ANALYSIS OF THE NONTRIVIAL ZEROS FOR THE CERTAIN DIRICHLET L-SERIES

XIAO-JUN YANG[1,2,3]

Abstract. In the present paper we propose a new approach for the generalized Riemann hypothesis in theoretical framework of the Dirichlet L-series. The Dirichlet's lambda function is used as the testing function to prove the generalized Riemann hypothesis. The obtained results can be also applied to consider the other classes of the Dirichlet L-series.

Keywords:
Riemann zeta-function, Dirichlet L-series, Dirichlet's lambda function, generalized Riemann hypothesis

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1. Introduction

The Dirichlet L-series has played an important role in analytic number theory due to the fact that primitive Dirichlet characters are one of the keys to the distribution of primes [1, 2, 3]. It has been applied to the Dirichlet series problems in the mathematical physics [4, 5].

Let $\mathcal{C}$ and $\mathcal{N}$ be the sets of the complex numbers and natural numbers. The Dirichlet L-series $L(s,\chi)$, proposed in 1837 by Dirichlet [3], is defined as [6]

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}, \tag{1}$$

where $\chi(m)$ is the Dirichlet character (mod $p>0$) [3], $s\in\mathcal{C}$ and $m\in\mathcal{N}$.

There exists a well-known fact that the Riemann zeta-function [7], Dirichlet's lambda function [8], Dirichlet's beta function [9] and Dirichlet's eta function [10, 11] are all Dirichlet L-series (see [12], p. 289). For example, the well-known Riemann zeta-function $\zeta(s)$ is defined as [7]

$$\zeta(s)=\sum_{m=1}^{\infty}m^{-s}, \tag{2}$$

which is related to the entire Riemann zeta-function [7]

$$\xi(s) = (s-1)\,\pi^{-s/2}\Gamma(s/2+1)\,\zeta(s)\,, \tag{3}$$

where $s \in \mathcal{C}$, $m \in \mathcal{N}$ and $Re(s) > 1$. It is well known that Equation (2) is an analytical continuation to the entire complex plane $s \in \mathcal{C}$ except for the simple pole $s = 1$ with residue 1 [13] and has the trivial zeros $s = 2\kappa$ for $\kappa \in \mathcal{N}$ [14].

Let $\mathcal{Z}$ be the set of the integral numbers and $i = \sqrt{-1}$.

The Dirichlet's eta function $\eta(s)$ is defined as (see [15], p.658)

$$\eta(s) = \sum_{m=1}^{\infty} \frac{(-1)^{k-1}}{m^s} = \left(1 - 2^{1-s}\right)\zeta(s)\,, \tag{4}$$

where $s \in \mathcal{C}$, $m \in \mathcal{N}$ and $Re(s) > 1$. It is pointed out that Equation (4) is an analytical continuation to the entire complex plane $s \in \mathcal{C}$ except for the simple pole $s = 1$ with residue 1, and has the pure imaginary number zeros $s = (2l\pi/\log 2)\,i + 1$ for $l \in \mathcal{Z}$ (see [16], p.160) and trivial zeros $s = 2\kappa$ for $\kappa \in \mathcal{N}$ [17, 18].

The Dirichlet's lambda function $\lambda(s)$ is the Dirichlet L-series, defined as (see [15], p.658)

$$\lambda(s) = \sum_{m=1}^{\infty} \frac{1}{(2m+1)^s} = \left(1 - 2^{-s}\right)\zeta(s)\,, \tag{5}$$

where $s \in \mathcal{C}$, $m \in \mathcal{N}$ and $Re(s) > 1$.

The Dirichlet's eta function $\eta(s)$ is connected with $\zeta(s)$ and $\lambda(s)$ by (see [15], p.658)

$$\frac{\lambda(s)}{2^s - 1} = \frac{\eta(s)}{2^s - 2} = \frac{\zeta(s)}{2^s} \tag{6}$$

and (see [15], p.658)

$$\zeta(s) + \eta(s) = 2\lambda(s)\,. \tag{7}$$

By (7) it is easily seen that, as one of the other classes of the Dirichlet L-series, the Dirichlet's lambda function $\lambda(s)$ may have the nontrivial zeros. The nontrivial zeros of the Dirichlet's lambda function $\lambda(s)$ imply the generalized Riemann hypothesis, which is represented as follows [19]:

The generalized Riemann hypothesis conjectures that neither the Riemann zeta function $\zeta(s)$ nor any Dirichlet L-series $L(s,\chi)$ has a zero with real part larger than $1/2$.

The generalized Riemann hypothesis can be connected with the Goldbach's conjecture [20] and the Diophantine equations [21], and is one of the important problems in the mathematics [17]. The criterion [22] and the number of zeros and poles for the Dirichlet L-series [23], hybrid bounds [24] and related functions [25, 26] were reported. As one of the classes of the Dirichlet L-series, the Dirichlet's lambda function $\lambda(s)$ can be expressed as follows:

Theorem 1. *(The stronger generalized Riemann hypothesis)*
The stronger generalized Riemann hypothesis that the Dirichlet's lambda function $\lambda(s)$ has the nontrivial zeros with the real part $1/2$.

The main target of the paper is to propose the sum and product representations for the Dirichlet's lambda function $\lambda(s)$, investigate the criterion for the existence of the nontrivial zeros of the Dirichlet's lambda function $\lambda(s)$, prove **Theorem 1** and give the applications in the presentations of the Bernoulli numbers.

The structure of the paper is designed as follows. In Section 2 we introduce the theory of the Riemann Ξ-function and entire Riemann zeta-function. In Section 3 we propose the integral, sum and product representations and the Turán inequalities for the Dirichlet's lambda function and the Dirichlet's lambda function at the critical line. In Section 4, we prove the stronger generalized Riemann hypothesis. Finally, we consider the applications in the presentations of the Bernoulli numbers.

2. New results for the special function related to the Riemann zeta-function

To begin with, we investigate the special functions related to the Riemann zeta-function.

2.1. Theory of the Riemann Ξ-function.

2.1.1. *The integral representations for the Riemann Ξ-function.* We denote the Riemann Ξ-function $\Xi(w)$ by [7]

$$\Xi(w) = 4\int_1^\infty \frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy} y^{-\frac{1}{4}} \cos\left(\frac{w}{2}\log y\right) dy, \tag{8}$$

where $w \in \mathcal{C}$.

The Jensen's integral representation for $\Xi(w)$ reads [27]

$$\Xi(w) = \int_0^\infty W(y)\cos(wy)\, dy, \tag{9}$$

where [28]

$$W(y) = \sum_{m=1}^\infty \left(8m^4\pi^2 e^{9y/2} - 12m^2\pi e^{5y/2}\right) e^{-m^2\pi e^{2y}} > 0. \tag{10}$$

2.1.2. *The series representations for the Riemann Ξ-function.* Here, we now investigate the series representations for the Riemann Ξ-function $\Xi(w)$.

Let $\mathcal{N}_0 = \mathcal{N} \cup 0$. The series representation for $\Xi(w)$ reads [14]

$$\Xi(w) = \sum_{u=1}^{\infty} a_u (-1)^u w^{2u}, \tag{11}$$

where $a_u > 0$ are the coefficients [18].

By (9) a_u are the Jensen coefficients [27, 29]

$$a_u = \frac{1}{(2u)!} \int_0^{\infty} W(y) y^{2n} dy. \tag{12}$$

With (9) a_u are the Pólya coefficients [29]

$$a_u = \frac{\Xi^{(2u)}(0)}{(2u)!}, \tag{13}$$

By (8) a_u are the Edwards's coefficients [30]

$$a_u = \frac{4}{(2u)!} \int_1^{\infty} \frac{d\left(y^{\frac{3}{2}} \psi^{(1)}(y)\right)}{dy} y^{-\frac{1}{4}} \left(\frac{\log y}{2}\right)^{2u} dy. \tag{14}$$

It is easily observed that the different representations of the coefficients a_u are equivalent [18] and satisfy the Turán inequalities [18, 31, 32]

$$(a_u)^2 - \left(\frac{2u-1}{2u+1}\right) a_{u-1} a_{u+1} > 0 \tag{15}$$

are always valid for any $u \in \mathcal{N}_0$.

It is clearly shown (11) is an alternating series.

2.1.3. *The series representations for the Riemann Ξ-function.* Assume that w_m are the real zeros of $\Xi(w)$, s_m are the nontrivial zeros of $\zeta(s)$ and $w_m = \vartheta_m > 0$. We now consider the infinite product formulas for $\Xi(x)$ as follows.

The infinite product of I type for $\Xi(w)$ reads [14]:

$$\Xi(w) = \Xi\left(\frac{1}{2}\right) \prod_{m=1}^{\infty} \left(1 - \frac{\frac{1}{2} + iw}{s_m}\right), \tag{16}$$

where $w \in \mathcal{C}$.

The infinite product of II type for $\Xi(w)$ can be expressed in the form [14]:

$$\Xi(w) = \Xi(0) \prod_{m=1}^{\infty} \left(1 - \frac{w}{w_m}\right), \tag{17}$$

where $w \in \mathcal{C}$.

The infinite product of III type for $\Xi(w)$ is suggested as [18]:

$$\Xi(w) = \Xi(0) \prod_{m=1}^{\infty} \left(1 - \frac{w^2}{(\vartheta_m)^2}\right), \tag{18}$$

where $w \in \mathcal{C}$.

Remark. Equation (18) was discovered by Titchmarsh (see [33], Equation (2.1), p.249) in the case of $\vartheta_m < 0$ and further reported in the case of $\vartheta_m > 0$ [18] since $\Xi(w)$ is an even function [13].

The infinite product of IV type for $\Xi(w)$ is represented by [18]:

$$\Xi(w) = \frac{e^{(1/2+iw)\hbar_0}}{2} \prod_{m=1}^{\infty} \left(1 - \frac{1/2 + iw}{s_m}\right) e^{(1/2+iw)/s_m}, \tag{19}$$

where $w \in \mathcal{C}$.

The infinite product of V type for $\Xi(w)$ can be written as [18]:

$$\Xi(w) = \Xi(0)\, e^{(1/2+iw)\hbar_0} \prod_{m=1}^{\infty} \left(1 - \frac{w}{w_m}\right) e^{(1/2+iw)/(1/2+iw_m)}, \tag{20}$$

where $w \in \mathcal{C}$.

Remark. By (21) and using the result of Edwards [30] one gives [18]

$$\Xi(w) = \mu \prod_{m=1}^{\infty} \left(1 - \frac{iw}{s_m - 1/2}\right). \tag{21}$$

Putting $s = 0$ into (21) we have (see [16], p.288; [18])

$$\mu = \Xi(0) \neq 0. \tag{22}$$

Thus, one writes (21) as [18]

$$\Xi(w) = \Xi(0) \prod_{m=1}^{\infty} \left(1 - \frac{iw}{s_m - 1/2}\right). \tag{23}$$

By Hardy theorem [34] and Turán inequalities [18, 31, 32], it is seen that $\Xi(w)$ has only the infinitely many real zeros for $w \in \mathcal{C}$ such that, by substituting $w = w_m$ into (23), we have [18]

$$\Xi(w_m) = \Xi(0) \prod_{m=1}^{\infty} \left(1 - \frac{iw_m}{s_m - 1/2}\right) = 0, \tag{24}$$

which leads to [18]

$$1-\frac{iw_m}{s_m-1/2}=0. \tag{25}$$

Hence, Equation (25) implies that

$$s_m=1/2+iw_m, \tag{26}$$

in other words that the Riemann conjecture is true [18].

Thus, (23) can be written as (17).

Equation (21) was written as [35]

$$\Xi\left(w\right)=\mu\prod_{m=1}^{\infty}\left(1-\frac{w}{w_m}\right), \tag{27}$$

and by (16), (22) and (26) there exist [18]

$$\mu=\Xi\left(0\right)=\Xi\left(\frac{1}{2}\right)\prod_{m=1}^{\infty}\left(1-\frac{1}{2s_m}\right) \tag{28}$$

and [18]

$$\Xi\left(\frac{1}{2}\right)=\Xi\left(0\right)\prod_{m=1}^{\infty}\left(1-\frac{1}{2w_m}\right). \tag{29}$$

It is well known that the real zeros w_m of $\Xi\left(w\right)$ can be fully solved with use of the Riemann-Siegel formula [36], discovered by Riemann and Siegel [37]. By (18) and (20) it is shown that $\Xi\left(w\right)$ is the even integral function of order $\rho=1$ and meromorphic continuation to the complex plane $w\in\mathcal{C}$ and that [18]

$$\lim_{u\to\infty}\sup\frac{\log\left(1/a_u\right)}{2u\left[\log\left(2u\right)\right]}=1 \tag{30}$$

always hold for $u\in\mathcal{N}$ and $w\in\mathcal{C}$.

Thus, there is the connection of $\Xi\left(w\right)$ with the Jensen's formula (see [38], p.12) since $\Xi\left(w\right)$ is the even integral function of order $\rho=1$ [18].

2.2. **Theory of the entire Riemann zeta-function.**

2.2.1. *The integral representations for the entire Riemann zeta-function.* Let us consider the connection of the Riemann zeta-function $\zeta\left(s\right)$ with the entire Riemann zeta-function $\xi\left(s\right)$, given as [7]

$$\xi\left(s\right)=\Lambda\left(s\right)\zeta\left(s\right), \tag{31}$$

where

$$\Lambda\left(s\right)=\left(s-1\right)\pi^{-s/2}\Gamma\left(s/2+1\right). \tag{32}$$

We not investigate the integral representations for the entire Riemann zeta-function $\xi(s)$.

The integral representation of I type for $\xi(s)$ can be given as [30]:

$$\xi(s) = 4\int\limits_{1}^{\infty} \frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy} y^{-\frac{1}{4}} \cosh\left[\frac{1}{2}(s-1/2)\log y\right] dy, \tag{33}$$

where $s \in \mathcal{C}$.

The integral representation of II type for $\xi(s)$ reads [18]:

$$\xi(s) = \int\limits_{0}^{\infty} W(y) \cosh\left[(s-1/2)\,y\right] dy, \tag{34}$$

where $s \in \mathcal{C}$.

2.2.2. *The series representations for the entire Riemann zeta-function.* We now present the series representations for the entire Riemann zeta-function $\xi(s)$.

The series representation of I type for $\xi(s)$ can be given as follows [30]:

$$\xi(s) = \sum_{u=0}^{\infty} a_u \left(s - \frac{1}{2}\right)^{2u} \tag{35}$$

where $s \in \mathcal{C}$ and

$$a_u = \frac{4}{(2u)!}\int\limits_{1}^{\infty} \frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy} y^{-\frac{1}{4}} \left(\frac{\log y}{2}\right)^{2u} dy.$$

The series representation of II type for $\xi(s)$ reads [18]:

$$\xi(s) = \sum_{u=0}^{\infty} a_u \left(s - \frac{1}{2}\right)^{2u} \tag{36}$$

where $s \in \mathcal{C}$ and

$$a_u = \frac{1}{(2u)!}\int\limits_{0}^{\infty} W(y)\, y^{2n} dy.$$

The series representation of III type for $\xi(s)$ can be suggested as [18]:

$$\xi(s) = \sum_{u=0}^{\infty} a_u \left(s - \frac{1}{2}\right)^{2u} \tag{37}$$

where $s \in \mathcal{C}$ and [18]

$$a_u = (-1)^u \frac{\xi^{(2u)}\left(\frac{1}{2}\right)}{(2u)!} \tag{38}$$

subjected to [18]

$$(-1)^u \frac{\xi^{(2u)}\left(\frac{1}{2}\right)}{(2u)!} = \frac{\Xi^{(2u)}(0)}{(2u)!}.$$

The Turán inequalities for $\xi(s)$ can be also written as [18, 31, 32]

$$(a_u)^2 - \left(\frac{2u-1}{2u+1}\right) a_{u-1} a_{u+1} > 0 \tag{39}$$

for any $u \in \mathcal{N}_0$ and $s \in \mathcal{C}$.

Thus, by (39) and Hardy theorem [34] it is not difficult to show that $\xi(s)$ has only the infinitely many nontrivial zeros.

2.2.3. *The infinite product representations for the entire Riemann zeta-function.* We now report the infinite product representations for the entire Riemann zeta-function $\xi(s)$ as follows.

Let

$$\hbar_0 = \log 2 + \frac{1}{2}\log\pi - 1 - \frac{1}{2}\gamma \tag{40}$$

and γ is the Euler's constant.

The infinite product of I type (Hadamard's infinite product) for $\xi(s)$ can be expressed as [39]

$$\xi(s) = \xi(0) \prod_{m=1}^{\infty} \left(1 - \frac{s}{s_m}\right), \tag{41}$$

where $s \in \mathcal{C}$.

The infinite product of II type for $\xi(s)$ reads [18]

$$\xi(s) = \xi(1/2) \prod_{m=1}^{\infty} \left(1 + \frac{i(s-1/2)}{w_m}\right), \tag{42}$$

where $s \in \mathcal{C}$.

The infinite product of III type (Weierstrass-Valiron product) for $\xi(s)$ is suggested as [40, 41]

$$\xi(s) = \xi(0) e^{s\hbar_0} \prod_{m=1}^{\infty} \left(1 - \frac{s}{s_m}\right) e^{s/s_m}, \tag{43}$$

where $s \in \mathcal{C}$.

The infinite product of IV type for $\xi(s)$ can be expressed as [18]

$$\xi(s) = \xi(1/2)\, e^{s\hbar_0} \prod_{m=1}^{\infty} \left(1 + \frac{i(s-1/2)}{w_m}\right) e^{s/(1/2+iw_m)}, \tag{44}$$

where $s \in \mathcal{C}$.

The infinite product of V type for $\xi(s)$ can be written as [18]

$$\xi(s) = \xi(1/2) \prod_{m=1}^{\infty} \left(1 - \frac{(s-1/2)^2}{(\vartheta_m)^2}\right), \tag{45}$$

where $s \in \mathcal{C}$.

Remark. By (44) and the result of Ivic (see [38], p.16) $\xi(s)$ is the integral function of order $\rho = 1$, which is in agreement with the result of Titchmarsh [42]. By the result of Ivic (see [38], p.16) and the fact that $\xi(s)$ is the integral function of order $\rho = 1$, there exists [18]

$$\lim_{u\to\infty} \sup \frac{2u\,[\log(2u)]}{\log(1/a_u)} = 1, \tag{46}$$

for $u \in \mathcal{N}$ and $s \in \mathcal{C}$. Thus, the Jensen's formula for $\xi(s)$ may be considered in theory of the Riemann zeta-function (see [38], p.12). The zeros of the entire Riemann zeta-function have been discussed in detail [18].

3. The theory of the Dirichlet's lambda function

In this section we consider the representations for the Dirichlet's lambda function for $s \in \mathcal{C}$

and $s = 1/2 + iw$ with $w \in \mathcal{C}$.

To find the connection of the Dirichlet's lambda function $\lambda(s)$ and entire Riemann zeta-function $\xi(s)$, we recall that

$$\xi(s) = \Lambda(s)\, \zeta(s), \tag{47}$$

and

$$\lambda(s) = \left(1 - 2^{-s}\right) \zeta(s), \tag{48}$$

which yields that

$$\lambda(s) = \left(1 - 2^{-s}\right) \xi(s) / \Lambda(s), \tag{49}$$

where

$$\Lambda(s) = (s-1)\, \pi^{-s/2} \Gamma(s/2+1). \tag{50}$$

It is easily seen that

$$\lambda(s) = \varpi(s)\, \xi(s), \tag{51}$$

where $s \in \mathcal{C}$ except for the simple pole $s = 1$ and

$$\varpi(s) = \left(1 - 2^{-s}\right) \Lambda(s) = \left(1 - 2^{-s}\right) / \left[(s-1)\,\pi^{-s/2}\Gamma(s/2+1)\right]. \tag{52}$$

Thus, Equation (51) is holomorphic in the entire complex plane $s \in \mathrm{C}$ except for the simple pole $s = 1$ with residue $1/2$.

3.0.1. *The integral representations for the Dirichlet's lambda function.* First we derive by (33) and (51) that

$$\begin{aligned} &\lambda(s) \\ &= \varpi(s)\,\xi(s) \\ &= 4\varpi(s) \int_1^\infty \frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy} y^{-\frac{1}{4}} \cosh\left[\tfrac{1}{2}(s-1/2)\log y\right] dy, \end{aligned} \tag{53}$$

which is the integral representation of type I for $\lambda(s)$, where $s \in \mathcal{C}$ except for the simple pole $s = 1$.

From (34) and (51) we show the integral representation of type II for $\lambda(s)$ by

$$\begin{aligned} &\lambda(s) \\ &= \varpi(s)\,\xi(s) \\ &= \varpi(s) \int_0^\infty W(y) \cosh\left[(s-1/2)\,y\right] dy, \end{aligned} \tag{54}$$

where $s \in \mathcal{C}$ except for the simple pole $s = 1$.

3.0.2. *The series representations for the Dirichlet's lambda function.* From (35) and (51) the series representation of type I for $\lambda(s)$ can be written in the form

$$\lambda(s) = \varpi(s) \sum_{u=0}^{\infty} a_u \left(s - \frac{1}{2}\right)^{2u} \tag{55}$$

where $s \in \mathcal{C}$ except for the simple pole $s = 1$ and

$$a_u = \frac{4}{(2u)!} \int_1^\infty \frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy} y^{-\frac{1}{4}} \left(\frac{\log y}{2}\right)^{2u} dy.$$

To derive (51) from (36) we get the series representation of type II for $\lambda(s)$ by

$$\lambda(s) = \varpi(s) \sum_{u=0}^{\infty} a_u \left(s - \frac{1}{2}\right)^{2u} \tag{56}$$

where

$$a_u = \frac{1}{(2u)!} \int_0^\infty W(y)\, y^{2n} dy.$$

Combining (37) and (51) it suffices to prove that

$$\lambda(s) = \varpi(s) \sum_{u=0}^{\infty} a_u \left(s - \frac{1}{2}\right)^{2u} \tag{57}$$

which is the series representation of type III for $\lambda(s)$, where

$$a_u = \frac{\Xi^{(2u)}(0)}{(2u)!}.$$

3.0.3. *The zeros and pole of the Dirichlet's lambda function.* Note that $\mathcal{Z}$ is the set of the integral numbers.

By (52) we have

$$\begin{aligned} \varpi(s) &= \tfrac{1-2^{-s}}{(s-1)\pi^{-s/2}\Gamma(s/2+1)} \\ &= \tfrac{\left(1-2^{-s}\right)\pi^{s/2}}{(s-1)\Gamma(s/2+1)} \\ &= \tfrac{\left(1-2^{-s}\right)\pi^{s/2}}{s-1} \cdot \tfrac{1}{\Gamma(s/2+1)} \\ &= \tfrac{\left(1-2^{-s}\right)e^{s/2\log\pi}}{s-1} \cdot e^{\gamma s/2} \prod_{k=1}^{\infty} \left(1 + \tfrac{s}{2k}\right) e^{-s/(2k)}, \end{aligned} \tag{58}$$

where

$$\frac{1}{\Gamma(s/2+1)} = e^{\gamma s/2} \prod_{k=1}^{\infty} \left(1 + \frac{s}{2k}\right) e^{-s/(2k)}. \tag{59}$$

By (58) it is easy to see that $\lambda(s)$ has simple pole $s = 1$ with residue $1/2$, the pure imaginary number zeros $s = (2\pi l/\log 2)\, i$ for $l \in \mathcal{Z}$ and the trivial zeros $s = 2\kappa$ for $\kappa \in \mathcal{N}_0$.

Applying the series representations for the Dirichlet's lambda function, we arrive at the Turán inequalities for $\lambda(s)$, given as

$$(a_u)^2 - \left(\frac{2u-1}{2u+1}\right) a_{u-1} a_{u+1} > 0 \tag{60}$$

for any $u \in \mathcal{N}_0$ and $s \in \mathcal{C}$.

Theorem 2. *(The existence theorem for the nontrivial zeros of $\lambda(s)$)*

The Dirichlet's lambda function $\lambda(s)$ has the infinitely many nontrivial zeros for the complex plane $s \in \mathcal{C}$ except for the simple pole $s = 1$, the pure imaginary number zeros $s = (2\pi l/\log 2)\, i$ for $l \in \mathcal{Z}$, and trivial zeros $s = 2\kappa$ for $\kappa \in \mathcal{N}_0$.

Proof. By (60) and Hardy theorem [34] $\lambda(s)$ has the infinite many nontrivial zeros.

Hence, the result follows. □

3.0.4. *The product representations for the Dirichlet's lambda function.* By (41) and (51) the product representation of I type for $\lambda(s)$ reads

$$\lambda(s) = \xi(0)\,\varpi(s)\prod_{m=1}^{\infty}\left(1-\frac{s}{s_m}\right), \tag{61}$$

where $s \in \mathcal{C}$ except for the simple pole $s = 1$.

Inserting (42) into (51) we illustrate the infinite product of II type for $\lambda(s)$ by

$$\lambda(s) = \xi(1/2)\,\varpi(s)\prod_{m=1}^{\infty}\left(1+\frac{i(s-1/2)}{w_m}\right), \tag{62}$$

where $s \in \mathcal{C}$ except for the simple pole $s = 1$.

Similarly, from (43) and (51) we show

$$\lambda(s) = \xi(0)\,\varpi(s)\,e^{s\hbar_0}\prod_{m=1}^{\infty}\left(1-\frac{s}{s_m}\right)e^{s/s_m}, \tag{63}$$

which is the infinite product of III type for $\lambda(s)$, where $s \in \mathcal{C}$ except for the simple pole $s = 1$.

With (44) and (51) the infinite product of IV type for $\lambda(s)$ can be represented as

$$\lambda(s) = \xi(1/2)\,\varpi(s)\,e^{s\hbar_0}\prod_{m=1}^{\infty}\left(1+\frac{i(s-1/2)}{w_m}\right)e^{s/(1/2+iw_m)}, \tag{64}$$

where $s \in \mathcal{C}$ except for the simple pole $s = 1$.

With use of (45) and (51) the infinite product of V type for $\lambda(s)$ can be expressed in the form

$$\lambda(s) = \xi(1/2)\,\varpi(s)\prod_{m=1}^{\infty}\left(1-\frac{(s-1/2)^2}{(\vartheta_m)^2}\right), \tag{65}$$

where $s \in \mathcal{C}$ except for the simple pole $s = 1$.

By (64) it is seen that $\lambda(s)$ is holomorphic in the entire complex plane $s \in \mathcal{C}$ except for the simple pole $s = 1$ with residue $1/2$, and has the pure imaginary number zeros $s = (2\pi l/\log 2)\,i$ for $l \in \mathcal{Z}$ and trivial zeros $s = 2\kappa$ for $\kappa \in \mathcal{N}_0$.

3.0.5. *The Dirichlet's lambda function at the critical line.* Let $\Phi(w)$ be the Dirichlet's lambda function at the critical line $s = 1/2 + iw$ by

$$\Phi(w) = \lambda(1/2+iw)\,. \tag{66}$$

Thus, there is

$$\Phi(w) = r(w)\,\Xi(w)\,, \tag{67}$$

where

$$\begin{aligned}&r(w)\\&=\varpi(1/2+iw)\\&=\left(1-2^{-(1/2+iw)}\right)/\left\{(iw-1/2)\,\pi^{-(1/2+iw)/2}\Gamma\left[(1/2+iw)/2+1\right]\right\}.\end{aligned}$$

From (53) and (54) we have

$$\Phi(w)=4r(w)\int_1^{\infty}\frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy}y^{-\frac{1}{4}}\cos\left(\frac{w}{2}\log y\right)dy \tag{68}$$

and

$$\Phi(w)=r(w)\int_0^{\infty}W(y)\cos(wy)\,dy. \tag{69}$$

Notice that by (69) we have the series representation for $\Phi(w)$, given as

$$\Phi(w)=r(w)\sum_{u=0}^{\infty}a_u(-1)^u w^{2u} \tag{70}$$

where

$$a_u=\frac{1}{(2u)!}\int_0^{\infty}W(y)\,y^{2n}dy. \tag{71}$$

With use of (60) the Turán inequalities for $\Phi(w)$ can be also expressed in the form

$$(a_u)^2-\left(\frac{2u-1}{2u+1}\right)a_{u-1}a_{u+1}>0, \tag{72}$$

where $u\in\mathcal{N}_0$ and $w\in\mathcal{C}$.

Let $r(w)\neq 0$ for $w\in\mathcal{R}$. Then we have the following result.

Corollary 1. *Let $w\in\mathcal{R}$. Then $\Phi(w)$ has the infinitely many nontrivial zeros.*

Proof. For $w\in\mathcal{R}$ one gives

$$r(w)=\left(1-2^{-(1/2+iw)}\right)/\left\{(iw-1/2)\,\pi^{-(1/2+iw)/2}\Gamma\left[(1/2+iw)/2+1\right]\right\}\neq 0. \tag{73}$$

With (67) and (73) it is seen that $\Phi(w)$ and $\Xi(w)$ are the same as the infinitely many nontrivial zeros.

We thus finish the proof. □

From (16), (17), (18), (19), (20) and (66) one gets

$$\Phi(w) = \Xi\left(\frac{1}{2}\right) r(w) \prod_{m=1}^{\infty}\left(1 - \frac{\frac{1}{2} + iw}{s_m}\right), \tag{74}$$

$$\Phi(w) = \Xi(0)\, r(w) \prod_{m=1}^{\infty}\left(1 - \frac{w}{w_m}\right), \tag{75}$$

$$\Phi(w) = \Xi(0)\, r(w) \prod_{m=1}^{\infty}\left(1 - \frac{w^2}{(\vartheta_m)^2}\right), \tag{76}$$

$$\Phi(w) = \frac{r(w)}{2} e^{(1/2+iw)\hbar_0} \prod_{m=1}^{\infty}\left(1 - \frac{1/2 + iw}{s_m}\right) e^{(1/2+iw)/s_m} \tag{77}$$

and

$$\Phi(w) = \Xi(0)\, r(w)\, e^{(1/2+iw)\hbar_0} \prod_{m=1}^{\infty}\left(1 - \frac{w}{w_m}\right) e^{(1/2+iw)/(1/2+iw_m)}, \tag{78}$$

where $w \in \mathcal{C}$.

Corollary 2. *Let $w \in \mathcal{R}$. The real zeros for $\Phi(w)$ read $w = w_m$ for $m \in \mathcal{N}$.*

Proof. Applying **Corollary 1**, Equation (75) and $r(w) \neq 0$ for $w \in \mathcal{R}$, one gives

$$\Phi(w_m) = 0, \tag{79}$$

which leads to

$$1 - \frac{w}{w_m} = 0. \tag{80}$$

Thus, the result follows. □

4. The proof of Theorem 1

Throughout in this paper $\lambda(s)$ is holomorphic in the entire complex plane $s \in \mathcal{C}$ except for the simple pole $s = 1$ with residue $1/2$.

To illustrate this result we consider

$$\lambda(s) = \xi(1/2)\, \varpi(s)\, e^{s\hbar_0} \prod_{m=1}^{\infty}\left(1 + \frac{i(s - 1/2)}{w_m}\right) e^{s/(1/2+iw_m)}, \tag{81}$$

where $s \in \mathcal{C}$.

It is well known that $\lambda(s)$ has the pure imaginary number zeros $s = (2\pi l/\log 2)\, i$ for $l \in \mathcal{Z}$ and trivial zeros $s = 2\kappa$ for $\kappa \in \mathcal{N}_0$.

Let $s \in \mathcal{C}$, $s \neq 1$, $s \neq (2\pi l/\log 2)\, i$ for $l \in \mathcal{Z}$ and $s \neq 2\kappa$ for $\kappa \in \mathcal{N}_0$. Then from (52) we have

$$\varpi(s) = \left(1 - 2^{-s}\right) / \left[(s-1)\, \pi^{-s/2} \Gamma(s/2+1)\right] \neq 0. \tag{82}$$

By **Theorem 2** $\lambda(s)$ has the infinitely many nontrivial zeros for the complex plane $s \in \mathcal{C}$ except for the simple pole $s = 1$, the pure imaginary number zeros $s = (2\pi l/\log 2)\, i$ for $l \in \mathcal{Z}$, and trivial zeros $s = 2\kappa$ for $\kappa \in \mathcal{N}_0$.

Substituting $s = s_m$ into (81) we have

$$\lambda(s_m) = \xi(1/2)\, \varpi(s_m)\, e^{s\hbar_0} \prod_{m=1}^{\infty} \left(1 + \frac{i(s_m - 1/2)}{w_m}\right) e^{s_m/(1/2+iw_m)} = 0. \tag{83}$$

Then applying (83) and $\varpi(s_m) \neq 0$ we get

$$1 + \frac{i(s_m - 1/2)}{w_m} = 0. \tag{84}$$

From (84) we get the following result:

$$s_m = 1/2 + iw_m. \tag{85}$$

Hence, the result follows.

As a direct result, we have the following corollary.

Corollary 3. *Let $s \in \mathcal{C}$, $s \neq 1$, $s \neq (2\pi l/\log 2)\, i$ for $l \in \mathcal{Z}$ and $s \neq 2\kappa$ for $\kappa \in \mathcal{N}_0$. Suppose that $\lambda(s)$ has the nontrivial zeros s_m, then $Re(s_m) = 1/2$.*

Proof. Since $s \in \mathcal{C}$, $s \neq 1$, $s \neq 0$, $s \neq (2\pi l/\log 2)\, i$ for $l \in \mathcal{Z}$, we have

$$1 - 2^{-s} \neq 0$$

and

$$s - 1 \neq 0.$$

Similarly, for $s \in \mathcal{C}$ and $s \neq 2\kappa$ with $\kappa \in \mathcal{N}$ we give

$$1/\Gamma(s/2+1) \neq 0.$$

Thus,

$$\varpi(s) = \left(1 - 2^{-s}\right) / \left[(s-1)\, \pi^{-s/2} \Gamma(s/2+1)\right] \neq 0. \tag{86}$$

By **Theorem 1** we have

$$\lambda(s) = \lambda(s_m) = 0 \tag{87}$$

such that

$$Re(s_m) = 1/2. \tag{88}$$

$\square$

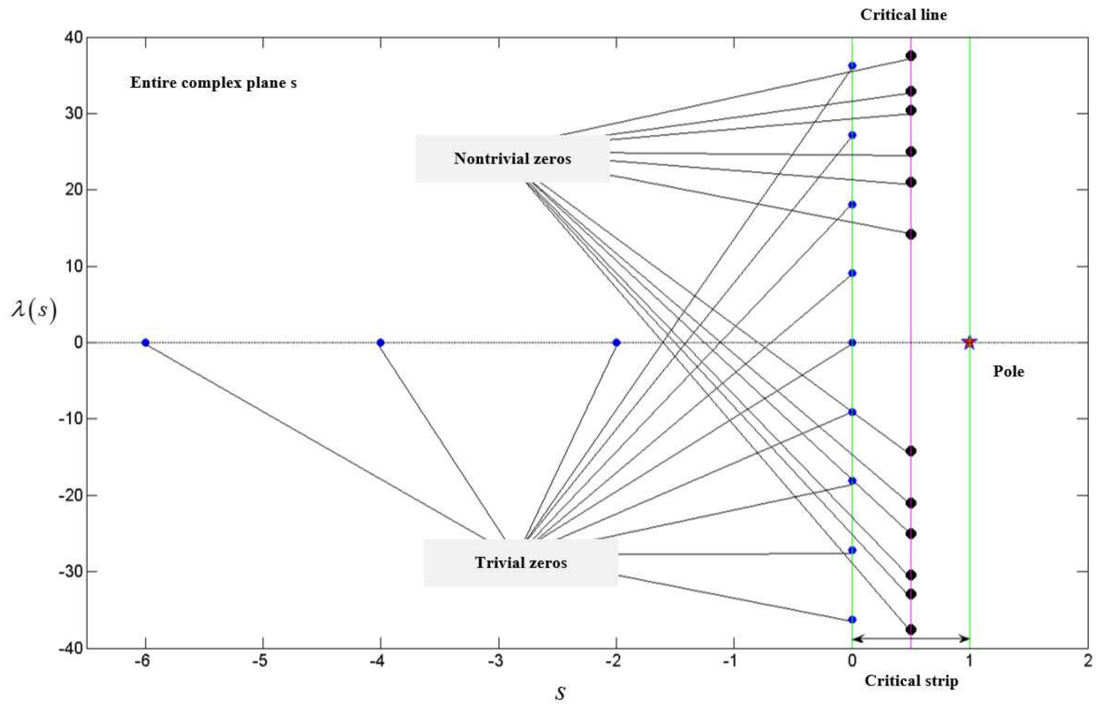

FIGURE 1. The blue dots represent the trivial zeros and pure imaginary number zeros which do not lie on the critical line. The black dots represent all nontrivial zeros which lie on the critical line $s = 1/2$ and in the critical strip $0 < Re\,(s) < 1$. There is a pole $s = 1$.

Corollary 4. *Let $s \in \mathcal{C}$, $s \neq 1$, $s \neq (2\pi l/\log 2)\, i$ for $l \in \mathcal{Z}$ and $s \neq 2\kappa$ for $\kappa \in \mathcal{N}_0$. Suppose that $N\,(T)$ is the number of the nontrivial roots of $\lambda\,(s) = 0$ in the region $0 < Re\,(s) < 1$ and $0 < x < T$, and $N_0\,(T)$ is the number of roots of $\Phi\,(w) = 0$ in the region $Re\,(s) = 1/2$ and $0 < x < T$, then there exists*

$$N\,(T) = N_0\,(T)\,. \tag{89}$$

Proof. By **Theorem 1** and **Corollary 3** the result follows. □

Remark. Corollary 4 is a generalized case of the Riemann zeta-functions [43, 44] and w_m can be confirmed by the Riemann-Siegel formula [36] and related results [45]. It is shown that the critical line for $\lambda\,(s)$ is $s = 1/2$ and that the critical strip for $\lambda\,(s)$ is $0 < Re\,(s) < 1$. They are related to the zeros of the Riemann Zeta-function on the critical line [46]. The some zeros (nontrivial zeros, trivial zeros and pure imaginary number zeros), pole, critical line and critical strip for $\lambda\,(s)$ are shown in Fig. 1.

5. Applications in the presentations of the Bernoulli numbers

In this section we give the alternative representations for the Bernoulli numbers.

By (2) the integral representation for the entire Riemann zeta-function $\xi(s)$ is expressed by [30]

$$\begin{aligned}&\xi(s)\\&=\Lambda(s)\zeta(s)\\&=4\int_1^\infty\frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy}y^{-\frac{1}{4}}\cosh\left[\tfrac{1}{2}(s-1/2)\log y\right]dy,\end{aligned}\tag{90}$$

where

$$\Lambda(s)=(s-1)\pi^{-s/2}\Gamma(s/2+1).\tag{91}$$

Let $k\in\mathcal{N}_0$.

Using the well-known identity (see [47], p.167)

$$\zeta(2k)=(-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}\ \ (k\in\mathcal{N}_0),\tag{92}$$

we present

$$\zeta(2k)=\frac{\xi(2k)}{\Lambda(2k)}=(-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}\ \ (k\in\mathcal{N}_0),\tag{93}$$

which leads, by (5), to

$$\lambda(2k)=\frac{1-2^{-2k}}{\Lambda(2k)}\xi(2k)=(-1)^{k+1}\left(1-2^{-2k}\right)\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}\ \ (k\in\mathcal{N}_0),\tag{94}$$

where B_{2k} are the Bernoulli numbers (see [47], p.52).

From (52) we have

$$\begin{aligned}&\varpi(2k)\\&=\left(1-2^{-2k}\right)/\left[(2k-1)\pi^{-k}\Gamma(k+1)\right]\\&=\tfrac{\left(1-2^{-2k}\right)\pi^k}{(2k-1)k!}.\end{aligned}\tag{95}$$

Now deriving (94) from (33) leads to

$$\begin{aligned}&\lambda(2k)\\&=4\varpi(2k)\int_1^\infty\frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy}y^{-\frac{1}{4}}\cosh\left[\tfrac{1}{2}(2k-1/2)\log y\right]dy\\&=(-1)^{k+1}\left(1-2^{-2k}\right)\tfrac{(2\pi)^{2k}}{2(2k)!}B_{2k}.\end{aligned}\tag{96}$$

With (95) and (96) we get
(97)

$$B_{2k}=\frac{(-1)^{k+1}}{2^{2k-3}\pi^{k}(2k-1)}\cdot\frac{(2k)!}{k!}\int\limits_{1}^{\infty}\frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy}y^{-\frac{1}{4}}\cosh\left[\frac{1}{2}(2k-1/2)\log y\right]dy.$$

From (94) and (54) we present

$$\begin{aligned}&\lambda(2k)\\&=\varpi(2k)\int\limits_{0}^{\infty}W(y)\cosh\left[(2k-1/2)y\right]dy\\&=(-1)^{k+1}\left(1-2^{-2k}\right)\tfrac{(2\pi)^{2k}}{2(2k)!}B_{2k}.\end{aligned}\tag{98}$$

From (98) and (95) we give

$$B_{2k}=\frac{(-1)^{k+1}}{2^{2k-1}\pi^{k}(2k-1)}\cdot\frac{(2k)!}{k!}\int\limits_{0}^{\infty}W(y)\cosh\left[(2k-1/2)y\right]dy.\tag{99}$$

Substituting (94) into (55) we have

$$\begin{aligned}&\lambda(2k)\\&=\varpi(2k)\sum_{u=0}^{\infty}a_u\left(2k-\tfrac{1}{2}\right)^{2u}\\&=(-1)^{k+1}\left(1-2^{-2k}\right)\tfrac{(2\pi)^{2k}}{2(2k)!}B_{2k},\end{aligned}\tag{100}$$

where

$$a_u=\frac{4}{(2u)!}\int\limits_{1}^{\infty}\frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy}y^{-\frac{1}{4}}\left(\frac{\log y}{2}\right)^{2u}dy.$$

Substituting (100) into (95) we obtain

$$B_{2k}=\frac{(-1)^{k+1}}{2^{2k-1}\pi^{k}(2k-1)}\cdot\frac{(2k)!}{k!}\sum_{u=0}^{\infty}a_u\left(2k-\frac{1}{2}\right)^{2u},\tag{101}$$

where

$$a_u=\frac{4}{(2u)!}\int\limits_{1}^{\infty}\frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy}y^{-\frac{1}{4}}\left(\frac{\log y}{2}\right)^{2u}dy.$$

In a similar way, combining (94) and (56), one gives

$$\begin{aligned}&\lambda\left(2k\right)\\&=\varpi\left(2k\right)\sum_{u=0}^{\infty}a_u\left(2k-\tfrac{1}{2}\right)^{2u}\\&=\left(-1\right)^{k+1}\left(1-2^{-2k}\right)\tfrac{\left(2\pi\right)^{2k}}{2(2k)!}B_{2k},\end{aligned}\tag{102}$$

where

$$a_u=\frac{1}{(2u)!}\int_0^{\infty}W\left(y\right)y^{2n}dy.$$

With use of (95) and (102) it would be enough to show that

$$B_{2k}=\frac{\left(-1\right)^{k+1}}{\left(2k-1\right)\pi^k2^{2k-1}}\cdot\frac{(2k)!}{k!}\sum_{u=0}^{\infty}a_u\left(2k-\frac{1}{2}\right)^{2u},\tag{103}$$

where

$$a_u=\frac{1}{(2u)!}\int_0^{\infty}W\left(y\right)y^{2n}dy.$$

By using (94) and (57) it is obvious that

$$\begin{aligned}&\lambda\left(2k\right)\\&=\varpi\left(2k\right)\sum_{u=0}^{\infty}a_u\left(2k-\tfrac{1}{2}\right)^{2u}\\&=\left(-1\right)^{k+1}\left(1-2^{-2k}\right)\tfrac{\left(2\pi\right)^{2k}}{2(2k)!}B_{2k},\end{aligned}\tag{104}$$

where

$$a_u=\frac{\Xi^{(2u)}\left(0\right)}{(2u)!}.$$

From (95) and (104) we suggest that

$$B_{2k}=\frac{\left(-1\right)^{k+1}}{\left(2k-1\right)\pi^k2^{2k-1}}\cdot\frac{(2k)!}{k!}\sum_{u=0}^{\infty}a_u\left(2k-\frac{1}{2}\right)^{2u},\tag{105}$$

where

$$a_u=\frac{\Xi^{(2u)}\left(0\right)}{(2u)!}.$$

To derive (61) from (94), one present

$$\begin{aligned}&\lambda(2k)\\&=\xi(0)\varpi(2k)\prod_{m=1}^{\infty}\left(1-\tfrac{2k}{s_m}\right)\\&=(-1)^{k+1}\left(1-2^{-2k}\right)\tfrac{(2\pi)^{2k}}{2(2k)!}B_{2k}.\end{aligned}\tag{106}$$

Further, it follows from (95) and (106) that

$$B_{2k}=\frac{(-1)^{k+1}\xi(0)}{(2k-1)\pi^k 2^{2k-1}}\cdot\frac{(2k)!}{k!}\prod_{m=1}^{\infty}\left(1-\frac{2k}{s_m}\right).\tag{107}$$

With (62) and (94) we have

$$\begin{aligned}&\lambda(2k)\\&=\xi(1/2)\varpi(2k)\prod_{m=1}^{\infty}\left(1+\tfrac{i(2k-1/2)}{w_m}\right)\\&=(-1)^{k+1}\left(1-2^{-2k}\right)\tfrac{(2\pi)^{2k}}{2(2k)!}B_{2k}.\end{aligned}\tag{108}$$

In view of (95) and (108) we derive that

$$B_{2k}=\frac{\xi(1/2)(-1)^{k+1}}{(2k-1)\pi^k 2^{2k-1}}\cdot\frac{(2k)!}{k!}\prod_{m=1}^{\infty}\left(1+\frac{i(2k-1/2)}{w_m}\right).\tag{109}$$

From (63) and (94) we observe that

$$\begin{aligned}&\lambda(2k)\\&=\xi(0)\varpi(2k)e^{2k\hbar_0}\prod_{m=1}^{\infty}\left(1-\tfrac{2k}{s_m}\right)e^{2k/s_m}\\&=(-1)^{k+1}\left(1-2^{-2k}\right)\tfrac{(2\pi)^{2k}}{2(2k)!}B_{2k}.\end{aligned}\tag{110}$$

By (95) and (110) this implies that

$$B_{2k}=\frac{(-1)^{k+1}\xi(0)}{(2k-1)\pi^k 2^{2k-1}}\cdot\frac{(2k)!}{k!}e^{2k\hbar_0}\prod_{m=1}^{\infty}\left(1-\frac{2k}{s_m}\right)e^{2k/s_m}.\tag{111}$$

By (64) and (94) one obtains

$$\begin{aligned}&\lambda(2k)\\&=\xi(1/2)\varpi(2k)e^{2k\hbar_0}\prod_{m=1}^{\infty}\left(1+\tfrac{i(2k-1/2)}{w_m}\right)e^{2k/(1/2+iw_m)}\\&=(-1)^{k+1}\left(1-2^{-2k}\right)\tfrac{(2\pi)^{2k}}{2(2k)!}B_{2k},\end{aligned}\tag{112}$$

where $k\in\mathcal{N}_0$.

Inserting (95) into (112) one has

$$B_{2k}=\frac{(-1)^{k+1}\xi(1/2)}{(2k-1)\pi^{k}2^{2k-1}}\cdot\frac{(2k)!}{k!}e^{2k\hbar_0}\prod_{m=1}^{\infty}\left(1+\frac{i(2k-1/2)}{w_m}\right)e^{2k/(1/2+iw_m)}. \tag{113}$$

With use of (65) and (94) we derive that

$$\begin{aligned}&\lambda(2k)\\&=\xi(1/2)\varpi(2k)\prod_{m=1}^{\infty}\left(1-\tfrac{(2k-1/2)^2}{(\vartheta_m)^2}\right)\\&=(-1)^{k+1}\left(1-2^{-2k}\right)\tfrac{(2\pi)^{2k}}{2(2k)!}B_{2k}.\end{aligned} \tag{114}$$

In view of (95) into (114) we write

$$B_{2k}=\frac{(-1)^{k+1}\xi(1/2)}{(2k-1)\pi^{k}2^{2k-1}}\cdot\frac{(2k)!}{k!}\prod_{m=1}^{\infty}\left(1-\frac{(2k-1/2)^2}{(\vartheta_m)^2}\right). \tag{115}$$

With the aid of the result of Srivastava and Choi (see [47], p. 81)

$$B_0=1 \tag{116}$$

we derive by (97), (99), (101), (107), (109) and (115) that

$$B_0=8\int_1^{\infty}\frac{d\left(y^{\frac{3}{2}}\psi^{(1)}(y)\right)}{dy}y^{-\frac{1}{4}}\cosh\left(\frac{\log y}{4}\right)dy=1, \tag{117}$$

$$B_0=2\int_0^{\infty}W(y)\cosh\left(\frac{y}{2}\right)dy=1, \tag{118}$$

$$B_0=2\sum_{u=0}^{\infty}\left(\frac{\Xi^{(2u)}(0)}{(2u)!}\cdot\frac{1}{2^{2u}}\right)=1, \tag{119}$$

$$B_0=2\xi(0)=1, \tag{120}$$

$$B_0=2\xi(1/2)\prod_{m=1}^{\infty}\left(1-\frac{i}{2w_m}\right)=1 \tag{121}$$

and

$$B_0=2\xi(1/2)\prod_{m=1}^{\infty}\left(1-\frac{1}{4(\vartheta_m)^2}\right)=1. \tag{122}$$

By the relation [7]

$$\Xi(w) = \xi\left(\frac{1}{2} + iw\right) \tag{123}$$

we have

$$\Xi(0) = \xi\left(\frac{1}{2}\right) \tag{124}$$

and

$$\Xi\left(\frac{i}{2}\right) = \xi(0). \tag{125}$$

By (11), (13) and (123) we present

$$\xi(0) = \Xi(i/2) = \sum_{u=1}^{\infty}\left(\frac{\Xi^{(2u)}(0)}{(2u)!}\cdot\frac{1}{2^{2u}}\right), \tag{126}$$

which implies that from (119) and (120) we derive that

$$2\xi(0) = 2\Xi(i/2) = 2\sum_{u=1}^{\infty}\left(\frac{\Xi^{(2u)}(0)}{(2u)!}\cdot\frac{1}{2^{2u}}\right) = B_0 = 1. \tag{127}$$

From (121) and (123) we have

$$2\xi(1/2)\prod_{m=1}^{\infty}\left(1-\frac{i}{2w_m}\right) = 2\Xi(0)\prod_{m=1}^{\infty}\left(1-\frac{i}{2w_m}\right) = B_0 = 1. \tag{128}$$

Thus from (127) there is

$$\xi(0) = \frac{1}{2}, \tag{129}$$

which is in agreement with the result (see [48], p.49).

In a similar way, from (128) we get

$$\prod_{m=1}^{\infty}\left(1-\frac{i}{2w_m}\right) = \frac{1}{2\xi(1/2)} = \frac{1}{2\Xi(0)} = \frac{\Xi(i/2)}{\Xi(0)}, \tag{130}$$

which is equivalent to the result [18] since (see [48], p.49)

$$\xi(0) = \xi(1) \tag{131}$$

and

$$\xi(0) = \Xi(i/2) = \frac{1}{2} \tag{132}$$

where [7]

$$\xi\left(1/2+iw\right)=\Xi\left(w\right). \tag{133}$$

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Email address: dyangxiaojun@163.com; xjyang@cumt.edu.cn

[1] School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China

[2] State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou 221116, China

[3] School of Mechanics and Civil Engineering, China University of Mining and Technology, Xuzhou 221116, China