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LETTER

Corrigendum to "*Existence and exponential decay for a nonlinear wave equation with nonlocal boundary conditions of 2N-point type*" [Math Meth Appl Sci. https://onlinelibrary.wiley.com/doi/10.1002/mma.6773]

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Abstract

This note presents corrections to our paper[1].

KEYWORDS:
Nonlinear wave equation, boundary conditions of $2N-$ point type, local existence, global existence, exponential decay

CORRIGENDUM TO THE PROOF OF THEOREM 4.1

In this section, we present corrections to Proof of Theorem 4.1 related to Lemma 4.3 in[1] (pp. 16), by the fact that Lemma 4.3 is wrong with the functional $I(t)$ defined by (74) of[1].

We first give corrections to the functionals $E(t)$, $J(t)$, $I(t)$ as in (74), (75) below. Next, we present corrections to the proof of Theorem 4.1 related to Lemma 4.3.

Let k, $\bar{\lambda} \in \mathbb{R}$, with $k > 0$ and $0 < \bar{\lambda} < \lambda$. Consider $g(t) = 2\bar{\lambda}e^{-2kt}$ $(t \geq 0)$ and $(g * u')(t) = \int_0^t g(t-s)\,\|u'(s)\|^2\,ds$, with $u \in C^1\left(\mathbb{R}_+; L^2\right)$.

We construct the following Lyapunov functional

$$\mathcal{L}(t) = E(t) + \delta\psi(t), \tag{73}$$

where

$$
\begin{aligned}
E(t) =& \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}(g * u')(t) + \frac{1}{2}\|u(t)\|_a^2 - \frac{1}{p}\|u(t)\|_{L^p}^p \\
=& \frac{1}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\left[(g * u')(t) + \|u(t)\|_a^2\right] + \frac{1}{p}I(t) \\
=& \frac{1}{2}\left\|u'(t)\right\|^2 + J(t), \\
J(t) =& \left(\frac{1}{2} - \frac{1}{p}\right)\left[(g * u')(t) + \|u(t)\|_a^2\right] + \frac{1}{p}I(t), \\
I(t) =& I(u(t)) = (g * u')(t) + \|u(t)\|_a^2 - \|u(t)\|_{L^p}^p,
\end{aligned}
\tag{74}
$$

$$
\psi(t) = \langle u(t), u'(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2 + \frac{\lambda_0}{2}u^2(0,t) + \frac{\lambda_1}{2}u^2(1,t), \tag{75}
$$

where $\delta > 0$ is chosen later.

Then we have the following theorem.

Theorem 4.1. *Assume that $(A_1^\infty) - (A_3^\infty)$ hold. Let $I(0) > 0$ and the initial energy $E(0)$ satisfy*

$$
\gamma^* = \frac{C_p^p}{\sqrt{\bar{\alpha}_1^p}} R_*^{p-2} < 1, \tag{76}
$$

where

$$
\begin{aligned}
R_*^2 =& \frac{2pr}{p-2}\left(E(0) + \frac{1}{2\varepsilon_0}\int_0^\infty \rho(s)ds\right), \\
\varepsilon_0 =& \min\{\lambda, \frac{1}{4}\mu_*\},\ r = \exp\left[\frac{1}{\varepsilon_0\bar{\alpha}_1}\frac{2p}{p-2}\sum\nolimits_{i=0}^1 \left\|\bar{H}_i\right\|_{L^2(\mathbb{R}_+)}^2\right], \\
\bar{H}_0(t) =& \left|\bar{h}_0(t)\right| + \sum\nolimits_{i=1}^{N-1}\left|\bar{h}_{0i}(t)\right|,\ \bar{H}_1(t) = \left|\bar{h}_1(t)\right| + \sum\nolimits_{i=1}^{N-1}\left|\bar{h}_{1i}(t)\right|, \\
\rho(t) =& \|F(t)\|^2 + g_0^2(t) + g_1^2(t),
\end{aligned}
\tag{77}
$$

and C_p is a constant satisfying the inequality $\|v\| \le C_p\|v\|_{H^1}$, for all $v \in H^1$.

Then, for $\bar{\lambda}_0^2$, $\bar{\lambda}_1^2$, $\left\|\bar{H}_0\right\|_{L^\infty(\mathbb{R}_+)}$, $\left\|\bar{H}_1\right\|_{L^\infty(\mathbb{R}_+)}$ sufficiently small, there exist positive constants C, γ such that

$$
E(t) \le C\exp(-\gamma t), \text{ for all } t \ge 0.
$$

Proof of Theorem 4.1.

First, we need the following lemmas.

Lemma 4.2. *Suppose that the assumptions of Theorem 4.1 hold. Then, $E(t)$ satisfies*

$$
\begin{aligned}
E'(t) \le& -\left(\lambda - \bar{\lambda} - \frac{1}{2}\varepsilon_1\right)\left\|u'(t)\right\|^2 - k(g * u')(t) - \left(\frac{1}{2}\mu_* - \varepsilon_1\right)\left(\left|u'(0,t)\right|^2 + \left|u'(1,t)\right|^2\right) \\
&+ \frac{1}{\varepsilon_1\bar{\alpha}_1}\sum\nolimits_{i=0}^1 \bar{H}_i^2(t)\|u(t)\|_a^2 + \frac{1}{2\varepsilon_1}\rho(t),
\end{aligned}
\tag{78}
$$

for all $\varepsilon_1 > 0$. Proof of Lemma 4.2. Multiplying (1) by $u'(x,t)$ and integrating over $[0,1]$, we get

$$
\begin{aligned}
E'(t) =& -\left(\lambda - \bar{\lambda}\right)\left\|u'(t)\right\|^2 - k(g * u')(t) \\
&- \left[\lambda_0\left|u'(0,t)\right|^2 + \lambda_1\left|u'(1,t)\right|^2 + \left(\bar{\lambda}_0 + \bar{\lambda}_1\right)u'(0,t)u'(1,t)\right] - \bar{a}_0(t; u(t), u'(t)) \\
&- \bar{a}_1(t; u(t), u'(t)) - g_0(t)u'(0,t) - g_1(t)u'(1,t) + \left\langle F(t), u'(t)\right\rangle,
\end{aligned}
\tag{79}
$$

where

$$
\begin{aligned}
\bar{a}_0(t; u, v) =& \left(\bar{h}_0(t)u(1) + \sum\nolimits_{i=1}^{N-1}\bar{h}_{0i}(t)u(\eta_i)\right)v(0), \\
\bar{a}_1(t; u, v) =& \left(\bar{h}_1(t)u(0) + \sum\nolimits_{i=1}^{N-1}\bar{h}_{1i}(t)u(\theta_i)\right)v(1), \text{ for all } u, v \in H^1,\ 0 \le t \le T.
\end{aligned}
$$

By Lemma 2.3, we have

$$
\lambda_0\left|u'(0,t)\right|^2 + \lambda_1\left|u'(1,t)\right|^2 + \left(\bar{\lambda}_0 + \bar{\lambda}_1\right)u'(0,t)u'(1,t) \ge \frac{1}{2}\mu_*\left[\left|u'(0,t)\right|^2 + \left|u'(1,t)\right|^2\right].
$$

On the other hand

$$\begin{aligned}
-\bar{a}_0(t;u(t),u'(t)) &= -\left(\bar{h}_0(t)u(1,t)+\sum\nolimits_{i=1}^{N-1}\bar{h}_{0i}(t)u(\eta_i(t),t)\right)u'(0,t)\\
&\leq \frac{1}{2}\varepsilon_1\left|u'(0,t)\right|^2+\frac{1}{\varepsilon_1\bar{\alpha}_1}\bar{H}_0^2(t)\left\|u(t)\right\|_a^2;\\
-\bar{a}_1(t;u(t),u'(t)) &= -\left(\bar{h}_1(t)u(0,t)+\sum\nolimits_{i=1}^{N-1}\bar{h}_{1i}(t)u(\theta_i(t),t)\right)u'(1,t)\\
&\leq \frac{1}{2}\varepsilon_1\left|u'(1,t)\right|^2+\frac{1}{\varepsilon_1\bar{\alpha}_1}\bar{H}_1^2(t)\left\|u(t)\right\|_a^2;\\
-g_0(t)u'(0,t) &\leq \frac{1}{2}\varepsilon_1\left|u'(0,t)\right|^2+\frac{1}{2\varepsilon_1}g_0^2(t);\\
-g_1(t)u'(1,t) &\leq \frac{1}{2}\varepsilon_1\left|u'(1,t)\right|^2+\frac{1}{2\varepsilon_1}g_1^2(t);\\
\left\langle F(t),u'(t)\right\rangle &\leq \frac{1}{2}\varepsilon_1\left\|u'(t)\right\|^2+\frac{1}{2\varepsilon_1}\left\|F(t)\right\|^2.
\end{aligned} \tag{80}$$

Combining (79), (80), it is clear that (78) holds. Lemma 4.2 is proved. □

Lemma 4.3. *Suppose that the assumptions of Theorem 4.1 hold. Then,* $I(t)>0$, $\forall t\geq 0$.

Proof of Lemma 4.3. By the continuity of $I(t)$ and $I(0)>0$, there exists $T_1>0$ such that

$$I(t)>0,\ \ \forall t\in[0,T_1], \tag{81}$$

this implies

$$J(t)=\frac{p-2}{2p}\left[(g*u')(t)+\|u(t)\|_a^2\right]+\frac{1}{p}I(t)\geq\frac{p-2}{2p}\|u(t)\|_a^2,\ \ \forall t\in[0,T_1]. \tag{82}$$

It follows from (81), (82) that

$$\|u(t)\|_a^2\leq\frac{2p}{p-2}J(t)\leq\frac{2p}{p-2}E(t),\ \ \forall t\in[0,T_1]. \tag{83}$$

Combining (78), (83) with $\varepsilon_1=\varepsilon_0\equiv\min\{\lambda,\frac{1}{4}\mu_*\}$ and using Gronwall's inequality we have

$$\|u(t)\|_a^2\leq\frac{2p}{p-2}E(t)\leq\frac{2pr}{p-2}\left(E(0)+\frac{1}{2\varepsilon_0}\int_0^\infty\rho(s)ds\right)\equiv R_*^2,\ \forall t\in[0,T_1], \tag{84}$$

where r as in (77).

Hence, it follows from (76), (84) that

$$\begin{aligned}
\|u(t)\|_{L^p}^p &\leq C_p^p\|u(t)\|_{H^1}^p\leq C_p^p\left(\frac{1}{\bar{\alpha}_1}\|u(t)\|_a^2\right)^{p/2}\\
&= \frac{C_p^p}{\sqrt{\bar{\alpha}_1^p}}\|u(t)\|_a^{p-2}\|u(t)\|_a^2\leq\gamma^*\|u(t)\|_a^2,\ \forall t\in[0,T_1].
\end{aligned} \tag{85}$$

Therefore $I(t)\geq(g*u')(t)+(1-\gamma^*)\|u(t)\|_a^2\geq 0$, $\forall t\in[0,T_1]$.

Now, we put $T_*=\sup\{T>0: I(u(t))>0,\ \forall t\in[0,T]\}$. If $T_*<+\infty$ then, by the continuity of $I(t)$, we have $I(T_*)\geq 0$.

If $I(T_*)>0$, by the same arguments as in the above part we can deduce that there exists $T_*'>T_*$ such that $I(t)>0$, $\forall t\in[0,T_*']$. We obtain a contradiction to the definition of T_*.

If $I(T_*)=0$, it follows that

$$0=I(T_*)\geq(g*u')(T_*)+(1-\gamma^*)\left\|u(T_*)\right\|_a^2\geq 0.$$

Therefore

$$u(T_*)=(g*u')(T_*)=0.$$

By the fact that the function $s\mapsto g(T_*-s)\|u'(s)\|^2$ is continuous on $[0,T_*]$ and $g(T_*-s)>0$, $\forall s\in[0,T_*]$, we have

$$(g*u)(T_*)=\int_0^{T_*}g(T_*-s)\left\|u'(s)\right\|^2ds=0,$$

it follows that $\|u'(s)\|=0$, $\forall s\in[0,T_*]$, it means that u is a constant function on $[0,T_*]$. Then, $u(0)=u(T_*)=0$. It leads to $I(0)=0$. We get in contradiction with $I(0)>0$.

Consequently, $T_*=+\infty$, i.e. $I(t)>0$, $\forall t\geq 0$. Lemma 4.3 is proved completely. □

Lemma 4.4. *Suppose that the assumptions of Theorem 4.1 hold. Then, there exist positive constants β_1, β_2 such that*

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t),\ \forall t \ge 0, \tag{86}$$

for δ is small enough.

Proof of Lemma 4.4. It is obviously that

$$\begin{aligned}
\mathcal{L}(t) \le\ & \frac{1+\delta}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2}-\frac{1}{p}\right)\left[(g*u')(t)+\|u(t)\|_a^2\right] + \frac{1}{p}I(t) \\
& + \frac{\delta}{2}\left[1+\lambda+2\left(\lambda_0+\lambda_1\right)\right]\|u(t)\|_{H^1}^2 \\
\le\ & \frac{1+\delta}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2}-\frac{1}{p}\right)(g*u')(t) + \frac{1}{p}I(t) \\
& + \left\{\left(\frac{1}{2}-\frac{1}{p}\right) + \frac{\delta}{2\bar{\alpha}_1}\left[1+\lambda+2\left(\lambda_0+\lambda_1\right)\right]\right\}\|u(t)\|_a^2 \\
=\ & (1+\delta)\frac{1}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2}-\frac{1}{p}\right)(g*u')(t) + \frac{1}{p}I(t) \\
& + \left(\frac{1}{2}-\frac{1}{p}\right)\left\{1 + \frac{\delta}{2\bar{\alpha}_1\left(\frac{1}{2}-\frac{1}{p}\right)}\left[1+\lambda+2\left(\lambda_0+\lambda_1\right)\right]\right\}\|u(t)\|_a^2 \\
\le\ & \beta_2 E(t),
\end{aligned}$$

where $\beta_2 = 1+\delta+\frac{\delta p}{\bar{\alpha}_1(p-2)}\left[1+\lambda+2\left(\lambda_0+\lambda_1\right)\right]$.

Similarly, we can prove that

$$\begin{aligned}
\mathcal{L}(t) \ge\ & \frac{1}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2}-\frac{1}{p}\right)\left[(g*u')(t)+\|u(t)\|_a^2\right] + \frac{1}{p}I(t) + \delta\langle u(t), u'(t)\rangle \\
\ge\ & \left(\frac{1}{2}-\frac{\delta}{2}\right)\left\|u'(t)\right\|^2 + \left(\frac{1}{2}-\frac{1}{p}\right)(g*u')(t) \\
& + \left(\frac{1}{2}-\frac{1}{p}-\frac{\delta}{2\bar{\alpha}_1}\right)\|u(t)\|_a^2 + \frac{1}{p}I(t) - \frac{\delta}{2\bar{\alpha}_1}\|u(t)\|_a^2 \\
\ge\ & \frac{1}{2}(1-\delta)\left\|u'(t)\right\|^2 + \left(\frac{1}{2}-\frac{1}{p}\right)(g*u')(t) \\
& + \left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\frac{\delta p}{(p-2)\bar{\alpha}_1}\right)\|u(t)\|_a^2 + \frac{1}{p}I(t) \\
\ge\ & \beta_1 E(t),
\end{aligned}$$

where $\beta_1 = \min\left\{1-\delta,\ 1-\frac{\delta p}{(p-2)\bar{\alpha}_1}\right\} > 0,$ and δ is small enough such that $0 < \delta < \min\{1; (1-\frac{2}{p})\bar{\alpha}_1\}$. Lemma 4.4 is proved completely. □

Lemma 4.5. *Suppose that the assumptions of Theorem 4.1 hold. Then, $\psi(t)$ satisfies*

$$\begin{aligned}
\psi'(t) \le\ & \left\|u'(t)\right\|^2 + (g*u')(t) - \frac{1}{2}I(t) + \frac{\varepsilon_2}{2}\left(\left|u'(0,t)\right|^2 + \left|u'(1,t)\right|^2\right) \\
& - \left\{\frac{1-\gamma^*}{2} - \frac{2}{\bar{\alpha}_1}\left[\varepsilon_2 + \frac{1}{2\varepsilon_2}\left(\bar{\lambda}_0^2+\bar{\lambda}_1^2\right) + \sum\nolimits_{i=0}^{1}\left\|\bar{H}_i\right\|_{L^\infty(\mathbb{R}_+)}\right]\right\}\|u(t)\|_a^2 \\
& + \frac{1}{\varepsilon_2}\rho(t),
\end{aligned} \tag{87}$$

for all $\varepsilon_1 > 0$.

Proof of Lemma 4.5. By multiplying (1) by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\psi'(t) &= \left\|u'(t)\right\|^2 + \|u(t)\|_{L^p}^p - \|u(t)\|_a^2 - \bar{a}_0(t;u(t),u(t)) - \bar{a}_1(t;u(t),u(t)) \\ &\quad - \bar{\lambda}_0 u'(1,t)u(0,t) - \bar{\lambda}_1 u'(0,t)u(1,t) - g_0(t)u(0,t) - g_1(t)u(1,t) + \langle F(t),u(t)\rangle \\ &= \left\|u'(t)\right\|^2 + (g * u')(t) - \frac{1}{2}I(t) - \frac{1}{2}I(t) \\ &\quad - \bar{a}_0(t;u(t),u(t)) - \bar{a}_1(t;u(t),u(t)) - \bar{\lambda}_0 u'(1,t)u(0,t) - \bar{\lambda}_1 u'(0,t)u(1,t) \\ &\quad - g_0(t)u(0,t) - g_1(t)u(1,t) + \langle F(t),u(t)\rangle.\end{aligned} \tag{88}$$

We have the following inequalities

$$\begin{aligned} I(t) &\geq (1-\gamma^*)\,\|u(t)\|_a^2; \\ \bar{\alpha}_1\,\|u(t)\|_{H^1}^2 &\leq \|u(t)\|_a^2 \leq \bar{\alpha}_2\,\|u(t)\|_{H^1}^2; \\ -\bar{a}_0(t;u(t),u(t)) &\leq 2\bar{H}_0(t)\,\|u(t)\|_{H^1}^2 \leq \frac{2}{\bar{\alpha}_1}\left\|\bar{H}_0\right\|_{L^\infty(\mathbb{R}_+)}\|u(t)\|_a^2; \\ -\bar{a}_1(t;u(t),u(t)) &\leq 2\bar{H}_1(t)\,\|u(t)\|_{H^1}^2 \leq \frac{2}{\bar{\alpha}_1}\left\|\bar{H}_1\right\|_{L^\infty(\mathbb{R}_+)}\|u(t)\|_a^2. \end{aligned} \tag{89}$$

On the other hand, for $\varepsilon_1 > 0$, we also have

$$\begin{aligned} -\bar{\lambda}_0 u'(1,t)u(0,t) &\leq \left|\bar{\lambda}_0\right|\sqrt{2}\left|u'(1,t)\right|\|u(t)\|_{H^1} \leq \frac{\varepsilon_2}{2}\left|u'(1,t)\right|^2 + \frac{1}{\varepsilon_2}\bar{\lambda}_0^2\frac{1}{\bar{\alpha}_1}\|u(t)\|_a^2; \\ -\bar{\lambda}_1 u'(0,t)u(1,t) &\leq \frac{\varepsilon_2}{2}\left|u'(0,t)\right|^2 + \frac{1}{\varepsilon_2}\bar{\lambda}_1^2\frac{1}{\bar{\alpha}_1}\|u(t)\|_a^2; \\ -g_0(t)u(0,t) &\leq \sqrt{2}\left|g_0(t)\right|\|u(t)\|_{H^1} \leq \frac{\varepsilon_2}{2\bar{\alpha}_1}\|u(t)\|_a^2 + \frac{1}{\varepsilon_2}g_0^2(t); \\ -g_1(t)u(1,t) &\leq \frac{\varepsilon_2}{2\bar{\alpha}_1}\|u(t)\|_a^2 + \frac{1}{\varepsilon_2}g_1^2(t); \\ \langle F(t),u(t)\rangle &\leq \|F(t)\|\,\|u(t)\| \leq \frac{\varepsilon_2}{\bar{\alpha}_1}\|u(t)\|_a^2 + \frac{1}{4\varepsilon_2}\|F(t)\|^2. \end{aligned} \tag{90}$$

Combining (88)-(90), it follows that (87) holds. Lemma 4.5 is proved. □

Now, we use the results obtained in Lemmas 4.2-4.5 to prove the following decay of the solution of (1)-(4).

It follows from (73)-(76), (78) and (87), that

$$\begin{aligned} \mathcal{L}'(t) \leq &-\left(\lambda - \bar{\lambda} - \frac{1}{2}\varepsilon_1 - \delta\right)\left\|u'(t)\right\|^2 - (k-\delta)(g*u')(t) \\ &- \left(\frac{1}{2}\mu_* - \varepsilon_1 - \frac{\delta\varepsilon_2}{2}\right)\left[\left|u'(0,t)\right|^2 + \left|u'(1,t)\right|^2\right] \\ &- \left\{\delta\left[\frac{1-\gamma^*}{2} - \frac{2}{\bar{\alpha}_1}\left(\varepsilon_2 + \frac{1}{2\varepsilon_2}\left(\bar{\lambda}_0^2 + \bar{\lambda}_1^2\right) + \sum\nolimits_{i=0}^{1}\left\|\bar{H}_i\right\|_{L^\infty(\mathbb{R}_+)}\right)\right]\right. \\ &\qquad\left. - \frac{1}{\varepsilon_1\bar{\alpha}_1}\sum\nolimits_{i=0}^{1}\left\|\bar{H}_i\right\|_{L^\infty(\mathbb{R}_+)}^2\right\}\|u(t)\|_a^2 \\ &- \frac{\delta}{2}I(t) + \left(\frac{1}{2\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\rho(t), \end{aligned} \tag{91}$$

for all $\delta, \varepsilon_1, \varepsilon_2 > 0$. Next, we shall choose $\varepsilon_1, \varepsilon_2, \delta$ with the following growth properties as follows

$$\begin{cases} 0 < \frac{\varepsilon_1}{2} < \lambda - \bar{\lambda},\ 0 < \varepsilon_1 < \frac{1}{2}\mu_*; \\ 0 < \frac{\delta\varepsilon_2}{2} < \frac{\varepsilon_2}{2} < \frac{1}{2}\mu_* - \varepsilon_1;\ \frac{2\varepsilon_2}{\bar{\alpha}_1} < \frac{1-\gamma^*}{2}; \\ \text{and } 0 < \delta < \min\left\{1;\ k;\ (1-\frac{2}{p})\bar{\alpha}_1;\ \lambda - \bar{\lambda} - \frac{1}{2}\varepsilon_1\right\}. \end{cases}$$

Then, if $\bar{\lambda}_0^2$, $\bar{\lambda}_1^2$, $\left\|\bar{H}_0\right\|_{L^\infty(\mathbb{R}_+)}$, $\left\|\bar{H}_1\right\|_{L^\infty(\mathbb{R}_+)}$ are sufficiently small, such that

$$\begin{aligned}&\frac{1}{\varepsilon_1\bar{\alpha}_1}\sum_{i=0}^{1}\left\|\bar{H}_i\right\|^2_{L^\infty(\mathbb{R}_+)}\\ &<\delta\left[\frac{1-\gamma^*}{2}-\frac{2}{\bar{\alpha}_1}\left(\varepsilon_2+\frac{1}{2\varepsilon_2}\left(\bar{\lambda}_0^2+\bar{\lambda}_1^2\right)+\sum_{i=0}^{1}\left\|\bar{H}_i\right\|_{L^\infty(\mathbb{R}_+)}\right)\right],\end{aligned} \tag{92}$$

we deduce from (86), (91) and (92) that there exists a positive constant $\gamma < \rho_2$ such that

$$\mathcal{L}'(t) \leq -\gamma\mathcal{L}(t)+\left(\frac{1}{2\varepsilon_1}+\frac{\delta}{\varepsilon_2}\right)\rho_1 e^{-\rho_2 t}, \quad \forall t \geq 0. \tag{93}$$

Combining (86) and (93), Theorem 4.1 is complete. □

Conflict of interest

This work does not have any conflicts of interest.

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□