[2]. Indeed, if $\vartheta_0 \in BUC(\mathbb{R}^n)$, then $\vartheta \in C([0,\infty); BUC(\mathbb{R}^n))$.

We can easily see that for $j \in \mathbb{N}$, there exists a positive constant C $(= \pi^{-j/2} < 1)$ such that $\|\partial_i^j e^{t\Delta}\vartheta_0\| \leq Ct^{-j/2}\|\vartheta_0\|$ for $t > 0$ and $1 \leq i \leq n$. So, $\vartheta(t) \in BUC^j(\mathbb{R}^n)$ for $j \in \mathbb{N}$ and $t > 0$, which implies that $\vartheta(t) \in C^\infty(\mathbb{R}^n)$ for $t > 0$. Moreover, $\vartheta \in C^\infty(\mathbb{R}^n \times (0,\infty))$ by using (H).

In what follows, we recall some properties and estimates for time-evolution operators. Consider the following problem:

$$(\mathrm{P_A}) \qquad \begin{cases} \partial_t\varphi = d\Delta\varphi - \psi(x,t)\varphi & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ \varphi|_{t=0} = \varphi_0 & \text{in} \quad \mathbb{R}^n. \end{cases}$$

Here, $\psi(x,t)$ is a given bounded function. We establish the time-local solvability of $(\mathrm{P_A})$ with upper bounds of $\varphi(t)$.

Lemma 1. Let $n \in \mathbb{N}$, $d, T > 0$ and $\psi \in L^\infty([0,T]; W^{1,\infty}(\mathbb{R}^n))$. If $\varphi_0 \in BUC(\mathbb{R}^n)$, then there exist a $T_* \in (0,T]$ and a time-local unique classical solution to $(\mathrm{P_A})$, having $\|\varphi(t)\| \leq \frac{4}{3}\|\varphi_0\|$ for $t \in [0,T_*]$. Moreover, if $\varphi_0 \geq 0$, then $\varphi \geq 0$.[3]

Proof. Although the proof is written, we give it here. The idea is to use the standard iteration[3]. Let $\varphi_1(t) := e^{dt\Delta}\varphi_0$, and let

$$\varphi_{\ell+1}(t) := e^{dt\Delta}\varphi_0 - \int_0^t e^{d(t-s)\Delta}\left[\psi\varphi_\ell\right](s)\,ds$$

for each $\ell \in \mathbb{N}$, successively. It is easy to see that $\|\varphi_\ell(t)\| \leq \frac{4}{3}\|\varphi_0\|$ for $t \in [0,T_*]$ with some $T_* > 0$ (independent of ℓ) and $\ell \in \mathbb{N}$. We can easily show that $\left\{\varphi_\ell\right\}_{\ell=1}^\infty$ is a Cauchy sequence in $C([0,T_*]; BUC(\mathbb{R}^n))$. So, the limit $\varphi := \lim_{\ell\to\infty}\varphi_\ell$ exists and satisfies $(\mathrm{P_A})$, having the estimate $\|\varphi(t)\| \leq \frac{4}{3}\|\varphi_0\|$ for $t \in [0,T_*]$. It is rather straightforward to obtain the uniqueness and regularity of φ. Moreover, the nonnegativity of φ easily follows from the maximum principle. □

Note that if $\|\varphi_0\| \leq L$ and $\sup_{0\leq t\leq T}\|\psi(t)\| \leq L$ with some $L > 0$, then we may derive the estimate $T_* \geq C/L$ with $C > 0$.

The solution to $(\mathrm{P_A})$ can be rewritten as $\varphi(t) = U(t,0)\varphi_0$, using time-evolution operators $\left\{U(t,s)\right\}_{t\geq s\geq 0}$ associated with $A := A(x,t) := d\Delta - \psi(x,t)$; see e.g. the book of Tanabe[8]. The boundedness of solutions φ implies that $\|U(t,0)\|_{L^\infty\to L^\infty} \leq \frac{4}{3}$ for $t \in [0,T_*]$, and then $\|U(t,s)\|_{L^\infty\to L^\infty} \leq \frac{4}{3}$ for $0 \leq s \leq t \leq T_*$. Here, we have used the notation of an operator-norm $\|\mathcal{O}\|_{X\to Y} := \sup_{x\in X}\|\mathcal{O}x\|_Y/\|x\|_X$.

4 | TIME-LOCAL SOLVABILITY

We give a proof of the time-local solvability on (P) in this section. Recall $\|\cdot\| := \|\cdot\|_\infty$.

Proposition 1. Assume that $n \in \mathbb{N}$, $d > 0$, and those other parameters are nonnegative. Let $u_0, v_0 \in BUC(\mathbb{R}^n)$, $w_0 \in BUC^1(\mathbb{R}^n)$, and $M := \max\{\|u_0\|, \|v_0\|, \|w_0\|, \|\partial_i w_0\|\}$. If u_0, v_0 and w_0 are nonnegative, then there exist a positive time T_0 and a triplet $\left(u, v, w\right)$ of time-local unique classical solutions to (P) in $C([0,T_0]; BUC(\mathbb{R}^n))$, having $0 \leq u(x,t), v(x,t), w(x,t) \leq 2M$ for $x \in \mathbb{R}^n$ and $t \in [0,T_0]$. Furthermore, $T_0 \geq C_*/(M^4 + 1)$ with some $C_* > 0$ independent of M.

Proof. For the sake of simplicity, we assume that all parameter is positive. Making the approximation sequences, we begin with (13). For $\ell \in \mathbb{N}$, we successively define $u_{\ell+1}$, $v_{\ell+1}$ and $w_{\ell+1}$ by (10) – (12). So, $\left(u_{\ell+1}, v_{\ell+1}, w_{\ell+1}\right)$ formally satisfies (7) – (9) for $x \in \mathbb{R}^n$ and $t > 0$ with nonnegative functions u_0, v_0, w_0, u_ℓ, v_ℓ and w_ℓ.

In what follows, we estimate u_ℓ, v_ℓ, w_ℓ, $\partial_i u_\ell$, $\partial_i v_\ell$ and $\partial_i w_\ell$. Put

$$\begin{aligned} K_{1,\ell} &:= \sup_{0\leq t\leq T}\|u_\ell(t)\|, & K_{2,\ell} &:= \sup_{0\leq t\leq T}\|v_\ell(t)\|, \\ K_{3,\ell} &:= \sup_{0\leq t\leq T}\|w_\ell(t)\|, & K_{4,\ell} &:= \sup_{0\leq t\leq T}t^{1/2}\|\partial_i u_\ell(t)\|, \\ K_{5,\ell} &:= \sup_{0\leq t\leq T}(dt)^{1/2}\|\partial_i v_\ell(t)\|, & K_{6,\ell} &:= \sup_{0\leq t\leq T}\|\partial_i w_\ell(t)\| \end{aligned}$$

for $T > 0$, $\ell \in \mathbb{N}$ and $1 \leq i \leq n$. To derive uniform estimates, we argue the induction of ℓ, taking T small.

$\underline{\ell=1}$ For $0 \le u_0(x), v_0(x), w_0(x) \le M$, by the maximum principle and the fact that $e^{t(d\Delta - m)} = e^{-mt}e^{dt\Delta}$, we easily see that

$$0 \le u_1(x,t) \le \|u_0\|, \quad 0 \le v_1(x,t) \le \|v_0\|, \quad 0 \le w_1(x,t) \le \|w_0\|$$

for $x \in \mathbb{R}^n$ and $t > 0$ by $m, \rho \ge 0$. In addition, it is also easy to obtain that

$$t^{1/2}\|\partial_i u_1(t)\| \le \|u_0\|, \quad (dt)^{1/2}\|\partial_i v_1(t)\| \le \|v_0\|, \quad \|\partial_i w_1(t)\| \le \|\partial_i w_0\|$$

for $t > 0$ and $1 \le i \le n$ by the estimate of the heat kernel. Here and hereafter, we replace the constant $C = \pi^{-1/2} < 1$ by 1, for the sake of simplicity. Thus, we have

$$K_{j,1} \le M \quad \text{for} \quad T > 0, \quad 1 \le j \le 6 \quad \text{and} \quad 1 \le i \le n. \tag{14}$$

$\underline{\ell=2}$ Before estimating u_2 and v_2, we will confirm bounds for time-evolution operators U_1 and V_1. By $u_1 \ge 0$ and (14), it holds that

$$\|\eta_1(t)\| \le M + \frac{\gamma M}{h} =: \overline{\eta}_1 \quad \text{with} \quad \eta_1(x,t) := u_1(x,t) + \frac{\gamma v_1(x,t)}{u_1(x,t)+h}$$

for $t > 0$. By Lemma 1, for $\left\{U_1(t,s)\right\}_{t \ge s \ge 0}$ with $A_1(x,t) := \Delta - \eta_1(x,t)$, we thus see that $0 \le U_1(t,s)u_0 \le \frac{4}{3}\|u_0\|$ for $x \in \mathbb{R}^n$ and $0 \le s \le t \le T_2'$ with some $T_2' > 0$ depending only on $\overline{\eta}_1$. So, by (10) with $\ell = 1$, we have

$$0 \le u_2(t) \le \|U_1(t,0)u_0\| + \int_0^t \|U_1(t,s)\zeta_1(s)\| ds \le 2M$$

with $\zeta_1(x,t) := u_1(x,t)$ and $0 \le \zeta_1(x,s) \le \overline{\zeta}_1 := M$, provided $0 \le s \le t \le T_2^{\dagger}$ with $T_2^{\dagger} := \min\left\{T_2', 1/2\right\}$. Similarly, since

$$\|\xi_1(t)\| \le m + M =: \overline{\xi}_1 \quad \text{with} \quad \xi_1(x,t) := m + v_1(x,t)$$

for $x \in \mathbb{R}^n$ and $t > 0$, if we define that $\left\{V_1(t,s)\right\}_{t \ge s \ge 0}$ is the time-evolution operator associated with $B_1(x,t) := d\Delta - \xi_1(x,t)$, then we see that $0 \le V_1(t,s)v_0 \le \frac{4}{3}\|v_0\|$ for $0 \le s \le t \le T_2^{\sharp}$ with some $T_2^{\sharp} > 0$ depending only on $\overline{\xi}_1$. So, by (11),

$$0 \le v_2(t) \le \|V_1(t,0)v_0\| + \int_0^t \|V_1(t,s)\chi_1(s)\| ds \le 2M$$

hold with $\chi_1(x,t) := \mu u_1(x,t)v_1(x,t)/\{u_1(x,t)+h\} + \alpha w_1(x,t)$ and $0 \le \chi_1(x,s) \le \overline{\chi}_1 := (\mu M/h + \alpha)M$, provided if $0 \le s \le t \le T_2^{\flat}$ with $T_2^{\flat} := \min\left\{T_2^{\dagger}, T_2^{\sharp}, h/(2\mu M + 2\alpha h)\right\}$. For the estimate of w_2, we obtain

$$0 \le w_2(t) \le \|e^{-\rho t}w_0\| + \int_0^t e^{-\rho(t-s)}\|\nu u_1 v_1/(u_1+h) + \theta v_1\| ds \le 2M$$

for $0 \le s \le t \le T_2^{\natural}$ with $T_2^{\natural} := \min\left\{T_2^{\flat}, h/(\nu M + h\theta)\right\}$. To derive the estimate for $\partial_i u_2$, we use the heat semigroup expression:

$$u_2(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left[\zeta_1 - \eta_1 u_2\right](s)\, ds,$$

rewriting (10). Hence, it holds that

$$t^{1/2}\|\partial_i u_2(t)\| \le \|u_0\| + t^{1/2}\int_0^t (t-s)^{-1/2}\left[\overline{\zeta}_1 + \overline{\eta}_1\|u_2\|\right] ds \le 2M$$

for $t \in (0, T_2^{\heartsuit}]$ with $T_2^{\heartsuit} := \min\{T_2^{\natural}, h/(2h+4hM+4\gamma M)\}$. As similar way, for $\partial_i v_2$, we appeal to the heat semigroup expression again:

$$(dt)^{1/2}\|\partial_i v_2(t)\| \leq (dt)^{1/2}\|\partial_i e^{dt\Delta} v_0\| + (dt)^{1/2} \int_0^t \|\partial_i e^{d(t-s)\Delta}\left[\chi_1 - \xi_1 v_2\right]\| ds$$

$$\leq \|v_0\| + t^{1/2} \int_0^t (t-s)^{-1/2}\left[\overline{\chi}_1 + \overline{\xi}_1 2M\right] ds \leq 2M$$

for $t \in (0, T_2^{\diamond}]$ with $T_2^{\diamond} := \min\{T_2^{\heartsuit}, h/(2\mu M + 2\alpha h + 4hm + 4hM)\}$. Furthermore,

$$\partial_i w_2(t) = e^{-\rho t}\partial_i w_0 + \int_0^t e^{-\rho(t-s)}\left[\frac{\nu h(\partial_i u_1)v_1 + \nu u_1(\partial_i v_1)(u_1+h)}{(u_1+h)^2} + \theta\partial_i v_1\right] ds$$

holds true, and this implies that

$$\|\partial_i w_2(t)\| \leq M + \int_0^t \left\{\frac{\nu h\sqrt{d}M + \nu M(M+h)}{h^2} + \theta\right\} M(ds)^{-1/2} ds \leq 2M$$

for $t \in [0, T_2]$ with

$$T_2 := \min\{T_2^{\diamond}, dh^4/[4\nu h\sqrt{d}M + 4\nu M^2 + 4\nu hM + 4h^2\theta]^2\}.$$

Therefore, it is shown that $u_2, v_2, w_2 \geq 0$ and

$$K_{j,2} \leq 2M \quad \text{for} \quad t \in (0, T_2], \quad 1 \leq j \leq 6 \quad \text{and} \quad 1 \leq i \leq n. \tag{15}$$

$\underline{\ell = 3}$ We stand for the time-evolution operator $\{U_2(t,s)\}_{t \geq s \geq 0}$ associated with

$$A_2(x,t) := \Delta - \eta_2(x,t) \quad \text{and} \quad \eta_2(x,t) := u_2(x,t) + \gamma v_2(x,t)/\{u_2(x,t)+h\}.$$

By Lemma 1, $U_2(t,s)u_0 \geq 0$ holds and $\|U_2(t,s)\|_{L^\infty \to L^\infty} \leq \frac{4}{3}$ for $0 \leq s \leq t \leq T_3'$ with some $T_3' > 0$, since $0 \leq \eta_2(x,t) \leq \overline{\eta} := 2M + 2\gamma M/h$ by (15). So, we get

$$0 \leq u_3(x,t) \leq \|U_2(t,0)u_0\| + \int_0^t \|U_2(t,s)\zeta_2(s)\| ds \leq 2M$$

for $x \in \mathbb{R}^n$ and $t \in [0, T_3^{\dagger}]$ with $T_3^{\dagger} := \min\{T_3', 1/4\}$. Here we have used that

$$0 \leq \zeta_2(x,t) := u_2(x,t) \leq \overline{\zeta} := 2M.$$

Similarly, we denote the time-evolution operator by $\{V_2(t,s)\}_{t \geq s \geq 0}$ associated with $B_2(x,t) := d\Delta - \xi_2(x,t)$. Since $0 \leq \xi_2(x,t) := m + v_2(x,t) \leq \overline{\xi} := m + 2M$, we seek that $V_2(t,s)v_0 \geq 0$ and $\|V_2(t,s)\|_{L^\infty \to L^\infty} \leq \frac{4}{3}$ for $0 \leq s \leq t \leq T_3^{\sharp}$ with some $T_3^{\sharp} > 0$ by Lemma 1. Hence, we can see that

$$0 \leq v_3(x,t) \leq \|V_2(t,0)v_0\| + \int_0^t \|V_2(t,s)\chi_2(s)\| ds \leq 2M$$

for $x \in \mathbb{R}^n$ and $t \in [0, T_3^{\flat}]$ with $T_3^{\flat} := \min\left\{T_3^{\dagger}, T_3^{\sharp}, h/(8\mu M + 4\alpha h)\right\}$. Here we have used

$$\begin{aligned} 0 \leq \chi_2(x,t) &:= \mu u_2(x,t)v_2(x,t)/\{u_2(x,t)+h\} + \alpha w_2(x,t) \\ &\leq \overline{\chi} := 4\mu M^2/h + 2\alpha M \end{aligned}$$

by (15). It is also easy to show that

$$0 \leq w_3(x,t) \leq \|w_0\| + \int_0^t \|\nu u_2 v_2/(u_2+h) + \theta v_2\| ds \leq 2M$$

for $x \in \mathbb{R}^n$ and $t \in [0, T_3^\natural]$ with $T_3^\natural := \min\left\{T_3^\flat, h/(4\nu M + 2h\theta)\right\}$. By the heat semigroup expression, we obtain that

$$t^{1/2}\|\partial_i u_3(t)\| \le \|u_0\| + t^{1/2}\int_0^t (t-s)^{-1/2}\left[\|\zeta_2\| + \|\eta_2 u_3\|\right] ds \le 2M$$

for $t \in (0, T_3^\heartsuit]$ with $T_3^\heartsuit := \min\{T_3^\natural, h/(4h + 8hM + 8\gamma M)\}$. As similar way, we derive

$$(dt)^{1/2}\|\partial_i v_3(t)\| \le \|v_0\| + t^{1/2}\int_0^t (t-s)^{-1/2}\left[\|\chi_2\| + \|\xi_2 v_3\|\right] ds \le 2M$$

for $t \in (0, T_3^\diamondsuit]$ with $T_3^\diamondsuit := \min\{T_3^\heartsuit, h/(4hm + 8hM + 8\mu M + 4\alpha h)\}$. For $\partial_i w_3$, see

$$\|\partial_i w_3(t)\| \le M + \int_0^t \left\|\frac{\nu h(\partial_i u_2)v_2 + \nu u_2(\partial_i v_2)(u_2 + h)}{h^2} + \theta\partial_i v_2\right\| ds \le 2M$$

for $t \in (0, T_0]$ with

$$T_0 := \min\{T_3^\diamondsuit, dh^4/[8\nu h\sqrt{d}M + 16\nu M^2 + 8\nu hM + 4h^2\theta]^2\}.$$

Note that the estimate $T_0 \ge C/(M^4 + 1)$ is yielded with some $C > 0$.

Therefore, we see that $u_3, v_3, w_3 \ge 0$ and

$$K_{j,3} \le 2M \quad \text{for} \quad t \in (0, T_0], \quad 1 \le i \le n \quad \text{and} \quad 1 \le j \le 6.$$

$\underline{\ell = 4, 5, \dots}$ Let $\ell \ge 4$. We assume that $u_\ell, v_\ell, w_\ell \ge 0$ and

$$K_{j,\ell} \le 2M \quad \text{for} \quad t \in (0, T_0], \quad 1 \le j \le 6 \quad \text{and} \quad 1 \le i \le n \tag{16}$$

hold true. We will compute estimates for $u_{\ell+1}$, $v_{\ell+1}$ and $w_{\ell+1}$. Note that $\eta_\ell \le \overline{\eta}$, $\zeta_\ell \le \overline{\zeta}$, $\xi_\ell \le \overline{\xi}$, and $\chi_\ell \le \overline{\chi}$ hold, independently of $\ell \ge 3$. So, as the same discussion in the case $\ell = 3$ above, we can see that $u_{\ell+1}, v_{\ell+1}, w_{\ell+1} \ge 0$ and

$$K_{j,\ell+1} \le 2M \quad \text{for} \quad t \in (0, T_0], \quad 1 \le j \le 6 \quad \text{and} \quad 1 \le i \le n.$$

The detail is omitted here. Hence, the nonnegativities of approximations and (16) hold true for all $\ell \in \mathbb{N}$.

We can see that $\left(u_\ell, v_\ell, w_\ell\right)$ are continuous in $t \in [0, T_0]$ for $\ell \in \mathbb{N}$. And also, it is easy to see that $\left\{u_\ell, v_\ell, w_\ell, t^{1/2}\partial_i u_\ell, t^{1/2}\partial_i v_\ell, \partial_i w_\ell\right\}_{\ell=1}^\infty$ are Cauchy sequences in $C([0, T_0]; BUC)$, choosing T_0 small again, if necessary. Let

$$\left(u, v, w, \hat{u}, \hat{v}, \hat{w}\right) := \lim_{\ell\to\infty}\left(u_\ell, v_\ell, w_\ell, t^{1/2}\partial_i u_\ell, t^{1/2}\partial_i v_\ell, \partial_i w_\ell\right)$$

in the topology of $C([0, T_0]; BUC)$. Obviously, the coincidences $\hat{u} = t^{1/2}\partial_i u$, $\hat{v} = t^{1/2}\partial_i v$ and $\hat{w} = \partial_i w$ hold by construction. Furthermore, it is also ensured that

$$0 \le u(x,t), v(x,t), w(x,t) \le 2M \quad \text{for} \quad x \in \mathbb{R}^n \quad \text{and} \quad t \in [0, T_0].$$

The uniqueness follows from (1) − (3) and Gronwall's inequality, directly. If fact, let $\left(u, v, w\right)$ and $\left(u^*, v^*, w^*\right)$ be solutions to (P) in $[0, T_0]$ with the same initial data, then $u \equiv u^*$, $v \equiv v^*$ and $w \equiv w^*$ simultaneously hold. Thanks to the boundedness of the first derivatives, it is easy to control the second derivatives in x of u and v for $t \in (0, T_0]$, as well as the first derivatives in t of solutions. So, we see that $\left(u, v, w\right)$ is a triplet of time-local unique classical solutions to (P). This completes the proof of Proposition 1. □

Remark 2. (i) If w_0 is smooth, then $\left(u, v, w\right)$ is smooth in x and t.
(ii) For $d = 0$, we can also get time-local well-posedness, if $v_0 \in BUC^1$.
(iii) The instability of the trivial solution $\left(0, 0, 0\right)$ is easily obtained. Moreover, by strong maximum principle for solutions to the heat equation, $u > 0$ for $x \in \mathbb{R}^n$ and $t \in (0, T_0]$, if $u_0 \not\equiv 0$. This means that $\operatorname{supp} u(t) = \mathbb{R}^n$ for any small $t > 0$, even if $\operatorname{supp} u_0$ is compact. That is, the propagation speed of solutions to (P) is infinite, as the same as the heat equation. In addition, $v > 0$ and $w > 0$ for $x \in \mathbb{R}^n$ and $t > 0$, if either $v_0 \not\equiv 0$ or $w_0 \not\equiv 0$.

5 | TIME-GLOBAL WELL-POSEDNESS

In this section, we will derive a priori bounds of solutions and their derivatives. To do so, our first task is to obtain upper bounds of solutions to (P) with large initial data. For the case when $\|u_0\| \le 1$, we will discuss in Remark 3 (ii) below and Section 5.

Proposition 2. Suppose the assumption of Proposition 1. If $\|u_0\| > 1$, then $0 < u < \|u_0\|$, $0 \le v \le \widetilde{v}$, and $0 \le w \le \widetilde{w}$ for $x \in \mathbb{R}^n$ and $t > 0$ with some $\widetilde{v}$ and $\widetilde{w} > 0$ depending on $|u_0\|$, $\|v_0\|$, and $\|w_0\|$, as long as the classical solutions exist.

Proof. If $v_0 \equiv 0$ and $w_0 \equiv 0$, then $v \equiv w \equiv 0$ for $t > 0$. Assume either $v_0 \not\equiv 0$ or $w_0 \not\equiv 0$. So, as seen in Remark 2 (iii), we have $u, v, w > 0$. For observing the behavior of u, we consider the following logistic equation:

$$\kappa' = (1-\kappa)\kappa, \quad \kappa(0) = \kappa_0 > 1, \tag{17}$$

where $\kappa_0 = \|u_0\|$.
By maximum principle, $u(x,t) \le \kappa(t)$ holds for $x \in \mathbb{R}^n$ and $t > 0$, as long as the classical solution u exists. Since

$$\kappa(t) = \kappa_0/(\kappa_0 + e^{-t} - \kappa_0 e^{-t}) < \kappa_0$$

for $t > 0$, it is clear that $u < \kappa_0$.

Next, we investigate on upper bounds of v and w. Let a pair $\sigma = \sigma(t)$ and $\omega = \omega(t)$ be solutions to

$$\begin{cases} \sigma' = \alpha\omega - (m_\star + \sigma)\sigma, & \sigma(0) = \sigma_0 := \|v_0\|, \\ \omega' = \theta_\star \sigma - \rho\,\omega, & \omega(0) = \omega_0 := \|w_0\|. \end{cases} \tag{18}$$

Here, $m_\star := m - \mu\kappa_0/(\kappa_0 + h)$ and $\theta_\star := \theta + \nu\kappa_0/(\kappa_0 + h)$. Since $(e^{\rho t}\omega)' = \theta_\star e^{\rho t}\sigma$, we have

$$\omega(t) = e^{-\rho t}\omega_0 + e^{-\rho t}\theta_\star \int_0^t e^{\rho s}\sigma(s)ds \le \omega_0 + (\theta_\star/\rho)\sup_{0\le s\le t}\sigma(s)$$

for $t > 0$. Inserting it into the first equation of (18), it holds that

$$\sigma' \le \alpha\left\{\omega_0 + (\theta_\star/\rho)\sup_{0\le s\le t}\sigma(s)\right\} - m_\star\sigma - \sigma^2$$

for $t > 0$. Therefore, we can see that $\sigma(t) < \widetilde{v} := \max\left\{\sigma_0, \bar{\sigma}\right\} + 1$ for $t > 0$, where

$$\bar{\sigma} := \alpha\theta_\star/2\rho - m_\star/2 + \sqrt{\alpha\omega_0 + (\alpha\theta_\star/\rho - m_\star)^2/4},$$

and $\bar{\sigma}$ satisfies $\alpha(\omega_0 + (\theta_\star/\rho)\bar{\sigma}) - m_\star\bar{\sigma} - \bar{\sigma}^2 = 0$. Indeed, if there exists some $t_\star > 0$ such that $\sigma(t_\star) = \widetilde{v} \ge \bar{\sigma} + 1$ and $\sigma(t) < \widetilde{v}$ for $t \in [0, t_\star)$, then $\sigma'(t_\star) \ge 0$. This contradicts $\sigma'(t_\star) < 0$. We can similarly deduce $\omega(t) \le \widetilde{w}$ holds for $t > 0$, where

$$\widetilde{w} := \max\left\{\omega_0, \theta_\star\widetilde{v}/\rho\right\} + 1.$$

Put $V := \sigma - v$ and $W := \omega - w$. Hence, $V(0) \ge 0$ and $W(0) \ge 0$. Also, we see

$$\begin{aligned} \partial_t V &= d\Delta V + \alpha W - mV + \mu\kappa_0\sigma/(\kappa_0 + h) - \mu uv/(u+h) - \sigma^2 + v^2 \\ &= d\Delta V + \alpha W - (m + \sigma + v)V + \frac{\mu}{(\kappa_0 + h)(u+h)}\left[(u+h)\kappa_0 V + hv(\kappa_0 - u)\right] \end{aligned}$$

and

$$\begin{aligned} \partial_t W &= \theta V - \rho W + \nu\kappa_0\sigma/(\kappa_0 + h) - \nu uv/(u+h) \\ &= \theta V - \rho W + \frac{\nu}{(\kappa_0 + h)(u+h)}\left[(u+h)\kappa_0 V + hv(\kappa_0 - u)\right]. \end{aligned}$$

We thus find the fact that $V \ge 0$ and $W \ge 0$ for $t > 0$, as the same discussion in the proof of Proposition 1. This implies that

$$v(x,t) \le \sigma(t), \quad w(x,t) \le \omega(t) \tag{19}$$

for x and t. Therefore, we conclude that $0 \le v \le \widetilde{v}$ and $0 \le w \le \widetilde{w}$. □

Remark 3. (i) By definitions of $\overline{v}$ and $\overline{w}$ are given in Theorem 2, it is clear that $\widetilde{v} \ge \overline{v}$ and $\widetilde{w} \ge \overline{w}$, if $|u_0\| \ge 1$. Besides, $\widetilde{v} \le \overline{v}$ and $\widetilde{w} \le \overline{w}$, if $|u_0\| \le 1$, $\|v_0\| \le \overline{v}$ and $\|w_0\| \le \overline{w}$; see Section 5.
(ii) Even if $\|u_0\| \le 1$, then the uniform bounds on v and w are obtained; $v \le \widetilde{v}$ and $w \le \widetilde{w}$ hold, replacing $m_\star$ by $m_1 := m - \mu/(1+h)$ and $\theta_\star$ by $\theta_1 := \theta + \nu/(1+h)$.

(iii) Although we can take the maximum values of the solutions to ODEs (18) by the comparison method (finding t as $\sigma'(t) = 0$ or $\omega'(t) = 0$), such critical points do not always give maximum values of solutions to PDEs, in general. So, we have used the technique of renormalization type above.

In what follows, we will give the a priori estimate for $\|\partial_i w(t)\|$, which may grow in t. As seen in Proposition 2, and by using definitions of $\overline{v}$ and $\overline{w}$ in Theorem 2, we prove that $0 \le u, v, w \le N$ as long as the classical solutions exist, if N is chosen as

$$N := \max\left\{1, \|u_0\|, \overline{v}, \widetilde{v}, \|v_0\|, \overline{w}, \widetilde{w}, \|w_0\|\right\}. \tag{20}$$

Proposition 3. Let T, $N > 0$. If $0 \le u, v, w \le N$ for $x \in \mathbb{R}^n$ and $t \in [0, T]$, then there exists a $C > 0$ (independent of N and T) such that

$$\|\partial_i w(t)\| \le \|\partial_i w_0\| + C(N^4 + N)\left(t^{1/2} + t^{3/2}\right), \quad t \in [0, T], \ \ 1 \le i \le n.$$

Proof. We first derive the estimate for $\partial_i u$. By (1), we have

$$\begin{aligned}\|\partial_i u(t)\| &\le \|u_0\| t^{-1/2} + \int_0^t (t-s)^{-1/2} \left\|(1-u)u - \frac{\gamma uv}{u+h}\right\| ds \\ &\le C(N^2 + N)(t^{-1/2} + t^{1/2})\end{aligned}$$

for $t \in [0, T]$ and $1 \le i \le n$ with some C. Similarly, by (2), we seek

$$\begin{aligned}\|\partial_i v(t)\| &\le \|v_0\|(dt)^{-1/2} + \int_0^t (dt - ds)^{-1/2} \left\|\frac{\mu uv}{u+h} + \alpha w - (m+v)v\right\| ds \\ &\le C(N^2 + N)(t^{-1/2} + t^{1/2})\end{aligned}$$

with some C. Finally, by (3) and estimates above, it turns out that

$$\begin{aligned}\|\partial_i w(t)\| &\le \|\partial_i w_0\| + \int_0^t \left\|\frac{\nu h(\partial_i u)v + \nu u(\partial_i v)(u+h)}{(u+h)^2} + \theta \partial_i v\right\| ds \\ &\le \|\partial_i w_0\| + C(N^4 + N) \int_0^t (s^{-1/2} + s^{1/2}) ds \\ &\le \|\partial_i w_0\| + C(N^4 + N)(t^{1/2} + t^{3/2})\end{aligned}$$

for $t \in [0, T]$ and $1 \le i \le n$ with some positive constant C depending on parameters, however, independent of N and T. $\square$

Note that the proof of Theorem 1 is now complete. In fact, Theorem 1 follows from Propositions 1, 2, 3 and $T_0 \ge C_*/(M^4+1)$ in Proposition 1, since we can extend the obtained unique solutions time-globally, repeating the construction.

6 | INVARIANT REGIONS

This section will be devoted to observing invariant regions. The proof of Theorem 2 (i) is easy, since $\left(1, 0, 0\right)$ is only one stable constant state. So, we skip it here.

We are now in position to give a proof of Theorem 2 (ii). The key step is to deduce a priori bounds of solutions, due to the maximum principle and comparison with solutions to the system of corresponding ordinary differential equations of κ, σ and ω given by (17) and (18). Let us recall the assumptions:

$$\begin{aligned}\overline{v} &:= \mu/(1+h) + \alpha(\nu + \theta + \theta h)/(\rho + \rho h) - m > 0, \\ \overline{w} &:= (\nu + \theta + \theta h)\overline{v}/(\rho + \rho h) > 0\end{aligned}$$

and $R := [0, 1] \times [0, \overline{v}] \times [0, \overline{w}]$.

Proof of Theorem 2 (ii). We first show that R is an invariant region. Let $\left(u_0, v_0, w_0\right) \in R$. By construction of time-local solutions in Proposition 1, the nonnegativity of solutions is clarified. Note that $\left(0, 0, 0\right)$ and $\left(1, 0, 0\right)$ are classical solutions in R. If

$u_0 \equiv 0$, then $u \equiv 0$, in addition, $v \in [0, \overline{v}]$ and $w \in [0, \overline{w}]$, since $v^\flat := \alpha\theta/\rho - m \leq \overline{v}$ and $w^\flat := \theta(\alpha\theta - m\rho)/\rho^2 \leq \overline{w}$. Also, it is easy to see that $v \equiv 0$ and $w \equiv 0$ hold for $t > 0$, provided if $v_0 \equiv 0$ and $w_0 \equiv 0$.

Let $u_0 \not\equiv 0$ and either $v_0 \not\equiv 0$ or $w_0 \not\equiv 0$. As seen in Remark 2 (iii), it is clear that the classical solutions u, v, w never touch 0, as long as they exist. Moreover, with $u_0 \leq 1$, we observe that $u(\tau) < 1$ for small $\tau > 0$ by the strong maximum principle. Similarly, it turns out that $v(\tau) < \overline{v}$ by $v_0 \leq \overline{v}$, as well as $w(\tau) < \overline{w}$. So, regarding τ as the initial time, we can assume $\left(u_0, v_0, w_0\right) \in R^\circ := R \setminus \partial R$, without loss of generality.

Put $\hat{t} \in (0, T_0]$ is the first time when u touches 1 at $\hat{x} \in \mathbb{R}^n$. We may assume $|\hat{x}| < \infty$ by Oleinik's argument; see e.g. [1]. Since $u(\hat{x}, \hat{t}) = 1$ is the local maximum, at $(\hat{x}, \hat{t})$ we see that $\partial_t u \geq 0$, $\Delta u \leq 0$, $(1 - u)u = 0$ and $-\gamma uv/(u + h) < 0$ by $v > 0$. This contradicts to that u is a solution to (P). Hence, u never touches 1.

The same argument works on v and w. Indeed, let $0 < u < 1$, $0 < w < \overline{w}$, and if there exists $(\check{x}, \check{t}) \in \mathbb{R}^n \times (0, T_0]$ such that $\check{t}$ is the first time when v touches $\overline{v}$ at $\check{x}$. So, at $(\check{x}, \check{t})$, we see that $\partial_t v \geq 0$, $d\Delta v \leq 0$ and

$$\frac{\mu uv}{u + h} + \alpha w - (m + v)v < \frac{\mu\overline{v}}{1 + h} + \alpha\overline{w} - (m + \overline{v})\overline{v} = 0.$$

So, v never touches $\overline{v}$. As the same as above, we can confirm that w never touches $\overline{w}$ as long as classical solutions exist. This means that the solutions always remains in $R^\circ \subset R$.

Next, we show the asymptotic behavior of solutions, briefly. Even if $\|u_0\| > 1$, by $u(x, t) \leq \kappa(t)$, then there exists a $T_\varepsilon^* > 0$ such that $\|u(t)\| < 1 + \varepsilon$ for $t > T_\varepsilon^*$. From this and the comparison $v(x, t) \leq \sigma(t)$, there exists $T_\varepsilon^\sharp > T_\varepsilon^*$ such that $\|v(t)\| < \overline{v} + \varepsilon$ for $t > T_\varepsilon^\sharp$. Finally, we can also show that there exists $T_\varepsilon > T_\varepsilon^\sharp$ such that $\|w(t)\| < \overline{w} + \varepsilon$ for $t > T_\varepsilon$, by the similar way. This completes the proof of Theorem 2 (ii). □

The proof of Theorem 2 (iii) is essentially similar to above. So, we omit it here.

Remark 4. The stability of non-trivial constant states to the system of corresponding ordinary differential equations can be easily obtained. For example, if

$$\mu = \nu = \frac{\gamma}{2},\ m = \theta = 0,\ \alpha = \rho = \frac{1}{4},\ \gamma = h + \frac{1}{2}$$

are chosen, then the stability of a constant state $\left(u, v, w\right) = \left(1/2, 1/2, 1/2\right)$ is bifurcared in h at 0. Indeed, the constant state $\left(1/2, 1/2, 1/2\right)$ is stable for any $h > 0$, while this is unstable for any $-1/2 < h < 0$. The authors believe that such stability is still valid for solutions to (P). For studying the Turing instability, we need to deal with more complicated situation, e.g. when μ and ν are functions of u.

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