# New Exact Solutions of Time Conformable Fractional Klein Kramer Equation

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**Abstract**

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## Abstract

The Klein Krames equation (KKE) stands for the probability distribution function (PDF) that describes the diffusion of particles subjected an external force. It is shown that the conformable fractional derivative (CFD) KKE can be reduced to the classical one's by using similarity transformations. Here, the objective of this work is to find the exact solutions of CFD-KKE.. To this issue, an approach is presented. It is based on transforming the KKE to a system of first order PDEs. The solutions are found by implementing extended unified method. It is found that, the integrability condition is that the external force is constant. The numerical results of the solutions are calculated and the are shown graphically.

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**Key Words:** Time Conformable Fractional Klein Kramer Equation; Unified method; Similarity transformations

## 1. Introduction

The KKE is a diffusion equation with velocity dependence advection and with variables coefficients in velocity and space. In [1,2] the diffusion description, is fully systematized, as it is an interesting and successful method complementary to transition state theory. The position and the velocity of the diffused particles are not deterministic as they are Brownian particles. The KKE has a wide applications in biological science, biophysics and chemical kinetics. It describes the cell migration as the paths of migrating cells resemble those of thermally driven Brownian particles [3].

In this respect experimental work revealed a precise spatial and relaxation time of multiple components of the cellular migration processes. In this context, immune defense, and the formation of tumor metastases are well known phenomena that rely on cell migration. In biophysics, anomalous diffusion processes have been observed in bacterial cytoplasm motion [5] and fluorescence indeterminacy in single enzymes [6-8].In chemistry reaction diffusion systems and autocatalytic reactions in chemical kinetics and in biochemistry are described by diffusion equations with advection and variable, or constant, coefficients. Stochastic or random states may be more relevant whenever the KKE is applicable.

Also, it serves as a mechanical approach to molecular interactions and reaction dynamics. It can be introduced to describe the diffusive-stochastic approach to reaction dynamics. The stochastic-diffusion description of chemical dynamics involves in theoretical and computational chemistry. A super diffusion increase of the mean squared displacement, non-Gaussian spatial probability distributions, and decays of the velocity auto correlations can help in interpreting phenomena [4]. Also it arises in many different physical system such as those which are generated by the presence of anomalously large particle displacements [9-18].

The KKE reads

$$\frac{\partial}{\partial t}K(t,x,v)=(-v\frac{\partial}{\partial x}+\frac{\partial}{\partial v}(\eta v-\frac{r(x)}{m})+B_0\frac{\partial^2}{\partial v^2})K(t,x,v), B_0=\frac{\eta T K_B}{m}, \tag{1}$$

where $W(z,v,t)$ is the distribution density function of particles, $r(x)$ is an external force field , $T$ is the absolute temperature, $K_B$ is the Boltzmann constant, , $m$ is the particle mass , $v$ its velocity and$\eta$ stands for the friction coefficient. We proceed the by introducing to the CFD.

## 2. Conformable fractional derivative

In this section, we present the definitions and their properties that will be used in this work, [25].

**Definition** Let $f:(0,\infty)\to R$ be a function, then its conformable fractional derivative of order $\beta$ is defined as

$${}_0^{CFD}D_t^{\beta}f(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon\beta^{-1}t^{1-\beta}\right)-f\left(t\right)}{\varepsilon},\ t>0. \tag{2}$$

If $f\epsilon C^1(\mathbb{R}^+)$, then ${}_0^{CFD}D_t^{\beta}f(t)=\beta^{-1}t^{1-\beta}f'(t)$. Then KKE can be rewritten in the CFD sense as

$${}_0^{CFD}D_t^{\beta}K(t,x,v)=\left(-v\frac{\partial}{\partial x}+\frac{\partial}{\partial v}(\eta v-\frac{r(x)}{m})+B_0\frac{\partial^2}{\partial v^2}\right)K\left(t,x,v\right), \tag{3}$$

which is rewritten

$$p(t)K_t(t,x,v)=\left(-v\frac{\partial}{\partial \mathrm{t}}+\frac{\partial}{\partial v}(\eta v-\frac{r(x)}{m})+B_0\frac{\partial^2}{\partial v^2}\right)\mathrm{K}\left(t,x,v\right), \tag{4}$$

where $p(t)=\beta^{-1}t^{1-\beta}$. Indeed (4) can takes two forms. By using the similarity transformations $K(t,x,v)=\tilde{K}(\tau_0,x,v)$ and $\tau_0=\int_0^t\frac{1}{p(s)}ds=t^{\beta}$, (4) turns to be (1), where $(W(z,v,t),t)\to(\tilde{W}(z,v,\tau),t\to\tau)$. The second form of (4) is

$$\tfrac{\partial}{\partial\tau}\tilde{K}(z,v,\tau)=\left(-v\tfrac{\partial}{\partial z}+\tfrac{\partial}{\partial v}(\eta v-\tfrac{r(x)}{m})+B_0\tfrac{\partial^2}{\partial v^2}\right)\tilde{K}(\tau_0,x,v), \tag{5}$$

which is an autonomous equation.

## 3. Solutions of the KKE.

In this section, we present the outlines of the method to find the exact solution of (1) (or (5)) as follows. We use the transformations:

$$\begin{aligned}\tilde{K}_v(\tau_0,x,z)&=F_0(\tau_0,x,v)\,\tilde{K}(\tau_0,x,v),\\ \tilde{K}_{\tau_0}(v,z,t)&=G_0(\tau_0,x,v)\,W(\tau_0,x,z),\\ \hat{K}_x(\tau_0,x,z)&=K_0(\tau_0,x,z)\,\tilde{K}(\tau_0,x,v).\end{aligned}$$

Therefore (1) is written in the form

$$G_0(\tau_0,x,v)+vK_0(\tau_0,x,z)-\eta vF_0(\tau_0,x,v)-\eta+\frac{r(x)}{m}F_0(\tau_0,x,v)$$
$$-B_0(F_{0v}(\tau_0,x,v)+F_0(\tau_0,x,v)^2)=0. \tag{6}$$

Now, in the following subsections, we investigate the exact solutions of (6) with the case of linear auxiliary equations (AE)and the case of quadratic auxiliary equation (AE)

3.1. **When AE is linear.** Here we use the extended unified method, where solutions are expressed in the form of rational function in an auxiliary function that satisfy an auxiliary equation [22-25].

Thus we assume that the solution of (6), takes the form

$$\tilde{K}(\tau_0,x,v)=\frac{w_1(v)g(\tau_0,x,v)+w_0(v)}{\alpha_1(v)g(\tau_0,x,v)+{}_0(v)}, \tag{7}$$

$$F_0(\tau_0,x,v)=\frac{\beta_1(v)g(\tau_0,x,v)+\beta_0(v)}{w_1(v)g(\tau_0,x,v)+w_0(v)}, \tag{8}$$

$$G_0(\tau_0,x,v)=\frac{\gamma_1(v)g(\tau_0,x,v)+\gamma_0(v)}{w_1(v)g(v,z,t)+w_0(v)}, \tag{9}$$

$$K_0(\tau_0,x,v)=\frac{r_1(v)g(\tau_0,x,v)+r_0(v)}{w_1(v)(\tau_0,x,v)+w_0(v)}, \tag{10}$$

together with the linear auxiliary equation

$$g_{\tau_0}(\tau_0,x,v)=\mu\,(c_1g(\tau_0,x,v)+c_0), \tag{11}$$

$$g_v(\tau_0,x,v)=h(v)\,(c_1g(\tau_0,x,v)+c_0), \tag{12}$$

$$g_x(\tau_0,x,v)=k(x)\,(c_1g(\tau_0,x,v)+c_0). \tag{13}$$

It is worth noticing that in (11)–(13), the compatibly equations, $g_{\tau_0 v}(\tau_0,x,v)=g_{v\tau_0}(\tau_0,x,v)$ , $g_{xv}(\tau_0,x,v)=g_{vx}(v,z,t)$ and $g_{\tau_0 x}(\tau_0,x,v)=g_{x\tau_0}(\tau_0,x,v)$ hold.

By inserting (7)–(10) into (6), and by using (11)–(13), we obtain a system of coupled PDEs of first order in $a_i,b_i,d_i,r_i,i=1,2$.

We have to use the compatibility equation $\beta_j'(v)-(\beta_j(v))'=0$, due to the calculations are not direct and also the obtained equations are nonlinear, and it appears that two equations can be obtained, for example, $a_j(v)$ and $a_j'(v)$, $j=0,1$. On other hand, we found that, the equations are not consistent unless $r(x)'=0$. So that, this last condition of integrability of

(1). In this case $q(z) = q_0$ and also, we find that $k(z) = k_0$. After some simplifications, we obtain the following system of equations

$$\alpha_0'(v) = \frac{1}{s_0(v)}\Big(\alpha_0(v)\left(-\beta_0(v) + p(v)w_1(v)c_0 + w_0'(v)\right) - w_0(v)p(v)\alpha_1(v)c_0\Big), \tag{14}$$

$$w_0'(v) = \beta_0(v) + p(v)\left(w_0(v)c_1 - s_1(v)c_0\right), \; s_1'(v) = b_1(v), \tag{15}$$

$$b_0'(v) = \frac{1}{m\,B_0}\Big(b_0(v)\left(B_0 m p(v)c_1 + r_0 - m\,v\,\eta\right) - m\Big(B_0 p(v)c_0 b_1(v) - (\mu + k_0 v)w_1(v)c_0 + (\eta + c_1\mu + c_1 k_0 v)w_0(v)\Big)\Big), \tag{16}$$

$$b_1'(v) = \frac{1}{m\,B}\left(b_1(v)\left(q_0 - m\,v\,\eta\right) - m\,\eta s_1(v)\right), \tag{17}$$

$$\gamma_0(v) = \frac{c_1\mu}{\alpha_1(v)}\left(w_1(v) - w_0(v)\alpha_1(v)\right), \; \gamma_1(v) = 0, \tag{18}$$

$$r_1(v) = 0, \; \alpha_0(v) = 1, \; \alpha 1(v) = c_1/c_0, \tag{19}$$

$$r_0(v) = k_o\left(w_1(v)c_o - w_0(v)c_1\right). \tag{20}$$

By rewriting the equation (15) as $\beta_1(v) = w_1'(v)$, the compatibly equation $\beta_1'(v) - (\beta_1(v))' = 0$ gives rise to

$$m\,\eta w_1(v) + (-r_0 + mv\eta)w_1'(v) + B\,m w_1''(v) = 0, \tag{21}$$

where (21) solves to

$$w_1(v) = A_2 e^{\frac{q_0 v}{m\,B} - \frac{v^2\eta}{2\,B}} + A_1\sqrt{\frac{\pi\,B}{2\eta}}\, e^{\left(\frac{r_0 v}{m\,B} - \frac{v^2\eta}{2\,B} - \frac{r_0^2}{2Bm^2\eta}\right)}\operatorname{erfi}\Big(\frac{-r_0 + m\,v\,\eta]}{\sqrt{2B_0\eta m}}\Big), \tag{22}$$

where $\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{y^2}dy$.

Taking into account (22), we can solve $\beta_1(v) = w_1'(v)$. Similarly, it can be rewritten (16) as follows:

$$\beta_0(v) = p(v)\left(w_0(v)c_1 - w_1(v)c_0\right) - w_0'(v). \tag{23}$$

According to the compatibly equation $\beta_0'(v) - (\beta_0(v))' = 0$ , we can get the following

$$r_0\eta\,(m\,v\,\eta - r_0)w_0(v) + m\,B_0\left(2A_1 B_0 m c_0 k_0 + r_0\eta w_0'(v)\right) + B_0 k_0 m(m\,\eta w_0(v) + (m\,v\,\eta - r_0)w_0'(v) + B_0 m w_0''(v) = 0, \tag{24}$$

$$(m\,v\,\eta - r_0)p(v) + B_0 c_1 m p(v)^2 + m(k_0 v + \mu - B_0 p'(v)) = 0. \tag{25}$$

By using

$$\mu=-\frac{2B_0k_0}{4mF_0\eta^3}(r_0^2\eta^2+4B_0m^2\,n\eta^3),\ c_1=-\frac{r_0\eta}{2B_0k_0m}, \tag{26}$$

the solutions of (**??**)–(25) are given by

$$\begin{aligned}w_0(v)&=e^{\frac{q_0v}{m\,B_0}-\frac{v^2\eta}{2\,B}}(2r_0\eta^{3/2}C-2A_1B_0^{3/2}c_0e^{\frac{-q_0^2}{2B_0m^2\eta}}\\&\times(k_0m\sqrt{\pi}erfi(\frac{-r_0+m\,v\,\eta]}{\sqrt{2B_0\eta}m})\\&+\sqrt{B_0}e^{\frac{-(k_0+\eta\,v)^2}{2Bk_0^2m^2\eta}}r_0\sqrt{2\pi}\eta C_2\mathrm{erfi}(\frac{-r_0+m\,v\,\eta]}{\sqrt{2B_0\eta}m})),\end{aligned} \tag{27}$$

$$\begin{aligned}p(v)&=\frac{P_1(v)}{Q_1},\ P_1(v)=-k_o\Big(-2m\,n\sqrt{2B_0\eta}C_0H_{n-1}\left(v\sqrt{\frac{\eta}{2B}}\right)\\&+r_0C_0H_n\left(v\sqrt{\frac{\eta}{2B}}\right)+2m\,n\,v\,\eta\,{}_1F_1\left(1-\frac{n}{2},\frac{3}{2},\frac{\eta v^2}{2B}\right)\\&+r_0{}_1F_1\left(-\frac{n}{2},\frac{1}{2},\frac{\eta v^2}{2B}\right)\Big),\end{aligned} \tag{28}$$

$$Q_1(v)=r_0\eta\left({}_1F_1\left(-\frac{n}{2},\frac{1}{2},\frac{\eta v^2}{2B}\right)+C_0H_n\left(v\sqrt{\frac{\eta}{2B}}\right)\right), \tag{29}$$

where ${}_1F_1\left(-\frac{n}{2},\frac{1}{2},\frac{\eta v^2}{2B}\right)$ and $H_n\left(v\sqrt{\frac{\eta}{2B}}\right)$ are the hyper- geometric and the Hermite functions respectively. The solution of the auxiliary equations is given by

$$g(\tau_0,x,v)=-\frac{c_0}{c_1}+C_3\,e^{c_1\int p(v)dv+\chi}, \tag{30}$$

where

$$\chi=k_0x-\frac{B_0k_0}{2mq_0\eta^3}\left(r_0^2\eta^2+4B_0m^2\,n\eta^3\right)\tau_0.$$

We can not evaluate $\int h(v)dv$ directly, so we assume that (see (28))

$$(m(v)+Q_1(v))'=P_1(v), \tag{31}$$

and after some calculations we obtain

$$
\begin{aligned}
m(v) = -\frac{1}{B_0}\Bigg( & - B_0\left(-B_0 q_0 \eta + 2B_0 k_0 m C_0\right) H_n\left(v\sqrt{\frac{\eta}{2B}}\right) \\
& + \frac{B_0^{3/2} C_0 q_0 k_0}{(1+n)\sqrt{2\eta}} H_{n+1}\left(v\sqrt{\frac{\eta}{2B}}\right) \\
& - 2B_0^2 k_0 m \left(-1 +_P F_Q\left(\{-\frac{n}{2}\},\{\frac{1}{2}\},\frac{\eta v^2}{2B}\right)\right) \\
& + B_0 v q_0 k_0 \,_P F_Q\left(\{-\frac{n}{2}\},\{\frac{3}{2}\},\frac{\eta v^2}{2B}\right)\Bigg),
\end{aligned} \tag{32}
$$

where $_PF_Q$ is the generalized-hyper-geometric function. Finally, we obtain

$$\int p(v)dv = \log(\mid m(v) + P_1(v) \mid). \tag{33}$$

In view of (18)-(33) and substituting for $s_i(v), a_i(v)$ into (7), we obtain the required solution. It is too lengthy to be produced here.

3.2. **When the AE is quadratic.** In this subsection we investigate the exact solution of (6) in the case of quadratic auxiliary equation. We assume that the solution of (6), by using (7)–(10), have the form together with quadratic auxiliary equation

$$g_t(\tau_0, x, v) = \mu\,(c_2 g(\tau_0, x, v)^2 + c_1), \tag{34}$$

$$g_v(\tau_0, x, v) = h(v)\,(c_2 g(\tau_0, x, v)^2 + c_1), \tag{35}$$

$$g_t(\tau_0, x, v) = k(z)\,(c_2 g(\tau_0, x, v)^2 + c_2). \tag{36}$$

By inserting (7)–(10) into (6), and using (34)–(36), we get a system of coupled PDEs of first order in $a_i, b, d_i, r_i, i = 1, 2$. Now, we have use the compatibly equation, $a_j'(v) - (a_j(v))' = 0$ for the same reason that we mentioned above. On the other hand, we found the equations are not consistent unless $r(x)' = 0$. Hence, this is the condition of integrability of (1). Thus, we take $r(x) = r_0$ and also, we find that $k(x) = k_0$. Thus, we have the following equations

$$\alpha_0'(v) = \frac{1}{w_0(v)}\alpha_0(v)(w_0'(v) - \beta_0(v)), \tag{37}$$

$$
\begin{aligned}
w_1'(v) = \frac{1}{w_0(v)}\Big( & (b_1(v) - c_1 p(v) w_1(v)) w_0(v) - \beta_0(v) w_1(v) \\
& + c_2 h p v) w_0(v)^2 + w_1(v) w_0'(v)\Big),
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
\beta_0'(v) = \frac{1}{B\, m s_0(v)} & (-B_0\, m \beta_0(v)^2 + \beta_0(v)(-B_0\, m w_0(v) \\
& + ((r_0 - m\, v\eta) w_0(v) + B_0\, m w_0'(v)))
\end{aligned} \tag{39}
$$

$$\begin{aligned}\beta_1'(v) = &\frac{1}{B\,ms_0(v)}(mw_0(v)(B_0c_2\beta_0(v)p(v)\\ &+(c_1k_0v-\eta+c_1\mu)w_1(v)-c_2(kov+\mu)w_0(v))\\ &-\beta_1(v)(B_0\,m\beta_0(v)+((-r_0+m\,v\eta\\ &+B_0\,c_1mp(v))w_0(v)-B_0\,mw_0'(v)))),\end{aligned} \tag{40}$$

together with the equations

$$\alpha_1(v)=\frac{c_2\alpha_0(v)}{c_1},\ \gamma_0(v)=0,\ r_0(v)=0, \tag{41}$$

$$\gamma_1(v)=\frac{c_1\mu}{\alpha 0(v)}(w_1(v)\alpha_0(v)-\alpha_1(v)w_0(v)), \tag{42}$$

$$r_1(v)=k_0(c_1w_1(v)-c_2w_0(v)),\ c_0=0. \tag{43}$$

We mention that the equations in (37)–(40) can not be integrated, in general, thus we are led to find particular solutions. To this end, we take

$$w_0'(v)=w_0(v)(\frac{-r_0+mv\eta}{B_0\,m}+c_1p(v)), \tag{44}$$

$$\beta_1(v)=-c_2p(v)w_0(v)+w_1'(v). \tag{45}$$

By using (37)–(40), we find

$$w_0(v)=A_0e^{\frac{v(-2q_0+m\,v\,\eta)}{2B}+c_1\int hpv)dv},\alpha_0(v)=\frac{P_1}{P_2},\ \beta 0(v)=\frac{P_3}{m\,B\,P_2} \tag{46}$$

$$P_1=A_1e^{\frac{v(-2q_0+m\,v\,\eta)}{2B_0}},\ P_2=2\sqrt{\eta}e^{\frac{q_0^2}{2B_0m^2\eta}}+\sqrt{B_0}C_0\sqrt{2\pi}\text{erfi}(\frac{-r_0+m\,v\,\eta}{\sqrt{2B_0}m\sqrt{\eta}}), \tag{47}$$

$$\begin{aligned}P_3 = &A_0e^{\frac{v(-2q_0+m\,v\,\eta)}{2B_0}}\Big(2\sqrt{\eta}e^{\frac{q_0^2}{2Bm^2\eta}}(r_0-m\,v\,\eta)\\ &+(q_0-m\,v\,\eta)\sqrt{B_0}C_0\sqrt{2\pi}\text{erfi}(\frac{-r_0+m\,v\,\eta}{\sqrt{2B_0}m\sqrt{\eta}})\\ &+m\,B_0e^{\frac{(q_0-m\,v\,\eta)^2}{2B_0m^2\eta}})\Big)\end{aligned} \tag{48}$$

The compatibly equation $(\beta_1(v))'-\beta_1'(v)=0$, gives rise to

$$\begin{aligned}&w_1(v)=\frac{Q_1}{Q_2},\\ &\quad Q_1=c_2(-2B_0\,m\beta_0(v)p(v)+w_0(v)((r_0-m\,v\,\eta)p(v)-B_0c_1mp(v)^2,\\ &\qquad +m\,(k_0v+\mu-B_0p'(v)))),\\ &\quad Q_2=m\,(-\eta+c_1(k_0v+\mu)).\end{aligned} \tag{49}$$

The compatibly equation yields an ODE of nonlinear second of $h(v)$ with variable coefficients. It can not be solved in general. Detailed calculations give rise to lengthy result to $h(v)$. But when taking the following equations

$$\begin{aligned} \mu &= \frac{(3Bc_{01}^2 k_0^2 m + 4c_1 r_0 k_0 \eta + 4m\eta^3)}{4c_1 m\eta^2}, \\ r_0 &= \frac{(-21B_0 c_1^2 k_0^2 m\,\eta - 16m\,\eta^4)}{32c_1 k_0 \eta^2}, \\ k_0 &= -\frac{4\eta^{3/2}}{\sqrt{B_0}c_1}. \end{aligned} \tag{50}$$

we obtain

$$\begin{aligned} p(v) &= -\frac{R}{Q}, \\ Q &= \sqrt{B_0}c_1(551B_0^2 - 608B_0 v^2\eta - 256v^4\eta^2), \\ R &= 8(\sqrt{B_0} - 4v\sqrt{\eta})\sqrt{\eta}(31B_0^{3/2} - 86B_0\, v\sqrt{\eta} + 112\sqrt{B_0}v^2\eta - 32v^3\eta^{3/2}). \end{aligned} \tag{51}$$

The solution of the auxiliary equations (34)–(36) is

$$g(v,z,t) = -\frac{c_1 e^{(13+6S\sqrt{3})t\,\eta + c_1 h_1(v) + k_0 z}}{-1 + c_2 e^{(13+6S\sqrt{3})t\,\eta + c_1 h_1(v) + k_0 z}}, h_1(v) = -\frac{Q_2}{76\sqrt{551}c1}, \tag{52}$$

$$\begin{aligned} Q_2 &= (2\sqrt{-19 + 4\sqrt{57}}(-361 + 21\sqrt{57}) \\ &\times \sqrt{B}\arctan(\frac{4v\sqrt{\eta}}{\sqrt{19 + 4\sqrt{57}\sqrt{B_0}}}) \\ &+ 2\sqrt{19 + 4\sqrt{57}}(361 + 21\sqrt{57})\sqrt{B}\mathrm{arctanh}(\frac{4v\sqrt{\eta}}{\sqrt{-19 + 4\sqrt{57}\sqrt{B}}}) \\ &+ 19\sqrt{29}(-16\sqrt{19}v\sqrt{\eta} - 15(\sqrt{3} - \sqrt{19})\sqrt{B_0} \\ &\times \log(-19B + 4\sqrt{57}B_0 - 16v^2\eta) \\ &+ 15(\sqrt{3} + \sqrt{19})\sqrt{B}\log(19B_0 + 4\sqrt{57}B_0 + 16v^2\eta)). \end{aligned} \tag{53}$$

By substituting from (46)-(53) into the first equation in (7), we get the required solution, $W(v,z,t)$. The results are too lengthy to be produced here.

## 4. Numerical Results

Numerical results are shown here for the solutions of the CFD- KKE, where the similarity transformations are used (cf. sec.2). The PDF is displayed ,against $v$ and $t$ when $z = const.$ in figures 1 (i) and (ii). In figures 1 (i) and (ii), the solution of (5) is displayed against $v$and $t$ , where .the parameters are taken, $m = 2.6, F_0 = 0.05, B = 0.7, \eta := 0.5, \tau =$

$t^{\beta}, A_1 = -0.05, B_0 = 0.05, B_1 = 0.03, k_0 = -0.5, c_0 = 2.3, A_2 = 0.07, B_2 = 0.09, B_3 := -0.09, z = 3, n = 10.$

After this figure we remark that for small values of t, the PDF is Gaussian in $v$ and is exponential distribution in $t$. While it is zero otherwise. We remark that there no significant effect of the CFD order. In in figures 2, (i) and (ii),The PDF is displayed against $z$ and $t$ when $v = const$. Figures 2, (i) and (ii), the solution of (5) is displayed against $z$and $t$ when $c_1 = 0.05, m = 2.6, B = 7, c_2 = 2, \eta = 0.5, \tau = t^{\beta};, A = 1.7, A_1 := 1.3, v = 3, B_0 = 0.05$. Here $F_0$ is evaluated by using (19). Here $F_0$ is evaluated by using (19).. After this figure we find the PDF is exponential in $t$. In figures 3 (i) and (ii) it is displayed against $v$ and z when $t = const$. Figures 3, (i) and (ii), the solution of (5) is displayed against $z$and $t$ when $c_1 = 0.05, m = 2.6, B = 7, c_2 = 2, \eta = 0.5, \tau = t^{\beta};, A = 1.7, A_1 := 1.3, t = 3, B_0 = 0.05$. Here $F_0$ is evaluated by using (19). Here $F_0$ is evaluated by using (19).. After this figure we find the PDF is Gaussian in $v$. The mean and mean square of the velocity are shown in figure 4. Figure 4. The mean and the mean square of the velocity are shown for the same caption as in Fig. 1.This figure shows that the mean and mean square of the velocity are mainly constant. The mean and mean square of the space variable are shown in figure 5. Figure 5 shows the mean and mean square of the space variable for the same caption as in Fig. 3.This figure shows that the mean and mean square of the space variable are mainly constant.

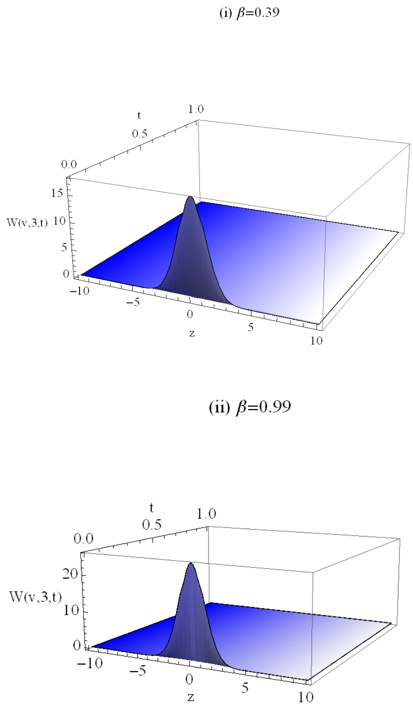


FIGURE 1. **Graph of the solution of (5) is displayed against** $v$**and** $t$ **for** $m = 2.6, F_0 = 0.05, B = 0.7, \eta := 0.5, \tau = t^{\beta}, A_1 = -0.05, B_0 = 0.05, B_1 = 0.03, k_0 = -0.5, c_0 = 2.3, A_2 = 0.07, B_2 = 0.09, B_3 := -0.09, z = 3, n = 10.$

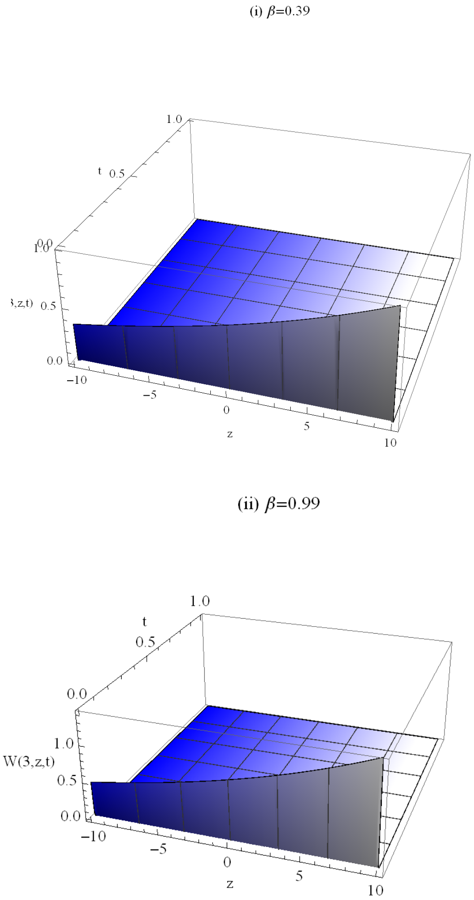


FIGURE 2. **Graph of the solution of (5) is displayed against** $z$ **and** $t$ **when** $c_1 = 0.05, m = 2.6, B = 7, c_2 = 2, \eta = 0.5, \tau = t^{\beta};, A = 1.7, A_1 := 1.3, v = 3, B_0 = 0.05.$

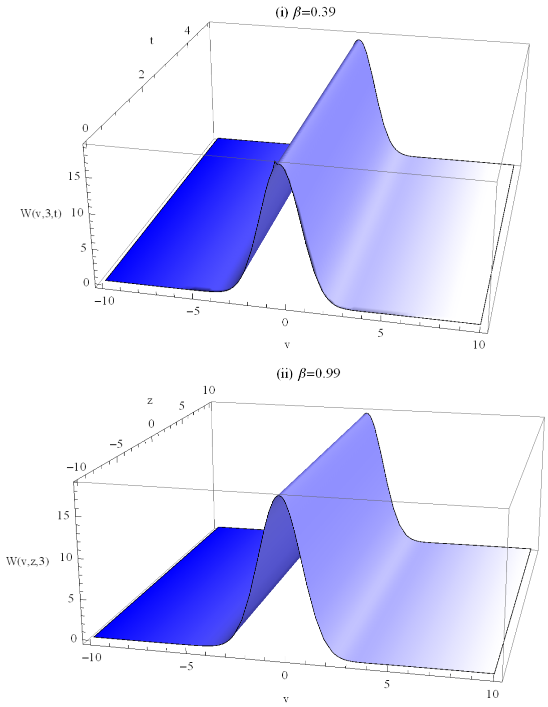


FIGURE 3. **Graph of the solution of (5) is displayed against** $z$ **and** $t$ **when** $c_1 = 0.05, m = 2.6, B = 7, c_2 = 2, \eta = 0.5, \tau = t^{\beta};, A = 1.7, A_1 := 1.3, t = 3, B_0 = 0.05.$

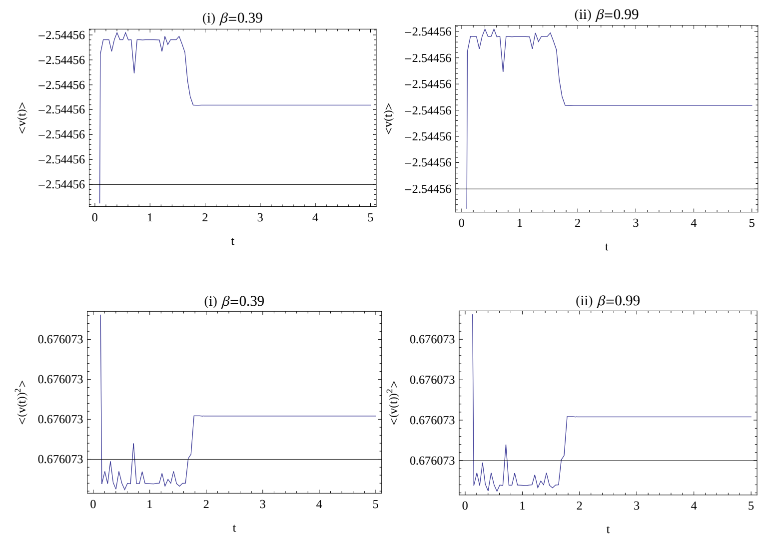


FIGURE 4. **Graph of the mean and the mean square of the velocity are shown for the same caption as in Figure 1.**

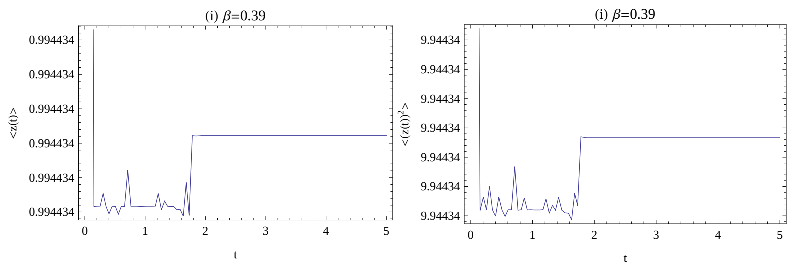


FIGURE 5. **Graph of the mean and mean square of the space variable are shown in figure 5.**

## 5. Conclusions

In this paper, a new approach is presented to solve linear PDEs with variable coefficients. This approach is based on converting the PDEs into a system of first-order PDEs. The exact solutions of the CFD-KKE are obtained via using the extended unified method. A variety of exact solutions are found by taking two cases for the auxiliary equations. The numerical evaluation of the solutions are evaluated and the are shown in figures. These figures show Gaussian or multiple Gaussian distributions.

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**Conflicts of Interest:** The author declares that he has no conflicts of interest.

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