ALMOST BS-COMPACT OPERATORS AND DOMINATION PROBLEM

MOHAMED HAJJI

ABSTRACT. Let X and Y be two Banach spaces. A bounded operator $T : X \longrightarrow Y$ is said to be a BS-compact operator whenever T sends Banach-Saks subsets of X onto norm compact sets of Y([20]). In this paper, our central focus is upon introducing the class of almost BS-compact operators. The paper rests essentially on two parts. The first is devoted to the connection of this new class of operators with classical notions of operators, such as BS-compact operators, AM-compact operators, and Dunfort-Pettis operators. The second part is dedicated to the domination problem within the framework of (almost) BS-compact operators.

Mathematics Subject Classification. 46B42, 47B60, 47B65.

1. Introduction

Let E and F be Banach lattices. Consider operators $0 \leq R \leq T : E \longrightarrow F$ such that T satisfies some property $\mathcal{P}$. The domination problem stands for finding conditions under which $\mathcal{P}$ will be inherited by R. In the particular case that $E = F$, it is interesting to investigate whether some power of R inherit $\mathcal{P}$. This is corresponds to the power domination problem.

Domination properties of operators on Banach lattices have whetted the interest and drawn the attention of multiple researchers. For instance, consult([2, 7, 8, 9, 15, 18, 21]).

As far as we are basically concerned with introducing the class of almost BS-compact operators. The manuscript relies on two intrinsic parts. The first part addresses the connection of this new class of operators with classical notions of operators, such as, BS-compact operators, AM-compact operators, or Dunfort-Pettis operators. However, the second part tackles the domination problem within the framework of (almost) BS-compact operators.

2. Preliminaries

Throughout this paper, X and Y will denote Banach spaces and E, F will denote Banach lattices. The positive cone of E will be expressed by $E_+ = \{x \in E; \quad 0 \leq x\}$. We will use the term operator, between two Banach spaces, to indicate a bounded linear mapping.

A bouded subset B of a Banach space X is called Banach-Saks if each sequence (x_n) in B has a subsequence (y_n), whose arithmetic means converge in norm. That is, there exists $x \in X$ such that:

$$\|\frac{1}{n}\sum_{k=1}^{n} y_k - x\| \longrightarrow 0.$$

Note that every Banach-Saks set is relatively weakly compact [26, Proposition 2.3]. The converse statement is not true in general ([6]). Recall that a bounded operator $T : X \longrightarrow Y$ is said to be a BS-compact operator whenever T sends Banach-Saks subsets of X onto norm compact sets of Y([20]). Clearly, every compact operator is BS-compact. The identity operator $I_{l_1} : l_1 \to l_1$ is BS-compact which is not compact. If X has the Banach-Saks property, these classes coincide .

A bounded subset A of a Banach lattice E is said to be L-weakly compact, if $\|x_n\| \longrightarrow 0$ for every disjoint sequence $(x_n)_n$ in the solid hull of A([27]). The solid hull of a subset A of a Banach lattice E is the set

$$\mathrm{Sol}(A) = \{x \in E : \exists a \in A \text{ with } |x| \leq |a|\}.$$

A characterization of L-weakly compact set is expressed as follows.

Lemma 2.1. [12, Lemma 2.4] *For every nonempty bounded subset $A \subset E$, the following assertions are equivalent.*

(1) *A is L-weakly compact.*

(2) *$f_n(x_n) \to 0$ for every sequence (x_n) of A and every disjoint sequence (f_n) of $B_{E'}$.*

Recall that a Banach lattice E is said to be order continuous if $\lim_\alpha \|x_\alpha\| = 0$ for every decreasing net $(x_\alpha)_\alpha$ in E such that $\wedge_\alpha x_\alpha = 0$. An element $e \in E$ is said to be a weak unit if for $h \in E$, $e \wedge h = 0$ implies $h = 0$. Note that every separable Banach lattice has a weak unit.

Departing from Theorem 1.b.14 in [24], we realize that an order continuous Banach lattice with a weak unit can be assumed to be included in $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ. From this perspective, we denote this inclusion by $j : E \hookrightarrow L_1(\Omega, \Sigma, \mu)$. Let X be a separable subspace of an order continuous Banach lattice E. It follows from Proposition 1.a.9 in [24] that E_X (E_X being the ideal generated by X) has a weak unit. Let (g_n) be a sequence of E. Then, we denote by $[g_n]$ the closed subspace spanned by the vectors (g_n). In terms of order continuous Banach lattices, the convergence of a bounded sequence is characterized as follows.

Lemma 2.2. *Let E be a Banach lattice with an order continuous norm, and $(g_n)_n$ be a bounded sequence in E. Then,*
$(g_n)_n$ is convergent in E if and only if it is L-weakly compact and $\|.\|_1-$convergent.

Proof. Since $[g_n]$ is a separable subspace of E, it follows from Proposition 1.a.9 in [24] that $E_{[g_n]}$ ($E_{[g_n]}$ being the ideal generated by $[g_n]$) has a weak unit. The rest of the proof follows from Lemma 1.4.2 in [31]. □

A Banach space E has the weak Banach-Saks property (or it is weakly Banach-Saks) if every weakly convergent sequence $(x_n)_n$ in E has a subsequence which is Cesàro convergent.

Theorem 2.3. (*Szlenk* [30]) *Let* (Ω, Σ, μ) *be a probability space. Then,* $L_1(\Omega, \Sigma, \mu)$ *is weakly Banach-Saks.*

3. Almost BS-compact operators

Relying upon [22], we state that a Banach lattice has the $(W1)$ property if for every relatively weakly compact subset A of E, the set $|A| := \{|a| : a \in A\}$ is again relatively weakly compact. Likewise, we define the $(BS1)$ property as follows.

Definition 3.1. A Banach lattice E has the property $(BS1)$ if for every Banach-Saks set $A \subset E$, the set $|A| := \{|a| : a \in A\}$ is also Banach-Saks.

Clearly, every Banach-Saks space has the $(BS1)$ property. An important example of a Banach lattice without property $(BS1)$ is $c_0(L_2[0,1])$, where $c_0(L_2[0,1])$, is the Banach space of all null sequences in $L_2[0,1]$, endowed with the supremum norm.

Example 3.2. Let $E = c_0(L_2[0,1])$. Referring to the Example page 108 in [28], there exists a relatively weakly compact subset A of E such that $|A|$ is not Banach-Saks. On the other side, since $L_2[0,1]$ has the uniform weak Banach-Saks property(see Theorem page 109 in [14]), it follows from Theorem 3 in [25] that E has the weak Banach-Saks property. As a matter of fact, A is Banach-Saks, which implies that E is not $(BS1)$ space.

The preceding example stands for the impetus urging us to define the class of almost BS-compact operators.

Definition 3.3. An operator T from a Banach lattice E into a Banach space Y is said to be almost BS-compact if T carries Banach-Saks subsets of E_+ onto relatively compact subsets of Y.

Note that every BS-compact operator is almost BS-compact. A linear operator T from a Banach lattice E to a Banach space Y is said to be AM-compact if it maps order bounded subset of E to a totally bounded subset of Y[15].

Theorem 3.4. *Let E be an order continuous Banach lattice and Y be a Banach space. Then, every almost BS-compact operator $T : E \to Y$ is AM-compact.*

Proof. It is enough to demonstrate that every order bounded subset of E is Banach-Saks. For this reason, let $(x_n)_n$ be a sequence in E satisfying $0 \leq x_n \leq y$ for all n and some $y \in E_+$. Since E is order continuous, it follows from Theorem 4.9 in [1] that $[0, y]$ is weakly compact. Thus, there exists a sequence $(x_{\phi(n)})_n$ of (x_n) such that $x_{\phi(n)} \xrightarrow{\sigma(E,E')} x$ for some $x \in E_+$. Since $X := [x_{\phi(n)}]$ is a separable subspace of E, it follows from Proposition 1.a.9 in [24] that $\overline{E_X}$ is an order ideal with a weak order unit. Therefore, it can be represented as a dense order ideal of $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ, such that the formal inclusion

$$j : \overline{E_X} \hookrightarrow L_1(\Omega, \Sigma, \mu)$$

is continuous ([24], Theorem 1.b.14). Thus, (jx_n) converges weakly to jx in $L_1(\Omega, \Sigma, \mu)$. At this stage of analysis, Theorem 2.3 combined with the Theorem in [16] reveals that there exists a subsequence (y_n) of $(x_{\phi(n)})$ such that $\frac{1}{n}\sum_{k=1}^{n} jy_k$ converges in norm to jx. On the other side, since E is order continuous and $0 \leq \frac{1}{n}\sum_{k=1}^{n} y_k \leq y$ for all n, we infer that $A = \{\frac{1}{n}\sum_{l=1}^{n} y_l, \quad n \in \mathbb{N}\}$ is an L-weakly compact subset of E (see Theorem 4.14 in [1]). According to Lemma 2.2, we have $\frac{1}{n}\sum_{k=1}^{n} y_k$ converges to $x \in E$. Which implies that $[0, y]$ is Banach-Saks. $\square$

Remark 3.5. It is noteworthy that the converse of Theorem 3.4 is not true in general. For instance, consider the identity operator $Id_{c_0} : c_0 \longrightarrow c_0$.It is obvious that Id_{c_0} is AM-compact. On the other side, the standard unit vectors of c_0 is Banach-Saks and has no convergent subsequence on c_0. Hence, Id_{c_0} is not Almost BS-compact.

The preceding theorem combined with Theorem 5.97 in [1] yields:

Corollary 3.6. *Let E be a Banach lattice with order continuous norm, and let F be an AL-space. Then, for a regular operator $T : E \to Y$, the following assertions are equivalent.*

(1) *The linear operator T is Dunford-Pettis.*
(2) *The linear operator T is AM-compact.*
(3) *The linear operator T is almost BS-compact.*
(4) *The linear operator T BS-compact.*

The notions of Almost BS-and BS-compact operators may coincide. The next result provides a condition for this to happen.

Theorem 3.7. *Let T be an operator from an order continuous Banach lattice E into a Banach space Y; if E has the $(BS1)$ property, then the following assertions are equivalent.*

(1) *The linear operator T is BS-compact.*
(2) *The linear operator T is almost BS-compact.*

Proof. (2) $\Longrightarrow$ (1). Let A be a Banach-Saks set of E, and let $(x_n)_n$ be a sequence in A. Since E has the $(BS1)$ property, it follows that $|A|$ is Banach-Saks. Therefore, by passing to a subsequence, we can assume that for some $x \in E_+$ we have

$$\lim_n \|\frac{1}{n}\sum_{k=1}^{n} |x_k| - x\| = 0.$$

To this extent, resting on our hypothesis, there exists a subsequence (z_n) of $(x_n)_n$ such that $T|z_n|$ converges in norm. Next, let $(h_n) \subset E_+$ be a disjoint sequence in the solid hull of $\{z_n, \quad n \in \mathbb{N}\}$. The weak compactness of A (by Proposition 2.3 in [26]) implies (by Theorem 4.34 in [1]) that $h_n \xrightarrow{\sigma(E,E')} 0$. Let's take a subsequence (w_n) of (h_n). Moving to a subsequence, we can assume that $0 \leq w_n \leq |z_n|$ holds for all n. In particular, for $n \in \mathbb{N}$ we have

$$0 \leq \frac{1}{n}\sum_{k=1}^{n} w_k \leq \frac{1}{n}\sum_{k=1}^{n} |z_k|.$$

Grounded on Lemma 2.2, we realize that $\{\frac{1}{n}\sum_{k=1}^{n} w_k, \quad n \in \mathbb{N}\}$ is L-weakly compact. Since $\frac{1}{n}\sum_{k=1}^{n} w_k \xrightarrow{\sigma(E,E')} 0$, it follows from Lemma 2.3 and Lemma 2.4 in [12] that $\frac{1}{n}\sum_{k=1}^{n} w_k \xrightarrow{\|\cdot\|} 0$. Without loss of generality, we assume that $Tw_n \xrightarrow{\|\cdot\|} 0$. Thus, the choice of (w_n) guarantees that $Th_n \xrightarrow{\|\cdot\|} 0$.

Let $\epsilon > 0$. Based on Theorm 4.36 in [1], there exists some $u \in E_+$ satisfying

$$\|T(|y_n| - u)^+\| \leq \epsilon,$$

for all n. From $|y_n| = |y_n| \wedge u + (|y_n| - u)^+$, it follows that

$$\{Ty_n, \quad n \in \mathbb{N}\} \subset T[-u, u] + \epsilon B_Y.$$

From this perspective, an easy application of Theorem 3.4 guarantees that $\{Ty_n, \quad n \in \mathbb{N}\}$ is relatively compact. Hence, TA is relatively compact. □

However, the following problem remains unresolved.

Problem 3.8. Is there an almost BS-compact operator that is not BS-compact?

Other properties of (Almost)BS-compact operators are provided by the following Theorem.

Theorem 3.9. *Let T be a BS-compact (rep. Almost BS-compact) operator from a Banach lattice E into a Banach space Y.*

1. *The class of all BS-compact (rep. Almost BS-compact) operators from E to Y is a closed subspace of $\mathcal{L}(E, Y)$.*
2. *If R is a bounded operator from Y into a Banach space Z, then RT is BS-compact (rep. Almost BS-compact).*
3. *If R is a bounded operator from a Banach space Z into E, then TR is almost BS-compact(rep. Almost BS-compact).*

Proof. (1) Let $(T_n)_n$ be a sequence of BS-(resp Almost BS-)compact operators from E to Y which satisfies $T_n \to T$ in $L(E, Y)$, and let A be a Banach-Saks subset of E (resp. E_+). Fix $\epsilon > 0$. Therefore, there exists N_0 such that

$$T(A) \subset T_{N_0}(A) + \epsilon B_Y.$$

Since $T_{N_0}(A)$ is a norm relatively compact subset of Y, it follows that $T(A)$ is also a relatively compact subset of Y. This reveals that T is BS-(resp Almost BS-)compact.

(2) Let A be a Banach-Saks subset of E (resp. E_+). Since T is a BS-compact (rep. Almost BS-compact) operator, it follows that $T(A)$ is a norm relatively compact subset of Y. Thus, $RT(A)$ is a norm relatively compact subset of Z(the linear operator R is bounded). Hence, RT is BS-(resp Almost BS-)compact.

□

A significant property of the order bounded disjoint sequence is included in the next proposition.

Proposition 3.10. *Let E be a normed riesz space and let (w_n) be an order bounded disjoint sequence of E_+. Then, $\lim_n \|\frac{1}{n}\sum_{i=1}^{n} w_i\| = 0$.*

Proof. Let $(w_n)_n$ be a positive disjoint sequence of E and let $x \in E_+$ such that $0 \leq w_n \leq x$ for all n. Since $\vee_{i=1}^{n} w_i = \sum_{i=1}^{n} w_i$ for all $n \in \mathbb{N}$, it follows that

$$0 \leq \frac{1}{n}\sum_{k=1}^{n} w_i = \frac{\vee_{i=1}^{n} w_i}{n} \leq \frac{x}{n},$$

which implies that

$$\|\frac{1}{n}\sum_{k=1}^{n} w_i\| \leq \frac{\|x\|}{n} \longrightarrow 0.$$

□

An operator T between a Banach lattice E and a Banach space Y is said to be order weakly compact if $T([-x, x])$ is relatively weakly compact for every positive element $x \in E$. Order weakly compact operators can be characterized as those operators which fail to be invertible on any sublattice isomorphic to c_0 with an order bounded unit ball (see Corollary 3.4.5 in [28]). The preceding proposition combined with Theorem 3.4.4 in [28] unveils that an almost BS-compact operator is order weakly compact.

Corollary 3.11. *Let E be a Banach lattice, and let Y be a Banach space. Then every almost BS-compact operator $T : E \to Y$ is order weakly compact.*

Proof. Let (w_n) be an order bounded disjoint sequence of E_+. It follows from Proposition 3.10 that $\lim_n \|\frac{1}{n}\sum_{i=1}^{n} w_i\| = 0$. Since T is almost BS-compact, then $\lim_n \|Tw_n\| = 0$. The rest of the proof follows from Theorem 5.57 in [1].

□

4. Domination results

Let $R : E \longrightarrow F$ be a positive operator between two Banach lattices dominated by a BS-compact operator (respectively, almost BS-compact) T. Is then R necessarily BS-compact (respectively, almost BS-compact) ? The answer is negative in general. The details are provided below.

Example 4.1. There exist two operators $0 \leq R \leq T : L_2[0,1] \longrightarrow l_\infty$ such that T is BS-compact but R is not almost BS-compact.

Proof. Let (r_n) denote the sequence of Rademacher functions on $[0,1]$. This means, $r_n(t) = \mathrm{Sgnsin}(2^n \pi t)$. Let $0 \leq R \leq T : L_2[0,1] \longrightarrow l_\infty$ be the positive operators defined in Example 3.1 of [1] by

$$Rf = (\int_0^1 f(x) r_1^+(x)dx, \int_0^1 f(x) r_2^+(x)dx, ...).$$

$$Tf = (\int_0^1 f(x)dx, \int_0^1 f(x)dx, \int_0^1 f(x)dx, ...).$$

Clearly, T is BS-compact. On the other side, referring to Example 2.7 in [5], we infer that R is not AM-compact. In particular, from Theorem 3.4 it follows that R is not almost BS-compact. □

4.1. **Power domination by BS-compact operators.** In this section, we tackle the power problem for BS-compact operators. To state our main result, we need the following Theorem.

Theorem 4.2. *([19], Theorem I.2) Let E_1 and E_2 be Banach lattices and consider operators $0 \leq R \leq T : E_1 \longrightarrow E_2$. Then, there exist a Banach lattice G, a lattice homomorphism $\phi : E_1 \longrightarrow G$ and operators $0 \leq R^G \leq T^G : G \longrightarrow E_2$, with $T = T^G\phi$ and $R = R^G\phi$, such that G is order continuous if and only if T is order weakly compact.*

In addition, we will need the next lemma.

Lemma 4.3. *Let E be an order continuous Banach lattice, and let (x_n) be a sequence of E such that $|x_n| \xrightarrow{\sigma(E,E')} 0$. Then, either $\lim_n \|x_n\| = 0$, or there is a subsequence (y_n) of (x_n) and a disjoint sequence $(w_n)_n \subset E$ such that*

$$\|y_n - w_n\| \longrightarrow 0.$$

Proof. Let (x_n) be a sequence of E such that $|x_n| \xrightarrow{\sigma(E,E')} 0$. Since $X := [x_n]$ is separable subspace of E, it follows from Proposition 1.a.9 in [24] that $\overline{E_X}$ is an order ideal with a weak order unit and Therefore can be represented as a dense order ideal of $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ, such that the formal inclusion

$$j : \overline{E_X} \hookrightarrow L_1(\Omega, \Sigma, \mu)$$

is continuous ([24], Theorem 1.b.14). Since $L_1(\mu)$ has the positive Schur property, then

$$\|jx_n\|_1 \longrightarrow 0. \tag{4.1}$$

According to Theorem 1.2.8 in [26], we have
(1) either $\|jx_n\|_1 \geq \delta\|x_n\|$ for some $\delta > 0$,
(2) or there is a subsequence (y_n) of (x_n) and a disjoint sequence $(w_n)_n \subset E$ such that

$$\|y_n - w_n\| \longrightarrow 0.$$

If $\|jx_n\|_1 \geq \delta\|x_n\|$, for some $\delta > 0$. Referring to (4.1), we deduce that $\|x_n\| \longrightarrow 0$, and the proof is complete. □

Theorem 4.4. *Let*

$$E_1 \xrightarrow[R_1]{T_1} E_2 \xrightarrow[R_2]{T_2} E_3 \xrightarrow[R_3]{T_3} E_4 \xrightarrow[R_4]{T_4} E_5$$

be operators between Banach lattices, such that $0 \leq R_i \leq T_i$, for $i = 1, 2, 3, 4$. If T_2, T_4 are BS-compact and T_1, T_3 are order weakly compact, then $R_4R_3R_2R_1$ is also BS-compact.

Proof. Since T_1, T_3 are order weakly compact, according to Theorem 4.2 , there exist order continuous Banach lattice G, a lattice homomorphism $\phi : E_1 \longrightarrow G$ and operators $0 \leq R_1^G \leq T_1^G : G \longrightarrow E_2$, with $R_1 = R_1^G \phi$ and $T_1 = T_1^G \phi$. Furthermore, there exist order continuous Banach lattice F, a lattice homomorphism $\psi : E_3 \longrightarrow F$ and operators $0 \leq R_3^F \leq T_3^F : F \longrightarrow E_4$, with $R_3 = R_3^F \psi$ and $T_3 = T_3^F \psi$. The proof will be developed through the following steps.

Step 1. The positive operator $\psi R_2 R_1 : E_1 \longrightarrow F$ is AM-compact. Indeed, by Theorem 3.4 we have that $T_2 T_1^G : G \longrightarrow E_3$ is an AM-compact operator, and hence $\psi T_2 T_1 : E_1 \longrightarrow F$ is also AM-compact. Since F has an order continuous norm and

$$0 \leq \psi R_2 R_1 \leq \psi T_2 T_1 : E_1 \longrightarrow F,$$

it follows from Thorem 5.10 in [1] that $\psi R_2 R_1 : E_1 \longrightarrow F$ is AM-compact.

Step 2. Let (x_n) be a bounded sequence of E_1 such that $\lim_n \frac{1}{n} \sum_{i=1}^{n} x_i = 0$. Then, there is a subsequence (z_n) of (x_n) such that $\{|\psi R_2 R_1 z_n|, \quad n \in \mathbb{N}\}$ is Banach-Saks.

Indeed, by Proposition 2.3 in [26], there exists a subsequence (y_n) such that $y_n \xrightarrow{\sigma(E_1,E_1')} 0$. It follows from Theorem 5.96 in [1] that $|\psi R_2 R_1 y_n| \xrightarrow{\sigma(F,F')} 0$ ($\psi R_2 R_1$: AM-compact). Since F is order continuous, it follows from Lemma 4.3 that $\lim_n \|\psi R_2 R_1 y_n\| = 0$ or there is a subsequence (z_n) of (y_n) and a disjoint sequence $(w_n)_n$ such that $\|\psi R_2 R_1 z_n - w_n\| \longrightarrow 0$. By passing to a subsequence, we can assume that

$$\sum_{n=1}^{+\infty} \|\psi R_2 R_1 z_n - w_n\| < +\infty.$$

Since $\{\psi R_2 R_1 z_n - \psi R_2 R_1 y, \quad n \in \mathbb{N}\}$ is Banach-Saks, it follows from Lemma 2.9 in [26] that $\{w_n, \quad n \in \mathbb{N}\}$ is also Banach-Saks. Note that for any $n \in \mathbb{N}$ and any choice of scalars we have

$$|\sum_{k=1}^{n} \alpha_k w_k| = |\sum_{k=1}^{n} |\alpha_k| w_k| = |\sum_{k=1}^{n} \alpha_k |w_k||.$$

Then, the basic sequence (w_n) is equivalent to the sequence $(|w_n|)$, and consequently from Fact 4.22 (ii) in [17] we infer that $\{|w_n|, \quad n \in \mathbb{N}\}$ is Banach-Saks. Subsequently, using the fact that

$$\sum_{n=1}^{+\infty} \||\psi R_2 R_1 z_n| - |w_n|\| < +\infty,$$

it follows from Lemma 2.9 in [26] that $\{|\psi R_2 R_1 z_n|, \quad n \in \mathbb{N}\}$ is Banach-Saks.

Step 3. The positive operator $R_4 R_3 R_2 R_1$ is BS-compact. To demonstrate this, let A be a Banach-Saks subset of E_+ and let (x_n) be a sequence of A. Then, there exist $z \in E_1$ and a subsequence (y_n) of (x_n) such that $\lim_n \frac{1}{n} \sum_{i=1}^{n} (y_i - z) = 0$. By step 2, there exists a subsequence (z_n) of (y_n) such that $\{|\psi R_2 R_1 (z_n - z)|, \quad n \in$

$\mathbb{N}\}$ is Banach-Saks. Since $T_4T_3^G$ is BS-compact and $|\psi R_2R_1z_n - \psi R_2R_1z| \xrightarrow{\sigma(F,F')} 0$, it follows (by passing to a subsequence) that

$$\lim_n \|T_4T_3^G|\psi R_2R_1z_n - \psi R_2R_1z|\| = 0.$$

The inequality $0 \leq |R_4R_3R_2R_1z_n - R_4R_3R_2R_1z| \leq T_4T_3^G|\psi R_2R_1z_n - \psi R_2R_1z|$ implies

$$\lim_n \|R_4R_3R_2R_1z_n - R_4R_3R_2R_1z|\| = 0.$$

Thus, $R_4R_3R_2R_1(A)$ is a relatively compact subset of E_5, and the proof of the theorem holds. □

As a consequence, we get what follows

Corollary 4.5. *Let E be a Banach lattice, and consider operators $0 \leq R \leq T : E \to E$. If T is BS-compact, then R^4 is also BS-compact. Moreover, if E has an order continuous norm, then R^2 is BS-compact.*

Proof. Since T is BS-compact, it follows from Corollary 3.11 that T is order weakly compact. Thus, it is sufficient to apply Theorem 4.4 to $E_i = E, R_i = R$ and $T_i = T$ for all i. □

The following question has been left unresolved.

Problem 4.6. Let E be a Banach lattice and $0 \leq R \leq T : E \to E$ with T is BS-compact. Is R^3 or R^2 BS-compact?

4.2. **Domination by almost BS-compact operators.** In this section, new domination results are displayed for almost BS-compact operators between Banach lattices. For this reason we need the following.

Theorem 4.7. [21] *Let E and F be Banach lattices each with a quasi-interior positive element. Let T be a positive operator $T : \longrightarrow F$ and let $A \subset E, B \subset F'$ be solid bounded sets. Suppose that whenever $(a_n)_n$ is disjoint in A_+ and $(b_n)_n$ is disjoint in B_+, then*

(1) $Ta_n \xrightarrow{\sigma(F,F')} 0$,
(2) $T'b_n \xrightarrow{\sigma(F',F)} 0$,
(3) $| < Ta_n, b_n > | \to 0$.

Suppose further that $R, S \in \mathcal{L}_r(E,F)$ satisfy $|S| \leq |R| \leq T \in \mathcal{L}_r(E,F'')$. Then, given $\epsilon > 0$ there exist central operators $M_1, ..., M_k \in \mathcal{L}_r(E)$, $L_1, ..., L_k \in \mathcal{L}_r(F)$ so that if

$$S_0 = \sum_{i=1}^{k} L_iRM_i,$$

then

$$< Sa - S_0a, b > \leq \epsilon, \quad a \in A, b \in B.$$

We shall also need the following lemma.

Lemma 4.8. *Let E be an order continuous Banach lattice and let $x \in E_+$. If $\{x_n, \quad n \in \mathbb{N}\}$ is a Banach-Saks sequence in E_+, then $\{|x_n - x|, \quad n \in \mathbb{N}\}$ is also Banach-Saks.*

Proof. Let $\{x_n, \quad n \in \mathbb{N}\}$ be a Banach-Saks subset of E_+. By passing to a subsequence, we can assume that

$$\lim_n \|\frac{1}{n}\sum_{i=1}^{n} x_i - y\| = 0.$$

From $|x_n - x| \leq x_n + x$, we infer that $\frac{1}{n}\sum_{i=1}^{n}|x_i - x| \leq \frac{1}{n}\sum_{i=1}^{n}(x_i + x)$. Since $(\frac{1}{n}\sum_{i=1}^{n}(x_i + x))_n$ converges in norm, it follows from Lemma 2.2 that

$$\{\frac{1}{n}\sum_{i=1}^{n}|x_i - x|, \quad n \in \mathbb{N}\},$$

is L-weakly compact. Arguing as in the proof of Lemma 4.9, $E_{[x_n - x]}$ can be represented as a dense ideal of $L_1(\mu)$ for some probability measure μ such that the formal inclusion

$$j : E_{[x_n - x]} \hookrightarrow L_1(\mu)$$

is continuous. Applying the Rosenthal'l_1 Theorem to the subsequence $(|x_n - x|)_n$,there is a subsequence $(z_n)_n$ of $(|x_n - x|)_n$, such that (1) either (z_n) is a weak Cauchy sequence or (2) (z_n) is equivalent to the standard basis $(e_n)_n$ of l_1. Suppose first that (z_n) is equivalent to the standard basis $(e_n)_n$ of l_1. Since $\{\frac{1}{n}\sum_{k=1}^{n} z_k; \quad n \in \mathbb{N}\}$ is an L-weakly compact subset of E, it follows from Proposition 3.6.5 in [28] that $\{\frac{1}{n}\sum_{k=1}^{n} e_k; \quad n \in \mathbb{N}\}$ is a relatively weakly compact subset of l_1. Since l_1 has the Schur property, it follows that $\{\frac{1}{n}\sum_{k=1}^{n} e_k; \quad n \in \mathbb{N}\}$ is a relatively compact subset of l_1. Therefore, e_n converges weakly to zero. From Theorem 4.32 in [1], we have $\lim_n \|e_n\|_1 = 0$. This contradicts the fact that $\|e_n\|_1 = 1$.

Then, (z_n) is weak Cauchy. According to Theorem 9.3.1 in [23], there exists some z'' such that $z_n \xrightarrow{\sigma(E'',E')} z''$. On the other side, since $\{\frac{1}{n}\sum_{k=1}^{n} z_k; \quad n \in \mathbb{N}\}$ is L-weakly compact, it follows from Proposition 3.6.5 in [28] that there is a subsequence $(t_n)_n$ of $(z_n)_n$ such that $\frac{1}{n}\sum_{k=1}^{n} t_k \xrightarrow{\sigma(E,E')} z \in E$. Consequently, $z'' = z \in E$. Hence, $t_n \xrightarrow{\sigma(E,E')} z$, and thus $jt_n \xrightarrow{\sigma(L_1(\mu),L_\infty(\mu))} jz$. Since $L_1(\mu)$ has the weak Banach-Saks property, then $\frac{1}{n}\sum_{k=1}^{n} jt_k \xrightarrow{\|\cdot\|_1} jz \in L_1(\mu)$. The rest of the proof follows from Lemma 2.2. ☐

The main result of this section is the following.

Theorem 4.9. *Let E, F be order continuous Banach lattices. If $0 \leq R \leq T : E \to F$ with T is almost BS-compact, then R is almost BS-compact.*

Proof. Let A be a Banach-Saks subset of E_+ and let (x_n) be a bounded sequence in A. By Proposition 2.3 in [26], we can assume without loss of generality that

$x_n \xrightarrow{\sigma(E,E')} y$, for some $y \in E$. Consider a subsequence (y_n) of (x_n) such that:

$$\|\frac{1}{n}\sum_{k=1}^{n} y_k - y\| \to 0. \tag{4.2}$$

Since T is almost BS-compact, by movinging to a subsequence, we can assume that Ty_n converges to Ty. Note that (by Lemma 4.8) the sequence $\{|y_n - y|, \quad n \in \mathbb{N}\}$ is Banach-Saks. Thus, there exists a subsequence of (y_n) (which we shall denote by (y_n) again) such that

$$\frac{1}{n}\sum_{k=1}^{n} |y_k - y|$$

is norm convergent in E. Next, let (h_n) be a positive disjoint sequence in the solid hull of $\{y_n - y, \;\; n \in \mathbb{N}\}$. Consider an arbitrary subsequence $(w_n)_n$ of $(h_n)_n$. By moving to a subsequence, we can state that

$$0 \leq w_n \leq |y_n - y|$$

holds for all n. In particular,

$$0 \leq \frac{1}{n}\sum_{k=1}^{n} w_k \leq \frac{1}{n}\sum_{k=1}^{n} |y_k - y|,$$

holds for all n. Since E is order continuous, it follows from Lemma 2.2 that $\{\frac{1}{n}\sum_{k=1}^{n} w_k, \;\; n \in \mathbb{N}\}$ is L-weakly compact.

Now, Proposition 2.3 in [26] combined with Theorem 4.34 in [1] depicts that $w_n \xrightarrow{\sigma(E,E')} 0$. Therefore, $\frac{1}{n}\sum_{k=1}^{n} w_k \xrightarrow{\sigma(E,E')} 0$. Departing from the L-weak compactness of $\{\frac{1}{n}\sum_{k=1}^{n} w_k, \;\; n \in \mathbb{N}\}$, Lemma 2.3 and Lemma 2.4 in [12], it follows that $\frac{1}{n}\sum_{k=1}^{n} w_k \xrightarrow{\|\cdot\|} 0$. Hence, grounded on our hypothesis, there exists a subsequence $(w_{n_i})_i$ of $(w_n)_n$ such that $Tw_{n_i} \xrightarrow{\|\cdot\|} 0$. Since (w_n) is an arbitrary subsequence of (h_n), $Th_n \xrightarrow{\|\cdot\|} 0$ also holds.

Next, let $x = \sum_{n=1}^{+\infty} \frac{1}{2^n}|y_n - y|$. Consider that I_x (resp J_x) is the ideal generated by x in E (resp Tx in F). At this level an easy argument demonstrates that $R(I_x) \subset J_x$ for every operators $0 \leq R \leq T : E \to F$.

Consider A =Sol$\{y_n - y; \;\; n \in \mathbb{N}\}$ and $B = B_{(J_x)'}$. Let (a_n) and (b_n) be two normalized positive disjoint sequences in A and B respectively. Based on the above discussion, we have $Ta_n \xrightarrow{\|\cdot\|} 0$, in particular $Ta_n \xrightarrow{\sigma(E,E')} 0$. On the other side, since F is order continuous, it follows from Corollary 2.4.3 in [28] that $b_n \xrightarrow{\sigma(E',E)} 0$. Thus, since T is bounded, we have

$$Tb_n \xrightarrow{\sigma(E',E)} 0.$$

Furthermore, since $| < Ta_n, b_n > | \leq \|Ta_n\|\|b_n\|$ and $\|Ta_n\| \to 0$, then

$$| < Ta_n, b_n > | \longrightarrow 0.$$

Note that all hypotheses of Theorem 4.7 are verified. Therefore, for every $\epsilon > 0$ there exist central operators $M_1, ..., M_k \in \mathcal{L}(I_x)$, $L_1, ..., L_k \in \mathcal{L}(J_x)$ such that

$$| < Ra - R_\epsilon a, b > | \leq \frac{\epsilon}{2}$$

for every $a \in A$ and $b \in B$, where $R_\epsilon = \sum_{i=1}^k L_i T M_i$. In particular, this implies that

$$\|R(y_n - y) - R_\epsilon(y_n - y)\| < \epsilon.$$

To complete the proof, it is enough to establish that $\lim_n \|R_\epsilon(y_n - y)\| = 0$. Note that

$$\begin{aligned} |R_\epsilon(y_n - y)| &= |\sum_{i=1}^k L_i T M_i(y_n - y)| \\ &\leq \sum_{i=1}^k |L_i T M_i(y_n - y)| \\ &\leq \sum_{i=1}^k |L_i| T |M_i|(|y_n - y|). \end{aligned}$$

Since T is Almost BS-compact, it follows from Lemma 4.8 (by moving to a subsequence) that $\sum_{i=1}^k |L_i| T |M_i|(|y_n - y|)$ converges in norm to some $f \in F$. Thus, by Lemma 2.2, $\{|R_\epsilon(y_n - y)|, \quad n \in \mathbb{N}\}$ is L-weakly compact. On the other side, since $y_n - y \xrightarrow{\sigma(E,E')} 0$ and T is AM-compact (see Theorem 3.4), it follows from Theorem 5.96 in [1] that $|R_\epsilon(y_n - y)| \xrightarrow{\sigma(F,F')} 0$. Lemma 2.3 and Lemma 2.4 in [12] are conducive to the conclusion that $|R_\epsilon(y_n - y)| \xrightarrow{\|\cdot\|} 0$. Let $N_0 \in \mathbb{N}$ such that

$$\|R_\epsilon(y_n - y)\| < \frac{\epsilon}{2},$$

holds for all $n \geq N_0$. Finally, for $n \geq N_0$ we have

$$\begin{aligned} \|R(y_n - y)\| &= \|R(y_n - y) - R_\epsilon(y_n - y) + R_\epsilon(y_n - y)\| \\ &\leq \|R(y_n - y) - R_\epsilon(y_n - y)\| + \|R_\epsilon(y_n - y)\| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

That is, $\lim_n \|Ry_n - Rx\| = 0$, and the proof of the theorem is complete. □

As a consequence, we get the following.

Corollary 4.10. *Let E be a Banach lattice, and consider operators $0 \leq R \leq T : E \to E$. If T is almost BS-compact, then R^3 is also almost BS-compact.*

Proof. Since T is BS-compact, it follows from Corollary 3.11 that T is order weakly compact. According to Theorem 4.2 , there exist an order continuous Banach lattice G, a lattice homomorphism $\phi : E \longrightarrow G$ and operators $0 \leq R^G \leq T^G : G \longrightarrow E$, with $R = R^G\phi$ and $T = T^G\phi$. Note that

$$0 \leq \phi R R^G \leq \phi T T^G : G \to G.$$

Since G is order continuous and ϕTT^G is almost BS-compact, it follows from Theorem 4.9 that ϕRR^G is almost BS-compact and consequently R^3 is almost BS-compact. □

The following question remains unanswered:

Problem 4.11. Let E be a Banach lattice and $0 \leq R \leq T : E \to E$ with T is almost BS-compact. Is R^2 almost BS-compact?

References

[1] Aliprantis, C.D., Burkinshaw, O.: *Positive Operators.* Springer, Berlin (2006)
[2] Aliprantis, C.D., Burkinshaw, O.: *Positive compact operators on Banach lattices.* Math. Z. $174, 289 - 298$ (1980)
[3] Aliprantis, C,D., Burkinshaw, O.: *On weakly compact operators on Banach lattices.* Proc. Amer. Math. Soc. 83 (1981)
[4] Aliprantis, C,D., Burkinshaw, O.: *Dunford-Pettis operators on Banach lattices.* Trans. Amer. Math. Soc. 274 (1982)
[5] Aqzzouz, B., Nouira, R., Zraoula, L.: *Compactness Properties for Operators Dominated by AM-compact Operators.* Trans. Amer. Math. Soc. 153(4), $1151 - 1157$ (2007)
[6] Baernstein, A.: *On reflexivity and summability II.* Studia Math. 42, $91 - 94$ (1972)
[7] Baklouti, H., Hajji, M.: *Domination problem on Banach lattices and almost weak compactness.* Positivity. 19. $797 - 805$ (2015)
[8] Baklouti, H., Hajji, M.: *Schur operators and domination problem.* Positivity. 21, $35 - 48$ (2017)
[9] Baklouti, H., Hajji, M.: *Disjointly improjective operators and domination problem.* Indagationes Mathematicae 28, $1175 - 1182$ (2017)
[10] Banach, S ., Saks, S.: *Sur la convergence forte dans les champs L_p.* Studia. Math. 2, $51 - 57$ (1930)
[11] Beauzamy, B.: *Propriété de BanachSaks,* ibid. 66, $227 - 235$ (1980)
[12] Bouras, K., Lhaimer, D., Moussa, M.: *On the class of almost L-weakly and almost M-weakly compact operators.* Positivity 22, $1433 - 1443$ (2018)
[13] Brunel, A., Sucheston, L.: *On J-convexity and some ergodic super-properties of Banach spaces*, Proc. Amer. Math. Soc. 204, $79 - 90$ (1975)
[14] Diestel, J.: *Sequences and series in Banach spaces,* Springer-Verlag, New York, 1984
[15] Dodds, P,G., Fremlin, D,H.: *Compact operators in Banach lattice*, Isr. J. Math. 34, $287 - 320$ (1979)
[16] Erdos, P., Magidor, M.: *A note on regular methods of summability and the Banach-Saks property*, Proc. Amer. Math. Soc. 59, $232 - 234$ (1976)
[17] Fabian, M., Habala, P., HAjek, P., Montesinos, V., Zizler, V.: *Banach space theory: basis for linear and nonlinear analysis*, Springer-Verlag, New York. (2011)
[18] Flores, J., Hernandez, F, L., Tradacete, P.: *Powers of operators dominated by strictly singular operators*, Proc. Q. J. Math. Soc. 59, $321 - 334$ (2008)
[19] Ghoussoub, N., Johnson, W. B.: *Factoring operators through Banach lattices not containing $C(0,1)$.* Math. Z. 194, $153 - 171$ (1987)
[20] Jarosz, K.: *Function Spaces: Proceedings of the Third Conference on Function Spaces*, Southern Illinois University at Edwardsville, May. $19 - 23$ (1998)
[21] Kalton, N., Saab, P.: *Ideal properties of regular operators between Banach lattices*, Illinois J. Math. 29, $382 - 400$ (1985)
[22] G, Groenewegen.: *On spaces of Banach lattice valued functions and measures*, PhD Thesis, Nijmegen University, 1982.
[23] Larsen, R.: *Functional analysis: An introduction*, Marcel Dekker, Inc., New York, 1973.
[24] Lindenstrauss, J., Tzafriri, L.: *Classical Banach space II. Function Spaces*, Springer, New york. (1979)

[25] Núñez, C. *Characterization of Banach Spaces of Continuous Vector Valued Functions with The Weak Banach-Saks Property,* Illinois Journal of Mathematics. $33(1), 27-41$ (1989).
[26] Lopez-Abad, J., Ruiz, C., Tradacete, P.: *The convex hull of a Banach-Saks set,* Journal of Functional Analysis. $266(4)(2014)2251-2280$
[27] Meyer-Nieberg, P.:*Uber Klassen Schwach Kompakter Operatoren in Banachverbanden.* Math. Z. 138, 145159 (1974)
[28] Meyer-Nieberg, P.: *Banach lattices*, Springer-Verlag, Berlin, Heidelberg, New York. (1991)
[29] Nishiura, T., Waterman, D.: *Reflexivity and summability,* Studia Math. 23, $53-57$ (1963)
[30] Szlenk, W.: *Sur les suites faiblement convergentes dans l'espace l*, Studia Math. 25, 337341 (1965)
[31] Tradacete, P.: *Factorization and domination properties of operators on Banach Latices*, Phd thesis, Universidad Complutense de Madrid. (2010)
[32] Weis, L.: *Banach lattices with the subsequence splitting property*, Proc. Am. Math. Soc.105, $87-96$ (1989)

Mohamed Hajji
Department of Mathematics and Computer Science, Issat Kasserine , BP 471, Kasserine, 1200, Tunisia
E-mail address: medhajji.issatkasserine@gmail.com