Steady states and Hopf bifurcation of a diffusive predator-prey model with prey harvesting and prey-taxis*

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Abstract

This paper is concerned with a predator-prey model with prey-taxis and linear prey harvesting under the homogeneous Neumann boundary condition. The stability of the unique positive constant solution of the predator-prey model without prey-taxis is derived. Also, the emergence of Hopf bifurcation is concluded by choosing the proper Hopf bifurcation parameters. Moreover, the existence of non-constant positive steady states is investigated by the introduce of prey-taxis. The conclusions show that prey harvesting and prey-taxis can enrich the dynamics.

Keywords: Predator-prey model; Steady states; Hopf bifurcation; Prey-taxis.

1 Introduction

In paper [1], based on the assumption that the prey exhibits herd behavior and the predator interacts with the prey along the outer corridor of the herd of prey, Braza proposed the predator-prey model with square root functional responses. Considering the spatial diffusion of populations, the model proposed by Braza was extended to a diffusive model in paper [2]. The diffusive predator-prey model is as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = u(1-u) - \sqrt{u}v, & (x,t) \in (-\infty,+\infty)\times(0,\infty), \\ \dfrac{\partial v}{\partial t} - d_2\Delta v = v(c\sqrt{u} - sv), & (x,t) \in (-\infty,+\infty)\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x = -\infty,+\infty,\ t\in(0,\infty), \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in (-\infty,+\infty), \end{cases} \tag{1}$$

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where u, v represent the populations of the prey and predator respectively; positive constants d_1 and d_2 are the random diffusion coefficients of prey and predator population respectively; positive constant c represents the rate of biomass conversion; positive constant s represents the scaled death rate. And in paper [3, 26], the pattern formation of system (1) has been studied.

Predator-prey models are basic differential equation models for describing the interactions between two species, and are of great interest to researchers in mathematics and ecology. Both the functional response and harvesting can affect dynamical properties of biological and mathematical models. For different species, constant harvesting [5, 6, 7, 13], proportional harvesting [8, 9], and nonlinear harvesting [10, 25] are currently investigated by many authors. In particular, in paper [14], results were obtained for optimal harvesting. In this paper, we introduce the linear harvesting term into the model (1), and consider the following model:

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = u(1-u) - \sqrt{u}v - hu, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v = v(c\sqrt{u} - sv), & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega,\ t \in (0,\infty), \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain in $\Re^{\aleph}$, $\aleph \geq 1$, n is the outward unit normal vector of the boundary of $\partial\Omega$ which we will assume is smooth. hu represents linear prey harvesting.

Bifurcation is a very important issue in dynamic system theory. It reflects the qualitative variation of the topology of the flow caused by changes in parameters. It has great significance both in mathematical theory and practical applications [30]. Hopf bifurcation has been widely investigated in [24, 31, 32, 33, 34]. In this paper, we will treat h as a Hopf bifurcation parameter, and demonstrate the importance role of the harvesting in the dynamical behaviour.

In addition to the random movements of predators and prey in space predation activities, there is also a chemotaxis phenomenon, that is, the spatiotemporal changes in the predator population density are also affected by the gradient of the prey population. The biochemotaxis model is not only used to describe the biological movement process at the micro-scale, but also applied to the study of population dynamics at the macro-scale. Due to the existence and important role of chemotaxis, more and more scholars have begun to carry out research. For example, in paper [11], the authors dealed with a prey-predator model with indirect prey-taxis; in paper [12], the global boundedness and stability of the predator-prey model with prey-taxis were obtained. And a prey-taxis equation was derived in paper [28] and was extended in [29]. Therefore, it is necessary to study the predator-prey model with prey-taxis term. Next, we will further study the above model of (2) with prey-taxis term, and the corresponding model is as follows:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - d_1 \Delta u = u(1-u) - \sqrt{u}v - hu, & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial v}{\partial t} - d_2 \Delta v + \nabla \cdot (\alpha v \nabla u) = v(c\sqrt{u} - sv), & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega,\ t \in (0,\infty), \\
u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \Omega,
\end{cases}
\tag{3}
$$

where α denotes the prey-tactic sensitivity. The term $\alpha v \nabla u$ gives the velocity by which predators move up the gradient of the prey. Motivated by the "volume-filling" mechanism [4, 15], we have

$$
\alpha = \alpha(v) = \begin{cases}
\chi(1 - \frac{v}{v_m}), & 0 < v < v_m, \\
0, & v \geq v_m,
\end{cases}
\tag{4}
$$

where χ and v_m are positive constants. In the following research, we mainly study the case $0 \leq v < v_m$.

The outline of this paper is as follows. In Section 2, after analyzing the characteristic equations, we conclude the stability of constant equilibrium solutions of problem (2). In Section 3, we research the existence of periodic solutions bifurcating from the unique positive constant solution of problem (2). We analyze the existence of the non-constant steady states of problem (3) by the fixed point index theory in Section 4. In Section 5, we will adapt simulations to carry out our conclusions.

Throughout the paper, μ_k denotes the eigenvalues of $-\Delta$ in Ω under the homogeneous Neumann boundary condition satisfying

$$0 = \mu_0 < \mu_1 \leq \mu_2 < \cdots \mu_k < \cdots < \infty.$$

2 Stability of equilibrium points of problem (2)

In this section we will study the stability of constant equilibrium points of problem (2).

It is easy to see that the trivial equilibrium point $(0,0)$ always exists. If $0 < h < 1$, semi-trivial equilibrium point $(1-h, 0)$ exists. Especially if

$$0 < h < 1 - \frac{c}{s}, \tag{5}$$

a unique positive constant solution $(u_\star, v_\star)$ also exists, where

$$u_\star = 1 - h - \frac{c}{s}, \quad v_\star = \frac{c}{s}\sqrt{u_\star}.$$

Theorem 2.1 *For problem (2),*

(1) $(0,0)$ *is unstable;*

(2) If $0 < h < 1$ *holds,* $(1-h, 0)$ *is unstable;*

(3)Assume that

$$0 < h < h_\star, \quad \text{and} \quad 1 - \frac{3c}{2s} > 0 \tag{6}$$

hold, where $h_\star$ *will be determined in the later, then* $(u_\star, v_\star)$ *is locally asymptotically stable.*

Proof In what follows, we will only prove the case (3), the similar method can be used to prove the other two cases. We consider the linearization near $(u_\star, v_\star)$ of problem (2):

Denote

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

It is easy to see that the Jacobi matrix at $(u_\star, v_\star)$ of problem (2) is as follows:

$$-\mu_k D + L_\star = \begin{pmatrix} -\mu_k d_1 + 1 - 2u_\star - \frac{v_\star}{2\sqrt{u_\star}} - h & -\sqrt{u_\star} \\ cv_\star \frac{1}{2\sqrt{u_\star}} & -\mu_k d_2 + c\sqrt{u_\star} - 2sv_\star \end{pmatrix}.$$

Hence, λ satisfies the characteristic equation:

$$\lambda^2 + B_{1k}\lambda + B_{2k} = 0,$$

where

$$B_{1k} = \mu_k d_1 + \mu_k d_2 - (1 - 2u_\star - \frac{v_\star}{2\sqrt{u_\star}} - h - sv_\star),$$

$$B_{2k} = [\mu_k d_1 - (1 - 2u_\star - \frac{v_\star}{2\sqrt{u_\star}} - h)](\mu_k d_2 + sv_\star) + \frac{1}{2}cv_\star.$$

Let $Z(h) = 1 - 2u_\star - \frac{v_\star}{2\sqrt{u_\star}} - h - sv_\star$, and through a series of calculations, we obtain

$$Z(h) = -1 + h + \frac{3c}{2s} - c\sqrt{1 - h - \frac{c}{s}},$$

and

$$Z(1-\frac{c}{s})=\frac{c}{s}>0,$$

$$Z(1-\frac{3c}{2s})=-c\sqrt{\frac{c}{s}}<0.$$

According to

$$Z'(h)=1-\frac{-c}{2\sqrt{1-h-\frac{c}{s}}}>0, \tag{7}$$

we have that $Z(h)$ monotonically increases with respect to h. Then there exists a $h_\star \in (1-\frac{3c}{2s}, 1-\frac{c}{s})$ satisfying $Z(h_\star)=0$. So we could acquire that if $h \in (0, h_\star)$, $Z(h)<0$ holds; if $h \in (h_\star, 1-\frac{c}{s})$, $Z(h)>0$ holds. Therefore, by (6), we have $B_{1k}>0$, $B_{2k}>0$ for all k, which imply that $(u_\star, v_\star)$ is locally asymptotically stable.

3 Hopf bifurcation of problem (2)

In this section we are going to analyze the conditions about the parameters under which the Hopf bifurcation occurs near the unique positive constant solution $(u_\star, v_\star)$ of problem (2). From Theorem 2.1, we know that if $h \in (0, h_\star)$, $(u_\star, v_\star)$ is locally asymptotically stable. So the possible Hopf bifurcation interval is $h \in [h_\star, 1-\frac{c}{s})$.

Denote

$$W = sv_\star(1-h-\frac{c}{s})-\frac{[d_1 s v_\star - d_2(-1+h+\frac{3c}{2s})]^2}{4d_1 d_2}.$$

Theorem 3.1 *For problem (2). If $h_\star \le h < 1-\frac{c}{s}$, $1-\frac{3c}{2s}>0$, and $W>0$ hold. Let Ω be a bounded smooth domain so that the spectral set $S=\{\mu_i\}$ satisfies*

(S1) *All the eigenvalues μ_i are simple for $i \ge 0$.*

Then there exists $n_0 \in \aleph$ such that $h^H_{n_0} < 1-\frac{c}{s} < h^H_{n_0+1}$, and for problem (2), there are (n_0+1) Hopf bifurcation points satisfying

$$h_\star = h_0^H < h_1^H < h_2^H < \cdots < h^H_{n_0} < 1-\frac{c}{s},$$

where $h_i^H = h^H(\mu_i)$, $i=0,1,\cdots,n_0$.

Moreover,

(1) The bifurcating periodic orbits from $h=h_0^H$ are spatially homogeneous, which coincide with the periodic orbits of the corresponding ODE system;

(2) The bifurcating periodic orbits from $h=h_i^H$ are spatially nonhomogeneous, $1 \le i \le n_0$.

Proof From problem (2), we define

$$\begin{aligned} T(h,\mu_k) &= -\mu_k(d_1+d_2)+(1-2u_\star-\tfrac{v_\star}{2\sqrt{u_\star}}-h-sv_\star),\\ D(h,\mu_k) &= [\mu_k d_1-(1-2u_\star-\tfrac{v_\star}{2\sqrt{u_\star}}-h)](\mu_k d_2+sv_\star)+\tfrac{1}{2}cv_\star,\\ H &= \{(h,\mu_k)\in(0,\infty)\times(0,\infty): T(h,\mu_k)=0\}. \end{aligned}$$

Then H is the Hopf bifurcation curve.

Let h^H be possible Hopf bifurcation value, by [16, 17], to identify h^H be the Hopf bifurcation point, we recall the following sufficient conditions:

(AH) There exists $i\in\aleph_0:=\aleph\cup\{0\}$ such that $T_i(h^H)=0$ and $D_i(h^H)>0$ hold, and as $i\neq j$, $T_j(h^H)\neq 0$ and $D_j(h^H)\neq 0$. And the unique pair of complex eigenvalues $\lambda(h)=\sigma(h)\pm i\omega(h)$ near the imaginary axis satisfy $\sigma'(h^H)\neq 0$, $\omega(h^H)>0$, where $T_i(h^H)=T(h^H,\mu_i)$, $D_i(h^H)=D(h^H,\mu_i)$.

A series of calculations are performed as follows: Let $T(h,\mu_i)=0$, that is $\mu_i(d_1+d_2)=Z(h)$, and due to (7), we have $\mu(h)$ monotonically increases with respect to h. When $i=0$, that is $\mu_0=0$, then $h=h_\star$. There exists n_0 such that $\mu_\star\in(\mu_{n_0},\mu_{n_0+1})$ satisfying $\mu_\star(d_1+d_2)=Z(1-\frac{c}{s})$. Hence there are (n_0+1) possible Hopf bifurcation points satisfying

$$h_\star=h_0^H<h_1^H<h_2^H<\cdots<h_{n_0}^H<1-\frac{c}{s}.$$

According to the above discussion, we get

$$\sigma'(h_i^H)=\frac{dRe\lambda(h)}{dh}|_{h=h_i^H}=\frac{1}{2}(1+\frac{c}{2\sqrt{1-h-\frac{c}{s}}})>0.$$

Next we will show that under some additional conditions, $D_j(h_i^H)>0$ holds for $0\le i\le n_0$ and $j\in\aleph_0$, then we must have $D_i(h_i^H)>0$ and $D_j(h_i^H)\neq 0$ for $0\le i\le n_0$ and $j\in\aleph_0$ as required in the condition **(AH)**.

We find

$$\begin{aligned} D_j(h) &= [\mu_j d_1-(1-2u_\star-\tfrac{v_\star}{2\sqrt{u_\star}}-h)](\mu_j d_2+sv_\star)+\tfrac{cv_\star}{2}\\ &= d_1d_2\mu_j^2+\mu_j G-sv_\star(-1+h+\tfrac{3c}{2s})+\tfrac{cv_\star}{2}\\ &= [\sqrt{d_1d_2}\mu_j+\tfrac{G}{2\sqrt{d_1d_2}}]^2-\tfrac{M^2}{4d_1d_2}-sv_\star(-1+h+\tfrac{3c}{2s})+\tfrac{cv_\star}{2}\\ &= [\sqrt{d_1d_2}\mu_j+\tfrac{G}{2\sqrt{d_1d_2}}]^2+W, \end{aligned}$$

where

$$G=d_1sv_\star-d_2(-1+h+\frac{3c}{2s}).$$

If $W > 0$, then we verify $D_j(h_i^H) > 0$, especially $D_i(h_i^H) > 0$.

Collecting the above analysis, we have that when $i \neq j$, $T_j(h_i^H) \neq 0$ holds, and

$$\sigma'(h_i^H) > 0, \quad \omega(h_i^H) = \sqrt{D_j(h_i^H)} > 0.$$

So the proof is accomplished by the Hopf bifurcation theorem in [16, 27].

4 Existence of non-constant positive steady-states of problem (3)

In the section, we mainly analyze the existence of non-constant positive steady-states of problem (3) with the assumption $\partial\Omega \in C^{2+\alpha}(0 < \alpha < 1)$, that is to say, we will deal with the following model:

$$\begin{cases} -d_1\Delta u = u(1-u) - \sqrt{u}v - hu, & x \in \Omega, \\ -d_2\Delta v + \nabla\cdot(\alpha v\nabla u) = v(c\sqrt{u} - sv), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{8}$$

Notice that the constant equilibrium solutions of the above model is the same with problem (3). By using the fixed point index theory to calculate the indexes, we provide some sufficient conditions for non-constant positive solutions of (8). We first introduce fixed point index theory as follows:

Let E be a Banach space and W be the natural positive cone of E. For $y \in W$, define W_y=$\{x \in E : y + kx \in W \text{for some } k > 0\}$ and S_y=$\{x \in \bar{W}_y : -x \in \bar{W}_y\}$. Let $y_\star$ be a fixed point of compact operator $A : W \to W$ and $L = A'(y_\star)$ be the F derivative of A at $y_\star$. We say that A' has property γ on $\bar{W}_{y_\star}$ if there exist $t \in (0,1)$ and $\omega \in \bar{W}_{y_\star} \setminus S_{y_\star}$ such that $\omega - tA'\omega \in S_{y_\star}$. For an open subset $U \subset W$, define $\mathrm{index}_W(A,U) = \mathrm{index}_W(A,U,W) = \deg_W(I-A,U,0)$, where I is the identity map. Furthermore, the fixed point index of A at $y_\star$ in W is defined by $\mathrm{index}_W(A,y_\star) = \mathrm{index}_W(A,y_\star,W) = \mathrm{index}_W(A,U_{y_\star},W)$, where $U(y_\star)$ is a small open neighborhood of $y_\star$ in W. Then the following lemma can be obtained from the result in [18, 19, 20, 21].

Lemma 4.1 *Assume that $I - L$ is invertible on $\bar{W}_{y_\star}$,*

(i) If L has property γ on $\bar{W}_{y_\star}$, then $index_W(A,y_\star) = 0$.

(ii) If L does not have property γ on $\bar{W}_{y_\star}$, then $index_W(A,y_\star) = (-1)^\delta$, where δ is the sum of algebraic multiplicities of the eigenvalues of L which are greater than 1.

The following theorem gives a priori bound for positive solutions of (8).

Theorem 4.1 *Any positive solution (u, v) of (8) satisfies*

$$u(x), v(x) \leq max\{1-h, \frac{c}{s}\sqrt{1-h}, v_m\} \quad in \quad \bar{\Omega},$$

where v_m is defined in (4).

Proof For the first equation of (8), define $u(x_0) = \max_{\bar{\Omega}} u$, we use the maximum principle of elliptic equation, we obtain

$$[u(1-u) - \sqrt{u}v - hu]|_{x=x_0} \geq 0,$$

then $1 - h - u(x_0) \geq 0$, so we get

$$u(x_0) \leq 1 - h.$$

For the second equation of (8), when $v \geq v_m$, by the maximum principle similarly, we have

$$v \leq \frac{c}{s}\sqrt{1-h}.$$

Hence,

$$u(x), v(x) \leq \max\{1-h, \frac{c}{s}\sqrt{1-h}, v_m\} \quad in \quad \bar{\Omega}.$$

This completes the proof.

Remark

(i) $E := C_n^1(\bar{\Omega}) \oplus C_n^1(\bar{\Omega})$, where $C_n^1(\bar{\Omega}) := \{\phi \in C^1(\bar{\Omega}) : \frac{\partial \phi}{\partial n} = 0 \ on \ \partial\Omega\}$;

(ii) $D := D_u \oplus D_v$, where $D_u := \{\phi \in C_n^1(\bar{\Omega}) : \phi < 1 - h + 1 \ in \ \bar{\Omega}\}$ and $D_v := \{\phi \in C_n^1(\bar{\Omega}) : \phi < \frac{c}{s}\sqrt{1-h} + 1 \ in \ \bar{\Omega}\}$;

(iii) $W := Q \oplus Q$, where $Q := \{\phi \in C_n^1(\bar{\Omega}) : \phi(x) \geq 0, x \in \bar{\Omega}\}$;

(iv) $D' := D \cap W$.

By Theorem 4.1, the standard regularity theory of elliptic equations, the embedding theorems and assumption $\partial\Omega \in C^{2+\alpha}(0 < \alpha < 1)$, we can obtain that $(u, v) \in C^2 \times C^2$ for elliptic system (8) in [23]. Thus there exists a positive constant M_1, such that $\|\nabla u\|_{C^1} \leq M_1$, $\|\nabla v\|_{C^1} \leq M_1$. According to the first equation of (8) and Theorem 4.1, we get $\|\Delta u\|_{C^1} \leq M_1$. Hence, there exists sufficiently large positive constant M such that $u(1-u) - \sqrt{u}v - hu + Mu$ and $-\chi(1 - \frac{2v}{v_m})\nabla v \cdot \nabla u - \alpha v \Delta u + v(c\sqrt{u} - sv) + Mv$ are monotone increasing functions with respect to u and v respectively.

Define a compact map $A : C^2(\bar{\Omega}) \oplus C^2(\bar{\Omega}) \to C^1(\bar{\Omega}) \oplus C^1(\bar{\Omega})$ by

$$A(u, v) = \begin{pmatrix} (-d_1\Delta + M)^{-1}(u(1-u) - \sqrt{u}v - hu + Mu) \\ \\ (-d_2\Delta + M)^{-1}(-\chi(1 - \frac{2v}{v_m})\nabla v \cdot \nabla u - \alpha v \Delta u + v(c\sqrt{u} - sv) + Mv) \end{pmatrix}.$$

By using a technical device developed by [18], we calculate the fixed point index of A over $\mathrm{Int}D'$ with respect to W.

Theorem 4.2 *$index_W(A, IntD') = 1$.*

Proof For $\theta \in (0,1)$, define a homotopy invariance $A_\theta : C^2(\bar\Omega) \oplus C^2(\bar\Omega) \to C^1(\bar\Omega) \oplus C^1(\bar\Omega)$ by

$$A_\theta(u,v) = \begin{pmatrix} (-d_1\Delta + M)^{-1}[\theta(u(1-u) - \sqrt{u}v - hu) + Mu] \\ \\ (-d_2\Delta + M)^{-1}[\theta(-\nabla\cdot(\alpha v\nabla u) + v(c\sqrt{u} - sv)) + Mv] \end{pmatrix}.$$

According to the proof of Theorem 4.1, it can be shown that any positive fixed point (u,v) of A_θ also satisfies

$$u(x), v(x) \le \max\{1-h, \frac{c}{s}\sqrt{1-h}, v_m\} \quad in \quad \bar\Omega$$

since $0 \le \theta \le 1$. Hence using homotopy invariance, it follows that

$$\mathrm{index}_W(A_1, \mathrm{Int}D') = \mathrm{index}_W(A_0, \mathrm{Int}D') = \mathrm{index}_Q(H_u, \mathrm{Int}D_u)\mathrm{index}_Q(H_v, \mathrm{Int}D_v),$$

where

$$H_u(u) = (-d_1\Delta + M)^{-1}(Mu),$$
$$H_v(v) = (-d_2\Delta + M)^{-1}(Mv),$$

so the spectral radius of $H_{u,0}$ and $H_{v,0}$ is as follows:

$$\mathrm{r}(H_{u,0}) = \mathrm{r}(H_{v,0}) = 1.$$

Using Lemma 13.1 in [22], we get

$$\mathrm{index}_Q(H_u, \mathrm{Int}D_u) = \mathrm{index}_Q(H_v, \mathrm{Int}D_v) = 1.$$

Hence, the proof is accomplished.

Next, we will calculate $\mathrm{index}_W(A,(0,0))$, $\mathrm{index}_W(A,(1-h,0))$, and $\mathrm{index}_W(A,(u_\star, v_\star))$, respectively.

Theorem 4.3 *$index_W(A,(0,0)) = 0$.*

Proof Through a series of analyses, we have $\bar W_{(0,0)} = W$, $S_{(0,0)} = \{(0,0)\}$ and

$$A'(0,0) = \begin{pmatrix} (-d_1\Delta + M)^{-1}(1-h+M) & 0 \\ \\ 0 & (-d_2\Delta + M)^{-1}M \end{pmatrix}.$$

Assume that $(I - A'(0,0))(\phi,\psi)^T = 0$ for $(\phi,\psi)^T \in W$, we get

$$\begin{cases} -d_1\Delta\phi = (1-h)\phi, & x \in \Omega, \\ -d_2\Delta\psi = 0, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\psi}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

It follows from the strong maximum principle and Hopf's Lemma that $(\phi,\psi)=(0,0)$. It shows that $(I-A'(0,0))$ is invertible on $\bar{W}_{(0,0)}$.

Let $(I-tA'(0,0))(1,0)^T=(0,0)^T$, that is,

$$1-t(-d_1\Delta+M)^{-1}(1-h+M)=0,$$

we obtain $t=\frac{M}{1-h+M}\in(0,1)$. Therefore, $A'(0,0)$ has property γ. The proof is accomplished.

Theorem 4.4 *When* $0<h<1$, $index_W(A,(1-h,0))=0$.

Proof By calculations, we have $\bar{W}_{(1-h,0)}=C_n^1(\bar{\Omega})\oplus Q$, $S_{(1-h,0)}=C_n^1(\bar{\Omega})\oplus\{0\}$ and

$$A'(1-h,0)=\begin{pmatrix}(-d_1\Delta+M)^{-1}(-1+h+M) & (-d_1\Delta+M)^{-1}(-\sqrt{1-h}) \\ \\ 0 & (-d_2\Delta+M)^{-1}(c\sqrt{1-h}+M)\end{pmatrix}.$$

Let $(I-A'(1-h,0))(\phi,\psi)^T=0$ for $(\phi,\psi)^T\in\bar{W}_{(1-h,0)}$, that is

$$\begin{cases}-d_1\Delta\phi+(1-h)\phi+\sqrt{1-h}\psi=0, & x\in\Omega,\\ -d_2\Delta\psi-c\sqrt{1-h}\psi=0, & x\in\Omega,\\ \dfrac{\partial\phi}{\partial n}=\dfrac{\partial\psi}{\partial n}=0, & x\in\partial\Omega.\end{cases}$$

By the strong maximum principle and Hopf's lemma, we have $(\phi,\psi)=(0,0)$. Hence, $(I-A'(1-h,0))$ is invertible on $\bar{W}_{(1-h,0)}$.

Assume that $(I-tA'(1-h,0))(0,1)^T\in S_{(1-h,0)}$, that is

$$1-t(-d_2\Delta+M)^{-1}(c\sqrt{1-h}+M)=0,$$

we get $t=\frac{M}{c\sqrt{1-h}+M}\in(0,1)$. Therefore, $A'(1-h,0)$ has property γ. The proof is accomplished.

Denote

$$\begin{aligned} N &= -1+h+\tfrac{3c}{2s},\\ F &= \alpha_\star v_\star\sqrt{u_\star}+d_1 s v_\star-d_2 N.\end{aligned}$$

Theorem 4.5 *Assume that (5) hold.*
(1) If

$$N<min\{\frac{1}{d_1},\frac{1}{d_2}\}(\alpha_\star v_\star\sqrt{u_\star}+d_1 s v_\star), \tag{9}$$

then $index_W(A, (u_\star, v_\star)) = 1$*;*
(2) If

$$N > \frac{1}{d_2}(\alpha_\star v_\star \sqrt{u_\star} + d_1 s v_\star) \tag{10}$$

and $F^2 - 4d_1 d_2 c u_\star^{\frac{3}{2}} > 0$*, then*

$$index_W(u_\star, v_\star) = \begin{cases} 1, & if \sum_{k=k_1+1}^{k_2} m_k \ is\ even, \\ -1, & if \sum_{k=k_1+1}^{k_2} m_k \ is\ odd, \end{cases}$$

where $\alpha_\star = \chi(1 - \frac{v_\star}{v_m})$, m_k is the multiplicity of μ_k, and k_1, k_2 will be determined in the later.

Proof (1) Observe that $\bar{W}_{(u_\star, v_\star)} = S_{(u_\star, v_\star)} = E$ and

$$A'(u_\star, v_\star) = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where

$$\begin{aligned} A_1 &= (-d_1\Delta + M)^{-1}(N + M), \\ A_2 &= (-d_1\Delta + M)^{-1}(-\sqrt{u_\star}), \\ A_3 &= (-d_2\Delta + M)^{-1}[\tfrac{\alpha_\star v_\star}{d_1} N + \tfrac{c^2}{2s}], \\ A_4 &= (-d_2\Delta + M)^{-1}(-\tfrac{\alpha_\star v_\star}{d_1}\sqrt{u_\star} - s v_\star + M). \end{aligned}$$

Let $A'(u_\star, v_\star)(u, v)^T = (u, v)^T$, then using the eigenfunction expansions (2.6) in [18] for u and v, we have

$$\begin{pmatrix} d_1\mu_k - N & \sqrt{u_\star} \\ -\frac{\alpha_\star v_\star}{d_1} N - \frac{c^2}{2s} & d_2\mu_k + \frac{\alpha_\star v_\star}{d_1}\sqrt{u_\star} + s v_\star \end{pmatrix} \begin{pmatrix} r_{kj} \\ \widetilde{r_{kj}} \end{pmatrix} = 0,$$

and

$$D(\mu_k) := det \begin{pmatrix} d_1\mu_k - N & \sqrt{u_\star} \\ -\frac{\alpha_\star v_\star}{d_1} N - \frac{c^2}{2s} & d_2\mu_k + \frac{\alpha_\star v_\star}{d_1}\sqrt{u_\star} + s v_\star \end{pmatrix},$$

that is

$$\begin{aligned} D(\mu_k) &= d_1 d_2 \mu_k^2 + [d_1(\tfrac{\alpha_\star v_\star}{d_1}\sqrt{u_\star} + s v_\star) - d_2 N]\mu_k - s v_\star N + \tfrac{c^2}{2s}\sqrt{u_\star} \\ &= d_1 d_2 \mu_k^2 + [d_1(\tfrac{\alpha_\star v_\star}{d_1}\sqrt{u_\star} + s v_\star) - d_2 N]\mu_k + 1 - h - \tfrac{c}{s}. \end{aligned}$$

We notice that $1 - h - \frac{c}{s} > 0$. Therefore, the solution of $D(\mu_k) = 0$ is as follows:
(i) If $F > 0$, $D(\mu_k) = 0$ has no positive solutions and $D(\mu_k) > 0$ for all $k \geq 0$;
(ii) If $F < 0$, and $F^2 - 4d_1 d_2 c u_\star^{\frac{3}{2}} < 0$, then $D(\mu_k) = 0$ has no positive solutions;
(iii) If $F < 0$, and $F^2 - 4d_1 d_2 c u_\star^{\frac{3}{2}} = 0$, then $D(\mu_k) = 0$ has two identical positive solutions;
(iv) If $F < 0$, and $F^2 - 4d_1 d_2 c u_\star^{\frac{3}{2}} > 0$, then $D(\mu_k) = 0$ has two different positive solutions.

Moreover, due to $\bar{W}_{(u_\star, v_\star)} = S_{(u_\star, v_\star)}$, thus $A'(u_\star, v_\star)$ does not have property γ on $\bar{W}_{(u_\star, v_\star)}$.

Now, according to the Lemma 4.1(ii) in this section, we use $\mathrm{index}_W(A, (u_\star, v_\star)) = (-1)^\delta$, where $\delta = \sum_{k \geq 0} \sum_{\lambda_k} m_{\lambda_k} m_k$, and m_{λ_k} is the multiplicity of λ_k as a positive root of $\det B(\lambda, \mu_k) = 0$, where $B(\lambda, \mu_k) = 0$ will be determined in the later.

To make the above purpose, we research for $\lambda > 0$ the eigenvalue problem

$$(A'(u_\star, v_\star) - I)(\phi, \psi)^T = \lambda(\phi, \psi)^T, \quad (\phi, \psi) \neq 0,$$

that is

$$\begin{cases} -d_1(\lambda + 1)\Delta\phi = (N - \lambda M)\phi - \sqrt{u_\star}\psi, \quad x \in \Omega, \\ -d_2(\lambda + 1)\Delta\psi = (\dfrac{\alpha_\star v_\star}{d_1} N + \dfrac{c^2}{2s})\phi - (\dfrac{\alpha_\star v_\star}{d_1} + s v_\star + \lambda M)\psi, \quad x \in \Omega, \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\psi}{\partial n} = 0, \quad x \in \partial\Omega, \\ \phi \neq 0, \psi \neq 0, \quad x \in \Omega. \end{cases} \tag{11}$$

Thus, we have

$$B(\lambda, \mu_k) = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where

$$\begin{aligned} B_1 &= d_1(\lambda + 1)\mu_k + \lambda M - N, \\ B_2 &= \sqrt{u_\star}, \\ B_3 &= -\tfrac{\alpha_\star v_\star}{d_1} N - \tfrac{c^2}{2s}, \\ B_4 &= d_2(\lambda + 1)\mu_k + \lambda M + \tfrac{\alpha_\star v_\star}{d_1}\sqrt{u_\star} + s v_\star. \end{aligned}$$

Next, consider the characteristic equation $\det(B(\lambda,\mu_k))=0$ for $k\geq 0$, that is

$$(d_1\mu_k+M)(d_2\mu_k+M)\lambda^2+[(d_1\mu_k+M)(d_2\mu_k+\tfrac{\alpha_\star v_\star}{d_1}\sqrt{u_\star}+sv_\star)$$

$$+(d_2\mu_k+M)(d_1\mu_k-N)]\lambda+D(\mu_k)=0.$$

If $N<\frac{1}{d_2}(\alpha_\star v_\star\sqrt{u_\star}+d_1sv_\star)$, then $D(\mu_k)=\det(B(0,\mu_k))>0$, that is to say, $I-A'(u_\star,v_\star)$ is invertible. And if $N<\frac{1}{d_1}(\alpha_\star v_\star\sqrt{u_\star}+d_1sv_\star)$, we conclusion that $\det(B(\lambda,\mu_k))=0$ has no positive solutions, thus $\delta=0$, that is $\mathrm{index}_W(A,(u_\star,v_\star))=1$.

(2) If $D(\mu_k)=0$ has two positive solutions, we assume that the two roots are $\mu_\star^-$ and $\mu_\star^+$ satisfying $\mu_\star^-\in(\mu_{k_1},\mu_{k_1+1})$ and $\mu_\star^+\in(\mu_{k_2},\mu_{k_2+1})$ for $1\leq k_1<k_2$. It follows that $D(\mu_k)\neq 0$ for $k\geq 0$, thus we have $I-A'(u_\star,v_\star)$ is invertible on $\bar{W}_{(u_\star,v_\star)}$.

First of all, we consider the case for $k=0$. If $k=0$, then $\mu_k=0$, so we get

$$\det(B(\lambda,0))=M^2\lambda^2+M[(\frac{\alpha_\star v_\star}{d_1}\sqrt{u_\star}+sv_\star)-N]\lambda+1-h-\frac{c}{s}=0.$$

Therefore, $\det(B(\lambda,0))=0$ may have no positive solutions; or two identical positive solutions; or two different positive solutions, thus $\delta=0$ *or* 2.

Next, we consider the case for $k\geq 1$. If $k_1+1\leq k\leq k_2$, $D(\mu_k)<0$, then $\det(B(\lambda,\mu_k))=0$ has one positive solution. If $1\leq k\leq k_1$ or $k\geq k_2+1$,$D(\mu_k)>0$, then $\det(B(\lambda,\mu_k))=0$ may have no positive solutions; or two identical positive solutions; or two different positive solutions, thus $\delta=0$ *or* 2. Finally, we conclusion that $\delta=\sum_{k=k_1+1}^{k_2}m_k+p$, p is an even number. The proof has been completed.

Theorem 4.6 *Assume that $1-\frac{3c}{2s}<h<1-\frac{c}{s}$ and $N<\frac{1}{d_1}(\alpha_\star v_\star\sqrt{u_\star}+d_1sv_\star)$ hold. If $\frac{N}{d_1}\in(\mu_{k^\star},\mu_{k^\star+1})$ for some $k^\star\geq 1$, then there exists a positive constant $d_2^\star$ such that*

$$index_W(u_\star,v_\star)=\begin{cases}1, & if\sum_{k=1}^{k^\star}m_k \ is\ even,\\ -1, & if\sum_{k=1}^{k^\star}m_k\ is\ odd\end{cases}$$

for $d_2\geq d_2^\star$.

Proof In the proof of Theorem 4.5, we get that $I-A'(u_\star,v_\star)$ is invertible on $\bar{W}(u_\star,v_\star)$, and $A'(u_\star,v_\star)$ does not have property γ, and thus we investigate the sum of algebraic multiplicities of the positive eigenvalues of $A'(u_\star,v_\star)-I$.

By calculating, we find that

$$\begin{aligned}\lim_{d_2\to\infty}\mu_\star^- &= \lim_{d_2\to\infty}\frac{-\alpha_\star v_\star\sqrt{u_\star}-d_1 s v_\star+d_2N-\sqrt{[\alpha_\star v_\star\sqrt{u_\star}+d_1 s v_\star-d_2N]^2-4d_1d_2(1-h-\frac{c}{s})}}{2d_1d_2}\\ &= 0,\\ \lim_{d_2\to\infty}\mu_\star^+ &= \lim_{d_2\to\infty}\frac{-\alpha_\star v_\star\sqrt{u_\star}-d_1 s v_\star+d_2N+\sqrt{[\alpha_\star v_\star\sqrt{u_\star}+d_1 s v_\star-d_2N]^2-4d_1d_2(1-h-\frac{c}{s})}}{2d_1d_2}\\ &= \frac{N}{d_1}.\end{aligned}$$

Thus there exists a positive constant $d_2^\star$ such that $\mu_\star^- < \mu_1$ and $\mu_{k^\star} < \mu_\star^+$ for $d_2 \geq d_2^\star$ since $\frac{N}{d_1} \in (\mu_{k^\star}, \mu_{k^\star+1})$. And we notice that if

$$d_2 > \tau_1 := \frac{(\alpha_\star v_\star\sqrt{u_\star}+d_1 s v_\star)\mu_1 + 1 - h - \frac{c}{s}}{N\mu_1 - d_1\mu_1^2},$$

then $D(\mu_1) < 0$, and if

$$d_2 > \tau_2 := \frac{(\alpha_\star v_\star\sqrt{u_\star}+d_1 s v_\star)\mu_{k^\star} + 1 - h - \frac{c}{s}}{N\mu_{k^\star} - d_1\mu_{k^\star}^2},$$

then $D(\mu_{k^\star}) < 0$.

Take a positive constant $d_2^\star$ with $d_2^\star > \max\{\tau_1, \tau_2\}$, then for each $1 \leq k \leq k^\star$ it is easy to see that the equation $\det(B(\lambda,\mu)) = 0$ has only one simple positive root since $D(\mu_k) < 0$ for $1 \leq k \leq k^\star$. On the other hand, for $k \geq k^\star + 1$, by $N < \frac{1}{d_1}(\alpha_\star v_\star\sqrt{u_\star} + d_1 s v_\star)$, the equation $\det(B(\lambda,\mu)) = 0$ has no positive root since $(d_1\mu_k + M)(d_2\mu_k + \frac{\alpha_\star v_\star}{d_1}\sqrt{u_\star} + s v_\star) + (d_2\mu_k + M)(d_1\mu_k - N) > 0$ and $D(\mu_k) > 0$ for $k \geq k^\star + 1$. Therefore we obtain $\delta = \sum_{k=1}^{k^\star} m_k + 0$ which derives the result.

Collecting the above analysis, we conclude the following two conclusions about the existence of non-constant positive solutions of problem (8).

Theorem 4.7 *Assume that (5), (10) and $F^2 - 4d_1d_2cu_\star^{\frac{3}{2}} > 0$ hold. If $\sum_{k=k_1+1}^{k_2} m_k$ is odd, then problem (8) has at least one non-constant positive solution.*

Proof Assume that problem (8) has no non-constant positive solution. Through the previous theorems, we have

$$\begin{aligned}1 &= \mathrm{index}_W(A, IntD')\\ &= \mathrm{index}_W(A,(0,0)) + \mathrm{index}_W(A,(1-h,0)) + \mathrm{index}_W(A,(u_\star,v_\star))\\ &= 0+0+(-1)\\ &= -1\end{aligned}$$

which gives a contradiction. The proof has been complicated.

Theorem 4.8 *Assume that* $1-\frac{3c}{2s}<h<1-\frac{c}{s}$ *and* $N<\frac{1}{d_1}\{\alpha_\star v_\star\sqrt{u_\star}+d_1 s v_\star\}$ *hold,* $\frac{N}{d_1}\in(\mu_{k^\star},\mu_{k^\star+1})$ *for some* $k^\star\geq 1$ *and* $\sum_{k=1}^{k^\star} m_k$ *is odd. Then there exists a positive constant* $d_2^\star$ *such that for* $d_2\geq d_2^\star$ *problem (8) has at least one non-constant positive solution.*

Proof The proof of the theorem is similar to that of Theorem 4.7. So we omit the proof.

5 Numerical Simulation

In this section, by using mathematical software Matlab, we show some numerical simulations to depict our theoretical analysis of the existence of homogeneous periodic solutions.

For problem (2), we choose that $d_1=1,\ d_2=0.8,\ h=0.2412,\ c=0.5,\ s=0.8,$ then we find that c,s satisfy $1-\frac{c}{s}>0, 1-\frac{3c}{2s}>0$, and h satisfies $0<h<1-\frac{c}{s}, W>0$ under the conditions in Theorem 3.1. Theorem 3.1 tell us that when $0<h<0.2437$, the unique positive constant solution is locally asymptotically stable, and problem (2) has a homogeneous Hopf bifurcation near $(u_\star,v_\star)$ with the first bifurcation value $h_0^H=0.2437$ when $0.2437\leq h<0.3750$. When $h=0.095<0.2437$, the local stability of $(u_\star,v_\star)$ is depicted in Fig 1, and the period solutions bifurcating from $(u_\star,v_\star)$ at $h_0^H\approx 0.2412$ are illustrated in Fig 2, respectively.

6 Conlusions

In this paper, we introduce the prey harvesting and prey-taixs into a predator-prey model. Using the Hopf bifurcation theorem, we investigate Hopf bifurcation by choosing prey harvesting parameter. And by the degree method and fixed point index theorem, we study the existence of non-negative steady states depongding on prey-taixs and predator's self-diffusion coefficient respectively. The results show that the introductions of prey harvesting and prey-taixs are necessary and they can enrich the dynamics.

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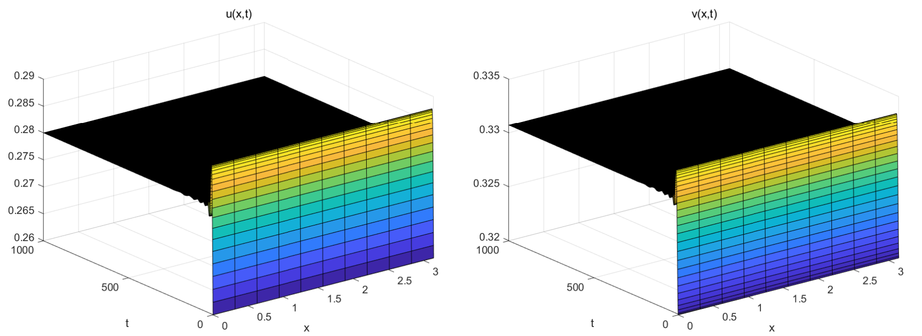

Figure 1: When $h = 0.095 < h_\star = 0.2437$, the unique positive constant solution $(u_\star, v_\star) = (0.28, 0.3307)$ with $(u_0, v_0) = (0.26, 0.32)$ is locally stable. Left: component u. Right: component v.

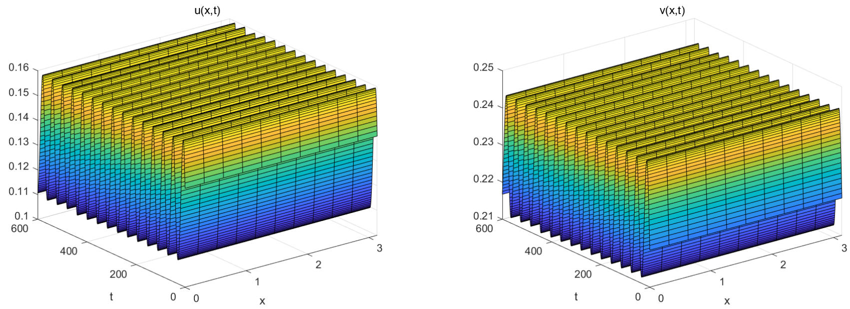

Figure 2: The homogeneous periodic solutions bifurcate from $(u_\star, v_\star) = (0.1338, 0.2286)$ with $(u_0, v_0) = (0.14, 0.22)$, when $h = 0.2412 \approx 0.2437 = h_0^H$. Left: component u. Right: component v.