New families of exact traveling wave solutions to the van der Waals p-system

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Abstract

In this article, we successfully construct various kinds of exact traveling wave solutions like, hyperbolic (kink and singular kink-shaped soliton), trigonometric (periodic and singular periodic) as well as rational function solutions for the best known mixed hyperbolic-elliptic system of conservation laws, namely the van der Waals gas system in the viscosity-capillarity regularization version by means of $(\frac{G'}{G^2})$-expansion and advanced $e^{-\phi(\tau)}$-expansion functions methods. These techniques are very useful and exceptionally helpful in a contrast with other analytical schemes, which show the effectiveness and the simplicity to discuss the exact solutions. 3D and contour figures are sketched in order to understand the physical movement of the gained results under the selections ofunknown parameters.

keyword: Exact solutions; van der Waals gas equation; $(\frac{G'}{G^2})$-expansion; advance $e^{-\phi(\tau)}$-expansion functions

1 Introduction

NLPDEs are widely used to discuus a number of physical and intricate phenomena that arises in several fields of nonlinear sciences like fluid mechanics, solid state physics, quantum field theory, hydro magnetic wave, bio-physics, biology, nonlinear optics, plasma physics and many others [1–8]. It has become an important bottom-line to explore the analytical solutions to these kinds of equations. In this manner, the basic focus to the experts is to extract the exact solutions. Due to this different computationl powerful strategies [9–18] have been designed for explaining behvior of NLPDEs by using various symbolic computation like Matlab, Mathematica, and Maple etc.

Exact solutions also help us to apprehend the intricacy of the phenomena, endorse the results of numerical analysis and explore the stability of these equations. Mixed-type hyperbolic systems of conservation laws have been employed to depict the various aspect of physical phenomena which find in different kinds of fluid dynamics and solids. For instance, the systems show the dynamical phase transitions in the propagating phase boundaries in solids and the van der Waals fluid [19]. Different numeric-analytic attempts on solving mixed systems have been carried out in previous studies [20–27]. The key rule of this contribution is to extract different solutions like hyperbolic, trigonometry and rationales by two integration schemes [28, 29] for the best known mixed-type hyperbolic-elliptic system of conservation laws, called as p-system in the form of van der Waals gas equations in

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the viscosity-capillarity version [30].

$$u_t + (p(v))_x = \eta u_{xx} - \omega\eta^2 u_{xxx} \quad \text{and} \quad v_t - u_x = 0, \tag{1}$$

where u_x^q denotes the qth partial derivative w.r.t x. $u(x,t)$ and $v(x,t)$ are the velocity and the volume respectively, while $p(v)$ represents the pressure of gas. By taking η and ω are constants which considered to be positive, whereas $\omega\eta^2$ represent the coefficient of interfacial capillarity. $\pm\sqrt{-p(v)}$ are equivalent eigenvalues of Eq. (1) which represents the one dimensional longitudinal isothermal motion in fluids or elastic bars. For various material problems or models, the mixed system of hyperbolic-elliptic type since the constitutive pressure function may not be monotone.

This piece of article has the following arrangement: In section 2, key points of the proposed methods. In section 3, applications and finally paper comes to end with conclusion in section 4.

2 Key points of the proposed methods

Let us consider the NLPDE as

$$S(\Phi, \Phi_t, \Phi_x, \Phi_{tt}, \Phi_{xt}, \Phi_{xx}, ...) = 0, \tag{2}$$

where $u = u(x,t)$ is an empirical function, S is a polynomial of $\Phi(x,t)$ and its partials derivatives in which higher order derivatives and nonlinear terms are involved. For finding the traveling wave solutions of Eq. (2), we introduce traveling wave transformation as: $\Phi(x,t) = u(\tau) \quad \text{and} \quad \tau = (x + \alpha t)$, where α represents the wave speed. After putting this transformation into Eq. (2), we get nonlinear ODE in the following form.

$$D(u, u', u'', u''', ...) = 0, \tag{3}$$

where $'$ denotes the derivative w.r.t τ.

2.1 Algorithm of Advanced $\mathrm{e}^{-\phi(\tau)}$-expansion function method

For this method, we take the travelling wave solution to the Eq. (3) in the following way.

$$u(\tau) = \sum_{i=0}^{N} a_i [e^{-\phi(\tau)}]^i \quad a_i \neq 0, \tag{4}$$

where a_i are constants to must be find and $\quad \phi = \phi(\tau) \quad$ holds the following ODE

$$\phi' + \lambda e^{\phi} + \mu e^{-\phi} = 0, \quad \lambda, \mu \in \Re \tag{5}$$

Eq. (5) has the following kinds of general solutions as follow:
Case I (Trigonometric function solution): When $\lambda\mu > 0$, then

$$\phi(\tau) = -\ln\left(\sqrt{\frac{\lambda}{\mu}} \tan\left(\sqrt{\lambda\mu}\,(\tau + c_0)\right)\right),$$

and,

$$\phi(\tau) \;=\; -\ln\Bigg(-\sqrt{\frac{\lambda}{\mu}}\cot\Big(\sqrt{\lambda\mu}\,(\tau+c_0)\Big)\Bigg).$$

Case II (Hyperbolic function solution): When $\lambda\mu < 0$ then,

$$\phi(\tau) \;=\; -\ln\Bigg(\sqrt{\frac{\lambda}{-\mu}}\tanh\Big(\sqrt{-\lambda\mu}\,(\tau+c_0)\Big)\Bigg),$$

and,

$$\phi(\tau) \;=\; -\ln\Bigg(\sqrt{\frac{\lambda}{-\mu}}\coth\Big(\sqrt{-\lambda\mu}\,(\tau+c_0)\Big)\Bigg).$$

Case III (Rational function solution): When $\lambda = 0 \quad \text{and} \quad \mu > 0$ then,

$$\phi(\tau) \;=\; -\ln\Bigg(-\frac{1}{\mu(\tau+c_0)}\Bigg),$$

Case IV (Rational function solution): When $\lambda \in \Re \quad \text{and} \quad \mu = 0$ then,

$$\phi(\tau) \;=\; -\ln\Big(\lambda(\tau+c_0)\Big),$$

Where $a_0, a_1, a_2, ...a_N, c_o$ are non-zero constants identified later. The positive integers N can be identified by taking the balance principle between the highest order derivatives and the highest degree of non-linear terms in Eq. (3). For detail see reference [28].

2.2 Algorithm of $(\frac{G'}{G^2})$-expansion method

Suppose that Eq. (3) has the solitary wave solution of the the form for $(\frac{G'}{G^2})$-expansion method

$$u(\tau) = a_0 + \sum_{n=1}^{N}\Bigg(\alpha_n\Big(\frac{G'}{G^2}\Big)^n + \beta_n\Big(\frac{G'}{G^2}\Big)^{-n}\Bigg),$$

where $G = G(\tau)$ holds

$$\Big(\frac{G'}{G^2}\Big)' = \zeta + \varphi\Big(\frac{G'}{G^2}\Big)^2,$$

with $\varphi \neq 0$, $\zeta \neq 1$ being integers. The unknown constants $a_0, \alpha_n, \beta_n (n = 1, 2, 3, \dots, N)$ must be found. The general solution of $(\frac{G'}{G^2})$ has three possibilities as enumerated below:

Case-1: Trigonometric function solutions:

If we take $\zeta\ \varphi > 0$, then

$$\Big(\frac{G'}{G^2}\Big) = \sqrt{\frac{\zeta}{\varphi}}\Bigg(\frac{E\cos\sqrt{\zeta\varphi}\tau + F\sin\sqrt{\zeta\varphi}\tau}{F\cos\sqrt{\zeta\varphi}\tau - E\sin\sqrt{\zeta\varphi}\tau}\Bigg), \tag{6}$$

Case-2: Hyperbolic function solutions:
If we have $\zeta\,\varphi < 0$, then

$$\left(\frac{G'}{G^2}\right) = -\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\tau) + E\cosh(2\sqrt{|\zeta\varphi|}\tau) + F}{E\sinh(2\sqrt{|\zeta\varphi|}\tau) + Ecosh(2\sqrt{|\zeta\varphi|}\tau) - F}\right), \tag{7}$$

Case-3: Rational function solutions:
When $\zeta = 0$, $\varphi \neq 0$ then rational solution can be written as

$$\left(\frac{G'}{G^2}\right) = \left(-\frac{E}{\varphi(E\tau + F)}\right), \tag{8}$$

where E and F are constants. Three types of the solution can be obtained by substituting the values of unknowns a_0, α_n, β_n $(n = 1, 2, 3, \ldots N)$. For detail [29].

3 Applications

For Eq. (1) with constitutive function $p(v) = v - v^3$, will be obtained in this section. By using the traveling wave variable $\tau = x + \alpha t$, Eq. (1) is carried into following ordinary differential system:

$$\alpha u' + (v - v^3)' = \eta u'' - \omega\eta^2 u''' \quad \text{and} \quad \alpha v' - u' = 0, \tag{9}$$

Once integrating of Eq. (9) w.r.t τ, and equating the integration constants to zero yields,

$$\alpha u + (v - v^3) = \eta u' - \omega\eta^2 u'' \quad \text{and} \quad \alpha v - u = 0. \tag{10}$$

3.1 Advanced $\mathrm{e}^{-\phi(\tau)}$-expansion function method

By balancing the highest order derivative and nonlinear term appear in u'' and u^3, as well as for v'' and v^3 yields, $n = 1$ implies the formal solutions:

$$u(\tau) = a_0 + a_1 e^{-\phi(\tau)} \quad \text{and} \quad v(\tau) = b_0 + b_1 e^{-\phi(\tau)}, \tag{11}$$

Substituting Eq. (11) and its derivative into Eq. (10), and the coefficients, with the same power of $-\phi(\tau)$, equating to zero and resultantly we have following set of algebraic equations. By using Mathematica, two clusters of solutions are obtained as follows:

Set-1

$$a_1 = -\frac{2\left(9a_0\eta^2\lambda\mu\omega + \sqrt{3}\right)}{9\eta\lambda}, \quad b_0 = -\frac{1}{\sqrt{3}}\ ,$$
$$b_1 = -\frac{1}{3}\sqrt{2}\sqrt{-3\sqrt{3}a_0\eta^2\mu^2\omega - \frac{\mu}{\lambda}}\ , \quad \alpha = -2\eta^2\lambda\mu\omega\ .$$

Set-2

$$a_1 = \alpha\eta\omega\sqrt{2\alpha\omega}\ , \quad b_0 = -\frac{\sqrt{\alpha\omega}}{3\sqrt{2}\mu}\ , \quad b_1 = \eta\omega\sqrt{2\alpha\omega}\ ,$$
$$\alpha = \frac{\psi \pm \sqrt{\psi^2 - 144\omega\mu^4}}{12\omega\mu^2}\ , \quad \psi = \omega^2\left(\omega - 12\eta^2\lambda\mu^3\right).$$

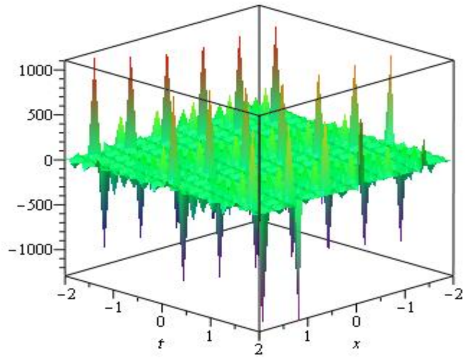
For **Set-1:** When $\lambda\mu > 0$, then we have the following trigonometric function solutions:

$$u_{1,1}(x,t) = a_0 - \frac{2\left(9a_0\eta^2\lambda\mu\omega + \sqrt{3}\right)}{9\eta\lambda}\left(\sqrt{\frac{\lambda}{\mu}}\tan\left(\sqrt{\lambda\mu}\ (\tau + c_o)\right)\right), \tag{12}$$

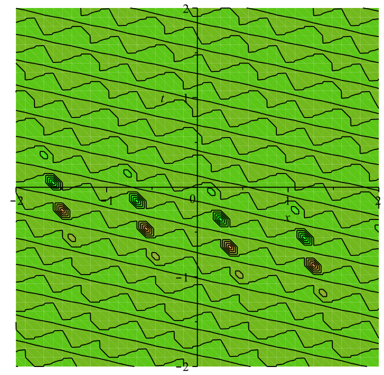
$$v_{1,1}(x,t) = -\frac{1}{\sqrt{3}} - \frac{1}{3}\sqrt{2}\sqrt{-3\sqrt{3}a_0\eta^2\mu^2\omega - \frac{\mu}{\lambda}}\left(\sqrt{\frac{\lambda}{\mu}}\tan\left(\sqrt{\lambda\mu}\ (\tau + c_o)\right)\right), \tag{13}$$

$$u_{1,2}(x,t) = a_0 + \frac{2\left(9a_0\eta^2\lambda\mu\omega + \sqrt{3}\right)}{9\eta\lambda}\left(\sqrt{\frac{\lambda}{\mu}}\cot\left(\sqrt{\lambda\mu}\ (\tau + c_o)\right)\right), \tag{14}$$

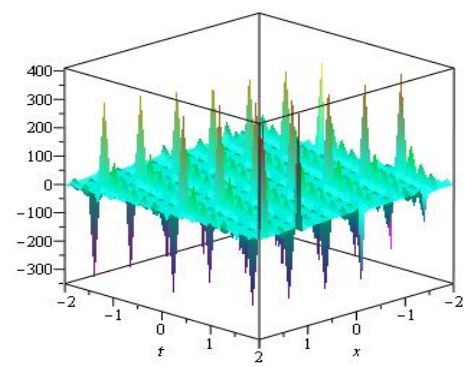
$$v_{1,2}(x,t) = -\frac{1}{\sqrt{3}} + \frac{1}{3}\sqrt{2}\sqrt{-3\sqrt{3}a_0\eta^2\mu^2\omega - \frac{\mu}{\lambda}}\left(\sqrt{\frac{\lambda}{\mu}}\cot\left(\sqrt{\lambda\mu}\ (\tau + c_o)\right)\right). \tag{15}$$



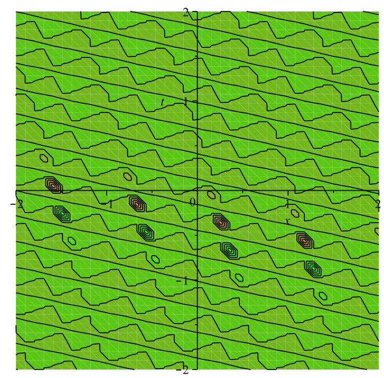
(a) $u_{1,1}(x,t)$



(b) $u_{1,1}(x,t)$



(c) $v_{1,1}(x,t)$



(d) $v_{1,1}(x,t)$

Figure 1: Graphics of the solution equations $u_{1,1}(x,t)$ and $v_{1,1}(x,t)$ for $a_0 = 2,\ \omega = 2,\ \eta = 3\ , \lambda = 3,\ \mu = 2$ and $c_0 = 2$.

For **Set-1:** When $\lambda\mu < 0$ then, we have the following hyperbolic function solutions:

$$u_{1,3}(x,t) = a_0 - \frac{2\left(9a_0\eta^2\lambda\mu\omega + \sqrt{3}\right)}{9\eta\lambda}\left(\sqrt{\frac{\lambda}{-\mu}}\tanh\left(\sqrt{-\lambda\mu}\ (\tau + c_o)\right)\right), \tag{16}$$

$$v_{1,3}(x,t) = -\frac{1}{\sqrt{3}} - \frac{1}{3}\sqrt{2}\sqrt{-3\sqrt{3}a_0\eta^2\mu^2\omega - \frac{\mu}{\lambda}}\left(\sqrt{\frac{\lambda}{-\mu}}\tanh\left(\sqrt{-\lambda\mu}\ (\tau + c_0)\right)\right), \tag{17}$$

$$u_{1,4}(x,t) = a_0 - \frac{2\left(9a_0\eta^2\lambda\mu\omega + \sqrt{3}\right)}{9\eta\lambda}\left(\sqrt{\frac{\lambda}{-\mu}}\coth\left(\sqrt{-\lambda\mu}\ (\tau + c_o)\right)\right), \tag{18}$$

$$v_{1,4}(x,t) = -\frac{1}{\sqrt{3}} - \frac{1}{3}\sqrt{2}\sqrt{-3\sqrt{3}a_0\eta^2\mu^2\omega - \frac{\mu}{\lambda}}\left(\sqrt{\frac{\lambda}{-\mu}}\coth\left(\sqrt{-\lambda\mu}\ (\tau + c_o)\right)\right). \tag{19}$$

For **Set-1:** When $\lambda \in \Re,\ \mu = 0$ then, we have the following rational function solution:

$$u_{1,5}(x,t) = a_0 - \frac{2\sqrt{3}}{9\eta\lambda}\Big(\lambda(\tau + c_0)\Big), \tag{20}$$

For **Set-2:** When $\lambda\mu > 0$, then we have the following trigonometric function solutions:

$$u_{2,1}(x,t) = a_0 + \alpha\eta\omega\sqrt{2\alpha\omega}\left(\sqrt{\frac{\lambda}{\mu}}\tan\left(\sqrt{\lambda\mu}\ (\tau + c_o)\right)\right), \tag{21}$$

$$v_{2,1}(x,t) = -\frac{\sqrt{\alpha\omega}}{3\sqrt{2}\mu} + \eta\omega\sqrt{2\alpha\omega}\left(\sqrt{\frac{\lambda}{\mu}}\tan\left(\sqrt{\lambda\mu}\ (\tau + c_o)\right)\right), \tag{22}$$

$$u_{2,2}(x,t) = a_0 - \alpha\eta\omega\sqrt{2\alpha\omega}\left(\sqrt{\frac{\lambda}{\mu}}\cot\left(\sqrt{\lambda\mu}\ (\tau + c_o)\right)\right), \tag{23}$$

$$v_{2,2}(x,t) = -\frac{\sqrt{\alpha\omega}}{3\sqrt{2}\mu} - \eta\omega\sqrt{2\alpha\omega}\left(\sqrt{\frac{\lambda}{\mu}}\cot\left(\sqrt{\lambda\mu}\ (\tau + c_o)\right)\right). \tag{24}$$

For **Set-2:** When $\lambda\mu < 0$ then, we have the following hyperbolic function solutions:

$$u_{2,3}(x,t) = a_0 + \alpha\eta\omega\sqrt{2\alpha\omega}\left(\sqrt{\frac{\lambda}{-\mu}}\tanh\left(\sqrt{-\lambda\mu}\ (\tau + c_o)\right)\right), \tag{25}$$

$$v_{2,3}(x,t) = -\frac{\sqrt{\alpha\omega}}{3\sqrt{2}\mu} + \eta\omega\sqrt{2\alpha\omega}\left(\sqrt{\frac{\lambda}{-\mu}}\tanh\left(\sqrt{-\lambda\mu}\ (\tau + c_o)\right)\right), \tag{26}$$

$$u_{2,4}(x,t) = a_0 + \alpha\eta\omega\sqrt{2\alpha\omega}\left(\sqrt{\frac{\lambda}{-\mu}}\coth\left(\sqrt{-\lambda\mu}\ (\tau + c_o)\right)\right), \tag{27}$$

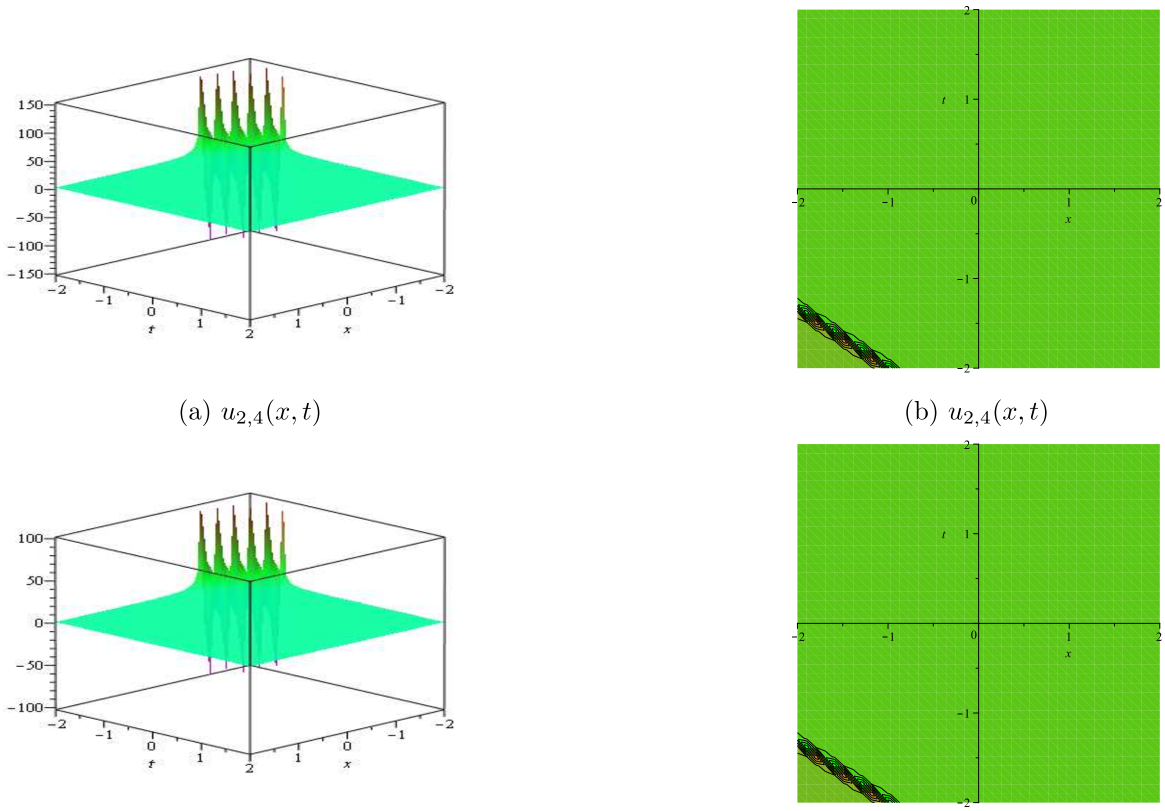
$$v_{2,4}(x,t) = -\frac{\sqrt{\alpha\omega}}{3\sqrt{2}\mu} + \eta\omega\sqrt{2\alpha\omega}\left(\sqrt{\frac{\lambda}{-\mu}}\coth\left(\sqrt{-\lambda\mu}\ (\tau + c_o)\right)\right). \tag{28}$$

For **Set-1:** When $\lambda = 0,\ \mu > 0$ then, we have the following rational function solution:

$$u_{2,5}(x,t) = a_0 + \alpha\eta\omega\sqrt{2\alpha\omega}\left(-\frac{1}{\mu(\tau + c_o)}\right), \tag{29}$$

$$v_{2,5}(x,t) = -\frac{\sqrt{\alpha\omega}}{3\sqrt{2}\mu} + \eta\omega\sqrt{2\alpha\omega}\left(-\frac{1}{\mu(\tau + c_o)}\right), \tag{30}$$

Where $\quad \alpha = \frac{\psi \pm \sqrt{\psi^2 - 144\omega\mu^4}}{12\omega\mu^2} \quad$ and $\quad \psi = \omega^2\left(\omega - 12\eta^2\lambda\mu^3\right).$



(a) $u_{2,4}(x,t)$ (b) $u_{2,4}(x,t)$

(c) $v_{2,4}(x,t)$ (d) $v_{2,4}(x,t)$

Figure 2: Graphics of the solution equations $u_{2,4}(x,t)$ and $v_{2,4}(x,t)$ for $a_0 = 1,\ \omega = 1,\ \eta = 2\ ,\lambda = 1,\ \mu = -1$ and $c_0 = 4$.

3.2 $(\frac{G'}{G^2})$-expansion method

Balancing the highest order derivative and nonlinear term appear in u'' and u^3, as well as for v'' and v^3 yields, $n = 1$ implies the formal solutions:

$$u(\tau) = a_0 + a_1\left(\frac{G'}{G^2}\right) + b_1\left(\frac{G'}{G^2}\right)^{-1} \quad \text{and} \quad v(\tau) = c_0 + c_1\left(\frac{G'}{G^2}\right) + d_1\left(\frac{G'}{G^2}\right)^{-1}, \tag{31}$$

Substituting Eq. (31) and its derivative into Eq. (10), and equating the coefficients, with the same power of $(\frac{G'}{G^2})$, to zero and resultantly we get the following set of algebraic equations. By using Mathematica, three clusters of solutions as obtained as follows:

Set-1

$$a_0 = 18c_0^3\omega\left(\frac{8c_0^2 - 2}{c_0^2(36\delta - 3) + 1} + 3\right)\ ,\ a_1 = -108c_0^3\eta\varphi\omega^2\ ,$$
$$b_1 = 0\ ,\quad c_1 = -6c_0\eta\varphi\omega\ ,\quad d_1 = 0\ ,\quad \alpha = \frac{-36c_0^2\delta + 3c_0^2 - 1}{18c_0^2\omega}.$$

Set-2

$$a_0 = 18c_0^3\omega\left(\frac{8c_0^2-2}{c_0^2(36\delta-3)+1}+3\right)\ ,\ a_1=0\ ,\ b_1=108c_0^3\zeta\eta\omega^2,$$
$$c_1=0\ ,\quad d_1=6c_0\zeta\eta\omega\ ,\quad \alpha=\frac{-36c_0^2\delta+3c_0^2-1}{18c_0^2\omega}.$$

Set-3

$$a_0=\frac{18c_0^3\omega\left(c_0^2(432\delta-1)+1\right)}{3c_0^2(48\delta\omega-1)+1}\ ,\ a_1=-108c_0^3\eta\varphi\omega^2,\ b_1=108c_0^3\zeta\eta\omega^2,$$
$$c_1=-6c_0\eta\varphi\omega,\quad d_1=6c_0\zeta\eta\omega,\quad \alpha=\frac{-144c_0^2\delta+3c_0^2-1}{18c_0^2\omega},$$
$$where \qquad \delta=\zeta\eta^2\varphi\omega^2.$$

For **Set-1** When $\zeta\varphi>0$, the trigonometric solution can be expressed as:

$$u_{1,1}(x,t)=18c_0^3\omega\left(\frac{8c_0^2-2}{c_0^2(36\delta-3)+1}+3\right)-108c_0^3\eta\varphi\omega^2\left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\ \tau)+F\sin(\sqrt{\zeta\varphi}\ \tau)}{F\cos(\sqrt{\zeta\varphi}\tau)-E\sin(\sqrt{\zeta\varphi}\ \tau)}\right)\right), \tag{32}$$

$$v_{1,1}(x,t)=c_0-6c_0\eta\varphi\omega\times\left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\ \tau)+F\sin(\sqrt{\zeta\varphi}\ \tau)}{F\cos(\sqrt{\zeta\varphi}\tau)-E\sin(\sqrt{\zeta\varphi}\ \tau)}\right)\right). \tag{33}$$

For **Set-1** When $\zeta\varphi<0$, the hyperbolic solution can be expressed as:

$$u_{1,2}(x,t)=18c_0^3\omega\left(\frac{8c_0^2-2}{c_0^2(36\delta-3)+1}+3\right)-108c_0^3\eta\varphi\omega^2 \times\left(-\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)+F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)-F}\right)\right), \tag{34}$$

$$v_{1,2}(x,t)=c_0-6c_0\eta\varphi\omega\left(-\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)+F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)-F}\right)\right), \tag{35}$$

For soliton solution, take $E=F$, we get singular kink-shaped soliton solution as:

$$u_{1,2}(x,t)=18c_0^3\omega\left(\frac{8c_0^2-2}{c_0^2(36\delta-3)+1}+3\right)-108c_0^3\eta\varphi\omega^2\left(\coth(\sqrt{|\zeta\varphi|}\ \tau)\right), \tag{36}$$

$$v_{1,2}(x,t)=c_0-6c_0\eta\varphi\omega\left(\coth(\sqrt{|\zeta\varphi|}\ \tau)\right). \tag{37}$$

For **Set-2** When $\zeta\varphi>0$, the trigonometric solution can be expressed

$$u_{2,1}(x,t)=18c_0^3\omega\left(\frac{8c_0^2-2}{c_0^2(36\delta-3)+1}+3\right)+108c_0^3\zeta\eta\omega^2 \times\left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\ \tau)+F\sin(\sqrt{\zeta\varphi}\ \tau)}{F\cos(\sqrt{\zeta\varphi}\tau)-E\sin(\sqrt{\zeta\varphi}\ \tau)}\right)\right)^{-1}, \tag{38}$$

$$v_{2,1}(x,t)=c_0+6c_0\zeta\eta\omega\times\left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\ \tau)+F\sin(\sqrt{\zeta\varphi}\ \tau)}{F\cos(\sqrt{\zeta\varphi}\tau)-E\sin(\sqrt{\zeta\varphi}\ \tau)}\right)\right)^{-1}. \tag{39}$$

For **Set-2** When $\zeta\varphi<0$, the hyperbolic solution can be expressed as:

$$u_{2,2}(x,t)=18c_0^3\omega\left(\frac{8c_0^2-2}{c_0^2(36\delta-3)+1}+3\right)+108c_0^3\zeta\eta\omega^2 \times\left(-\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)+F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)-F}\right)\right)^{-1}, \tag{40}$$

$$v_{2,2}(x,t)=c_0+6c_0\zeta\eta\omega\left(-\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)+F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)-F}\right)\right)^{-1}, \tag{41}$$

For soliton solution, take $E=F$, we get kink-shaped soliton as:

$$u_{2,2}(x,t)=8c_0^3\omega\left(\frac{8c_0^2-2}{c_0^2(36\delta-3)+1}+3\right)+108c_0^3\zeta\eta\omega^2\left(\tanh(\sqrt{|\zeta\varphi|}\ \tau)\right), \tag{42}$$

$$v_{2,2}(x,t)=c_0+6c_0\zeta\eta\omega\left(\tanh(\sqrt{|\zeta\varphi|}\ \tau)\right). \tag{43}$$

For **Set-3** When $\zeta\varphi>0$, the trigonometric solution can be written as:

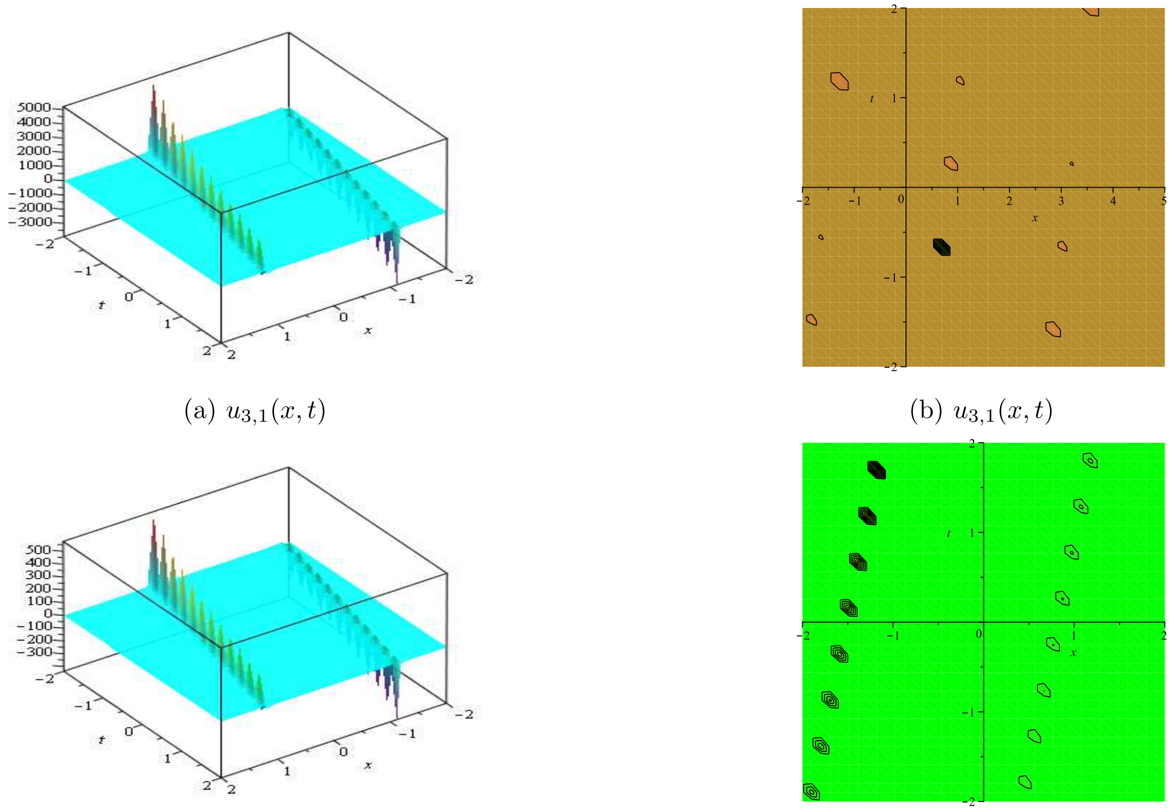
$$u_{3,1}(x,t)=\frac{18c_0^3\omega\left(c_0^2(432\delta-1)+1\right)}{3c_0^2(48\delta\omega-1)+1}-108c_0^3\eta\varphi\omega^2 \times\left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\ \tau)+F\sin(\sqrt{\zeta\varphi}\ \tau)}{F\cos(\sqrt{\zeta\varphi}\ \tau)-E\sin(\sqrt{\zeta\varphi}\ \tau)}\right)\right)+108c_0^3\zeta\eta\omega^2 \times\left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\ \tau)+F\sin(\sqrt{\zeta\varphi}\ \tau)}{F\cos(\sqrt{\zeta\varphi}\ \tau)-E\sin(\sqrt{\zeta\varphi}\ \tau)}\right)\right)^{-1}, \tag{44}$$

$$v_{3,1}(x,t)=c_0-6c_0\eta\varphi\omega\left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\ \tau)+F\sin(\sqrt{\zeta\varphi}\ \tau)}{F\cos\sqrt{\zeta\varphi}\ \tau)-E\sin(\sqrt{\zeta\varphi}\ \tau)}\right)\right) +6c_0\eta\zeta\omega\left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\ \tau)+F\sin(\sqrt{\zeta\varphi}\ \tau)}{F\cos(\sqrt{\zeta\varphi}\ \tau)-E\sin(\sqrt{\zeta\varphi}\ \tau)}\right)\right)^{-1}, \tag{45}$$

By choosing, $E=F$, we have periodic solution as:

$$u_{3,1}(x,t)=\frac{18c_0^3\omega\left(c_0^2(432\delta-1)+1\right)}{3c_0^2(48\delta\omega-1)+1}-216c_0^3\eta\sqrt{\zeta\varphi}\omega^2\left(\tan(2\sqrt{\zeta\varphi}\ \tau)\right), \tag{46}$$

$$v_{3,1}(x,t)=c_0-12c_0\eta\sqrt{\zeta\varphi}\omega\left(\tan(2\sqrt{\zeta\varphi}\ \tau)\right). \tag{47}$$



(a) $u_{3,1}(x,t)$ (b) $u_{3,1}(x,t)$

(c) $v_{3,1}(x,t)$ (d) $v_{3,1}(x,t)$

Figure 3: Graphics of the solution equations $u_{3,1}(x,t)$ and $v_{3,1}(x,t)$ for $E=F,\ \omega=1,\ \eta=1.5\ ,\zeta=1.2,\ \varphi=1.5$ and $c_0=0.5$.

For **Set-3** When $\zeta\varphi<0$, the hyperbolic solution can be written as:

$$u_{3,2}(x,t)=\frac{18c_0^3\omega\left(c_0^2(432\delta-1)+1\right)}{3c_0^2(48\delta\omega-1)+1}-108c_0^3\eta\varphi\omega^2 \times\Bigg(-\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)+F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)-F}\right)+108c_0^3\zeta\eta\omega^2 \times\Bigg(-\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)+F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)-F}\right)\Bigg)\Bigg)^{-1}, \quad (48)$$

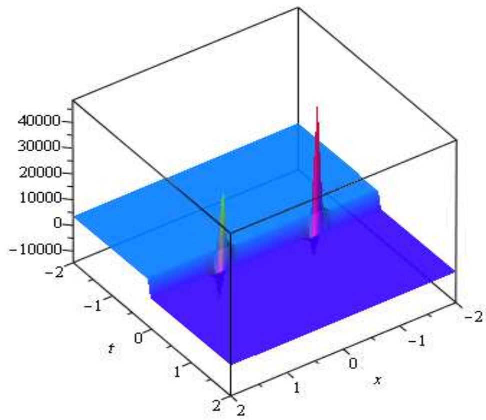
$$v_{3,2}(x,t)=c_0-6c_0\eta\varphi\omega\Bigg(-\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)+F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)-F}\right) +6c_0\eta\zeta\omega\times\Bigg(-\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)+F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \tau)+E\cosh(2\sqrt{|\zeta\varphi|}\ \tau)-F}\right)\Bigg)\Bigg)^{-1}, \quad (49)$$

For soliton solution, take $E=F$, we get singular kink-shaped soliton solution as:

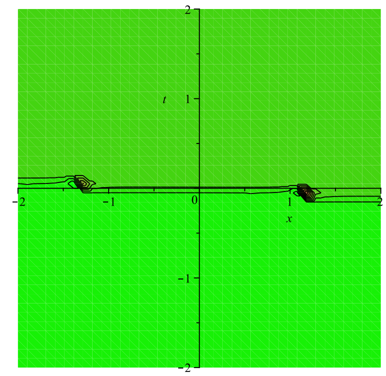
$$u_{3,2}(x,t)=\frac{18c_0^3\omega\left(c_0^2(432\delta-1)+1\right)}{3c_0^2(48\delta\omega-1)+1}-216c_0^3\eta\sqrt{\zeta\varphi}\omega^2\Big(\tanh(\sqrt{|\zeta\varphi|}\ \tau)-\coth(\sqrt{|\zeta\varphi|}\ \tau)\Big), \quad (50)$$

$$v_{3,2}(x,t)=c_0-12c_0\eta\sqrt{\zeta\varphi}\omega\Big(\tanh(\sqrt{|\zeta\varphi|}\ \tau)-\coth(\sqrt{|\zeta\varphi|}\ \tau)\Big), \quad (51)$$

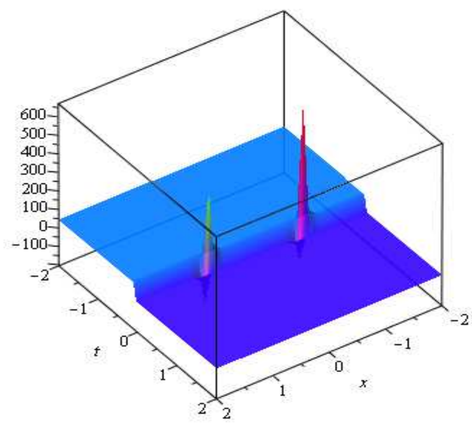
where, $\tau = (x + \alpha t).$



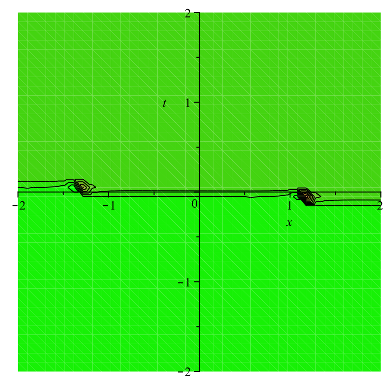
(a) $u_{3,2}(x,t)$



(b) $u_{3,2}(x,t)$



(c) $v_{3,2}(x,t)$



(d) $v_{3,2}(x,t)$

Figure 4: Graphics of the solution equations $u_{3,2}(x,t)$ and $v_{3,2}(x,t)$ for $E = F,\ \omega = 1,\ \eta = 2\ ,\zeta = -2,\ \varphi = 1.5$ and $c_0 = 2$.

4 Conclusion

In this work, we have investigated diverse traveling wave solutions like, hyperbolic (kink and singular kink-shaped soliton), trigonometric as well as rational function solutions to van der waals gas equation via $(\frac{G'}{G^2})$ and advanced $\mathrm{e}^{-\phi(\tau)}$-expansion function schemes. These various kinds of the solutions are favourable for explaining diverse non-linear physical phenomena. The calculations also reveals us the importance of these methods to achieve the solutions in a more broad manner. We plot some of our obtained solutions Figures [1-4] show the solitary and contour profiles of these solutions. The earned solitary solutions discuss the physical features of this model. At least, the study describes that the applied methods are very, reliable, consistent, efficient, and much more practical to obtain the exact solitary wave solutions for complicated PDE,s in many fields like engineering, mathematical biology, physics, chemistry and vice versa.

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