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RESEARCH ARTICLE

Stochastically-induced dynamics of earthquakes

Eugenya V. Makoveeva* | Ivan N. Tsvetkov | Lev B. Ryashko

aboratory of Multi-Scale Mathematical Modeling, Laboratory of Climate and Environmental Physics, Institute of Natural Sciences and Mathematics, Ural Federal University, Ekaterinburg, Russian Federation

Correspondence

*Eugenya V. Makoveeva, Laboratory of Multi-Scale Mathematical Modeling, Laboratory of Climate and Environmental Physics, Institute of Natural Sciences and Mathematics, Ural Federal University, Ekaterinburg, Russian Federation. Email: e.v.makoveeva@urfu.ru

Motivated by an important geophysical application, we analyze the nonlinear dynamics of the number of earthquakes per unit time in a given Earth's surface area. At first, we consider a dynamical model of earthquakes describing their rhythmic behaviour with time delays. This model comprises different earthquake scenarios divided into three types (A, B, and C) accordingly to various system dynamics. We show that the deterministic system contains stable equilibria and a limit cycle whose size drastically depends on the production rate α of earthquakes and their time-delay effect. As this takes place, the frequency of earthquakes possesses an oscillatory behaviour dependent on α. To study the role of α in more detail, we have introduced a white Gaussian noise in the governing equation. First of all, we have shown that the dynamical system is stochastically excitable, i.e. it excites larger-amplitude noise-induced fluctuations in the frequency of earthquakes. In addition, these large-amplitude stochastic fluctuations can alternate with small-amplitude fluctuations over time. In other words, the frequency of earthquakes can change its amplitude in an irregular manner under the influence of white noise. Another important effect is how close the current value of α is to its bifurcation point. The closer this value is, the less noise generates large-amplitude fluctuations in the earthquake frequency.

KEYWORDS:

earthquake frequency, nonlinear dynamics of earthquakes, phase diagram, bifurcation points, white noise, applied mathematical modeling

1 | INTRODUCTION

It is well known that many natural phenomena are controlled by non-linear processes (e.g. feedback effects or time lags) in which input influences (e.g., initial or boundary conditions) are not proportional to output processes. In terms of nonlinear dynamics, this means that the system solution at some point in time t_2 depends on the system state at the previous point in time $t_1 < t_2$. However, this dependence cannot be expressed by a simple proportion. Moreover, it is well known that the evolution of dynamical systems is described by a system of differential equations. In the case of nonlinear equations, however, there are possibilities for various dynamical scenarios of their solutions, even in the deterministic case (e.g., bifurcations of solutions). In the case of stochastic governing equations, however, the uncertainty in the system solution increases. In the presence of noise or even several noises of different intensities, the variety of possible stochastic-induced scenarios increases as well. Note that there is almost always a need to study a dynamical system under the influence of one or more noises because they model either random fluctuations that always exist in nature or variations in system parameters that under normal conditions are assumed to be constant quantities or are given by deterministic processes.

The presence of noise in a system can lead to a number of interesting and seemingly unexpected dynamic scenarios. In this connection, such evolutionary behaviours as noise-induced transitions between system attractors[1,2,3,4,5], chaotic behaviour[6,7], stochastically-induced resonance[8,9] and multiresonance[10] can be mentioned as good examples of irregular evolution of a dynamical system. It is important to emphasize here that the dynamic features mentioned do not only apply to idealized systems but are ubiquitous in the analysis of real natural processes and physical phenomena.[11,12,13,14,15,16,17,18]

In this article, we study the nonlinear dynamics of earthquake events based on a model with time delays. This paper is organized as follows. The nonlinear model is presented in section 2. Here we explain the physics of the model and discuss its characteristic deterministic scenarios. Deterministic peculiarities in more detail are considered in section 3. Here we plot the bifurcation diagrams and time series varying the main decision variables. The next paragraph 4 is devoted to the question of how stochastic fluctuations change an evolutionary behaviour of earthquakes occurrence. Here the random states and time series are analyzed at different noise intensities. Our conclusions are given in the last section 5.

2 | MATHEMATICAL MODEL

Let us consider seismic activity in a particular area on the Earth's surface. Let $n(t)$ designates the number of earthquakes per unit time (frequency of events). As the presence of earthquakes activates the Earth's crust and releases the strain energy, their growth rate can be modeled by a simplest logistic equation[19]

$$\frac{dn}{dt} = n(t)\left[\alpha - \beta n(t)\right], \tag{1}$$

where the constants α and β represent the growth rate of earthquakes and the degree of their reduction. Its solution is given by

$$n(t) = \frac{\alpha n_0}{\beta n_0 + \left(\alpha - \beta n_0\right)\exp\left[-\alpha\left(t - t_0\right)\right]}, \tag{2}$$

where the initial frequency $n(t_0) = n_0$.

A typical behaviour given by expression (2) is shown in figure 1 (type A). As is easily seen this solution corresponds to aftershock series of large earthquakes and earthquake swarms of large scale[20].

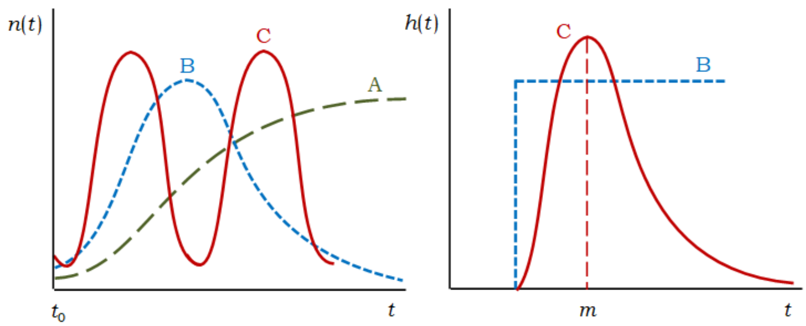

FIGURE 1 A scheme of earthquake types based on dynamical equations (1) and (3). Type A ($h = 0$): a logistic-type curve describing aftershock series of large earthquakes and earthquake swarms of large scale. Type B ($\beta = 0$ and $h = $ const): earthquake frequency, which describes volcanic swarms, grows at initial time and then it gradually reduces with time. Type C (expression (5) for $h(t)$): earthquake frequency, which describes earthquake swarms, contains time-delay effects and represents periodic or rhythmic behaviour.

Equation (1) can be extended to describe a wider range of earthquakes using an integral term with memory

$$\frac{dn}{dt} = n(t)\left[\alpha - \beta n(t) - \int_{-\infty}^{t} n(s)h(t-s)ds\right], \tag{3}$$

where $h(t)$ is responsible for the hysteresis effects of earthquake events. So, for instance, if the impact of earthquake events accumulates with time, $n(t)$ initially grows and then gradually reduces with time, we arrive at $\beta = 0$ and $h(t) = \mathrm{const}$. This case shown in figure 1 as type B describes a certain type of volcanic swarms[21,22].

Paying our attention to time-delay effects appearing, for example, due to inelastic properties of the Earth's crust, we should chose a model dependence $h(t)$. So, to derive a simplest time-delayed logistic law, we put $\beta = 0$, $h(t) = k\delta(t - m)$ and arrive at

$$\frac{dn}{dt} = n(t)\left[\alpha - kn(t - m)\right], \tag{4}$$

where $\delta(\cdot)$ is the Dirac delta function, and k is constant. This equation expresses periodic oscillations illustrated in figure 1 as type C.

The model proposed by Nakajima[23] describes more complex processes with a time lag using a smooth function

$$h(t) = k\left[\exp\left(-\gamma_1 t\right) - \exp\left(-\gamma_2 t\right)\right], \; \gamma_2 > \gamma_1. \tag{5}$$

Here the exponential contributions express the energy dissipation and activation with corresponding decaying rates γ_1 and γ_2. In this case, a time-delay effect occurs and the dynamical system possesses periodic bursts or rhythms of earthquakes, which are observed in earthquake swarms[24,25].

Let us rewrite equation (3) in terms of the following dynamical system

$$\begin{aligned} \dot{x} &= (\alpha - c\beta x - kz)x, \\ \dot{y} &= -\gamma_2 y + cx, \\ \dot{z} &= -\gamma_1 z + (\gamma_2 - \gamma_1)y, \end{aligned} \tag{6}$$

where dots above the variables denote the time derivatives,

$$\begin{aligned} x(t) &= n(t), \quad y(t) = c\int_{-\infty}^{t} n(s)\exp\left[-\gamma_2(t-s)\right] ds, \\ z(t) &= \int_{-\infty}^{t} n(s)\left\{\exp\left[-\gamma_1(t-s)\right] - \exp\left[-\gamma_2(t-s)\right]\right\} ds, \end{aligned} \tag{7}$$

and c is constant. The main variable here is $x(t)$ showing the number of earthquakes per unit time. To analyze the system dynamics, we fix below parameters $\beta = 1$ day^{-1}, $c = 0.01$, $k = 20$ day^{-2}, $\gamma_1 = 1$ day^{-1} after[19] and study deterministic and stochastic scenarios varying positive γ_2 and α.

3 | DETERMINISTIC BEHAVIOUR

At first, let us analyse a deterministic scenario of equations (6) in more detail. The system (6) possesses two equilibria: $M_0(0,0)$ and $M_1(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{\alpha\gamma_1\gamma_2}{c(\beta\gamma_1\gamma_2 + k(\gamma_2 - \gamma_1))}, \quad \bar{y} = \frac{\alpha\gamma_1}{\beta\gamma_1\gamma_2 + k(\gamma_2 - \gamma_1)}, \quad \bar{z} = \frac{\alpha(\gamma_2 - \gamma_1)}{\beta\gamma_1\gamma_2 + k(\gamma_2 - \gamma_1)}.$$

For M_0, eigenvalues of the Jacobi matrix are $(\alpha, -\gamma_1, -\gamma_2)$, so M_0 is unstable for any positive α.

For M_1, the Jacobi matrix is

$$J = \begin{bmatrix} -c\beta\bar{x} & 0 & -k\bar{x} \\ c & -\gamma_2 & 0 \\ 0 & \gamma_2 - \gamma_1 & -\gamma_1 \end{bmatrix}.$$

Eigenvalues of the matrix J are roots of the following cubic equation:

$$\lambda^3 + (c\beta\bar{x} + \gamma_1 + \gamma_2)\lambda^2 + (c\beta\bar{x}(\gamma_1 + \gamma_2) + \gamma_1\gamma_2)\lambda + c\beta\bar{x}\gamma_1\gamma_2 + k\bar{x}c(\gamma_2 - \gamma_1) = 0.$$

The Routh–Hurwitz criterion gives necessary and sufficient conditions of the exponential stability of the equilibrium M_1:

$$(c\beta\bar{x} + \gamma_1 + \gamma_2)(c\beta\bar{x}(\gamma_1 + \gamma_2) + \gamma_1\gamma_2) - c\beta\bar{x}\gamma_1\gamma_2 - k\bar{x}c(\gamma_2 - \gamma_1) > 0.$$

This inequality is quadratic in the parameter α:

$$g(\alpha) = p\alpha^2 + q\alpha + r > 0,$$

where

$$p = (\beta s)^2(1 + \gamma_1 + \gamma_2), \quad q = s(\beta(\gamma_1 + \gamma_2)^2 - k(\gamma_2 - \gamma_1)), \quad r = (\gamma_1 + \gamma_2)\gamma_1\gamma_2, \quad s = \frac{\gamma_1\gamma_2}{\beta\gamma_1\gamma_2 + k(\gamma_2 - \gamma_1)}.$$

Bifurcation borders separating parametric zones of stability/instability of M_1 can be found explicitly from the equation $g(\alpha) = 0$. These borders are shown by blue in Fig. 2 for $\beta = 1$ day^{-1}, $c = 0.01$, $k = 20$ day^{-2}, $\gamma_1 = 1$ day^{-1} in the (α, γ_2)-plane. Between these borders, the deterministic system (6) exhibits stable limit cycles.

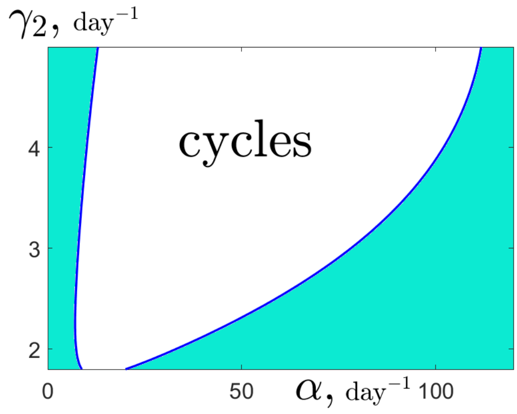

FIGURE 2 Two-parametric bifurcation diagram of the deterministic system (6) with $\beta = 1$ day^{-1}, $c = 0.01$, $k = 20$ day^{-2}, $\gamma_1 = 1$ day^{-1}. By light blue, zones of stability of the equilibrium M_1 are shown.

Details of the transformation "equilibrium→cycle→equilibrium" under variation of α can be seen in Fig. 3 , where attractors of the system (6) are shown for $\gamma_2 = 2$ day^{-1} versus parameter α. Here, values of $\alpha_1 = 7.33$ day^{-1} and $\alpha_2 = 33$ day^{-1} mark the Andronov-Hopf bifurcations, and the system exhibits limit cycles in the interval $\alpha_1 < \alpha < \alpha_2$. In Fig. 4 , these limit cycles and time series of x-coordinate are shown for different α.

Note that initially, as α (production rate of earthquakes) increases, the size of the limit cycle and the amplitude of oscillations x (frequency of earthquakes) increase too (see Fig. 4). A further increase in α leads to a decrease in these quantities. Qualitatively, this effect of α can be easily analyzed based on the analytical solution (2) (which describes only the limiting type A of earthquakes). Indeed, since n (and x) is directly proportional to α, an increase in one parameter means an increase in the other. However, as α increases, the denominator of formula (2) begins to play a role, an increase in which decreases n (and x). In the case of very large α the numerator and denominator of formula (2) practically compensate each other and a further increase in α has almost no effect on the system behaviour. Also, note that periodic rhythms, shown in figure 4 (panel b), completely correspond to oscillations schematically illustrated in figure 1 (type C of earthquake dynamics).

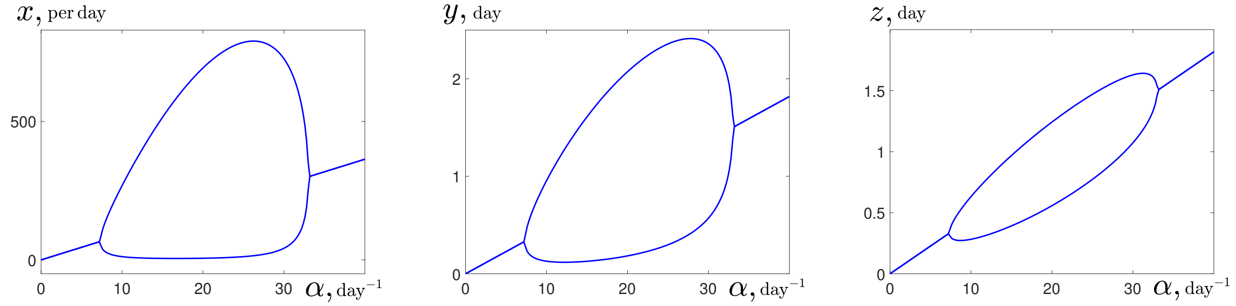

FIGURE 3 Bifurcation diagrams of the deterministic system (6) with $\gamma_2 = 2$ day^{-1}.

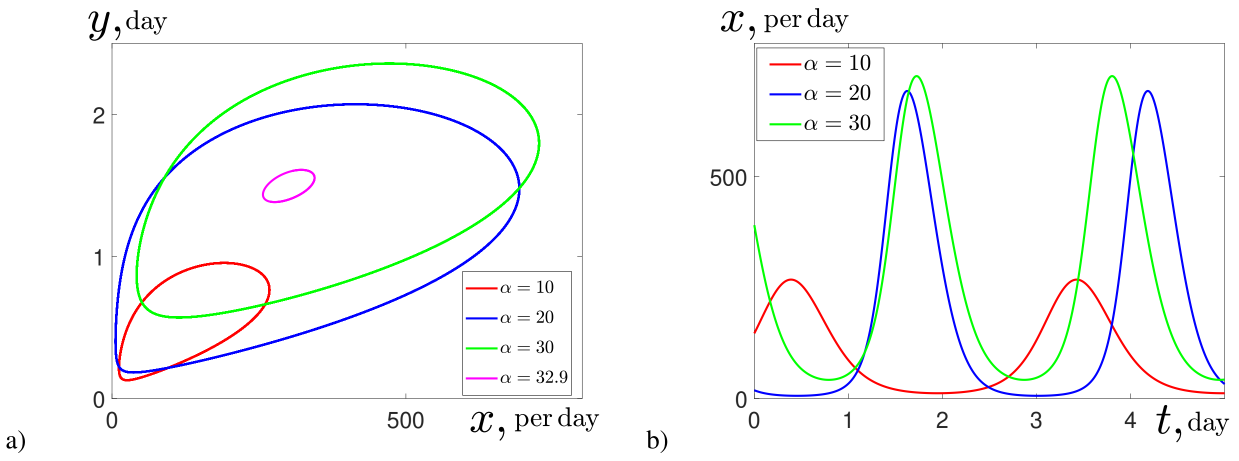

FIGURE 4 Self-oscillations in the deterministic system (6) with $\gamma_2 = 2$ day^{-1}: a) limit cycles, b) x-time series.

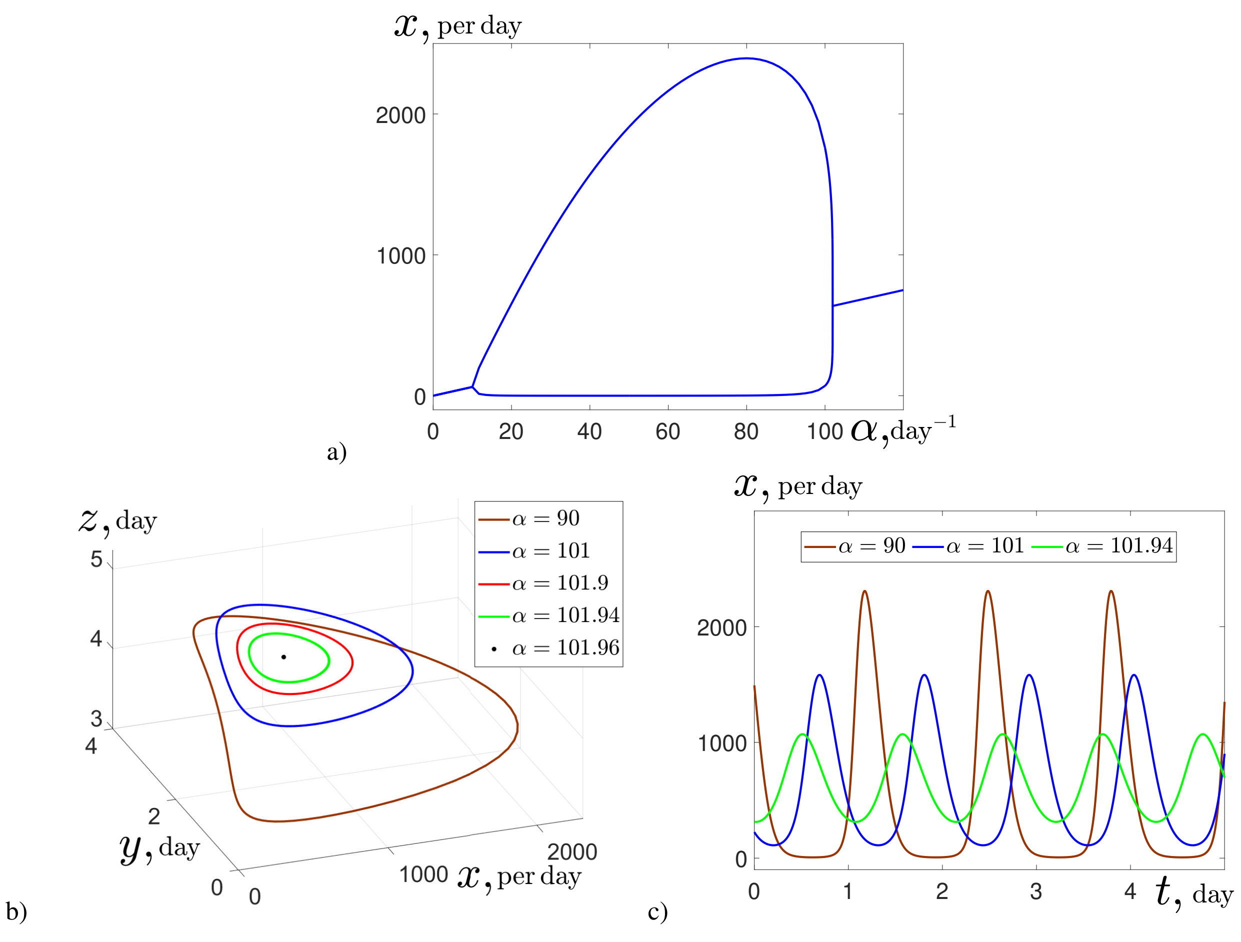

FIGURE 5 Deterministic system (6) with $\gamma_1 = 1$ day^{-1}, $\gamma_2 = 4$ day^{-1}: a) bifurcation diagram, b) limit cycles, c) time series.

Let us consider how the behavior of the deterministic system (6) changes with variation of the parameter γ_2. For $\gamma_2 = 4$ day^{-1}, in the bifurcation diagram (see Fig. 5 a), one can see that α-zone of self-oscillations is larger than for $\gamma_2 = 2$ day^{-1}: cycles are stable for $10.043 < \alpha < 101.957$. It should be noted that the amplitude of self-oscillations is more than two times greater in this parametric zone. Moreover, the transition from cycles to the stable equilibrium near $\alpha = 101.957$ day^{-1} occurs very sharply:

here, for very small changes of α, the size of the cycle changes significantly in a stepwise manner. In Fig. 5 b,c, we demonstrate the orbits of limit cycles and time series of x-coordinates for $\gamma_2 = 4$ day^{-1} and different α.

4 | STOCHASTIC MODEL

Let us now analyze the role of one main system parameter - α (the production rate of earthquakes) on nonlinear dynamics. To do this, we introduce stochastic fluctuations in the form of white Gaussian noise $\xi(t)$ and replace α with $\alpha + \varepsilon\xi(t)$, where ε stands for the noise intensity. As a result, the deterministic system (6) transforms as

$$\begin{aligned} \dot{x} &= (\alpha + \varepsilon\xi(t) - c\beta x - kz)x, \\ \dot{y} &= -\gamma_2 y + cx, \\ \dot{z} &= -\gamma_1 z + (\gamma_2 - \gamma_1)y. \end{aligned} \tag{8}$$

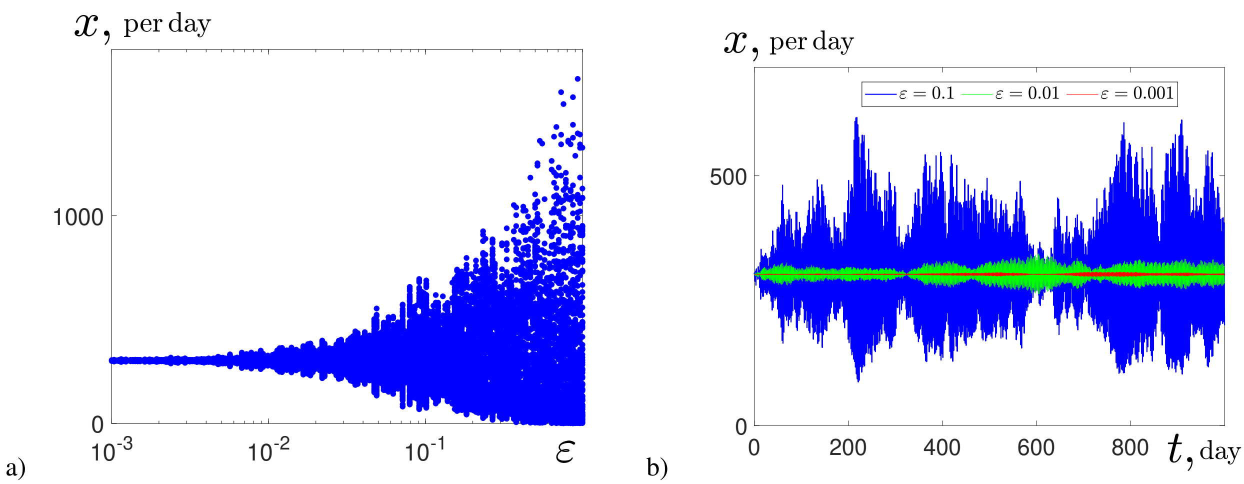

FIGURE 6 Stochastic system with $\alpha = 33.3$ day^{-1}, $\gamma_1 = 1$ day^{-1}, $\gamma_2 = 2$ day^{-1}: a) random states of the system (8) versus noise intensity; b) time series of random solutions for different values of noise intensity.

In Fig. 6 a, we show random states of the system (8) for $\gamma_2 = 2$ day^{-1} and $\alpha = 33.3 > \alpha_2$. For $\alpha = 33.3$ day^{-1}, the deterministic system has the stable equilibrium M_1. As can be seen, with increasing ε, the system exhibits stochastic excitement with generation of large-amplitude stochastic oscillations. In time series (see Fig. 6 b), one can see the alternation of small- and large-amplitude stochastic oscillations.

Consider now how noise impacts the equilibrium M_1 near the bifurcation point $\alpha_2 = 101.957$ day^{-1}. In Fig. 7 a, we show random states of the stochastic system (8) starting from the stable equilibrium M_1 for different α. As can be seen, with increasing ε, the system exhibits the phenomenon of stochastic excitement with the generation of large-amplitude stochastic oscillations with a larger amplitude than for $\gamma_2 = 2$ day^{-1} (compare with Fig. 6 a). As one can see, the closer α to the bifurcation point α_2, the smaller noise generates the large-amplitude oscillations. In Fig. 7 b, we show plots of the variance D of x-coordinates of random states. In Fig. 7 c,d, we illustrate the time series of a stochastic system for two values of α in the vicinity of α_2. Here, one can see the generation of large-amplitude oscillations by noise.

5 | CONCLUSION

The main idea of this paper is to describe the temporal evolution of earthquake sequences using a phenomenological population nonlinear model. Such an approach is essentially analogous to the non-linear dynamics of various biological and physical processes and phenomena studied earlier[26,27,28,29,30,31,32,33,34]. The model under consideration, containing positive and negative

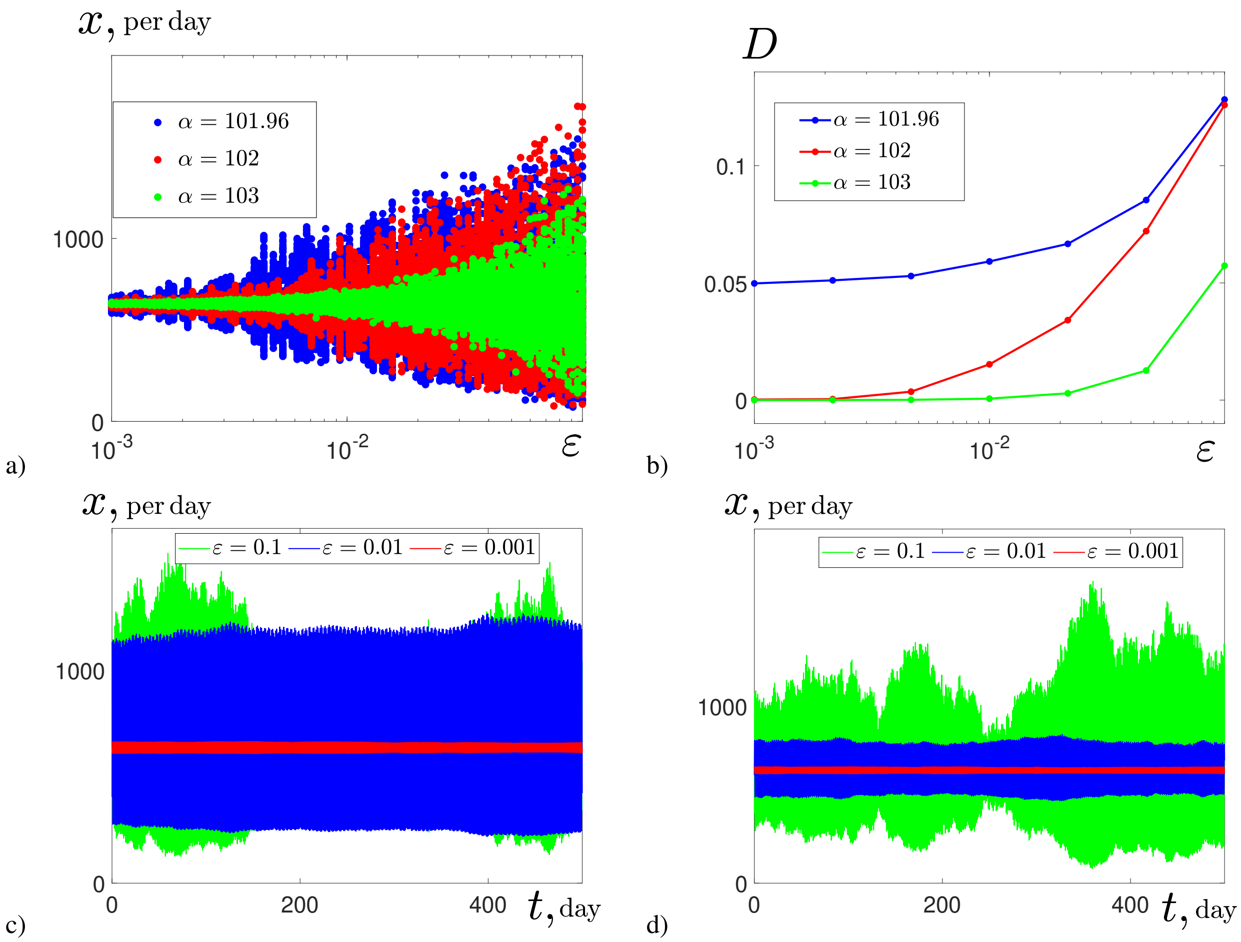

FIGURE 7 Stochastic system with $\gamma_1 = 1$ day^{-1}, $\gamma_2 = 4$ day^{-1}: a) random states of the system (8) versus noise intensity, b) variance of random states, c) time series for $\alpha = 101.96$ day^{-1}, d) time series for $\alpha = 102$ day^{-1}.

feedback, is simple enough to understand the factors being included. As a result, we conclude that earthquake sequences can be modeled by fairly simple equations, although the dynamics of each earthquake may be different and depend on a variety of geophysical factors.

In summary, the model under consideration includes the following impacts influencing the temporal variation of earthquakes frequency: (i) the production rate of earthquakes and the degree of their reduction, (ii) time-delay effects, and (iii) stochastic forcing. Physically, the model describes aftershock series of large earthquakes and earthquake swarms (type A), volcanic swarms (type B), and periodic patterns of bursts (rhythms) of earthquakes met in earthquake swarms (type C). This division into three types is a conditional simplification (it inherits the dynamical properties of the simplified types of the governing equation of population dynamics). The general equation, of course, contains a richer variety of evolutionary scenarios.

It is also of interest to investigate the presence of noise in other parameters of the system (8) or the simultaneous presence of noises of different intensities in several parameters of this system. The presence of additive noises in the right-hand sides of equations (8) is also an important issue that could alter the nonlinear dynamics of the earthquake time sequence. Such studies and their detailed comparison with experiments represent important challenges for future research.

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Author contributions

The authors contributed equally to the present research article.

Conflict of interest

The authors declare no potential conflict of interests.

ORCID

Eugenya V. Makoveeva https://orcid.org/0000-0002-2499-3196
Ivan N. Tsvetkov https://orcid.org/0000-0002-6228-1384
Lev B. Ryashko https://orcid.org/0000-0002-0817-3753

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