General decay and blow-up results for a nonlinear pseudoparabolic equation with Robin-Dirichlet conditions

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Abstract. *This paper is devoted to the study of a nonlinear pseudoparabolic equation in an annular with Robin-Dirichlet conditions. At first, by applying the standard Faedo-Galerkin method, we prove existence and uniqueness results. Next, using concavity method, we prove blow-up results for solutions when the initial energy is nonnegative or negative, then we also establish the lifespan for the equation via finding the upper bound and the lower bound for the blow-up times. Finally, we give a sufficient condition for the global existence and decay of weak solutions.*

Keywords: *Nonlinear pseudoparabolic equation; Faedo-Galerkin method; Blow-up; Lifespan; The global existence and decay of weak solutions.*
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1 Introdution

In this paper, we consider the following initial boundary problem for a nonlinear pseudoparabolic equation containing viscoelastic terms

$$u_t - \left(\mu\left(t\right) + \alpha\left(t\right)\frac{\partial}{\partial t}\right)Au + \int_0^t g\left(t-s\right)Au\left(x,s\right)ds = f\left(x,t,u\right), \quad \left(x,t\right) \in \left(1,R\right)\times\left(0,\infty\right), \tag{1.1}$$

with Robin-Dirichlet conditions

$$u_x\left(1,t\right) - h_1 u\left(1,t\right) = u\left(R,t\right) = 0, \tag{1.2}$$

and the initial condition

$$u\left(x,0\right) = \tilde{u}_0\left(x\right), \tag{1.3}$$

where $Au = u_{xx} + \frac{1}{x}u_x$, $R > 1$, $h_1 \geq 0$ are given constants and μ, α, g, f, $\tilde{u}_0$ are given functions satisfying conditions specified later.

The pseudoparabolic equation

$$u_t - u_{xxt} = F\left(x, t, u, u_x, u_{xx}, u_{xt}\right),\ 0 < x < 1,\ t > 0 \tag{1.4}$$

with the initial condition $u(x, 0) = \tilde{u}_0(x)$ and with the different boundary conditions, has been extensively studied by many authors, see for example [9]-[20], [23]-[25], [28]-[37], [41], [44]-[49] among others and the references given therein. In these works, many results about existence, asymptotic behavior, blow-up and decay of solutions were obtained.

An important special case of the model (1.4) is the Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \tag{1.5}$$

it was studied by Amick et al. in [2] with $\nu > 0$, $\alpha = 1$, $x \in \mathbb{R}$, $t \geq 0$, in which the solution of (1.5) with initial data in $L^1 \cap H^2$ decays to zero in L^2 norm *as* $t \to +\infty$. With $\nu > 0$, $\alpha > 0$, $x \in [0, 1]$, $t \geq 0$, the model has the form (1.5) was also investigated earlier by Bona and Dougalis [8], where uniqueness, global existence and continuous dependence of solutions on initial and boundary data were established and the solutions were shown to depend continuously on $\nu \geq 0$ and on $\alpha > 0$. The results obtained in [2] were developed by many authors, such as by Zhang for equations of the form

$$u_t - \nu u_{xx} - u_{xxt} - u_x + u^m u_x = 0, \tag{1.6}$$

where $m \geq 0$, see [25], [47].

The linear version of (1.4) was first studied by S.L. Sobolev [40] in 1954. Therefore, the equation of the form (1.4) is also called a Sobolev type equation. Mathematical study of pseudo-parabolic equations goes back to works of Showalter (see [36]-[38]) in the seventies, since then, numerous of interesting results about linear and nonlinear pseudoparabolic equations have been obtained. It is also well known that the work [38] is the first paper on nonlinear pseudoparabolic equation. These equations appear in the study of various problems of hydrodynamics, thermodynamics and filtration theory, see [25] and the references given therein.

The nonlinear pseudoparabolic equations of type (1.1) are related to frameworks of mathematical models in engineering and physical sciences on second-grade or third-grade fluid flows, see [3], [4], [19], [20], [23], [33], [42] and references therein. In [19], some unsteady flow problems of a second-grade fluid were also considered. The flows are generated by the sudden application of a constant pressure gradient or by the impulsive motion of a boundary. Here, the velocities of the flows are described by the partial differential equations and exact analytic solutions of these differential equations are obtained. Suppose that the second grade fluid is in a circular cylinder and is initially at rest, and the fluid starts suddenly due to the motion of the cylinder parallel to its length. The axis of the cylinder is chosen as the z-axis. Using cylindrical polar coordinates, the governing partial differential equation is

$$\begin{cases} \frac{\partial w}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) w(r,t) - Nw,\ 0 < r < a,\ t > 0, \\ w(a,t) = W,\ t > 0, \\ w(r,0) = 0,\ 0 \leq r < a, \end{cases} \tag{1.7}$$

where $w(r, t)$ is the velocity along the z-axis, ν is the kinematic viscosity, α is the material parameter, and N is the imposed magnetic field. In the boundary and initial conditions, W is the constant velocity at $r = a$ and a is the radius of the cylinder. Besides, it is well known that the nonlinear pseudoparabolic equations of type (1.1) also describe a variety of important physical processes such as the seepage of homogeneous fluids through a fissured rock [5], the unidirectional propagation of nonlinear, dispersive, long waves [6], [8] and the aggregation of populations [28].

There were also many profound works on the initial value problems of high order nonlinear pseudoparabolic equations, for example, we refer to two typical papers [12], [48]. In [48], Zhao and

Xuan studied the following pseudoparabolic equation of fourth-order

$$u_t - \alpha u_{xx} - \gamma u_{xxt} + \beta u_{xxxx} + f(u)_x = 0, \;\; x \in \mathbb{R}, \;\; t \geq 0. \tag{1.8}$$

They obtained the existence of the global smooth solutions for the initial value problem of (1.8) and discussed the convergence of solutions as $\beta \to 0$. In [12], Y. Cao et al. established the global existence of classical solutions and the blow-up in a finite time for the viscous diffusion equation of higher order

$$\begin{cases} u_t + k_1 u_{xxxx} - k_2 u_{txx} - \left(\Phi\left(u_x\right)\right)_x + A(u) = 0, \; 0 < x < 1, \; t > 0, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \; t > 0, \\ u(x,0) = u_0(x), \; 0 < x < 1, \end{cases} \tag{1.9}$$

where $k_1 > 0$, $k_2 > 0$ and $\Phi(s)$, $A(s)$ are appropriately smooth, $u_0 \in C^{1+\beta}$ with $\beta \in (0,1)$ and $u_0(0) = u_0(1) = u_{0xx}(0) = u_{0xx}(1) = 0$.

On the other hand, a numerous of nonlocal pseudoparabolic (or parabolic) equations with nonlocal terms or nonlocal boundary conditions have been widely studied in the last few decades, we refer to [9], [10], [15]-[18], [27], [35], [41], [49] and the references cited therein. In [9], Bouziani studied the solvability of solutions for the nonlinear pseudoparabolic equation

$$u_t - \frac{\partial}{\partial x}\left(a(x,t)u_x\right) - \eta \frac{\partial^2}{\partial t \partial x}\left(a(x,t)u_x\right) = f\left(x,t,u,u_x\right), \;\; \alpha < x < \beta, \;\; 0 < t < T, \tag{1.10}$$

subject to the initial condition

$$u(x,0) = u_0(x), \;\; \alpha \leq x \leq \beta, \tag{1.11}$$

and the nonlocal boundary condition

$$u(\alpha,t) = \int_\alpha^\beta u(x,t)dx = 0, \tag{1.12}$$

with $u_0(\alpha) = \displaystyle\int_\alpha^\beta u_0(x)dx = 0$. In [15], Dai and Huang considered the well-poseness and solvability of solutions for the nonlinear pseudoparabolic equation

$$u_t + \left(a(x,t)u_{xt}\right)_x = F\left(x,t,u,u_x,u_{xx}\right), \;\; \alpha < x < \beta, \;\; 0 < t < T, \tag{1.13}$$

with the initial condition (1.11) and the nonlocal moment boundary conditions

$$\int_\alpha^\beta u(x,t)dx = \int_\alpha^\beta x u(x,t)dx = 0, \;\; 0 \leq t \leq T. \tag{1.14}$$

In [35], Shang and Guo proved the existence, uniqueness, regularities of the global strong solution and gave some conditions of the nonexistence of global solution for the nonlinear pseudoparabolic equation with Volterra integral term

$$u_t - f(u)_{xx} - u_{xxt} - \int_0^t \lambda(t-s)\left(\sigma\left(u(x,s),u_x(x,s)\right)\right)_x ds = f\left(x,t,u,u_x\right), \;\; 0 < x < 1, \;\; t > 0. \tag{1.15}$$

In [27], the following initial boundary value problem for a nonlinear heat equation with viscoelastic was considered

$$u_t - \frac{\partial}{\partial x}\left[\mu(x,t)u_x\right] + \int_0^t g(t-s)\frac{\partial}{\partial x}\left[\mu(x,t)u_x\right]ds = f(u) + f_1(x,t), \;\; 0 < x < 1, \;\; t > 0, \tag{1.16}$$

and existence, uniqueness, regularity, blow-up and exponential decay estimates were established. Zhu et al. [49] studied the exponent decay behavior and blow-up phenomena of weak solutions for a class of pseudoparabolic equations with a nonlocal term

$$\begin{cases} u_t - a\Delta u_t - \Delta u + u = bu\Phi_u + u^{p-1}u,\ (x,t) \in \Omega \times (0,+\infty), \\ u = 0,\ (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x),\ x \in \Omega, \end{cases} \tag{1.17}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $1 < p < 5$, and Φ_u is a Newtonian potential

$$\Phi_u(x) = \frac{1}{4\pi}\int_\Omega \frac{u^2(y)}{|x-y|}dy,\ \ x \in \mathbb{R}^3. \tag{1.18}$$

We note more that, in many mathematical literatures related to parabolic or pseudoparabolic equations, many efforts with using a variety of methods have been devoted to the study of blow-up properties if the solutions blow up, see [22], [29]-[31], [41] and the references cited therein. In [22], Li et al. used a differential inequality technique to derive the lower bound for the blow-up time when the blow-up occurs. In [41], the authors proved the results of global existence and finite time blow-up for the solutions and obtained the upper bound for the blow-up time of the following problem with a linear memory term and a nonlinear source term

$$\begin{cases} u_t - \Delta u - \Delta u_t + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^{p-2}\, u, \text{ in } \Omega \times (0,T), \\ u = 0, \text{ on } \partial\Omega \times (0,T), \\ u(0) = u_0, \text{ in } \Omega, \end{cases} \tag{1.19}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$, $p > 2$, $T \in (0,\infty]$, $u_0 \in H^1(\Omega)$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive nonincreasing function. The concavity method and the improved potential method were used to have the upper bound for the blow-up time with initial data at arbitrary energy level.

Motivated by the above mentioned works, because of mathematical context, we study the existence, uniqueness, blow-up and general decay of solutions for the problem (1.1)-(1.3). In this paper, we will apply the Faedo-Galerkin method and techniques used in [7] to treat a nonlinear Volterra integral inequality and then the existence of weak solutions will be proved. Also under suitable conditions on the initial values and the given functions f, g, using the improved lemma (Lemma 4.4) given in [43], we obtain the upper bound and the lower bound of the blow-up time when the initial energy is nonnegative or negative but small, and then the lifespan of the solution is solved. Moreover, we prove a general decay of the energy function for the global solution. This paper consists of five sections. In Section 2, we present preliminaries. In Section 3, we prove the existence and uniqueness results. In Section 4, we obtain the existence of solutions which blow up in finite time with initial data at suitable energy levels. This section also derives the lifespan for the equation considered via finding the upper bound and the lower bound for the blow-up times. Finally, Section 5 is devoted to the proof of a sufficient condition for the global existence and decay of weak solutions.

2 Preliminary results and notations

In order to prove our main results specifically, we shall introduce some definitions and notations with some properties as follows. At first, we set $\Omega = (1,R)$, $Q_T = \Omega \times (0,T)$. Let us omit the definitions of the usual function spaces and denote them by the notations

$$L^p = L^p(\Omega),\ W^{k,p} = W^{k,p}(\Omega),\ H^k = W^{k,2},\ k \in \mathbb{Z}_+,\ 1 \leq p \leq \infty.$$

We use $\|\cdot\|$, $(\cdot,\cdot)$ as the norm and the associated scalar product on L^2 respectively. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. Let $L^p(0,T;X)$, $1 \le p \le \infty$, be the Banach space of measurable functions $u:(0,T) \longrightarrow X$ such that $\|u\|_{L^p(0,T;X)} < \infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\displaystyle\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p}, & \text{if } 1 \le p < \infty \\ \operatorname*{ess\,sup}_{t\in(0,T)} \|u(t)\|_X, & \text{if } p = \infty. \end{cases}$$

On H^1, we shall use the following norm:

$$\|v\|_{H^1} = \sqrt{\|v\|^2 + \|v_x\|^2}. \tag{2.1}$$

We put

$$V = \left\{v \in H^1 : v(R) = 0\right\}. \tag{2.2}$$

Then V is a closed subspace of H^1 and on V, two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v_x\|$ are equivalent norms. Note that L^2, H^1 are also the Hilbert spaces with the corresponding scalar products

$$\langle u,v\rangle = \int_1^R xu(x)\,v(x)\,dx, \ \ \langle u,v\rangle + \langle u_x,v_x\rangle, \tag{2.3}$$

respectively. The norms in L^2 and H^1 induced by the corresponding scalar products in (2.3) are denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. We can prove that V is continuously and densely embedded in L^2. Identifying L^2 with $\left(L^2\right)'$ (the dual of L^2), we have

$$V \hookrightarrow L^2 \equiv \left(L^2\right)' \hookrightarrow V',$$

where all the injections are continuous and dense. We remark more that the notation $\langle\cdot,\cdot\rangle$ is also used for the pairing between V and V'.

In what follows, we state the lemmas, they are useful for the proofs in the next sections.

Lemma 2.1. *The following inequalities are true*

$$\begin{aligned} &\text{(i)} \ \ \|v\| \le \|v\|_0 \le \sqrt{R}\,\|v\|, \ \forall v \in L^2, \\ &\text{(ii)} \ \ \|v\|_{H^1} \le \|v\|_1 \le \sqrt{R}\,\|v\|_{H^1}, \ \forall v \in H^1. \end{aligned} \tag{2.4}$$

Lemma 2.2. *The imbedding $H^1 \hookrightarrow C\left(\overline{\Omega}\right)$ is compact and*

$$\|v\|_{C(\overline{\Omega})} \le \alpha_0 \|v\|_{H^1}, \ \forall v \in H^1, \tag{2.5}$$

with $\alpha_0 = \sqrt{\frac{1+\sqrt{1+16(R-1)^2}}{2(R-1)}}$.

Lemma 2.3. *The imbedding $V \hookrightarrow C\left(\overline{\Omega}\right)$ is compact and*

$$\begin{aligned} &\text{(i)} \ \ \|v\|_{C(\overline{\Omega})} \le \sqrt{R-1}\,\|v_x\|, \ \forall v \in V, \\ &\text{(ii)} \ \ \|v\| \le \tfrac{R-1}{\sqrt{2}}\,\|v_x\|, \ \forall v \in V, \\ &\text{(iii)} \ \ \|v\|_0 \le \sqrt{\tfrac{R}{2}}\,(R-1)\,\|v_x\|_0, \ \forall v \in V. \end{aligned} \tag{2.6}$$

Remark 2.1. On L^2, two norms $v \mapsto \|v\|$ and $v \mapsto \|v\|_0$ are equivalent. So are two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v\|_1$ on H^1, and four norms $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v\|_1$, $v \mapsto \|v_x\|$ and $v \mapsto \|v_x\|_0$ on V.

For convenience, we denote by $a(\cdot,\cdot)$ the symmetric bilinear form on $V \times V$, that is

$$a(u,v) = \langle u_x, v_x \rangle + h_1 u(1) v(1) = \int_1^R x u_x(x) v_x(x)\, dx + h_1 u(1) v(1), \ \forall u,v \in V, \tag{2.7}$$

with $h_1 \geq 0$ is a given constant and $\|v\|_a = \sqrt{a(v,v)}$.

We then have the following lemmas.

Lemma 2.4. *The symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.7) is continuous on $V \times V$ and coercive on V. Moreover, we have*

$$\begin{aligned} &\text{(i)} \ \ |a(u,v)| \leq C_1 \|u_x\|_0 \|v_x\|_0, \ \forall u,v \in V, \\ &\text{(ii)} \ \ a(v,v) \geq \|v_x\|_0^2, \ \forall v \in V, \end{aligned} \tag{2.8}$$

with $C_1 = 1 + h_1(R-1)$.

Lemma 2.5. *There exists the Hilbert orthonormal base $\{w_j\}$ of L^2 consisting of the eigenfunctions w_j corresponding to the eigenvalue λ_j such that*

$$\begin{cases} 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots, \ \lim\limits_{j\to\infty} \lambda_j = \infty, \\ a(w_j, v) = \lambda_j \langle w_j, v \rangle, \ \forall v \in V, \ \forall j \in \mathbb{N}. \end{cases}$$

Furthermore, the sequence $\left\{ \lambda_j^{-1/2} w_j \right\}$ is also the Hilbert orthonormal base of V with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we have w_j satisfying the following boundary value problem

$$\begin{cases} -A w_j = \lambda_j w_j, \ in \ \Omega, \\ w_{jx}(1) + h_1 w_j(1) = w_j(R) = 0, \ w_j \in V \cap C^\infty(\bar{\Omega}), \ \forall j \in \mathbb{N}. \end{cases}$$

The proof of Lemma 2.5 can be found in [39] with $H = L^2$ and bilinear form $a(\cdot,\cdot)$ defined by (2.7).

Lemma 2.6. *Assume that $\mathcal{O}$ is closed set of $\left(\mathbb{R}^N, \|\cdot\|_*\right)$ and $f \in C(\mathcal{O};\mathbb{R})$. Then there is a continuous non-decreasing function $\Phi_f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that*

$$|f(x)| \leq \Phi_f\left(\|x\|_*\right), \ \forall x \in \mathcal{O}. \tag{2.9}$$

Proof of Lemma 2.6. First, we assume that $f \in C\left(\mathbb{R}^N;\mathbb{R}_+\right)$. With $r > 0$, we denote

$$B_r = \left\{ x \in \mathbb{R}^N : \|x\|_* < r \right\}, \ \bar{B}_r = \left\{ x \in \mathbb{R}^N : \|x\|_* \leq r \right\}. \tag{2.10}$$

We set

$$\overline{\varphi}_f(r) = \begin{cases} \sup\limits_{x \in \bar{B}_r} f(x), & \text{if } r > 0, \\ f(0), & \text{if } r = 0. \end{cases}$$

It is clear that $\overline{\varphi}_f(r) \geq 0$ for all $r \geq 0$ and $\overline{\varphi}_f$ is non-decreasing in $\mathbb{R}_+$. We claim that $\overline{\varphi}_f \in C(\mathbb{R}_+;\mathbb{R}_+)$. Indeed,

(i) We prove that $\overline{\varphi}_f$ continuous from right at 0.

For all $\varepsilon > 0$, by $f \in C\left(\mathbb{R}^N;\mathbb{R}_+\right)$, there exists $\delta > 0$ such that

$$|f(x) - f(0)| \leq \varepsilon, \ \forall x \in \bar{B}_\delta. \tag{2.11}$$

From (2.11), we have

$$f(x) \leq f(0) + \varepsilon = \overline{\varphi}_f(0) + \varepsilon, \ \forall x \in \bar{B}_\delta. \tag{2.12}$$

By definition of $\overline{\varphi}_f$ and (2.12), we have

$$\overline{\varphi}_f(0) \le \overline{\varphi}_f(r) \le \overline{\varphi}_f(\delta) \le \overline{\varphi}_f(0) + \varepsilon, \ \forall r \in [0, \delta].$$

Therefore $\overline{\varphi}_f$ continuous from right at 0.

(ii) For all $r_0 > 0$, we will prove that $\overline{\varphi}_f$ continuous at r_0.

(ii.a) We prove that $\overline{\varphi}_f$ continuous from left at r_0.

First, we define

$$\varphi_f(r) = \begin{cases} \sup\limits_{x \in B_r} f(x), & \text{if } r > 0, \\ f(0), & \text{if } r = 0. \end{cases}$$

It is obviously to see that $\varphi_f(r) \le \overline{\varphi}_f(r)$ for all $r \ge 0$. Fixed $r > 0$, by definition of $\overline{\varphi}_f$, we can assume that

$$\bar{\varphi}_f(r) = \sup_{x \in \bar{B}_r} f(x) = \max_{x \in \bar{B}_r} f(x) = f(x_0),$$

with certain $x_0 \in \bar{B}_r$. We define the sequence $\{x_m\}$ by $x_m = \left(1 - \frac{1}{m}\right) x_0$ for all $m \in \mathbb{N}$. We will have $\{x_m\} \subset B_r$ and $\lim\limits_{m \to \infty} x_m = x_0$. By definition of φ_f and continuity of f, we get

$$\varphi_f(r) \ge \lim_{m \to \infty} f(x_m) = f(x_0) = \overline{\varphi}_f(r).$$

It is clear that φ_f is non-decreasing in $\mathbb{R}_+$. For all $\varepsilon > 0$, by definition of φ_f, there exists $y_0 \in B_{r_0}$ such that

$$\varphi_f(r_0) - \varepsilon < f(y_0) \le \varphi_f(r_0). \tag{2.13}$$

Put $\delta = r_0 - \|y_0\|_* > 0$. For all $r \in (r_0 - \delta, r_0]$, we have

$$\varphi_f(r_0) - \varepsilon < f(y_0) \le \overline{\varphi}_f(\|y_0\|_*) = \varphi_f(\|y_0\|_*) \le \varphi_f(r) \le \varphi_f(r_0). \tag{2.14}$$

From (2.14), we have

$$\overline{\varphi}_f(r_0) - \varepsilon < \overline{\varphi}_f(r) \le \overline{\varphi}_f(r_0), \ \forall r \in (r_0 - \delta, r_0]. \tag{2.15}$$

Therefore $\overline{\varphi}_f$ continuous from left at r_0.

(ii.b) We prove that $\overline{\varphi}_f$ continuous from right at r_0.

By $f \in C\left(\mathbb{R}^N; \mathbb{R}_+\right)$, we have f is uniform continuous on $\bar{B}_{2r_0}$. For all $\varepsilon > 0$, there exists $\delta \in \left(0, \frac{r_0}{2}\right)$ such that

$$|f(x) - f(y)| \le \varepsilon, \ \forall x, y \in \bar{B}_{2r_0}, \ \|x - y\|_* < \delta. \tag{2.16}$$

For all $r \in [r_0, r_0 + \delta)$, by definition of $\overline{\varphi}_f$, there exists $x_r \in \bar{B}_r$, $y_r = \frac{r_0}{r} x_r \in \bar{B}_{r_0}$ such that $\overline{\varphi}_f(r) = f(x_r)$ and

$$f(x_r) - f(y_r) \le \varepsilon \Longleftrightarrow f(x_r) \le f(y_r) + \varepsilon. \tag{2.17}$$

From (2.17), for all $r \in [r_0, r_0 + \delta)$, we have

$$\overline{\varphi}_f(r_0) \le \overline{\varphi}_f(r) \le \overline{\varphi}_f(\|y_r\|_*) + \varepsilon \le \overline{\varphi}_f(r_0) + \varepsilon. \tag{2.18}$$

Therefore $\overline{\varphi}_f$ continuous from right at r_0.

Now, with $f \in C(\mathcal{O}; \mathbb{R})$, by Tietze extension theorem, there exists $\bar{f} \in C\left(\mathbb{R}^N; \mathbb{R}\right)$ such that $\bar{f}\big|_{\mathcal{O}} = f$. We put

$$\Phi_f(r) = \overline{\varphi}_{|\bar{f}|}(r), \ \forall r \ge 0. \tag{2.19}$$

For all $x \in \mathcal{U}$, we have

$$|f(x)| = \left|\bar{f}(x)\right| \le \overline{\varphi}_{|\bar{f}|}(\|x\|_*) \le \Phi_f(\|x\|_*). \tag{2.20}$$

Finally, it is obvious that $\Phi_f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous non-decreasing function. Lemma 2.6 is proved. □

Remark 2.2. Lemma 2.6 is a slight improvement of a result used in [27, Appendix 1, p. 2734] with $N = 1$ and $f \in C(\mathbb{R}; \mathbb{R})$.

Lemma 2.7. *Let $x : [0, T] \longrightarrow \mathbb{R}_+$ be a continuous function satisfying the inequality*

$$x(t) \leq M + \int_0^t k(s)\,\omega(x(s))\,ds, \ \forall t \in [0, T],$$

where $M \geq 0$, $k : [0, T] \longrightarrow \mathbb{R}_+$ is continuous and $\omega : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous and nondecreasing.

Set

$$\Psi(u) = \int_0^u \frac{dy}{\omega(y)}, \ u \geq 0.$$

(i) *If* $\displaystyle\int_0^\infty \frac{dy}{\omega(y)} = \infty$ *then*

$$x(t) \leq \Psi^{-1}\left(\Psi(M) + \int_0^t k(s)\,ds\right), \ \forall t \in [0, T].$$

(ii) *If* $\displaystyle\int_0^\infty \frac{dy}{\omega(y)} < \infty$ *then there exists $T_* \in (0, T]$ such that*

$$x(t) \leq \Psi^{-1}\left(\Psi(M) + \int_0^t k(s)\,ds\right), \ \forall t \in [0, T_*],$$

where

$$\int_0^{T_*} k(s)\,ds \leq \int_0^\infty \frac{dy}{\omega(y)}.$$

Proof of Lemma 2.7. See [7].

3 The existence and uniqueness theorem

In this section, we shall study the existence and uniqueness of a weak solution for Prob. (1.1)-(1.3).

Definition 3.1. A function u is called a weak solution of Prob. (1.1)-(1.3) on $(0, T)$ if and only if the function u belongs to the following functional space

$$W_T = \left\{u \in C([0, T]; V) : u' \in L^2(0, T; V)\right\}, \tag{3.1}$$

and satisfies the following variational problem:

$$\begin{aligned} \langle u'(t), v\rangle + \alpha(t)\,a(u'(t), v) + \mu(t)\,a(u(t), v) \\ = \int_0^t g(t-s)\,a(u(s), v)\,ds + \langle f[u](t), v\rangle, \ \forall v \in V, \end{aligned} \tag{3.2}$$

such that

$$u(0) = \tilde{u}_0, \tag{3.3}$$

where

$$f[u](x, t) = f(x, t, u(x, t)). \tag{3.4}$$

In order to get the existence results, we consider the following hypotheses.

(A_1) $\tilde{u}_0 \in V$;

(A_2) $\mu \in C^1(\mathbb{R}_+)$ and there exists the positive constant μ_* such that $\mu(t) \geq \mu_*$ for all $t \geq 0$;

(A_3) $\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and there exists the positive constant α_* such that $\alpha(t) \geq \alpha_*$ for all $t \geq 0$;

(A_4) $g \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$;

(A_5) $f \in C\left(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}\right)$.

Theorem 3.2. *Let $(A_1)-(A_5)$ hold. Then we have*

(Case 1) *In case of* $\displaystyle\int_0^\infty \frac{dy}{1+y+\Phi_f^2\left(\sqrt{y}\right)} = \infty$, *Prob.* (1.1)-(1.3) *has a global weak solution $u \in W_T$ for all $T > 0$;*

(Case 2) *In case of* $\displaystyle\int_0^\infty \frac{dy}{1+y+\Phi_f^2\left(\sqrt{y}\right)} < \infty$, *there is $T_* > 0$ such that Prob.* (1.1)-(1.3) *has a local weak solution $u \in W_{T_*}$.*

Furthermore, if in addition the hypotheses $(\bar{A}_3)$, $(\bar{A}_5)$ as follows

$(\bar{A}_3)$ *$\alpha \in C^1(\mathbb{R}_+)$ with the property $\alpha(t) \geq \alpha_*$ for all $t \geq 0$, where α_* is the positive constant;*

$(\bar{A}_5)$ *For all $M > 0$, there exists $\ell_M > 0$ such that*

$$|f(x,t,u_1) - f(x,t,u_2)| \leq \ell_M |u_1 - u_2|, \ \forall x \in \overline{\Omega}, \ t \geq 0, \ u_1, \ u_2 \in [-M, M],$$

then the solution obtained in the above cases is unique.

Moreover, denoting by T_∞ the maximal existence time of the solution u for the Prob. (1.1)-(1.3), *the following alternatives hold*

(Alt1) $T_\infty = \infty$;

or

(Alt2) $T_\infty < \infty$ *and* $\lim\limits_{t \to T_\infty^-} \|u(t)\|_a = \infty$.

Remark 3.1.

(i) If $T_\infty = \infty$, we say that the solution u is global;

(ii) If $T_\infty < \infty$, we then have $\lim\limits_{t \to T_\infty^-} \|u(t)\|_a = \infty$, we say that the solution u blows up in finite time and that T_∞ is the blow-up time.

Proof of Theorem 3.2. Based on the Faedo-Galerkin method, this proof consists of five steps.

Step 1. Finite-dimesional approximations.

Consider the basis $\{w_j\}$ for V as in Lemma 2.5. We find an approximate solution of Prob. (1.1)-(1.3) in the form

$$u_m(t) = \sum\nolimits_{j=1}^m c_{mj}(t) w_j, \tag{3.5}$$

where the coefficients $c_{m1}, \cdots, c_{mm}$ satisfy the system of integro-differential equations

$$\begin{aligned} &\langle u_m'(t), w_j\rangle + \alpha(t) a(u_m'(t), w_j) + \mu(t) a(u_m(t), w_j) \\ &\qquad = \int_0^t g(t-s) a(u_m(s), w_j) ds + \langle f[u_m](t), w_j\rangle, \ \forall j = \overline{1,m}, \end{aligned} \tag{3.6}$$

with the initial conditions

$$u_m(0) = u_{0m} = \sum\nolimits_{j=1}^m \alpha_j w_j = \sum\nolimits_{j=1}^m \frac{a(\tilde{u}_0, w_j)}{\lambda_j} w_j \to \tilde{u}_0 \text{ strongly in } V \text{ as } m \to \infty. \tag{3.7}$$

It is clear that, for each m, there exists a solution u_m of the form (3.5) which satisfies (3.6) and (3.7) almost everywhere on $t \in [0, T_m]$, for some $T_m > 0$ that is sufficiently small. In what follows, we present a brief proof that a solution of (3.6)-(3.7) of the form (3.5) exists. It is obvious that the system (3.6)-(3.7) can be rewritten in the vectorial form

$$c_m'(t) + \mu(t) A_m(t) c_m(t) = A_m(t) \int_0^t g(t-s) c_m(s) ds + \mathcal{F}(c_m(t)), \tag{3.8}$$

with the initial condition

$$c_m(0) = \alpha_m, \tag{3.9}$$

where

$$\begin{cases} c_m(t) = (c_{m1}(t), \cdots, c_{mm}(t))^T, \ \alpha_m = (\alpha_{m1}, \cdots, \alpha_{mm})^T, \\ A_m(t) = \left[\dfrac{\lambda_j \delta_{ij}}{1+\lambda_j \alpha(t)}\right]_{i,j=\overline{1,m}}, \ \mathcal{F}(c_m(t)) = (\mathcal{F}_1(c_m(t)), \cdots, \mathcal{F}_m(c_m(t)))^T, \\ \mathcal{F}_j(c_m(t)) = \dfrac{1}{1+\lambda_j \alpha(t)} \langle f[u_m](t), w_j \rangle, \ \forall j = \overline{1,m}, \end{cases} \tag{3.10}$$

which is also equivalent to the integral equation

$$\begin{aligned} c_m(t) &= \alpha_m - \int_0^t \mu(s) A_m(s) c_m(s)\, ds \\ &= \int_0^t A_m(s)\, ds \int_0^s g(s-\tau) c_m(\tau)\, d\tau + \int_0^t \mathcal{F}(c_m(s))\, ds. \end{aligned} \tag{3.11}$$

The integral equation (3.11) can be solved by applying Schauder's fixed point theorem. Therefore, there exists u_m of the form (3.5) which satisfies (3.6) and (3.7) almost everywhere on $t \in [0, T_m]$, where $T_m > 0$ is sufficiently small.

Step 2. A priori estimate.

Multiplying the j^{th} equation of (3.6) by $c'_{mj}(t)$ and summing over j and afterwards integrating with respect to time variable on $[0, t]$. After some rearrangements, we get

$$\begin{aligned} S_m(t) = S_m(0) &+ \int_0^t \mu'(s) \|u_m(s)\|_a^2\, ds + 2\int_0^t d\tau \int_0^\tau g(\tau - s)\, a\left(u_m(s), u'_m(\tau)\right) ds \\ &+ 2\int_0^t \left\langle f[u_m](s), u'_m(s)\right\rangle ds \\ &= S_m(0) + J_1 + J_2 + J_3, \end{aligned} \tag{3.12}$$

with

$$S_m(t) = \mu(t) \|u_m(t)\|_a^2 + 2\int_0^t \left(\|u'_m(s)\|_0^2 + \alpha(s) \|u'_m(s)\|_a^2\right) ds. \tag{3.13}$$

Given $T > 0$, $\varepsilon > 0$, that will be fixed later, we estimate the terms J_1, J_2, J_3 of (3.12) as follows.

First term J_1.

$$J_1 = \int_0^t \mu'(s) \|u_m(s)\|_a^2\, ds \le \int_0^t |\mu'(s)| \|u_m(s)\|_a^2\, ds \le \frac{\|\mu'\|_{C([0,T])}}{\mu_*} \int_0^t S_m(s)\, ds. \tag{3.14}$$

Second term J_2.

$$\begin{aligned} J_2 &= 2\int_0^t d\tau \int_0^\tau g(\tau - s)\, a\left(u_m(s), u'_m(\tau)\right) ds \\ &\le 2\int_0^t d\tau \int_0^\tau |g(\tau - s)| \left|a\left(u_m(s), u'_m(\tau)\right)\right| ds \\ &\le 2\int_0^t \left\|u'_m(\tau)\right\|_a d\tau \int_0^\tau |g(\tau - s)| \|u_m(s)\|_a ds \\ &\le \int_0^t \left[\varepsilon \left\|u'_m(\tau)\right\|^2 + \frac{1}{\varepsilon}\left(\int_0^\tau |g(\tau - s)| \|u_m(s)\|_a ds\right)^2\right] d\tau \\ &\le \int_0^t \left[\varepsilon \left\|u'_m(\tau)\right\|^2 + \frac{\|g\|_{L^2(0,T)}}{\varepsilon} \int_0^\tau \|u_m(s)\|_a^2\, ds\right] d\tau \\ &\le \frac{\varepsilon}{\mu_*} S_m(t) + \frac{T\|g\|_{L^2(0,T)}}{\varepsilon} \int_0^t S_m(s)\, ds. \end{aligned} \tag{3.15}$$

Third term J_3. It is known that

$$|u_m(x,t)| \le \|u_m(t)\|_{C(\overline{\Omega})} \le \sqrt{R-1}\|u_{mx}(t)\|_0 \le \sqrt{R-1}\|u_m(t)\|_a \le \sqrt{\frac{R-1}{\mu_*}S_m(t)}. \tag{3.16}$$

By Lemma 2.6, we have

$$\|f[u_m](t)\|_0^2 = \int_1^R xf^2[u_m](x,t)\,dx \le \frac{R^2-1}{2}\Phi_f^2\left(R+T+\sqrt{\frac{R-1}{\mu_*}S_m(t)}\right). \tag{3.17}$$

Therefore

$$\begin{aligned} J_3 &= 2\int_0^t \left\langle f[u_m](s), u_m'(s)\right\rangle ds \le 2\int_0^t \left|\left\langle f[u_m](s), u_m'(s)\right\rangle\right| ds \\ &\le 2\int_0^t \|f[u_m](s)\|_0 \left\|u_m'(s)\right\|_0 ds \\ &\le \frac{1}{\varepsilon}\int_0^t \|f[u_m](s)\|_0^2\,ds + \varepsilon\int_0^t \left\|u_m'(s)\right\|_0^2 ds \\ &\le \frac{R^2-1}{2\varepsilon}\int_0^t \Phi_f^2\left(R+T+\sqrt{\frac{R-1}{\mu_*}S_m(s)}\right)ds + \frac{\varepsilon}{2}S_m(t). \end{aligned} \tag{3.18}$$

We continue to estimate the term $S_m(0)$. By means of the convergences in (3.7), we can deduce the existence of a constant $\bar{S}_0 > 0$ such that

$$S_m(0) = \mu(0)\|u_{0m}\|_a^2 \le \bar{S}_0, \ \forall m \in \mathbb{N}. \tag{3.19}$$

Choosing $\varepsilon = \frac{2+\mu_*}{\mu_*} > 0$, from (3.12), (3.14), (3.15), (3.18), (3.19), there exists $M_T > 0$, it is a constant independent of m such that

$$S_m(t) \le 2\bar{S}_0 + M_T\int_0^t \omega(S_m(s))\,ds, \tag{3.20}$$

where

$$M_T = \frac{\|\mu'\|_{C([0,T])}}{\mu_*} + \frac{2T\|g\|_{L^2(0,T)} + R^2 - 1}{2\varepsilon}, \ \omega(S) = 1 + S + \Phi_f^2\left(R+T+\sqrt{\frac{R-1}{\mu_*}S}\right). \tag{3.21}$$

By the convergence of the integrals $\int_0^\infty \frac{dy}{\omega(y)}$ and $\int_0^\infty \frac{dy}{1+y+\Phi_f^2(\sqrt{y})}$, applying Lemma 2.7, we deduce from (3.20) that

Case1. If $\int_0^\infty \frac{dy}{1+y+\Phi_f^2(\sqrt{y})} = \infty$ then

$$S_m(t) \le \Psi^{-1}\left(\Psi\left(2\bar{S}_0\right) + M_T t\right) \le \Psi^{-1}\left(\Psi\left(2\bar{S}_0\right) + M_T T\right) \le C_T, \ \forall m \in \mathbb{N}, \ t \in [0,T]. \tag{3.22}$$

Case2. If $\int_0^\infty \frac{dy}{1+y+\Phi_f^2(\sqrt{y})} < \infty$ then

$$S_m(t) \le \Psi^{-1}\left(\Psi\left(2\bar{S}_0\right) + M_T t\right) \le \Psi^{-1}\left(\Psi\left(2\bar{S}_0\right) + M_T T_*\right) \le C_T, \ \forall m \in \mathbb{N}, \ t \in [0,T_*], \tag{3.23}$$

where $T_* \in (0,T]$ is chosen such that $T_* M_T \le \int_0^\infty \frac{dy}{\omega(y)}$.

This allows one to take the constant $T_m = T$ or $T_m = T_*$ for all $m \in \mathbb{N}$. In what follows, we will write T for both T and T_*.

Step 3. Passage to the limit.

From (3.13) and (3.22) (or (3.23)), we have

$$\|u_m\|_{L^\infty(0,T;V)} \leq \frac{C_T}{\mu_*}, \ \|u_m'\|_{L^2(0,T;V)} \leq \frac{C_T}{2\alpha_*}, \ \forall m \in \mathbb{N}. \tag{3.24}$$

From (3.24), we deduce the existence of a subsequence of $\{u_m\}$ denoted by the same symbol such that

$$\begin{aligned} &u_m \to u \text{ weakly* in } L^\infty(0,T;V), \\ &u_m' \to u' \text{ weakly in } L^2(0,T;V). \end{aligned} \tag{3.25}$$

By the compactness lemma of Lions ([21], p. 57) we can deduce from (3.25) the existence of a subsequence still denoted by $\{u_m\}$, such that

$$u_m \to u \text{ strongly in } L^2(Q_T) \text{ and } u_m(x,t) \to u(x,t) \text{ a.e. } (x,t) \in Q_T. \tag{3.26}$$

By the continuity of f, we have

$$f[u_m](x,t) = f(x,t,u_m(x,t)) \to f(x,t,u(x,t)) \text{ a.e. } (x,t) \in Q_T. \tag{3.27}$$

Besides, we also have

$$|f[u_m](x,t)| \leq \sup_{(x,t,u)\in\overline{\Omega}\times[0,T]\times[-C_T,C_T]} |f(x,t,u)| = \overline{C}_T, \ \forall m \in \mathbb{N}. \tag{3.28}$$

Consequently, it follows from the dominated convergence theorem that

$$f[u_m] \to f(u) \text{ strongly in } L^2(Q_T). \tag{3.29}$$

Combining (3.7), (3.25) and (3.29), it is enough to pass to the limit in (3.6) and (3.7) to show that u satisfies (3.2) and (3.3). In addition, from (3.25), we have $u \in W(T)$. Hence, the proof of the existence of a weak solution is complete.

Step 4. Uniqueness of the solution.

Suppose u_1, u_2 are two solutions of Prob. (1.1)-(1.3) on the interval $[0,T]$ such that u_1, $u_2 \in W_T$. Then $u = u_1 - u_2 \in W_T$ satisfies

$$\begin{aligned} &\langle u'(t), v\rangle + \alpha(t)\, a(u'(t), v) + \mu(t)\, a(u(t), v) - \int_0^t g(t-s)\, a(u(s), v)\, ds \\ = \ &\langle f[u_1](t) - f[u_2](t), v\rangle, \ \forall v \in V, \end{aligned} \tag{3.30}$$

and

$$u(0) = 0. \tag{3.31}$$

Taking $v = u(t)$ in (3.30) and integrating with respect to t, we obtain

$$\begin{aligned} \varrho(t) \ = \ &-\int_0^t \left(2\mu(s) - \alpha'(s)\right) \|u(s)\|_a^2\, ds \\ &+2\int_0^t ds \int_0^s g(s-\tau)\, a(u(\tau), u(s))\, d\tau + 2\int_0^t \langle f[u_1](s) - f[u_2](s), u(s)\rangle\, ds, \end{aligned} \tag{3.32}$$

where

$$\varrho(t) = \|u(t)\|_0^2 + \alpha(t)\, \|u(t)\|_a^2. \tag{3.33}$$

As in Step 2, we can easily estimate all terms on the right hand side of (3.32) to obtain

$$\varrho(t) \leq D_T \int_0^t \varrho(s)\, ds,\ \forall t \in [0, T], \tag{3.34}$$

where $D_T > 0$. By Gronwall's lemma, (3.34) leads to $\varrho(t) \equiv 0$; i.e., $u_1 \equiv u_2$.

Step 5. The alternative statement.

The last statement of Theorem 3.2 is proven by a standard continuation argument. Indeed, let $[0, T_\infty)$ be a maximal existence interval on which the solution of Prob. (1.1)-(1.3) exists. Suppose that $T_\infty < \infty$. We prove that $\lim_{t \to T_\infty^-} \|u(t)\|_a = \infty$ (Proof by Contradiction).

Indeed, assume there exists $M_0 > 0$ and $\{t_m\} \subset (0, T_\infty)$ such that $\lim_{m \to \infty} t_m = T_\infty$ and $\|u(t_m)\|_a \leq M_0$ for all $m \in \mathbb{N}$. As we have proved above, for each $m \in \mathbb{N}$, there exists a unique weak solution of Prob. (1.1)-(1.3) with initial data $u(t_m)$ on $[t_m, t_m + \delta]$ with $\delta > 0$ independent of $m \in \mathbb{N}$. Thus, we can get $T_\infty < t_m + \delta$ for $m \in \mathbb{N}$ sufficiently large and so, we obtain a contradiction to the maximality of T_∞. The proof of Theorem 3.2 is finished. $\square$

4 Blow-up and lifespan of solutions

Our main objective of this section is to show that the weak solution of Prob. (1.1)-(1.3) blows up at finite time at $\alpha(t) \equiv \alpha$, $f(x, t, u) \equiv K(x, t) f(u)$. We will consider the blow-up property when the initial energy is negative or nonnegative. We note more that this property still depends on the variety conditions of the relaxation function g.

Let we first state the blow-up result when the initial energy is negative. In this case, we make the following assumptions.

(A_2') $\mu \in C^1(\mathbb{R}_+)$ such that $\mu(t) \geq \mu_* > 0$, $\mu'(t) \leq 0$ for all $t \geq 0$;

(A_3') $\alpha > 0$;

(K_1) $K,\ K_t \in C(\overline{\Omega} \times \mathbb{R}_+)$ such that

(i) $K(x, t) \geq 0$ for all $(x, t) \in \overline{\Omega} \times \mathbb{R}_+$,

(ii) $K_t(x, t) \geq 0$ for all $(x, t) \in \overline{\Omega} \times \mathbb{R}_+$;

(F_1) $f \in C^1(\mathbb{R})$ and there exists the constant $p > 2$ such that

$$uf(u) \geq pF(u) = p\int_0^u f(z)\, dz \geq 0,\ \forall u \in \mathbb{R};$$

(G_1) $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ satisfies

(i) $g(t) \geq 0$ for all $t \geq 0$,

(ii) $g'(t) \leq 0$ for all $t \geq 0$,

(iii) $\tilde{g}_\infty \leq \dfrac{p(p-2)\mu_*}{p(p-2)+1}$ where $\tilde{g}(t) = \int_0^t g(s)\, ds$ and $\tilde{g}_\infty = \int_0^\infty g(s)\, ds$.

Let we define the following functionals

$$E(t) = \frac{1}{2}(g \star u)(t) + \frac{1}{2}(\mu(t) - \tilde{g}(t)) \|u(t)\|_a^2 - \int_1^R xK(x, t) F(u(x, t))\, dx, \tag{4.1}$$

where

$$(g \star u)(t) = \int_0^t g(t-s) \|u(t) - u(s)\|_a^2\, ds, \tag{4.2}$$

and

$$\rho(t) = \frac{1}{2}\|u(t)\|_0^2 + \frac{\alpha}{2}\|u(t)\|_a^2. \tag{4.3}$$

Lemma 4.1. *Assume that* (A_1), (A_2'), (A_3'), (K_1), (F_1) *and* (G_1) *hold. Then we have*

$$\frac{d}{dt}\left[E(t)+\int_0^t\left(\left\|u'(s)\right\|_0^2+\alpha\left\|u'(s)\right\|_a^2\right)ds\right]\leq 0. \tag{4.4}$$

Moreover, the following energy inequality holds

$$E(t)+\int_0^t\left(\left\|u'(s)\right\|_0^2+\alpha\left\|u'(s)\right\|_a^2\right)ds\leq E(0). \tag{4.5}$$

Proof of Lemma 4.1. By multiplying the equation in (1.1) by $xu_t(x,t)$, integrating over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\left[E(t)+\int_0^t\left(\|u'(s)\|_0^2+\alpha\|u'(s)\|_a^2\right)ds\right]&=\frac{1}{2}\mu'(t)\|u(t)\|_a^2\\ -\frac{1}{2}g(t)\|u(t)\|_a^2+\frac{1}{2}(g'\star u)(t)&-\int_1^R xK_t(x,t)F(u(x,t))dx,\end{aligned} \tag{4.6}$$

for any regular solution u. We can extend (4.6) to weak solutions by using density arguments. Combining (A_2'), (A_3'), (K_1), (F_1) and (G_1), the result of Lemma 4.1 is obtained. $\square$

Theorem 4.2. *Assume that the assumptions* (A_2'), (A_3'), (K_1), (F_1) *and* (G_1) *hold. Then, for any initial conditions* $\tilde{u}_0\in V$ *such that* $E(0)<0$, *the weak solution of the Prob.* (1.1)-(1.3) *blows up at finite time and the lifespan* T_∞ *of the solution* u *satisfies*

$$T_\infty\leq-\frac{8(p-1)\rho(0)}{(p-2)^2pE(0)}=T_\infty^{\max}. \tag{4.7}$$

Furthermore, if in addition the following assumptions

(K_1') $K(\cdot,T_\infty^{\max})\in C(\overline{\Omega})$, $K(\cdot,T_\infty^{\max})\neq 0$;

(F_1') (i) *There exists the constant* $d_2>p$ *such that* $uf(u)\leq d_2F(u)$ *for all* $u\in\mathbb{R}$,

(ii) $\displaystyle\int_0^\infty\frac{zdz}{z^2+\mathbb{F}(z)}<\infty$,

where

$$\mathbb{F}(r)=\varphi_F(r)=\begin{cases}\sup\limits_{|u|\leq r}F(u), & \text{if } r>0,\\ F(0)=0, & \text{if } r=0.\end{cases}$$

Then, the blow-up time T_∞ *satisfies*

$$T_\infty\geq\int_{\sqrt{2(R-1)\alpha^{-1}\rho(0)}}^{\infty}\frac{zdz}{\Psi_1(z)}=T_\infty^{\min}, \tag{4.8}$$

with

$$\Psi_1(z)=\frac{1}{2\alpha}\left[\mu(0)z^2+2(R-1)(1+d_2)\left\|\sqrt{K(T_\infty^{\max})}\right\|_0^2\mathbb{F}(z)\right]. \tag{4.9}$$

Proof of Theorem 4.2. By last statement in Theorem 3.2, it is enough to prove that the solution obtained here is not a global solution in $\mathbb{R}_+$. Indeed, by contradiction, we will assume that weak solutions exist in the whole interval $\mathbb{R}_+$.

For $T_0>0$, $\beta>0$ and $\tau>0$ specified later, we define the auxiliary functional

$$\begin{aligned}M:[0,T_0]&\longrightarrow\mathbb{R}\\ t\longmapsto M(t)&=2\int_0^t\rho(s)ds+2(T_0-t)\rho(0)+\beta(t+\tau)^2.\end{aligned} \tag{4.10}$$

By direct computation, we achieve that

$$\begin{aligned} M'(t) &= 2\rho(t) - 2\rho(0) + 2\beta(t+\tau) \\ &= 2\int_0^t \left\langle u'(s), u(s)\right\rangle ds + 2\alpha \int_0^t a\left(u'(s), u(s)\right) ds + 2\beta(t+\tau), \end{aligned} \tag{4.11}$$

and

$$M''(t) = 2\left\langle u'(t), u(t)\right\rangle + 2\alpha a\left(u'(t), u(t)\right) + 2\beta. \tag{4.12}$$

Since (4.10) and (4.11), we see that $M(t) > 0$ for all $t \in [0, T_0]$ and $M'(0) = 2\beta\tau > 0$.
By multiplying the equation in (1.1) by $xu(x,t)$, and then integrating over Ω, we obtain

$$M''(t) = 2\beta - 2\mu(t)\|u(t)\|_a^2 + 2\int_0^t g(t-s)\, a(u(s), u(t))\, ds + 2\langle K(t) f(u(t)), u(t)\rangle. \tag{4.13}$$

From (4.13), we have

$$M''(t) M(t) = 2M(t)\left[\beta - \mu(t)\|u(t)\|_a^2 + \int_0^t g(t-s)\, a(u(s), u(t))\, ds + \langle K(t) f(u(t)), u(t)\rangle\right]. \tag{4.14}$$

Now, we put

$$\theta(t) = \left(2\int_0^t \rho(s)\, ds + \beta(t+\tau)^2\right)\left(\int_0^t \left(\left\|u'(s)\right\|_0^2 + \alpha\left\|u'(s)\right\|_a^2\right) ds + \beta\right). \tag{4.15}$$

By the fact that

$$\begin{aligned} \left(\int_0^t \left\langle u'(s), u(s)\right\rangle ds\right)^2 &\le \int_0^t \left\|u'(s)\right\|_0^2 ds \int_0^t \|u(s)\|_0^2\, ds, \\ \left(\alpha\int_0^t a\left(u'(s), u(s)\right) ds\right)^2 &\le \alpha^2 \int_0^t \left\|u'(s)\right\|_a^2 ds \int_0^t \|u(s)\|_a^2\, ds, \\ 2\beta(t+\tau)\int_0^t \left\langle u'(s), u(s)\right\rangle ds &\le \beta(t+\tau)^2 \int_0^t \left\|u'(s)\right\|_0^2 ds + \beta\int_0^t \|u(s)\|_0^2\, ds, \\ 2\beta\alpha(t+\tau)\int_0^t a\left(u'(s), u(s)\right) ds &\le \alpha\beta(t+\tau)^2 \int_0^t \left\|u'(s)\right\|_a^2 ds + \alpha\beta\int_0^t \|u(s)\|_a^2\, ds, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} & 2\alpha\int_0^t \left\langle u'(s), u(s)\right\rangle ds \int_0^t a\left(u'(s), u(s)\right) ds \\ \le\ & \alpha\int_0^t \|u(s)\|_0^2\, ds \int_0^t \left\|u'(s)\right\|_a^2 ds + \alpha\int_0^t \left\|u'(s)\right\|^2 ds \int_0^t \|u(s)\|_a^2\, ds, \end{aligned} \tag{4.17}$$

so we imply from (4.15)-(4.17) that

$$\theta(t) \ge \frac{1}{4}\left|M'(t)\right|^2, \text{ for all } t \in [0, T_0]. \tag{4.18}$$

Therefore. by direct computation, we get

$$2pM(t)\left(\int_0^t \left(\left\|u'(s)\right\|^2 + \alpha\left\|u'(s)\right\|_a^2\right) ds + \beta\right) \ge 2p\theta(t) \ge \frac{p}{2}\left|M'(t)\right|^2. \tag{4.19}$$

From (4.18) and (4.19), we get

$$M''(t) M(t) - \frac{p}{2}\left|M'(t)\right|^2 \ge 2M(t) D(t), \tag{4.20}$$

with

$$D(t) = \beta - \mu(t)\|u(t)\|_a^2 + \int_0^t g(t-s)\,a(u(s),u(t))\,ds \tag{4.21}$$
$$+\langle K(t) f(u(t)), u(t)\rangle - p\left(\int_0^t \left(\|u'(s)\|^2 + \alpha\|u'(s)\|_a^2\right) ds + \beta\right).$$

We can easily estimate the third and fourth terms on the right hand side of (4.21) as follows

$$\int_0^t g(t-s)\,a(u(s),u(t))\,ds \geq -\frac{p}{2}(g\star u)(t) + \left(1-\frac{1}{2p}\right)\tilde{g}(t)\|u(t)\|_a^2, \tag{4.22}$$

and

$$\langle K(t) f(u(t)), u(t)\rangle \geq p\int_1^R xK(x,t)\,F(u(x,t))\,dx. \tag{4.23}$$

It implies from (4.5), (4.21)-(4.23) that

$$\begin{aligned} D(t) &\geq \beta - \mu(t)\|u(t)\|_a^2 - \frac{p}{2}(g\star u)(t) + \left(1-\frac{1}{2p}\right)\tilde{g}(t)\|u(t)\|_a^2 \\ &\quad + p\int_1^R xK(x,t)\,F(u(x,t))\,dx - p\left(\int_0^t \left(\|u'(s)\|^2 + \alpha\|u'(s)\|_a^2\right) ds + \beta\right) \\ &= (1-p)\beta - p\left[E(t) + \int_0^t \left(\|u'(s)\|_0^2 + \alpha\|u'(s)\|_a^2\right) ds\right] \\ &\quad + \left[\left(\frac{p}{2}-1\right)\mu(t) + \left(1-\frac{1}{2p}-\frac{p}{2}\right)\tilde{g}(t)\right]\|u(t)\|_a^2 \\ &\geq (1-p)\beta - pE(0) + \left[\left(\frac{p}{2}-1\right)\mu_* + \left(1-\frac{1}{2p}-\frac{p}{2}\right)\tilde{g}_\infty\right]\|u(t)\|_a^2 \\ &\geq (1-p)\beta - pE(0). \end{aligned} \tag{4.24}$$

Choosing β, $0<\beta\leq \frac{pE(0)}{1-p}$, from (4.20), (4.21) and (4.24), we have

$$M(t) \geq \left[\left(1-\frac{p}{2}\right)M^{\frac{-p}{2}}(0)\,M'(0)t + M^{1-\frac{p}{2}}(0)\right]^{-\frac{2}{p-2}},\ \forall t\in[0\ T_0]. \tag{4.25}$$

If we choose $\tau > \frac{2\rho(0)}{(p-2)\beta}$ and $T_0 \geq \frac{\beta\tau^2}{(p-2)\beta\tau - 2\rho(0)}$, we will have

$$T_* = -\frac{M^{1-\frac{p}{2}}(0)}{\left(1-\frac{p}{2}\right)M^{\frac{-p}{2}}(0)\,M'(0)} = \frac{2M(0)}{(p-2)\,M'(0)} = \frac{2T_0\rho(0)+\beta\tau^2}{(p-2)\,\beta\tau} \in (0,T_0]. \tag{4.26}$$

From (4.25), we get $\lim\limits_{t\to T_*^-} M(t) = \infty$. This is a contradiction. Consequently, the solution blows up at finite time.

Now, we will seek the upper bound for T_∞. It is clear to see that

$$T_\infty \leq \frac{2T_\infty\rho(0)+\beta\tau^2}{(p-2)\,\beta\tau} \Longleftrightarrow T_\infty \leq \frac{\beta\tau^2}{(p-2)\,\beta\tau - 2\rho(0)},\ \forall(\beta,\tau)\in\tilde{\Omega}(\tilde{u}_0), \tag{4.27}$$

where

$$\tilde{\Omega}(\tilde{u}_0) = \left\{(\beta,\tau)\in\mathbb{R}_+^2 : 0<\beta\leq\frac{pE(0)}{1-p},\ \tau > \frac{2\rho(0)}{(p-2)\,\beta}\right\}. \tag{4.28}$$

For all $(\beta,\tau)\in\tilde{\Omega}(\tilde{u}_0)$, we have

$$\begin{aligned}\frac{\beta\tau^2}{(p-2)\,\beta\tau-2\rho\,(0)} &\geq \frac{\beta\left(\frac{4\rho(0)}{(p-2)\beta}\right)^2}{(p-2)\,\beta\frac{4\rho(0)}{(p-2)\beta}-2\rho\,(0)} \\ &= \frac{8\rho\,(0)}{\beta(p-2)^2}\geq\frac{8\rho\,(0)}{\dfrac{pE\,(0)}{1-p}(p-2)^2}=-\frac{8\,(p-1)\,\rho\,(0)}{p(p-2)^2E\,(0)}.\end{aligned}\tag{4.29}$$

From (4.27) and (4.29), we get $T_\infty\leq-\dfrac{8\,(p-1)\,\rho\,(0)}{p(p-2)^2E\,(0)}=T_\infty^{\max}$.

Next, we seek a lower bound for the blow-up time T_∞ for the solution u. We have

$$\rho'\,(t)=-\mu\,(t)\,\|u\,(t)\|_a^2+\langle K\,(t)\,f\,(u\,(t))\,,u\,(t)\rangle+\int_0^t g\,(t-s)\,a\,(u\,(s)\,,u\,(t))\,ds. \tag{4.30}$$

We can easily estimate third term on the right hand side of (4.30) as follows

$$\int_0^t g\,(t-s)\,a\,(u\,(s)\,,u\,(t))\,ds\leq\frac{1}{2}\,(g\star u)\,(t)+\frac{3}{2}\tilde{g}\,(t)\,\|u\,(t)\|_a^2\,. \tag{4.31}$$

In order to continue process, by using Lemma 2.6 with $\mathcal{O}=\mathbb{R}$, $N=1$, $f=F\geq0$, we note that the function $\mathbb{F}:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is nondecreasing satisfying $F\,(u)\leq\mathbb{F}\,(|u|)$, for all $u\in\mathbb{R}$.

Combining (4.5), (4.30) and (4.31), it leads to

$$\begin{aligned}\rho'\,(t)&\leq-\mu\,(t)\,\|u\,(t)\|_a^2+\langle K\,(t)\,f\,(u\,(t))\,,u\,(t)\rangle+\frac{1}{2}\,(g\star u)\,(t)+\frac{3}{2}\tilde{g}\,(t)\,\|u\,(t)\|_a^2 \\ &=E\,(t)+\frac{1}{2}\mu\,(t)\,\|u\,(t)\|_a^2-2\,(\mu\,(t)-\tilde{g}\,(t))\,\|u\,(t)\|_a^2+\int_1^R xK\,(x,t)\,F\,(u\,(x,t))\,dx \\ &+\langle K\,(t)\,f\,(u\,(t))\,,u\,(t)\rangle \\ &\leq\frac{1}{2}\mu\,(0)\,\|u\,(t)\|_a^2+(1+d_2)\int_1^R xK\,(x,t)\,F\,(u\,(x,t))\,dx \\ &\leq\frac{\mu\,(0)}{\alpha}\rho\,(t)+(1+d_2)\int_1^R xK\,(x,T_\infty^{\max})\,\mathbb{F}\,(|u\,(x,t)|)\,dx \\ &\leq\frac{\mu\,(0)}{\alpha}\rho\,(t)+(1+d_2)\left\|\sqrt{K\,(T_\infty^{\max})}\right\|_0^2\mathbb{F}\left(\sqrt{R-1}\|u\,(t)\|_a\right) \\ &\leq\frac{\mu\,(0)}{\alpha}\rho\,(t)+(1+d_2)\left\|\sqrt{K\,(T_\infty^{\max})}\right\|_0^2\mathbb{F}\left(\sqrt{2\,(R-1)\,\alpha^{-1}\rho\,(t)}\right) \\ &=\frac{\alpha}{R-1}\Psi_1\left(\sqrt{2\,(R-1)\,\alpha^{-1}\rho\,(t)}\right).\end{aligned}\tag{4.32}$$

From (4.32), we get

$$t\geq\int_{\sqrt{2(R-1)\alpha^{-1}\rho(0)}}^{\sqrt{2(R-1)\alpha^{-1}\rho(t)}}\frac{zdz}{\Psi_1(z)}. \tag{4.33}$$

Hence, we derive the lower bound for T_∞, by (4.33), as follows

$$T_\infty\geq\int_{\sqrt{2(R-1)\alpha^{-1}\rho(0)}}^{\infty}\frac{zdz}{\Psi_1(z)}=T_\infty^{\min}. \tag{4.34}$$

The proof is complete. □

Next, we state the blow-up result when the initial energy is nonnegative. In this case, we make the following assumptions

(K_2) $K,\, K_t \in C\left(\overline{\Omega}\times\mathbb{R}_+\right)$ such that

(i) $0 \le K(x,t) \le K_1$ for all $(x,t)\in\overline{\Omega}\times\mathbb{R}_+$ with $K_1>0$,

(ii) $K_t(x,t)\ge 0$ for all $(x,t)\in\overline{\Omega}\times\mathbb{R}_+$;

(F_2) $f\in C^1(\mathbb{R})$ and there exist constants $\overline{d}_2>0$, $d_2>p>2$, $q_i>p$ for all $i=\overline{1,N}$, satisfying

(i) $uf(u)\ge pF(u)\ge 0$ for all $u\in\mathbb{R}$,

(ii) $uf(u)\le d_2F(u)\le d_2\overline{d}_2\left(|u|^p+\sum_{i=1}^N |u|^{q_i}\right)$ for all $u\in\mathbb{R}$.

First, by (4.1), (K_2) and (F_2), we have

$$\begin{aligned}
E(t) &= \frac{1}{2}(g\star u)(t)+\frac{1}{2}\left[\mu(t)-\tilde{g}(t)\right]\|u(t)\|_a^2-\int_1^R xK(x,t)F(u(x,t))\,dx \\
&\ge \frac{1}{2}\left[\mu(t)-\tilde{g}(t)\right]\|u(t)\|_a^2-RK_1\overline{d}_2\int_1^R\left(|u(x,t)|^p+\sum_{i=1}^N|u(x,t)|^{q_i}\right)dx \\
&= \frac{1}{2}\left[\mu(t)-\tilde{g}(t)\right]\|u(t)\|_a^2-RK_1\overline{d}_2\left(\|u(t)\|_{L^p}^p+\sum_{i=1}^N\|u(t)\|_{L^{q_i}}^{q_i}\right) \\
&\ge \frac{1}{2}\left[\mu(t)-\tilde{g}(t)\right]\|u(t)\|_a^2-RK_1\overline{d}_2\left(\widetilde{D}_p^p\|u(t)\|_a^p+\sum_{i=1}^N\widetilde{D}_{q_i}^{q_i}\|u(t)\|_a^{q_i}\right) \\
&\ge \frac{1}{2}y^2(t)-RK_1\overline{d}_2\left(\widetilde{D}_p^pL^{-\frac{p}{2}}y^p(t)+\sum_{i=1}^N\widetilde{D}_{q_i}^{q_i}L^{-\frac{q_i}{2}}y^{q_i}(t)\right) \\
&= \mathcal{H}(y(t)),
\end{aligned} \tag{4.35}$$

where

$$L=\mu_*-\tilde{g}_\infty=\mu_*-\int_0^\infty g(s)\,ds>0,\ \widetilde{D}_p=\sup_{0\ne v\in V}\frac{\|v\|_{L^p}}{\|v\|_a},\ y(t)=\sqrt{\mu(t)-\tilde{g}(t)}\|u(t)\|_a, \tag{4.36}$$

and

$$\begin{aligned}
&\mathcal{H}:\mathbb{R}_+\longrightarrow\mathbb{R} \\
&\lambda\longmapsto\mathcal{H}(\lambda)=\tfrac{\lambda^2}{2}-RK_1\overline{d}_2\left(\widetilde{D}_p^pL^{-\frac{p}{2}}\lambda^p+\sum_{i=1}^N\widetilde{D}_{q_i}^{q_i}L^{-\frac{q_i}{2}}\lambda^{q_i}\right).
\end{aligned} \tag{4.37}$$

Before stating our main results, we give a useful lemma as follows. The proof of this lemma is not difficult, so we omit it.

Lemma 4.3.

(i) *The equation $\mathcal{H}'(\lambda)=0$ has a unique positive solution λ_0 satisfies*

$$1-RK_1\overline{d}_2\left(p\widetilde{D}_p^pL^{-\frac{p}{2}}\lambda_0^{p-2}+\sum_{i=1}^N q_i\widetilde{D}_{q_i}^{q_i}L^{-\frac{q_i}{2}}\lambda_0^{q_i-2}\right)=0; \tag{4.38}$$

(ii) $\mathcal{H}(0)=0,\ \lim_{\lambda\to\infty}\mathcal{H}(\lambda)=-\infty$;

(iii) $\mathcal{H}'(\lambda)>0$ *if* $\lambda\in(0,\lambda_0)$ *and* $\mathcal{H}'(\lambda)<0$ *if* $\lambda\in(\lambda_0,\infty)$.

The following lemma will play an essential role in this paper, and it is similar to the lemma used firstly by E. Vitillaro in [43].

Lemma 4.4. *Assume that $E(0)<\mathcal{H}(\lambda_0)$. Then*

(i) *If $\|\tilde{u}_0\|_a<\dfrac{\lambda_0}{\sqrt{\mu(0)}}$ then exists $\hat{\lambda}_1\in[0,\lambda_0)$ such that*

$$y(t)\le\hat{\lambda}_1,\ \forall t\in[0,T_\infty); \tag{4.39}$$

(ii) *If* $\|\tilde{u}_0\|_a > \dfrac{\lambda_0}{\sqrt{\mu(0)}}$ *and* $E(0) \geq 0$ *then exists* $\hat{\lambda}_2 \in (\lambda_0, \infty)$ *such that*

$$y(t) \geq \hat{\lambda}_2, \ \forall t \in [0, T_\infty). \tag{4.40}$$

Proof of Lemma 4.4. Sine $0 \leq E(0) < \mathcal{H}(\lambda_0)$, there are constants $0 \leq \hat{\lambda}_1 < \lambda_0 < \hat{\lambda}_2$ such that

$$\mathcal{H}(\hat{\lambda}_1) = \mathcal{H}(\hat{\lambda}_2) = E(0). \tag{4.41}$$

First, we assume that $\|\tilde{u}_0\|_a < \dfrac{\lambda_0}{\sqrt{\mu(0)}}$. We have

$$\mathcal{H}(\hat{\lambda}_1) = E(0) \geq \mathcal{H}(y(0)) = \mathcal{H}\left(\sqrt{\mu(0)}\|\tilde{u}_0\|_a\right). \tag{4.42}$$

By Lemma 4.3, from (4.42), we get $\|\tilde{u}_0\|_a \leq \dfrac{\hat{\lambda}_1}{\sqrt{\mu(0)}}$. We claim that $y(t) \leq \hat{\lambda}_1$ for all $t \in [0, T_\infty)$. Suppose, by contradiction, there exists $t_0 \in [0, T_\infty)$ such that $y(t_0) > \hat{\lambda}_1$.

By the continuity of y, without loss of generality, we may assume that $y(t_0) \in \left(\hat{\lambda}_1, \lambda_0\right)$. By Lemma 4.1 and 4.3, we get

$$E(t_0) \geq \mathcal{H}(y(t_0)) > \mathcal{H}(\hat{\lambda}_1) = E(0), \tag{4.43}$$

this is an contradiction, because of $E(t_0) \leq E(0)$.

Now, we assume that $\|\tilde{u}_0\|_a > \dfrac{\lambda_0}{\sqrt{\mu(0)}}$. We have

$$\mathcal{H}(\hat{\lambda}_2) = E(0) \geq \mathcal{H}(y(0)) = \mathcal{H}\left(\sqrt{\mu(0)}\|u_0\|_a\right). \tag{4.44}$$

By Lemma 4.3, from (4.44), we get $\|\tilde{u}_0\|_a \geq \dfrac{\hat{\lambda}_2}{\sqrt{\mu(0)}}$. We claim that $y(t) \geq \hat{\lambda}_2$ for all $t \in [0, T_\infty)$. By the same arguments as above, we suppose by contradiction that there exists $t_0 \in [0, T_\infty)$ such that $y(t_0) < \hat{\lambda}_2$. By the continuity of y, without loss of generality, we may assume that $y(t_0) \in \left(\lambda_0, \hat{\lambda}_2\right)$. By Lemmas 4.1 and 4.3, we get

$$E(t_0) \geq \mathcal{H}(y(t_0)) > \mathcal{H}(\hat{\lambda}_2) = E(0), \tag{4.45}$$

in contradiction with $E(t_0) \leq E(0)$. Lemma 4.4 is proved. □

Theorem 4.5. *Assume that the assumptions* (A_2'), (A_3'), (K_2), (F_2) *and* (G_1) *hold. Then, for any initial conditions* $\tilde{u}_0 \in V$ *such that* $\|\tilde{u}_0\|_a > \dfrac{\lambda_0}{\sqrt{\mu(0)}}$ *and*

$$0 \leq E(0) < \min\left\{\mathcal{H}(\lambda_0), \ \frac{\left[(p-1)^2 L - \mu_*\right]\hat{\lambda}_2^2}{2p^2\mu(0)}\right\},$$

the weak solution of the Prob. (1.1)-(1.3) *blows up at finite time and the lifespan* T_∞ *of the solution* u *is defined by*

$$T_\infty \leq \frac{16p(p-1)\mu(0)\rho(0)}{(p-2)^2\left[\left((p-1)^2 L - \mu_*\right)\hat{\lambda}_2^2 - 2p^2\mu(0)E(0)\right]} = T_\infty^{\max}, \tag{4.46}$$

and

$$T_\infty \geq \int_{\rho(0)}^{\infty} \frac{dz}{\Psi_2(z)} = T_\infty^{\min}, \tag{4.47}$$

where

$$\Psi_2(z) = E(0) + \frac{\mu(0)}{\alpha} z + (1+d_2) R K_1 \overline{d}_2 \left[\widetilde{D}_p^p \left(\frac{2}{\alpha}\right)^{\frac{p}{2}} z^{\frac{p}{2}} + \sum\nolimits_{i=1}^{N} \widetilde{D}_{q_i}^{q_i} \left(\frac{2}{\alpha}\right)^{\frac{q_i}{2}} z^{\frac{q_i}{2}} \right]. \tag{4.48}$$

Proof of Theorem 4.5. By the same method in the proof of Theorem 4.2, with $T_0 > 0$, $\beta > 0$ and $\tau > 0$ are chosen later, we define the functional

$$\begin{aligned} M &: [0, T_0] \longrightarrow \mathbb{R} \\ t &\longmapsto M(t) = 2\int_0^t \rho(s)\,ds + 2(T_0 - t)\rho(0) + \beta(t+\tau)^2 . \end{aligned} \tag{4.49}$$

From (4.24) and Lemma 4.4, we get

$$\begin{aligned} D(t) &\geq (1-p)\beta - pE(0) + \left[\left(\frac{p}{2} - 1\right)\mu_* + \left(1 - \frac{1}{2p} - \frac{p}{2}\right) G_\infty \right] \frac{\hat{\lambda}_2^2}{\mu(0)} \\ &= (1-p)\beta - pE(0) + \frac{\left[(p-1)^2 L - \mu_*\right]\hat{\lambda}_2^2}{2p\mu(0)}. \end{aligned} \tag{4.50}$$

Choosing β, $0 < \beta \leq \frac{\left[(p-1)^2 L - \mu_*\right]\hat{\lambda}_2^2 - 2p^2\mu(0)E(0)}{2p(p-1)\mu(0)}$, it follows from (4.20), (4.21) and (4.50) that

$$M(t) \geq \left[\left(1 - \frac{p}{2}\right) M^{\frac{-p}{2}}(0)\, M'(0) t + M^{1-\frac{p}{2}}(0) \right]^{-\frac{2}{p-2}}, \ \forall t \in [0, T_0]. \tag{4.51}$$

If we choose $\tau > \frac{2\rho(0)}{(p-2)\beta}$ and $T_0 \geq \frac{\beta\tau^2}{(p-2)\beta\tau - 2\rho(0)}$, we will have

$$T_* = -\frac{M^{1-\frac{p}{2}}(0)}{\left(1 - \frac{p}{2}\right) M^{\frac{-p}{2}}(0)\, M'(0)} = \frac{2M(0)}{(p-2)M'(0)} = \frac{2T_0\rho(0) + \beta\tau^2}{(p-2)\beta\tau} \in (0, T_0]. \tag{4.52}$$

From (4.52), we get $\lim\limits_{t \to T_*^-} M(t) = \infty$. This is also a contradiction, hence the solution blows up at finite time.

As in proof of Theorem 4.2, we have

$$T_\infty \leq \frac{16p(p-1)\mu(0)\rho(0)}{(p-2)^2 \left[\left((p-1)^2 L - \mu_* \right) \hat{\lambda}_2^2 - 2p^2 \mu(0) E(0) \right]} = T_\infty^{\max}. \tag{4.53}$$

Finally, we seek a lower bound for the blow-up time T_∞ for the solution u. We have

$$\begin{aligned} \rho'(t) &\leq E(0) + \frac{\mu(0)}{\alpha}\rho(t) + (1+d_2)\int_1^R xK(x,t) F(u(x,t))\,dx \\ &\leq E(0) + \frac{\mu(0)}{\alpha}\rho(t) + (1+d_2) R K_1 \overline{d}_2 \int_1^R \left(|u(x,t)|^p + \sum_{i=1}^{N} |u(x,t)|^{q_i} \right) dx \\ &= E(0) + \frac{\mu(0)}{\alpha}\rho(t) + (1+d_2) R K_1 \overline{d}_2 \left(\|u(t)\|_{L^p}^p + \sum_{i=1}^{N} \|u(t)\|_{L^{q_i}}^{q_i} \right) \\ &\leq E(0) + \frac{\mu(0)}{\alpha}\rho(t) + (1+d_2) R K_1 \overline{d}_2 \left(\widetilde{D}_p^p \|u(t)\|_a^p + \sum_{i=1}^{N} \widetilde{D}_{q_i}^{q_i} \|u(t)\|_a^{q_i} \right) \\ &\leq E(0) + \frac{\mu(0)}{\alpha}\rho(t) + (1+d_2) R K_1 \overline{d}_2 \left[\widetilde{D}_p^p \left(\frac{2}{\alpha}\right)^{\frac{p}{2}} \rho^{\frac{p}{2}}(t) + \sum_{i=1}^{N} \widetilde{D}_{q_i}^{q_i} \left(\frac{2}{\alpha}\right)^{\frac{q_i}{2}} \rho^{\frac{q_i}{2}}(t) \right] \\ &= \Psi_2(\rho(t)). \end{aligned} \tag{4.54}$$

From (4.54), we get

$$T_\infty \geq \int_{\rho(0)}^{\infty} \frac{dz}{\Psi_2(z)} = T_\infty^{\min}. \tag{4.55}$$

Theorem 4.5 is proved. □

5 Global existence and decay of solutions

To give the theorem on global existence and decay, we make following assumption

(G_3) $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ satisfies

(i) $g(t) \geq 0$ for all $t \geq 0$,

(ii) $L = \mu_* - \int_0^\infty g(s)\,ds > 0$,

(iii) There exists a function $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$g'(t) \leq -\xi(t) g(t), \ \xi'(t) \leq 0, \ \forall t \geq 0, \ \int_0^\infty \xi(s)\,ds = \infty.$$

We have the following lemmas.

Lemma 5.1. *Suppose that $\tilde{u}_0 \in V$ and the assumptions (A_2'), (A_3'), (K_2), (F_2), (G_3) hold. Then the energy functional $E(t)$ satisfies*

$$E'(t) \leq -\frac{1}{2}\xi(t)(g \star u)(t) - \int_1^R xK(x,t) F(u(x,t))\,dx \leq 0. \tag{5.1}$$

Proof of Lemma 5.1. It is similar to proof of Lemma 4.1, we can use the similar computation to obtain this result. □

Lemma 5.2. *Assume that the assumptions (A_2'), (A_3'), (K_2), (F_2) and (G_3) hold. Then, for any initial conditions $\tilde{u}_0 \in V$ such that $\|\tilde{u}_0\|_a < \frac{\lambda_0}{\sqrt{\mu(0)}}$, $E(0) < \mathcal{H}(\lambda_0)$, the weak solution of the Prob.* (1.1)-(1.3) *is global on $\mathbb{R}_+$.*

Proof of Lemma 5.2. It is sufficients to combine the last statement in Theorem 3.1 and the estimate given in Lemma 4.4. Then, Lemma 5.2 is proved. □

We define the following functionals:

$$I(t) = (\mu(t) - \tilde{g}(t)) \|u(t)\|_a^2 - p\int_1^R xK(x,t) F(u(x,t))\,dx, \tag{5.2}$$

and

$$\mathcal{L}(t) = E(t) + \rho(t). \tag{5.3}$$

Lemma 5.3. *Assume that the assumptions (A_2'), (A_3'), (K_2), (F_2) and (G_3) hold. Then, for any initial conditions $\tilde{u}_0 \in V$ such that $\|\tilde{u}_0\|_a < \frac{\lambda_0}{\sqrt{\mu(0)}}$, $E(0) < \mathcal{H}(\lambda_0)$, there exist positive constants β_1, β_2 such that*

$$\beta_1 E_1(t) \leq \mathcal{L}(t) \leq \beta_2 E_1(t), \ \forall t \geq 0, \tag{5.4}$$

where

$$E_1(t) = (g \star u)(t) + \|u(t)\|_a^2. \tag{5.5}$$

Proof of Lemma 5.3. From (5.2), (A_2'), (A_3'), (K_2), (F_2) and (4.36), we have

$$\eta_* \|u(t)\|_a^2 \leq I(t) \leq \mu(0) \|u(t)\|_a^2, \ \forall t \geq 0, \tag{5.6}$$

where

$$\eta_* = L\left[1 - pRK_1\overline{d}_2\left(\widetilde{D}_p^p L^{-\frac{p}{2}}\hat{\lambda}_1^{p-2} + \sum_{i=1}^{N}\widetilde{D}_{q_i}^{q_i} L^{-\frac{q_i}{2}}\hat{\lambda}_1^{q_i-2}\right)\right] > 0. \tag{5.7}$$

Note that

$$\begin{aligned}\mathcal{L}(t) &= E(t) + \rho(t)\\ &= \frac{1}{2}(g \star u)(t) + \left(\frac{1}{2} - \frac{1}{p}\right)(\mu(t) - \tilde{g}(t))\|u(t)\|_a^2 + \frac{I(t)}{p} + \rho(t)\\ &\geq \frac{1}{2}(g \star u)(t) + \frac{(p-2)L}{2p}\|u(t)\|_a^2 \geq \beta_1 E_1(t),\end{aligned}$$

where $\beta_1 = \frac{1}{2}\min\left\{1, \frac{(p-2)L}{p}\right\}$.

On the other hand, we have

$$\begin{aligned}\mathcal{L}(t) &= E(t) + \rho(t)\\ &= \frac{1}{2}(g \star u)(t) + \left(\frac{1}{2} - \frac{1}{p}\right)(\mu(t) - \tilde{g}(t))\|u(t)\|_a^2 + \frac{I(t)}{p} + \frac{\alpha}{2}\|u(t)\|_a^2 + \frac{1}{2}\|u(t)\|_0^2\\ &\leq \frac{1}{2}(g \star u)(t) + \frac{1}{2}\left[\mu(0) + \alpha + \frac{R(R-1)^2}{2}\right]\|u(t)\|_a^2 \leq \beta_2 E_1(t),\end{aligned}$$

where $\beta_2 = \frac{1}{2}\max\left\{1, \mu(0) + \alpha + \frac{R(R-1)^2}{2}\right\}$. Thus, Lemma 5.3 is proved. $\square$

Finally, we have the following theorem.

Theorem 5.4. *Assume that the assumptions (A_2'), (A_3'), (K_2), (F_2) and (G_3) hold. Then, for any initial conditions $\tilde{u}_0 \in V$ such that $\|\tilde{u}_0\|_a < \dfrac{\lambda_0}{\sqrt{\mu(0)}}$, $E(0) < \mathcal{H}(\lambda_0)$ and*

$$\eta_* + \frac{p}{d_2}\mu_* - \mu(0) > 0, \tag{5.8}$$

there exists $C_* > 0$, $\gamma_* > 0$ such that

$$E_1(t) \leq C_* \exp\left(-\gamma_* \int_0^t \xi(s)\, ds\right), \ \forall t \geq 0. \tag{5.9}$$

Proof of Theorem 5.4. First, we have

$$-\mu(t)\|u(t)\|_a^2 \leq -\mu_*\|u(t)\|_a^2, \tag{5.10}$$

$$\int_0^t g(t-s)\, a(u(s), u(t))\, ds \leq \frac{p}{4(d_2 - p)}(g \star u)(t) + \frac{d_2}{p}\tilde{g}(t)\|u(t)\|_a^2, \tag{5.11}$$

and

$$\begin{aligned}\langle K(t) f(u(t)), u(t)\rangle &\leq d_2 \int_1^R xK(x,t)\, F(u(x,t))\, dx \\ &= \frac{d_2}{p}\left[(\mu(t) - \tilde{g}(t))\|u(t)\|_a^2 - I(t)\right]\\ &\leq \frac{d_2}{p}(\mu(0) - \tilde{g}(t))\|u(t)\|_a^2 - \frac{d_2\eta_*}{p}\|u(t)\|_a^2.\end{aligned} \tag{5.12}$$

From (4.30), (5.10)-(5.12), we have

$$\begin{aligned}\rho'(t) &\leq \frac{p}{4(d_2-p)}(g\star u)(t) - \frac{d_2}{p}\left(\eta_* + \frac{p}{d_2}\mu_* - \mu(0)\right)\|u(t)\|_a^2 \\ &= \frac{p}{2(d_2-p)}(g\star u)(t) - \frac{p}{4(d_2-p)}(g\star u)(t) - \frac{d_2}{p}\left(\eta_* + \frac{p}{d_2}\mu_* - \mu(0)\right)\|u(t)\|_a^2 \\ &\leq \frac{p}{2(d_2-p)}(g\star u)(t) - \beta_3 E_1(t),\end{aligned} \tag{5.13}$$

where $\beta_3 = \min\left\{\frac{p}{4(d_2-p)}, \frac{d_2}{p}\left(\eta_* + \frac{p}{d_2}\mu_* - \mu(0)\right)\right\} > 0$.

Combine (5.3), Lemma 5.1 and (5.13), it gives that

$$\mathcal{L}'(t) \leq \rho'(t) \leq \frac{p}{2(d_2-p)}(g\star u)(t) - \beta_3 E_1(t). \tag{5.14}$$

From (5.14), we obtain

$$\xi(t) L'(t) \leq \frac{p}{2(d_2-p)}\xi(t)(g\star u)(t) - \beta_3\xi(t)E_1(t) \leq -\frac{p}{d_2-p}E'(t) - \beta_3\xi(t)E_1(t). \tag{5.15}$$

We define the functional

$$L_1(t) = \xi(t)\mathcal{L}(t) + \frac{p}{d_2-p}E(t). \tag{5.16}$$

By direct computation, it implies that

$$\begin{aligned}L_1'(t) &= \xi'(t)\mathcal{L}(t) + \xi(t)\mathcal{L}'(t) + \frac{p}{d_2-p}E'(t) \\ &\leq -\beta_3\xi(t)E_1(t) \leq -\frac{\beta_3}{\beta_2}\xi(t)\mathcal{L}(t) \\ &\leq -\frac{\beta_3}{\beta_2}\left[\xi(0) + \frac{p}{d_2-p}\right]^{-1}\xi(t)L_1(t) = -\gamma_*\xi(t)L_1(t).\end{aligned} \tag{5.17}$$

Also by directly integrating (5.17), we deduce

$$E_1(t) \leq \frac{d_2-p}{p\beta_1}L_1(t) \leq C_*\exp\left(-\gamma_*\int_0^t \xi(s)\,ds\right), \ \forall t \geq 0, \tag{5.18}$$

where $C_* = \frac{d_2-p}{p\beta_1}L_1(0)$. Theorem 5.4 is proved. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this article. They read and approved the final manuscript.

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