A class of new modulus-based matrix splitting methods for linear complementarity problem*

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Abstract

In this paper, to economically and fast solve the linear complementarity problem, based on a new equivalent fixed-point form of the linear complementarity problem, we establish a class of new modulus-based matrix splitting methods, which is different from the previously published works. Some sufficient conditions to guarantee the convergence of this new iteration method are presented. Numerical examples are offered to show the efficacy of this new iteration method. Moreover, the comparisons on numerical results show the computational efficiency of this new iteration method advantages over the corresponding modulus method, the modified modulus method and the modulus-based Gauss-Seidel method.

Keywords: Linear complementarity problem; matrix splitting; iteration method; convergence

AMS classification: 90C33, 65F10,65F50,65G40

1 Introduction

As a very useful tool, the linear complementarity problem (LCP) often plays a key role in diverse fields of scientific computing and engineering applications, such as convex quadratic programming, bimatrix games, the free boundary problem, the contact problem, upon price problem, nonnegative constrained least squares problems, market equilibrium problems, see [1–6]. Essentially, the LCP is to find a vector that meets a kind of the inequality systems, i.e., finding a vector $z \in \mathbb{R}^n$ meets

$$v = Az + q \geq 0,\ z \geq 0 \text{ and } z^T v = 0, \tag{1.1}$$

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where $A \in \mathbb{R}^{n\times n}$ is a given matrix and $q \in \mathbb{R}^n$ is a given vector.

For solving the LCP (1.1), some direct methods (such as the pivot method in [4], the Bard-type method [7]) can be considered, it is noted that these direct methods are often restricted because of the roundoff error and the problem size. In particular, for the sparse and large system matrix A, these direct methods are short of ability to keep the sparsity of the system matrix A intact and require too many pivots so that they may be seriously challenged.

To avoid these disadvantages of direct methods, the alternative is to employ iteration methods for solving the LCP (1.1). As is known, the advantage of iteration methods over direct methods is the memory and the computational requirement. Not only that, the former can be easily executed on high-performance computer than the later. Naturally, in actual implementations, we often use iteration methods instead of direct methods for solving the LCP (1.1).

To construct the fast and economical iteration methods, one of popular and highly sought-after approaches is necessary to reformulate the LCP (1.1) as a certain equation, that is to say, the solution of this equation must be the same as the LCP (1.1). For this reason, some efficient equivalent forms of the LCP (1.1) have been developed. For example, in [8], let $z_+ \in \mathbb{R}^n$ with $(z_+)_i = \max\{0, x_i\}, i = 1, 2, \ldots, n$, then Mangasarian presented the following equivalent form

$$z = (z - \omega\Omega(Az + q))_+, \text{with } \omega > 0, \tag{1.2}$$

where $\Omega \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, and proposed the projected Jacobi overrelaxation, the projected SOR method and the projected symmetric SOR method. Making use of the equivalent form (1.2) to design iteration methods, one can also see [9–15] for more details. In fact, the equivalent form (1.2) belongs to a kind of fixed point schemes.

The second equivalent form of the LCP (1.1) is obtained by $z = \frac{|x|+x}{2}$ and $v = (\omega\Omega)^{-1}(|x| - x)$ for the LCP (1.1), and is of the form

$$x = \frac{|x| + x}{2} - \frac{\omega\Omega}{2}(A\frac{|x| + x}{2} + q), \text{with } \omega > 0, \tag{1.3}$$

where $\Omega \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix. One can see [16] for the corresponding iteration methods. Therewith, AL-said and Noor [17–19] extended further the work in [16]. Of course, the equivalent form (1.3) is a class of fixed point schemes as well.

Recently, making use of $z = \frac{|x|+x}{\gamma}$ and $v = \frac{\Omega}{\gamma}(|x| - x)$ and $A = M - N$ for the LCP (1.1), Bai in [20] given the following general equivalent form

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q, \text{with } \gamma > 0, \tag{1.4}$$

where $\Omega \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix and first designed a class of modulus-based matrix splitting iteration methods. This general equivalent form (1.4) covers the published works in [2, 21–24]. This kind of modulus-based matrix splitting iteration methods was regarded as a powerful method of solving the LCP (1.1). Other deformations of the equivalent form (1.4), one can see [25–33] for more details. In addition, this ideology has been successfully expanded to other complementarity problems, inculding the nonlinear complementarity problem [34–37], the implicit complementarity problem [38], the quasi-complementarity problem [39] and the horizontal linear complementarity problem [40].

In this paper, without making use of change of variable, by directly using the inequality systems of the LCP (1.1), a new equivalent form, fixed-point form, of the LCP (1.1) is obtained, which is different from the previously published works in [2,8,16,20–24]. This new equivalent form allows us to design a new class of iteration methods for solving the LCP (1.1). Here, we call a class of new modulus-based matrix splitting iteration methods. This class of new modulus-based matrix splitting iteration methods not only inherits the virtues of the presented modulus-based methods, but also generates many relaxed versions. We discuss the convergence property of this kind of new iteration methods and give its some convergence conditions under suitable assumptions. In addition, numerical examples are also provided to verify that this kind of new iteration methods are feasible and overmatch the corresponding modulus method, the modified modulus method and the classical modulus-based matrix splitting iteration methods in aspects of the computational efficiency.

The layout of this paper is organized below. In Section 2, for the sake of discussion in the rest of this paper, some necessary definitions, notations and well-known lemmas are provided. In Section 3, a class of new modulus-based matrix splitting iteration methods is established by the new equivalent fixed-point form of the LCP (1.1). And its convergence conditions are given in depth in Section 4. Numerical comparison the proposed methods with the modulus method, the modified modulus method and the classical modulus-based matrix splitting iteration methods are reported in Section 5. Finally, in Section 6, we draw some remarks to end this paper.

2 Preliminaries

In this section, we briefly introduce some necessary definitions, notations, and the well-known lemmas, which are used in the sequel discussions.

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. Then it is named as an M-matrix if $A^{-1} \geq 0$ and

$$a_{ij} = \begin{cases} > 0 \text{ for } i = j, \\ \leq 0 \text{ for } i \neq j, \end{cases} \quad i, j = 1, 2, \ldots, n;$$

an H-matrix if matrix $\langle A\rangle = (\langle a\rangle_{ij})$ is an M-matrix, where

$$\langle a\rangle_{ij} = \begin{cases} |a_{ij}| \text{ for } i = j, \\ -|a_{ij}| \text{ for } i \neq j, \end{cases} \quad i, j = 1, 2, \ldots, n;$$

(matrix $\langle A\rangle$ is called the comparison matrix of matrix A); an H_+-matrix if A is an H-matrix with $\mathrm{diag}(A) > 0$; a P-matrix if all of its principle minors are positive [3, 41] or the LCP (1.1) has a unique solution [1, 20]. In addition, $A = M - N$ is an M-splitting if M is a nonsingular M-matrix and $N \geq 0$; an H-splitting if $\langle M \rangle - |N|$ is an M-matrix with $|N| = (|n_{ij}|) \in \mathbb{R}^{n\times n}$. As is known, if $A = M - N$ is an M-splitting and A is a nonsingular M-matrix, then $\rho(M^{-1}N) < 1$, where $\rho(\cdot)$ indicates the spectral radius of the matrix, see [3, 41].

Lemma 2.1 [26] *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ with $a_{ij} \geq 0$. If there exists $u \in \mathbb{R}^n$ with $u > 0$ such that $Au < u$, then $\rho(A) < 1$.*

Lemma 2.2 [42] *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix, D be the diagonal part of the matrix A, and $A = D - B$. Then matrices A and $|D|$ are nonsingular, $|A^{-1}| \leq \langle A \rangle^{-1}$ and $\rho(|D|^{-1}|B|) < 1$.*

Lemma 2.3 [3, 42] *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix and $B \in \mathbb{R}^{n\times n}$ be a Z-matrix with $A \leq B$. Then B is an M-matrix.*

3 New modulus-based matrix splitting method

In this section, we introduce a class of new modulus-based matrix splitting methods for solving the LCP (1.1). To this end, we give a new lemma, see Lemma 3.1.

Lemma 3.1 *Let $a, b \in \mathbb{R}$. Then*

$$a \geq 0, b \geq 0, ab = 0 \tag{3.1}$$

if and only if

$$a + b = |a - b|. \tag{3.2}$$

This result carries immediately over to vectors in $\mathbb{R}^n$.

Proof. We first prove that (3.2)$\Rightarrow$ (3.1). By calculation, clearly,

$$a + b = |a - b| \Rightarrow (a + b)^2 = |a - b|^2 \Rightarrow a^2 + 2ab + b^2 = a^2 - 2ab + b^2 \Rightarrow ab = 0.$$

In addition, if $a = 0$, then from (3.2) we have $b = |b| \geq 0$. It follows that $ab = 0$. Similarly, if $b = 0$, then $ab = 0$ as well.

Next, we prove that (3.1) $\Rightarrow$ (3.2). By (3.1), we have

$$(a + b)^2 = (a - b)^2.$$

Taking the square root calculation for the above equation, we have

$$|a + b| = |a - b|.$$

Noting that $a \geq 0, b \geq 0$, it follows that (3.2) is valid.

This result is also true for vectors in $\mathbb{R}^n$. □

Based on Lemma 3.1, a new equivalent expression of the LCP (1.1) can be obtained, which is described below.

Theorem 3.1 *Let Ω be a positive diagonal matrix. Then the LCP (1.1) is equal to*

$$(\Omega + A)z = |(A - \Omega)z + q| - q. \tag{3.3}$$

Proof. Obviously, the LCP (1.1) is equal to

$$w = Az + q \geq 0,\ \ \Omega z \geq 0 \text{ and } (\Omega z)^T w = 0,$$

where Ω is a positive diagonal matrix. Here, we take $a = Az + q$ and $b = \Omega z$ for Lemma 3.1 and get

$$(\Omega + A)z + q = |(A - \Omega)z + q|.$$

Therefore, the proof of Theorem 3.1 is completed. □

When $\Omega = I$ in Theorem 3.1, the result in Theorem 3.1 reduces to a restate of the equivalence of LCP (1.1) with the min-equation, which was considered in [43, 44].

Theorem 3.1 implies that once we obtain the value of z by solving the implicit fixed-point equation (3.3), this z is the solution of LCP (1.1) as well. To rapidly and economically obtain the solution of the implicit fixed-point equation (3.3), one of the more popular strategies is to construct the iteration method by the efficient matrix splitting of the related matrix A. Based on this, we take $A = M - N$ as a matrix splitting of matrix $A \in \mathbb{R}^{n\times n}$ in (3.3). Then from (3.3) we have

$$(\Omega + M)z = Nz + |(A - \Omega)z + q| - q. \tag{3.4}$$

This new equivalent expression (3.4) of the LCP (1.1) is different from the perviously works in [2, 8, 16, 20–24]. Based on Eq. (3.4), we can naturally establish the following iteration method, which is named as a class of new modulus-based matrix splitting iteration methods for the LCP (1.1), see Method 3.1.

Method 3.1 *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, and matrix $\Omega + M$ be nonsingular, where Ω is a positive diagonal matrix. Given a non-negative initial vector $z^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, 2, \ldots$ until the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ converge, compute $z^{(k+1)} \in \mathbb{R}^n$ by solving the linear system*

$$(\Omega + M)z^{(k+1)} = Nz^{(k)} + |(A - \Omega)z^{(k)} + q| - q. \tag{3.5}$$

In [20], Bai ingeniously designed the modulus-based matrix splitting iteration methods, see Method 3.2.

Method 3.2 [20] *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, and matrix $\Omega + M$ be nonsingular, where Ω is a positive diagonal matrix. Given an initial vector $x^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, 2, \ldots$ until the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ converge, compute $z^{(k+1)} \in \mathbb{R}^n$*

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}),\ \ \gamma > 0.$$

where $x^{(k+1)}$ is obtained by solving the linear system

$$(\Omega + M)x^{(k+1)} = Nx^{(k)} + (\Omega - A)|x^{(k)}| - \gamma q. \tag{3.6}$$

Investigating Method 3.1 and Method 3.2, these two methods have resemblances, but Method 3.1 does not belong to Method 3.2, vice versa.

In addition, the new modulus-based matrix splitting iteration methods provides a new general framework for solving the LCP (1.1). Based on the matrix splitting technique, some new modulus-based relaxation methods are obtained as well. Specially, we express the system matrix A as

$$A = D - L - U,$$

where $D = \text{diag}(A)$, L and U, respectively, are the strictly lower upper triangular matrices of A. Then

(a) when $M = A$, $\Omega = I$, $N = 0$, from Method 3.1 we have

$$(I + A)z^{(k+1)} = |(A - I)z^{(k)} + q| - q,$$

which is called as the new modulus (NMOD) method.

(b) when $M = A$, $N = 0$, $\Omega = \alpha I$, from Method 3.1 we have

$$(\alpha I + A)z^{(k+1)} = |(A - \alpha I)z^{(k)} + q| - q,$$

which is called as the new modified modulus (NMMOD) iteration method.

(c) when $M = D$, $N = L + U$, from Method 3.1 we have

$$(\Omega + D)z^{(k+1)} = (L + U)z^{(k)} + |(A - \Omega)z^{(k)} + q| - q,$$

which is called as the new modulus-based Jacobi (NMJ) iteration method

(d) when $M = D - L$, $N = U$, from Method 3.1 we have

$$(\Omega + D - L)z^{(k+1)} = Uz^{(k)} + |(A - \Omega)z^{(k)} + q| - q,$$

which is called as the new modulus-based Gauss-Seidel (NMGS) iteration method.

(e) when $M = \frac{1}{\alpha}D - L$, $N = (\frac{1}{\alpha} - 1)D + U$, from Method 3.1 we have

$$(\alpha\Omega + D - \alpha L)z^{(k+1)} = [(1 - \alpha)D + \alpha Uz^{(k)}] + \alpha(|(A - \Omega)z^{(k)} + q| - q),$$

which is called as the new modulus-based SOR (NMSOR) iteration method.

(f) when $M = \frac{1}{\alpha}(D - \beta L)$, $N = \frac{1}{\alpha}((1 - \alpha)D + (\alpha - \beta)L + \alpha U)$, from Method 3.1 we have

$$(\alpha\Omega + D - \beta L)z^{(k+1)} = [(1 - \alpha)D + (\alpha - \beta)L + \alpha Uz^{(k)}] + \alpha(|(A - \Omega)z^{(k)} + q| - q),$$

which is called as the new modulus-based AOR (NMAOR) iteration method.

4 Convergence analysis

In this section, some sufficient conditions are given to guarantee the convergence of Method 3.1 under suitable conditions.

First, we give a general convergence condition of Method 3.1, see Theorem 4.1.

Theorem 4.1 *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$ with A being a P-matrix, and matrix $\Omega+M$ be nonsingular, where Ω is a positive diagonal matrix. If $\rho(T) < 1$, where*

$$T = |(\Omega + M)^{-1}|(|N| + |A - \Omega|),$$

then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ produced by Method 3.1 converges to the unique solution $z^* \in \mathbb{R}^n_+$ of the LCP (1.1) for a non-negative initial vector.

Proof. Let z^* is a solution of the LCP (1.1). Then from (3.4) we obtain

$$(\Omega + M)z^* = Nz^* + |(A - \Omega)z^* + q| - q. \tag{4.1}$$

Based on (3.5) and (4.1), note that matrix $\Omega + M$ is nonsingular, we can obtain

$$x^{(k+1)} - x^* = (\Omega + M)^{-1}(N(x^{(k)} - x^*) + |(A - \Omega)z^{(k)} + q| - |(A - \Omega)z^* + q|). \tag{4.2}$$

This indicates that

$$\begin{aligned}
|x^{(k+1)} - x^*| =& |(\Omega + M)^{-1}(N(x^{(k)} - x^*) + |(A - \Omega)z^{(k)} + q| - |(A - \Omega)z^* + q|)| \\
\le& |(\Omega + M)^{-1}N(x^{(k)} - x^*)| \\
&+ |(\Omega + M)^{-1}(|(A - \Omega)z^{(k)} + q| - |(A - \Omega)z^* + q|))| \\
\le& |(\Omega + M)^{-1}N| \cdot |x^{(k)} - x^*| \\
&+ |(\Omega + M)^{-1}| \cdot ||(A - \Omega)z^{(k)} + q| - |(A - \Omega)z^* + q|| \\
\le& |(\Omega + M)^{-1}N| \cdot |x^{(k)} - x^*| \\
&+ |(\Omega + M)^{-1}| \cdot |(A - \Omega)z^{(k)} + q - (A - \Omega)z^* - q| \\
=& |(\Omega + M)^{-1}N| \cdot |x^{(k)} - x^*| + |(\Omega + M)^{-1}| \cdot |(A - \Omega)(z^{(k)} - z^*)| \\
\le& |(\Omega + M)^{-1}| \cdot |N| \cdot |x^{(k)} - x^*| + |(\Omega + M)^{-1}| \cdot |A - \Omega| \cdot |z^{(k)} - z^*| \\
=& |(\Omega + M)^{-1}|(|N| + |A - \Omega|)|z^{(k)} - z^*| \\
=& T|x^{(k)} - x^*|.
\end{aligned}$$

Obviously, when $\rho(T) < 1$, for a non-negative initial vector, the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ produced by Method 3.1 converges to the unique solution $z^* \in \mathbb{R}^n_+$ of the LCP (1.1). □

Since

$$|A - \Omega| = |M - N - \Omega| \le |M - \Omega| + |N|,$$

Corollary 4.1 can be obtained.

Corollary 4.1 *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$ with A being a P-matrix, and matrix $\Omega+M$ be nonsingular, where Ω is a positive diagonal matrix. If $\rho(\bar{T}) < 1$, where*

$$\bar{T} = |(\Omega + M)^{-1}|(2|N| + |M - \Omega|),$$

then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ produced by Method 3.1 converges to the unique solution $z^* \in \mathbb{R}^n_+$ of the LCP (1.1) for a non-negative initial vector.

Using $\|\cdot\|_2$ for (4.2), it is easy to obtain Corollary 4.2.

Corollary 4.2 *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$ with A being a P-matrix, and matrix $\Omega + M$ be nonsingular, where Ω is a positive diagonal matrix. Let*

$$\eta(R) = \|(\Omega + M)^{-1}\|_2(\|N\|_2 + \|A - \Omega\|_2)$$

and

$$\bar{\eta}(R) = \|(\Omega + M)^{-1}\|_2(2\|N\|_2 + \|M - \Omega\|_2).$$

If $\eta(R) < 1$ or $\bar{\eta}(R) < 1$, then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ produced by Method 3.1 converges to the unique solution $z^* \in \mathbb{R}^n_+$ of the LCP (1.1) for a non-negative initial vector.

Next, we consider the convergence condition of Method 3.1 when the system matrix A is an H_+-matrix. To this end, Lemma 4.1 is required.

Lemma 4.1 *Let $A = M - N$ be an H-splitting of the matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, where A is an H_+-matrix. Then matrix $D - |B|$ is an M-matrix, where $D = diag(A)$.*

Proof. Since $A = M - N$ be an H-splitting, then $\langle M\rangle - |N|$ is an M-matrix. By calculate, we obtain

$$d_{ii} = a_{ii} = m_{ii} - n_{ii} = |m_{ii} - n_{ii}| \geq |m_{ii}| - |n_{ii}|$$

and

$$-|b_{ij}| = -|a_{ij}| = -|m_{ij} - n_{ij}| \geq -|m_{ij}| - |n_{ij}|.$$

Thus,

$$\langle M\rangle - |N| \leq D - |B|.$$

Based on Lemma 2.3, matrix $D - |B|$ is an M-matrix. □

Based on Lemma 4.1, Theorem 4.2 can be obtained.

Theorem 4.2 *Let $A = M - N$ be an H-splitting of the matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, where A is an H_+-matrix. Let the diagonal matrix $\Omega \geq D$, where $D = diag(A)$. Then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ produced by Method 3.1 converges to the unique solution $z^* \in \mathbb{R}^n_+$ of the LCP (1.1) for a non-negative initial vector.*

Proof. Since $\langle M\rangle - |N|$ is an M-matrix and

$$\langle M\rangle - |N| \leq \langle M\rangle,$$

matrix $\langle M\rangle$ is an M-matrix from Lemma 2.3. By Lemma 2.2 and $\Omega \geq D$, we can obtain that $\Omega + M$ is an H_+-matrix and

$$|(\Omega + M)^{-1}| \leq (\langle M\rangle + \Omega)^{-1}.$$

Further, based on Lemma 4.1, we get

$$\begin{aligned}
T &\leq (\langle M\rangle + \Omega)^{-1}(|N| + |A - \Omega|) \\
&= (\langle M\rangle + \Omega)^{-1}(\langle M\rangle + \Omega - \langle M\rangle - \Omega + |N| + |A - \Omega|) \\
&= (\langle M\rangle + \Omega)^{-1}(\langle M\rangle + \Omega - \langle M\rangle - \Omega + |N| + |D - \Omega| + |B|) \\
&= I - (\langle M\rangle + \Omega)^{-1}(\langle M\rangle - |N| + \Omega - |D - \Omega| - |B|) \\
&= I - (\langle M\rangle + \Omega)^{-1}(\langle M\rangle - |N| + D - |B|) \\
&\leq I - 2(\langle M\rangle + \Omega)^{-1}(\langle M\rangle - |N|).
\end{aligned}$$

Since $\langle M\rangle - |N|$ is an M-matrix, there exists a positive vector u such that

$$(\langle M\rangle - |N|)u > 0.$$

Therefore,

$$Tu \leq (I - 2(\langle M\rangle + \Omega)^{-1}(\langle M\rangle - |N|))u < u.$$

Based on Lemma 2.1, we can obtain that $\rho(T) < 1$. Based on Theorem 4.1, the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ produced by Method 3.1 converges to the unique solution $z^* \in \mathbb{R}^n_+$ of the LCP (1.1) for a non-negative initial vector. □

Theorem 4.3 *Let $A = D - L - U = D - B$ and $\rho := \rho(D^{-1}|B|)$, where $A \in \mathbb{R}^{n\times n}$ is an H_+-matrix. Assume that the diagonal matrix Ω satisfies $\Omega \geq D$. Then if the parameters α and β satisfy*

$$0 \leq \max\{\alpha, \beta\}\rho < \min\{1, \alpha\}. \tag{4.3}$$

then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ produced by NMAOR converges to the unique solution $z^* \in \mathbb{R}^n_+$ of the LCP (1.1) for a non-negative initial vector.

Proof. First, Under the condition (4.3), noting that $\rho(D^{-1}|B|) < 1$, it is easy to obtain that $\min\{1, \alpha\}I - \max\{\alpha, \beta\}D^{-1}|B|$ is M-matrix.

Second, by calculate, we have

$$(1 + \alpha - |1 - \alpha|) = 2\min\{1, \alpha\}$$

and

$$\begin{aligned}|\alpha B-\beta L|+\alpha|B|+\beta|L| &= |\alpha L+\alpha U-\beta L|+\alpha|U|+\alpha|L|+\beta|L|\\ &= (|\alpha-\beta|+\alpha+\beta)|L|+2\alpha|U|\\ &\le 2\max\{\alpha,\beta\}|B|.\end{aligned}$$

Based on the above results, we take

$$M=\frac{1}{\alpha}(D-\beta L) \text{ and } N=\frac{1}{\alpha}[(1-\alpha)D+(\alpha-\beta)L+\alpha U].$$

Then

$$\begin{aligned}T=&|\alpha\Omega+D-\beta L|^{-1}(|(1-\alpha)D+(\alpha-\beta)L+\alpha U|+\alpha|\Omega-A|)\\ \le&\langle\alpha\Omega+D-\beta L\rangle^{-1}(|(1-\alpha)D+(\alpha-\beta)L+\alpha U|+\alpha|\Omega-A|)\\ =&(\alpha\Omega+D-\beta|L|)^{-1}(\alpha\Omega+D-\beta|L|-(\alpha\Omega+D-\beta|L|)\\ &+|(1-\alpha)D+(\alpha-\beta)L+\alpha U|+\alpha|\Omega-A|)\\ =&I-(\alpha\Omega+D-\beta|L|)^{-1}(\alpha\Omega+D-\beta|L|\\ &-|(1-\alpha)D+(\alpha-\beta)L+\alpha U|-\alpha|\Omega-D+B|)\\ =&I-(\alpha\Omega+D-\beta|L|)^{-1}(\alpha\Omega+D-\beta|L|-|1-\alpha|D\\ &-|\alpha B-\beta L|-\alpha(\Omega-D)-\alpha|B|)\\ =&I-(\alpha\Omega+D-\beta|L|)^{-1}((1+\alpha)-|1-\alpha|)D\\ &-|\alpha B-\beta L|-\beta|L|-\alpha|B|)\\ =&I-(\alpha\Omega+D-\beta|L|)^{-1}D(\min\{1,\alpha\}I-\max\{\alpha,\beta\}D^{-1}|B|).\end{aligned}$$

The rest proof is similar to the proof of Theorem 4.2, which is omitted. □

5 Numerical experiments

In this section, we employ four numerical examples to show the convergence behaviors of Method 3.1.

To show the advantages of Method 3.1, we contrast Method 3.1 with Method 3.2. The published works in [33,45] pointed out that, among the modulus-based relaxation versions of Method 3.2, the modulus-based Gauss-Seidel (MGS) is best when $\Omega = D$ under the certain conditions. Based on this, with regard to the comparison of the modulus-based relaxation versions of Method 3.1 and Method 3.2, we consider the NMGS method and the MGS method. Other testing methods are considered as well, see Table 1.

All the testing methods are performed in MATLAB 2016b. In addition, we choose $x^{(0)} = (1,0,1,0,\dots,1,0,\dots)^T \in \mathbb{R}^n$ as the initial vectors for these testing methods. All the testing methods are stopped once the number of iteration are larger than 500 or the norm of residual

Abbreviation	Method
MOD	The modulus method in [2]
MMOD	The modified modulus method in [24]
NMOD	The new modulus method
NMMOD	The new modified modulus method

Table 1: Abbreviations of other testing methods.

vectors (RES) is less than 10^{-5}. Here, we still use $\mathrm{RES}(z^{(k)}) = \| \min(Az^{(k)} + q, z^{(k)})\|_2$ as the residual vectors, see [20]. The iteration parameter α used in the NMMOD and MMOD methods is chosen to be 2; $\gamma = 2$ for the MGS method in [20]. In the following tables, 'IT' denotes the iteration steps and 'CPU' denotes the elapsed CPU time in seconds.

Example 5.1 ([20]). Let the LCP (1.1) be given by $q = -Az^*$ and $A = \hat{A} + \mu I$, where

$$\hat{A} = \mathrm{tridiag}(-I, T, -I) = \begin{bmatrix} T & -I & 0 & \cdots & 0 & 0 \\ -I & T & -I & \cdots & 0 & 0 \\ 0 & -I & T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & T & -I \\ 0 & 0 & 0 & \cdots & -I & T \end{bmatrix} \in \mathbb{R}^{n\times n}$$

with

$$T = \mathrm{tridiag}(-1, 4, -1) = \begin{bmatrix} 4 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 4 \end{bmatrix} \in \mathbb{R}^{m\times m},$$

and

$$z^* = (1, 2, 1, 2, \ldots, 1, 2, \ldots)^T \in \mathbb{R}^n$$

is the unique solution of the LCP (1.1).

Tables 2-4 list the numerical results (including IT, CPU and RES) of three groups of methods for Example 5.1 with $\mu = 6$. In total, these numerical results in Tables 2-4 tell us that all the testing methods can rapidly calculate a satisfactory approximation to the solution of LCP (1.1). In addition, with the increasing of the problem size n, all the number of iterations and the CPU times of these six testing methods increase. For each group, we have the following facts:

- Table 2 show that the new modulus method requires less iteration steps and CPU times than the modulus method. That is to say, the computational efficiency of the new modulus method is superior to the modulus method in [2].

	n	16^2	32^2	64^2	128^2
MOD	IT	96	99	101	104
	CPU	0.0060	0.0153	0.1061	0.8052
	RES	9.8434e-6	9.2656e-6	9.9375e-6	9.0756e-6
NMOD	IT	83	86	88	91
	CPU	0.0072	0.0171	0.1171	0.9609
	RES	9.5787e-6	8.9764e-6	9.5820e-6	8.7156e-6

Table 2: Numerical comparison of NMOD and MOD for Example 5.1.

	n	16^2	32^2	64^2	128^2
MMOD	IT	48	49	51	52
	CPU	0.0043	0.0092	0.0574	0.4085
	RES	9.0226e-6	9.8760e-6	7.7848e-6	8.2570e-6
NMMOD	IT	42	43	45	46
	CPU	0.0053	0.0110	0.0686	0.4739
	RES	8.8414e-6	9.6971e-6	7.6996e-6	8.2292e-6

Table 3: Numerical comparison of NMMOD and MMOD for Example 5.1.

	n	16^2	32^2	64^2	128^2
MGS	IT	15	16	16	17
	CPU	0.0031	0.0040	0.0067	0.0199
	RES	4.3687e-6	3.6549e-6	8.1157e-6	5.6681e-6
NMGS	IT	15	16	16	17
	CPU	0.0039	0.0049	0.0075	0.0264
	RES	4.3687e-6	3.6549e-6	8.1157e-6	5.6681e-6

Table 4: Numerical comparison of NMGS and MGS for for Example 5.1.

- Table 3 show that the new modified modulus method requires less iteration steps and CPU times than the modified modulus method. That is to say, the new modified modulus method precedes the modified modulus method in [24]. Moreover, both overmatches the new modulus method as well as the modulus method.

- Table 4 show that the iteration steps and the norm of relative errors of the new modulus-based Gauss-Seidel method are the same as the modulus-based Gauss-Seidel method. Whereas, the former costs the less CPU times than the latter. In terms of computational efficiency, the former outperforms the latter.

- Among these testing methods, from these numerical results in Tables 2, 3 and 4, compared with five methods, the new modulus-based Gauss-Seidel method is more competitive.

Example 5.2 ([20]). Let the LCP (1.1) be given by $q = -Az^*$ and $A = \hat{A} + \mu I$, where

$$\hat{A} = \text{tridiag}(-1.5I, T, -0.5I) = \begin{bmatrix} T & -0.5I & 0 & \cdots & 0 & 0 \\ -1.5I & T & -0.5I & \cdots & 0 & 0 \\ 0 & -1.5I & T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & T & -0.5I \\ 0 & 0 & 0 & \cdots & -1.5I & T \end{bmatrix} \in \mathbb{R}^{n\times n}$$

with

$$T = \text{tridiag}(-1.5, 4, -0.5) = \begin{bmatrix} 4 & -0.5 & 0 & \cdots & 0 & 0 \\ -1.5 & 4 & -0.5 & \cdots & 0 & 0 \\ 0 & -1.5 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -0.5 \\ 0 & 0 & 0 & \cdots & -1.5 & 4 \end{bmatrix} \in \mathbb{R}^{m\times m},$$

and

$$z^* = (1, 2, 1, 2, \ldots, 1, 2, \ldots)^T \in \mathbb{R}^n$$

is the unique solution of the LCP (1.1).

	n	16^2	32^2	64^2	128^2
MOD	IT	98	101	104	106
	CPU	0.0057	0.0148	0.0989	0.7611
	RES	9.7768e-6	9.3137e-6	8.6578e-6	9.2393e-6
NMOD	IT	78	81	83	86
	CPU	0.0073	0.0180	0.1235	0.9466
	RES	9.9373e-6	9.0754e-6	9.5640e-6	8.5977e-6

Table 5: Numerical comparison of NMOD and MOD for Example 5.2.

Tables 5-7 list the numerical results (including IT, CPU and RES) for three groups of methods for Example 5.2 with the different problem sizes of n and $\mu = 6$. These numerical results in Tables 5-7 further verify the observed results obtained from Tables 2-4. That is to say, the computational efficiency of the new modulus-based Gauss-Seidel method is superior to the modified-based Gauss-Seidel method, and these two modulus-based Gauss-Seidel methods have the highest computational efficiency than other four testing methods. Among these methods, the new modulus-based Gauss-Seidel method is more competitive.

Example 5.3 (Black-Scholes American option pricing). We consider the Black-Scholes American option pricing, which was presented in [46]. The price $u(x, t)$ of American put

	n	16^2	32^2	64^2	128^2
MMOD	IT	49	50	52	53
	CPU	0.0046	0.0096	0.0583	0.4201
	RES	9.0068e-6	9.4490e-6	7.9444e-6	8.4616e-6
NMMOD	IT	43	45	46	47
	CPU	0.0054	0.0105	0.0650	0.4786
	RES	9.6228e-6	7.8648e-6	8.4706e-6	9.0390e-6

Table 6: Numerical comparison of NMMOD and MMOD for Example 5.2.

	n	16^2	32^2	64^2	128^2
MGS	IT	13	14	15	15
	CPU	0.0030	0.0039	0.0064	0.0176
	RES	7.5880e-6	5.6490e-6	3.6901e-6	7.7841e-6
NMGS	IT	13	14	15	15
	CPU	0.0037	0.0045	0.0072	0.0231
	RES	7.5880e-6	5.6490e-6	3.6901e-6	7.7841e-6

Table 7: Numerical comparison of NMGS and MGS for for Example 5.2.

options meets

$$(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2})(u(x,t)-g(x,t))=0, \frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\geq 0, u(x,t)-g(x,t)\geq 0, \tag{5.1}$$

with $u(x,0)=g(x,0)$, $\lim_{x\to\pm\infty}u(x,t)=\lim_{x\to\pm\infty}g(x,t)$, $(x,t)\in(-\infty,+\infty)\times[0,T]$. Here, we limit $x\in[a,b]$ and choose the values of a and b based on the approach in [46]. Using the forward difference scheme for time t and implicit difference scheme for the price x to discretize (5.1), we can obtain

$$w:=Az-q\geq 0, z-g\geq 0 \text{ and } w^T(z-g)=0,$$

with $A=\text{tridiag}(-\tau,1+2\tau,-\tau)$ and $\tau=\frac{\Delta t}{(\Delta x)^2}$, Δt denotes the time step and Δx denotes the price step. In our computations, we take $g=0.5z^*$, and $z^*=(1,0,1,0,\ldots,1,0,\ldots)^T$. The vector q is to be adjusted such that $q=Az^*-w^*$, where $w^*=(0,1,0,1,\ldots,0,1,\ldots)^T$.

Tables 8, 9 and 10 list the numerical results (including IT, CPU and RES) for three groups of methods for Example 5.3 with the different problem sizes of n and $\tau=1$. These numerical results further shows that the new modulus-based Gauss-Seidel method and the modulus-based Gauss-Seidel method require the less iteration steps and CPU times than any four methods. The iteration steps of the new modulus-based Gauss-Seidel method are the same as the modulus-based Gauss-Seidel method, but the CPU times of the former is less than the later. Among these methods, the new modulus-based Gauss-Seidel method can be top-priority when these six testing methods are used to solve the LCP (1.1).

	n	6000	8000	10000	12000
MOD	IT	16	16	16	16
	CPU	0.0090	0.0106	0.0121	0.0139
	RES	8.1835e-6	8.1835e-6	8.1835e-6	8.1835e-6
NMOD	IT	16	16	16	16
	CPU	0.0093	0.0110	0.0132	0.0161
	RES	5.4764e-6	5.4764e-6	5.4764e-6	5.4764e-6

Table 8: Numerical comparison of NMOD and MOD for for Example 5.3.

	n	6000	8000	10000	12000
MMOD	IT	18	18	18	18
	CPU	0.0097	0.0115	0.0134	0.0156
	RES	6.9146e-6	7.9861e-6	8.9300e-6	9.7832e-6
NMMOD	IT	18	18	18	18
	CPU	0.0099	0.0119	0.0149	0.0161
	RES	4.0682e-6	4.6986e-6	5.2539e-6	5.7559e-6

Table 9: Numerical comparison of NMMOD and MMOD for for Example 5.3.

	n	6000	8000	10000	12000
MGS	IT	14	14	14	14
	CPU	0.0071	0.0081	0.0093	0.0107
	RES	6.5415e-6	7.5547e-6	8.4472e-6	9.2540e-6
NMGS	IT	14	14	15	15
	CPU	0.0085	0.0095	0.0131	0.0159
	RES	8.6290e-6	9.9653e-6	1.7964e-6	1.9679e-6

Table 10: Numerical comparison of NMGS and MGS for for Example 5.3.

	n	16^2	32^2	64^2	128^2
MOD	IT	56	58	59	61
	CPU	0.0055	0.0112	0.0767	0.5443
	RES	8.8120e-6	8.3490e-6	9.7299e-6	9.6267e-6
NMOD	IT	70	72	73	74
	CPU	0.0061	0.0136	0.0855	0.6600
	RES	9.3320e-6	7.9144e-6	9.2424e-6	9.4256e-6

Table 11: Numerical comparison of NMOD and MOD for for Example 5.4.

Example 5.4 (continuous optimal control problem). We consider the quasi-variational inequality problem (QIP) from the continuous optimal control problem, which was intro-

	n	16^2	32^2	64^2	128^2
MMOD	IT	31	32	33	34
	CPU	0.0040	0.0081	0.0432	0.3073
	RES	9.9028e-6	9.6831e-6	9.7952e-6	9.4645e-6
NMMOD	IT	37	38	39	40
	CPU	0.0049	0.0090	0.0507	0.3625
	RES	8.4407e-6	8.5787e-6	7.9215e-6	8.1788e-6

Table 12: Numerical comparison of NMMOD and MMOD for for Example 5.4.

	n	16^2	32^2	64^2	128^2
MGS	IT	8	9	9	9
	CPU	0.0028	0.0034	0.0120	0.0055
	RES	4.9946e-6	1.2809e-6	5.2554e-6	2.6158e-6
NMGS	IT	9	9	9	9
	CPU	0.0037	0.0044	0.0174	0.0080
	RES	1.8299e-6	3.0380e-6	7.6522e-6	4.7956e-6

Table 13: Numerical comparison of NMGS and MGS for for Example 5.4.

duced in [47]: find $z \in K(z)$, such that

$$(v - z)^T (Az + F(z)) \geq 0, \text{ for any } v \in K(z), \tag{5.2}$$

where $K(z) = \phi(z) + K \subset \mathbb{R}^n$, K is a positive cone in $\mathbb{R}^n$, $\phi(z)$ and $F(z)$, respectively, are the implicit obstacle function and mapping from $\mathbb{R}^n$ to itself. Then, the QIP (5.2) can be formulated as the LCP (1.1), where the matrix A is the same as the matrix A of Example 5.1, and $F(z) = q = (-1, 1, -1, 1, \ldots, -1, 1, \ldots)^T$. In our computations, we take $v = 2z$. Similarly, we present some numerical results, see Tables 11, 12 and 13. From these numerical results in Tables 11, 12 and 13, we can still draw a conclusion shows that the new modulus-based Gauss-Seidel method can be top-priority as well.

In total, from these numerical results in Tables 2-13, we can see that the new modulus-based matrix splitting iteration methods for the LCP (1.1) is with good performance, that is to say, it is feasible and competitive, compared with the modulus-based matrix splitting iteration methods.

6 Conclusion

In this paper, making use of the inequality systems of the LCP (1.1), a new implicit fixed-point equation is obtained. Based on this, we have established a class of new modulus-based matrix splitting iteration methods, which is different from the pervious published works. The convergence property of this new iteration method has been investigated. Some sufficient conditions are given to guarantee the convergence of new modulus-based matrix splitting methods. Numerical experiments are given to illustrate the performance of this new iteration method. Moreover, the numerical comparisons show that this new modulus-based matrix splitting iteration method can compare most favorably with the classical modulus-based matrix splitting iteration method, and the new modulus and modified modulus methods excels the original modulus and modified modulus methods, respectively.

In addition, from the structure of new modulus-based matrix splitting iteration methods, this new iteration method can be extended to solve other complementarity problem, such as the nonlinear complementarity problem, the implicit complementarity problem, the quasi-complementarity problem, the horizontal linear complementarity problem and the vertical linear complementarity problem. These continuity works can be made in the future.

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