Multiple solutions for a class of non-cooperative critical nonlocal equation system with variable exponents

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Abstract

In this paper, we consider a class of non-cooperative critical nonlocal equation system with variable exponents of the form:

$$\begin{cases} -(-\Delta)^s_{p(\cdot,\cdot)}u - |u|^{p(x)-2}u = F_u(x,u,v) + |u|^{q(x)-2}u, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s_{p(\cdot,\cdot)}v + |v|^{p(x)-2}v = F_v(x,u,v) + |v|^{q(x)-2}u, & \text{in } \mathbb{R}^N, \\ u, v \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^N), \end{cases}$$

where $\nabla F = (F_u, F_v)$ is the gradient of a C^1-function $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}^+$ with respect to the variable $(u,v) \in \mathbb{R}^2$. We also assume that$\{x \in \mathbb{R}^N : q(x) = p_s^*(x)\} \neq \emptyset$, here $p_s^*(x) = Np(x,x)/(N - sp(x,x))$ is the critical Sobolev exponent for variable exponents. With the help of the Limit index theory and the concentration-compactness principles for fractional Sobolev spaces with variable exponents, we establish the existence of infinitely many solutions for the problem under the suitable conditions on the nonlinearity.

Keywords: Fractional $p(\cdot)$-Laplacian; Limit index; Fractional Sobolev spaces with variable exponents; Concentration-compactness principles; Variational method.

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1 Introduction and main results

In recent years, problems involving nonlocal operators have gained a lot of attentions due to their occurrence in real-world applications, such as, the thin obstacle problem, optimization, finance, phase transitions and also in pure mathematical research, such as, minimal surfaces, conservation laws (for more details see for example [2, 12] and the references therein). The celebrated work of Nezza et al. [16] provides the necessary functional set-up to study these nonlocal problems using variational method. We refer [40] and references therein for more details on problems involving semi-linear fractional Laplace

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operator. In continuation to this, the problems involving quasilinear nonlocal fractional p-Laplace operator are extensively studied by many researchers including Squassina, Palatucci, Mosconi, Rădulescu et al. (see [41, 42]), where the authors studied various aspects, such as existence, multiplicity and regularity of the solutions of the quasilinear nonlocal problem involving fractional p-Laplace operator.

On the other hand, in the recent years, the investigation on problems about differential equations and variational problems involving $p(\cdot)$-growth conditions has been the center of attention because they can be presented as a model for many physical phenomena that arise in the research of elastic mechanics, electrorheological fluids, image processing, etc. We refer the readers to [15, 17] and the references therein. The Lebesgue-Sobolev spaces related to the $p(\cdot)$-Laplacian are called variable exponent Lebesgue-Sobolev spaces and were studied in [20, 29].

While this was happening, it is a natural question to investigate which results can be recovered when the $p(\cdot)$-Laplacian is changed into the fractional $p(\cdot)$-Laplacian. In this regard, Kaumann et al.[28] recently introduced a new class of fractional Sobolev spaces with variable exponents, and elliptic problems involving the fractional $p(\cdot)$-Laplacian have been investigated [3]. The authors in [4] gave some further elementary properties both on this function space and the related nonlocal operator. As applications, they investigated the existence of solutions for equations involving the fractional $p(\cdot)$-Laplacian by employing the critical point theory in [1]. Very recently, Ho and Kim [25] obtained fundamental embedding for the new fractional Sobolev spaces with variable exponent that is a generalization of well-known fractional Sobolev spaces. Using this, they demonstrated a priori bounds and multiplicity of solutions of some nonlinear elliptic problems involving the fractional $p(\cdot)$-Laplacian. We refer to [44, 45] fractional Sobolev spaces with variable exponents and the corresponding nonlocal equations with variable exponents.

To the authors' best knowledge, though most properties of the classical fractional Sobolev spaces have been extended to the fractional Sobolev spaces with variable exponents, there are few results for the critical Sobolev type imbedding for these spaces. The critical problem was initially studied in the seminal paper by Brezis-Nirenberg [10], which treated for Laplace equations. Since then there have been extensions of [10] in many directions. Elliptic equations involving critical growth are delicate due to the lack of compactness arising in connection with the variational approach. For such problems, the concentration-compactness principles introduced by P.L. Lions [38, 39] and its variant at infinity [6, 7, 14] have played a decisive role in showing a minimizing sequence or a Palais-Smale sequence is precompact. By using these concentration-compactness principles or extending them to the Sobolev spaces with fractional order or variable exponents, many authors have been successful to deal with critical problems involving p-Laplacian or $p(\cdot)$-Laplacian or fractional p-Laplacian, see e.g., [8, 9, 11, 13, 21, 23, 25, 32, **?**, 34, 35, 46] and references therein. Recently, Ho and Kim [26] proved the concentration-compactness principles for fractional Sobolev spaces with variable exponents and obtained the existence of many solutions for a class of critical nonlocal problems with variable exponents.

The present paper is devoted to the solvability of non-cooperative critical nonlocal equation system with variable exponents:

$$\begin{cases} -(-\Delta)^s_{p(\cdot,\cdot)}u - |u|^{p(x)-2}u = F_u(x,u,v) + |u|^{q(x)-2}u, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s_{p(\cdot,\cdot)}v + |v|^{p(x)-2}v = F_v(x,u,v) + |v|^{q(x)-2}u, & \text{in } \mathbb{R}^N, \\ u, v \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\nabla F = (F_u, F_v)$ is the gradient of a C^1-function $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}^+$ with respect to the variable $(u,v) \in \mathbb{R}^2$, $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ is symmetric, i.e., $p(x,y) = p(y,x)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $q \in C(\mathbb{R}^N)$

satisfies

$$p(x) := p(x,x) < q(x) \le p_s^*(x) := \frac{Np(x,x)}{N - sp(x,x)} \quad \text{for all } x \in \mathbb{R}^N.$$

The main aim of this paper is to obtain the existence results of a sequence of infinitely many solutions to the problem (1.1). The strategy of the proof for these assertions is based on the applications of the Limit index theory, which were initially introduced by Li [31] for local problems with subcritical growth condition in bounded domains, in view of the variational nature of the problem considered. We also refer the works related to those papers [27, 36, 37]. Motivated by the contribution cited above, we shall study the existence of solutions for (1.1) with the help of the Limit index theory. We can see that there are two main difficulties in considering our problem. Firstly, problem (1.1) involves critical nonlocal which prevents us from applying the methods as before. To overcome the challenge we use the concentration-compactness principles for fractional Sobolev spaces with variable exponents due to [26] in order to prove the $(PS)_c$ condition at special levels c. The second difficulty is that the energy functional associated to the problem is strongly indefinite in the sense that it is neither unbounded from below or from above on any subspace of finite codimension. Therefore, one cannot apply the symmetric mountain pass theorem on the energy functional. To our best knowledge, there are no existence results about the critical nonlocal problems with variable exponents (1.1).

In the rest of this paper, we always assume that the variable exponents p, q and the function f satisfy the following assumptions:

($\mathcal{P}$) $p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is uniformly continuous and symmetric such that

$$1 < \underline{p} := \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) \le \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) =: \overline{p} < \frac{N}{s};$$

there exists $\varepsilon_0 \in (0, \frac{1}{2})$ such that $p(x,y) = \overline{p}$ for all $x, y \in \mathbb{R}^N$ satisfying $|x-y| < \varepsilon_0$ and $\sup_{y\in\mathbb{R}^N} p(x,y) = \overline{p}$ for all $x \in \mathbb{R}^N$; and $|\{x \in \mathbb{R}^N : p_*(x) \neq \overline{p}\}| < \infty$, where $p_*(x) := \inf_{y\in\mathbb{R}^N} p(x,y)$ for $x \in \mathbb{R}^N$.

($\mathcal{Q}$) $q : \mathbb{R}^N \to \mathbb{R}$ is uniformly continuous such that $p_*(x) \le q(x) \le \overline{p}_s^*$ for all $x \in \mathbb{R}^N$ and $\mathcal{C} := |\{x \in \mathbb{R}^N : q(x) = \overline{p}_s^*\}| \neq \emptyset$.

($\mathcal{E}_\infty$) There exist $\lim\limits_{|x|,|y|\to\infty} p(x,y) = \bar{p}$ and $\lim\limits_{|x|\to\infty} q(x) = q_\infty$ for $\bar{p}$ given by ($\mathcal{P}$) and some $q_\infty \in (1,\infty)$.

($\mathcal{F}$) (F$_1$) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^+)$, here $\mathbb{R}^+ = \{x \in \mathbb{R} | x \ge 0\}$; and there exist two positive constants $C_1, C_2 > 0$, the function r with $r \in C(\mathbb{R}^N, \mathbb{R}^+)$, $\inf_{x\in\mathbb{R}^N}[q(x) - r(x)] > 0$ and $r^- > \overline{p}$ such that

$$|F_s(r,s,t)| + |F_t(r,s,t)| \le C_1(x)|s|^{r(x)-1} + C_2(x)|t|^{r(x)-1}.$$

(F$_2$) there exist $\overline{p} < \theta < q^-$ such that $0 < \theta F(r,s,t) \le sF_s(r,s,t) + tF_t(r,s,t)$, for any $(r,s,t) \in (\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^+)$.

(F$_3$) $sF_s(x,s,t) \ge 0$ for all $(x,s,t) \in \mathbb{R}^N \times \mathbb{R}^2$.

(F$_4$) $F(x,s,t) = F(x,-s,-t)$ for all $(x,s,t) \in \mathbb{R}^N \times \in \mathbb{R}^2$.

The main result of this paper is as follows.

Theorem 1.1. *Let ($\mathcal{P}$), ($\mathcal{Q}$) and ($\mathcal{E}_\infty$) hold. If F satisfies* (F$_1$)–(F$_4$) *are fulfilled. Then problem* (1.1) *possesses infinitely many solutions.*

The rest of our paper is organized as follows. In Section 2, we briefly review some properties of the Sobolev spaces with fractional order or variable exponents. Moreover, we introduce the Limit Index Theory due to Li [31]. In Section 3, we prove the Palais-Smale condition at some special energy levels by using the concentration-compactness principles for fractional Sobolev spaces with variable exponents. The proof of the main result Theorem 1.1 is given in Section 4.

2 Fractional Sobolev spaces and Limit Index Theory

This section will be divided into three parts. First, we briefly review the definitions and list some basic properties of the Lebesgue spaces. Second, we recall and we establish some qualitative properties of the new fractional Sobolev spaces with variable exponent. Finally, we recall the Limit Index Theory due to Li [31].

2.1 Variable exponent Lebesgue spaces and fractional Sobolev spaces

In this subsection, we recall some useful properties of variable exponent spaces. For more details we refer the reader to [17, 18, 29], and the references therein.

Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h_+ = \sup_{x\in\overline{\Omega}} h(x) \quad \text{and} \quad h_- = \inf_{x\in\overline{\Omega}} h(x).$$

We can introduce the variable exponent Lebesgue space as follows:

$$L^{p(\cdot)}(\Omega) = \big\{u : u \text{ is a measurable real-valued functionsuch that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\big\},$$

for $p \in C_+(\Omega)$. Defining the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(\cdot)} = \inf\big\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\big\},$$

then the space $L^{p(x)}(\Omega)$ is a Banach space, we call it a generalized Lebesgue space.

Proposition 2.1. *[17, 19]* (i) *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{p_*(x)}(\Omega)$, where $1/p_*(x) + 1/p(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p_*(x)}(\Omega)$, we have*

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p_*^-}\Big)|u|_{p(\cdot)}|v|_{p_*(\cdot)}; \tag{2.1}$$

(ii) *If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.*

Proposition 2.2. *[17, 19] The mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by*

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

Then the following relations hold:

$$\begin{aligned}
|u|_{p(\cdot)} < 1\ (=1;\ >1) \ &\Leftrightarrow\ \rho_{p(\cdot)}(u) < 1\ (=1;\ >1),\\
|u|_{p(\cdot)} > 1 \ &\Rightarrow\ |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+},\\
|u|_{p(\cdot)} < 1 \ &\Rightarrow\ |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-},\\
|u_n - u|_{p(\cdot)} \to 0 \ &\Leftrightarrow\ \rho_{p(\cdot)}(u_n - u) \to 0.
\end{aligned}$$

Let $s \in (0,1)$ and $p \in (1,\infty)$ be constants. Define the fractional Sobolev space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \iint_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty \right\}$$

endowed with norm

$$\|u\|_{s,p,\Omega} := \left(u \in L^p(\Omega) : \int_\Omega |u(x)|^p dx + \iint_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}.$$

We recall the following crucial imbeddings:

Proposition 2.3. *[16] Let $s \in (0,1)$ and $p \in (1,\infty)$ be such that $sp < N$. It holds that*
(i) $W^{s,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *if Ω is bounded and $1 \le q < \frac{Np}{N-sp} =: p_s^*$;*
(ii) $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ *if Ω is bounded and $p \le q \le \frac{Np}{N-sp} =: p_s^*$.*

2.2 Fractional Sobolev spaces with variable exponent

In this subsection, we recall the fractional Sobolev spaces with variable exponents that was first introduced in [28], and was then refined in [25]. Furthermore, we will obtain a critical Sobolev type imbedding on these spaces.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. In the following, for brevity, we write $p(x)$ instead of $p(x,x)$ and with this notation, $p \in C_+(\bar{\Omega})$. Define

$$W^{s,p(\cdot,\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy < +\infty \right\}$$

endowed with the norm

$$\|u\|_{s,p,\Omega} := \inf \left\{ \lambda > 0 : M_\Omega(\frac{u}{\lambda}) < 1 \right\},$$

where

$$M_\Omega(u) := \int_\Omega |u|^{p(x)}\, dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy.$$

Then $W^{s,p(\cdot,\cdot)}(\Omega)$ is a separable reflexive Banach space (see [3, 4, 28]). On $W^{s,p(\cdot,\cdot)}(\Omega)$, we also make use of the following norm

$$|u|_{s,p,\Omega} := \|u\|_{L^{p(\cdot)}(\Omega)} + [u]_{s,p,\Omega},$$

where

$$[u]_{s,p,\Omega} := \inf \left\{ \lambda > 0 : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dx\, dy < 1 \right\}.$$

Note that $\|\cdot\|_{s,p,\Omega}$ and $|\cdot|_{s,p,\Omega}$ are equivalent norms on $W^{s,p(\cdot,\cdot)}(\Omega)$ with the relation

$$\frac{1}{2}\|u\|_{s,p,\Omega} \leq |u|_{s,p,\Omega} \leq 2\|u\|_{s,p,\Omega}, \quad \forall\ u \in W^{s,p(\cdot,\cdot)}(\Omega). \tag{2.2}$$

Remark 2.1. *It is clear that if p satisfies ($\mathcal{P}$), then $p(x,x) = \overline{p}$ for all $x \in \mathbb{R}^N$. Hence, by Theorem 3.3 in [26], we have*

$$W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow L^{\overline{p}_s^*(\cdot)}(\mathbb{R}^N). \tag{2.3}$$

On the other hand, by ($\mathcal{P}$) we have that for any $u \in L^{\overline{p}}(\mathbb{R}^N)$,

$$\begin{aligned}
\int_{x\in\mathbb{R}^N} |u|^{p_*(x)}dx &= \int_{p_*(x)=\overline{p}} |u|^{p_*(x)}dx + \int_{p_*(x)\neq\overline{p}} |u|^{p_*(x)}dx \\
&\leq \int_{p_*(x)=\overline{p}} |u|^{\overline{p}}dx + \int_{p_*(x)\neq\overline{p}} \left[1+|u|^{\overline{p}}\right]dx \\
&= \left|\{x \in \mathbb{R}^N : p_*(x) \neq \overline{p}\}\right| + \int_{x\in\mathbb{R}^N} |u|^{\overline{p}}dx < \infty.
\end{aligned}$$

Hence, $L^{\overline{p}}(\mathbb{R}^N) \subset L^{p_*(\cdot)}(\mathbb{R}^N)$. From this and (2.3) *we obtain*

$$W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow L^{t(\cdot)}(\mathbb{R}^N). \tag{2.4}$$

for any $t \in C(\mathbb{R}^N)$ satisfying $p_*(x) \leq t(x) \leq \overline{p}_s^*$ for all $x \in \mathbb{R}^N$. In particular, ($\mathcal{Q}$) yields

$$S_q := \inf_{u\in W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|}{\|u\|_{L^{q(\cdot)}(\mathbb{R}^N)}} \tag{2.5}$$

In what follows, when Ω is understood, we just write $\|\cdot\|_{s,p}$, $|\cdot|_{s,p}$ and $[\cdot]_{s,p}$ instead of $\|\cdot\|_{s,p,\Omega}$, $|\cdot|_{s,p,\Omega}$ and $[\cdot]_{s,p,\Omega}$, respectively. We also denote the ball in $\mathbb{R}^N$ centered at z with radius ε by $B_\varepsilon(z)$ and denote the Lebesgue measure of a set $E \subset \mathbb{R}^N$ by $|E|$. For brevity, we write B_ε and B_ε^c instead of $B_\varepsilon(0)$ and $\mathbb{R}^N \setminus B_\varepsilon(0)$, respectively.

Proposition 2.4. *([25]) On $W^{s,p(\cdot,\cdot)}(\Omega)$ it holds that*
(i) for $u \in W^{s,p(\cdot,\cdot)}(\Omega)$, $\lambda = \|u\|_{s,p}$ if and only if $M_\Omega(\frac{u}{\lambda}) = 1$;
(ii) $M_\Omega(u) > 1(= 1; < 1)$ if and only if $\|u\|_{s,p} > 1(= 1; < 1)$, respectively;
(iii) if $\|u\|_{s,p} \geq 1$, then $\|u\|_{s,p}^{p^-} \leq M_\Omega(u) \leq \|u\|_{s,p}^{p^+}$;
(iv) if $\|u\|_{s,p} < 1$, then $\|u\|_{s,p}^{p^+} \leq M_\Omega(u) \leq \|u\|_{s,p}^{p^-}$.

Theorem 2.1. *(Subcrtitical imbeddings, [25]). It holds that*
(i) $W^{s,p(\cdot,\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\Omega)$, if Ω is a bounded Lipschitz domain and $r \in C_+(\overline{\Omega})$ such that $r(x) < \frac{Np(x)}{N-sp(x)} =: p_s^*(x)$ for all $x \in \overline{\Omega}$;
(ii) $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow L^{r(\cdot)}(\mathbb{R}^N)$ for any uniformly continuous function $r \in C_+(\mathbb{R}^N)$ satisfying $p(x) \leq r(x)$ for all $x \in \mathbb{R}^N$ and $\inf_{x\in\mathbb{R}^N}(p_s^*(x) - r(x)) > 0$;
(iii) $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L_{loc}^{r(\cdot)}(\mathbb{R}^N)$ for any $r \in C_+(\mathbb{R}^N)$ satisfying $r(x) < p_s^*(x)$ for all $x \in \mathbb{R}^N$.

2.3 Limit Index Theory

In this section, we recall the Limit Index Theory due to Li [31]. In order to do that, we introduce the following definitions.

Definition 2.1. *[31, 43] The action of a topological group G on a normed space Z is a continuous map*

$$G \times Z \to Z : [g, z] \mapsto gz$$

such that

$$1 \cdot z = z, \quad (gh)z = g(hz) \quad z \mapsto gz \ \text{ is linear}, \quad \forall\, g, h \in G.$$

The action is isometric if

$$\|gz\| = \|z\|, \quad \forall\, g \in G, \quad z \in Z.$$

And in this case Z is called the G-space.
The set of invariant points is defined by

$$\mathrm{Fix}(G) := \{z \in Z : gz = z, \ \forall\, g \in G\}.$$

A set $A \subset Z$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : Z \to R$ is invariant $\varphi \circ g = \varphi$ for every $g \in G$, $z \in Z$. A map $f : Z \to Z$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Suppose that Z is a G-Banach space, that is, there is a G isometric action on Z. Let

$$\Sigma := \{A \subset Z : A \text{ is closed and } gA = A, \forall\, g \in G\}$$

be a family of all G-invariant closed subsets of Z, and let

$$\Gamma := \left\{h \in C^0(Z, Z) : h(gu) = g(hu), \ g \in G\right\}$$

be the class of all G-equivariant mappings of Z. Finally, we call the set

$$O(u) := \{gu : g \in G\}$$

the G-orbit of u.

Definition 2.2. *[31] An index for (G, Σ, Γ) is a mapping $i : \Sigma \to \mathcal{Z}_+ \cup \{+\infty\}$ (where $\mathcal{Z}_+$ is the set of all nonnegative integers) such that for all $A, B \in \Sigma$, $h \in \Gamma$, the following conditions are satisfied:*
(1) $i(A) = 0 \Leftrightarrow A = \emptyset$;
(2) *(Monotonicity)* $A \subset B \Rightarrow i(A) \leq i(B)$;
(3) *(Subadditivity)* $i(A \cup B) \leq i(A) + i(B)$;
(4) *(supervariance)* $i(A) \leq i(\overline{h(A)}), \forall\, h \in \Gamma$;
(5) *(Continuity) If A is compact and $A \cap Fix(G) = \emptyset$, then $i(A) < +\infty$ and there is a G-invariant neighbourhood N of A such that $i(\overline{N}) = i(A)$;*
(6) *(Normalization) If $x \notin Fix(G)$, then $i(O(x)) = 1$.*

Definition 2.3. *[5] An index theory is said to satisfy the d-dimensional property if there is a positive integer d such that*

$$i(V^{dk} \cap S_1) = k$$

for all dk-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \cap Fix(G) = \{0\}$, where S_1 is the unit sphere in Z.

Suppose that U and V are G-invariant closed subspaces of Z such that

$$Z = U \oplus V,$$

where V is infinite dimensional and

$$V = \overline{\bigcup_{j=1}^{\infty} V_j},$$

where V_j is a dn_j-dimensional G-invariant subspace of V, $j = 1, 2, \cdots$, and $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$. Let

$$Z_j = U \oplus V_j,$$

and $\forall\ A \in \Sigma$, let

$$A_j = A \oplus Z_j.$$

Definition 2.4. *[31] Let i be an index theory satisfying the d-dimensional property. A limit index with respect to (Z_j) induced by i is a mapping*

$$i^{\infty} : \Sigma \to \mathcal{Z} \cup \{-\infty, +\infty\}$$

given by

$$i^{\infty}(A) = \limsup_{j \to \infty} (i(A_j) - n_j).$$

Proposition 2.5. *[31] Let $A, B \in \Sigma$. Then i^{∞} satisfies:*
(1) *$A = \emptyset \Rightarrow i^{\infty} = -\infty$;*
(2) *(Monotonicity) $A \subset B \Rightarrow i^{\infty}(A) \leq i^{\infty}(B)$;*
(3) *(Subadditivity) $i^{\infty}(A \cup B) \leq i^{\infty}(A) + i^{\infty}(B)$;*
(4) *If $V \cap \mathrm{Fix}(G) = \{0\}$, then $i^{\infty}(S_{\rho} \cap V) = 0$, where $S_{\rho} = \{z \in Z : \|z\| = \rho\}$;*
(5) *If Y_0 and $\widetilde{Y_0}$ are G-invariant closed subspaces of V such that $V = Y_0 \oplus \widetilde{Y_0}$, $\widetilde{Y_0} \subset V_{j_0}$ for some j_0 and $\dim(Y_0) = dm$, then $i^{\infty}(S_{\rho} \cap Y_0) \geq -m$.*

Definition 2.5. *[43] A functional $I \in C^1(Z, R)$ is said to satisfy the condition $(PS)_c^*$ if any sequence $\{u_{n_k}\}$, $u_{n_k} \in Z_{n_k}$ such that*

$$I(u_{n_k}) \to c, \quad dI_{n_k}(u_{nk}) \to 0, \quad \text{as } k \to \infty$$

possesses a convergent subsequence, where Z_{n_k} is the n_k-dimensional subspace of Z, $I_{n_k} = I|_{Z_{n_k}}$.

Theorem 2.2. *[31] Assume that*
(B_1) *$I \in C^1(Z, R)$ is G-invariant;*
(B_2) *There are G-invariant closed subspaces U and V such that V is infinite dimensional and $Z = U \oplus V$;*
(B_3) *There is a sequence of G-invariant finite dimensional subspaces*

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots, \quad \dim(V_j) = dn_j,$$

such that $V = \overline{\cup_{j=1}^{\infty} V_j}$;
(B_4) *There is an index theory i on Z satisfying the d-dimensional property;*
(B_5) *There are G-invariant subspaces Y_0, $\widetilde{Y_0}$, Y_1 of V such that $V = Y_0 \oplus \widetilde{Y_0}$, $Y_1, \widetilde{Y_0} \subset V_{j_0}$ for some j_0 and $\dim(\widetilde{Y_0}) = dm < dk = \dim(Y_1)$;*

(B_6) *There are α and β, $\alpha < \beta$ such that f satisfies $(PS)_c^*$, $\forall\ c \in [\alpha, \beta]$;*

$$(B_7)\quad \begin{cases} (a)\ \textit{either}\ \mathrm{Fix}(G) \subset U \oplus Y_1, \quad \textit{or} \quad \mathrm{Fix}(G) \cap V = \{0\}, \\ (b)\ \textit{there is}\ \rho > 0\ \textit{such that}\ \forall\ u \in Y_0 \cap S_\rho, f(z) \geq \alpha, \\ (c)\ \forall\ z \in U \oplus Y_1,\ f(z) \leq \beta, \end{cases}$$

if i^∞ is the limit index corresponding to i, then the numbers

$$c_j = \inf_{i^\infty(A) \geq j} \sup_{z \in A} f(u), \quad -k+1 \leq j \leq -m,$$

are critical values of f, and $\alpha \leq c_{-k+1} \leq \cdots \leq c_{-m} \leq \beta$. Moreover, if $c = c_l = \cdots = c_{l+r}$, $r \geq 0$, then $i(\mathbb{K}_c) \geq r+1$, where $\mathbb{K}_c = \{z \in Z : df(z) = 0, f(z) = c\}$.

3 Verification of $(PS)_c$ condition

In this section, we perform a careful analysis of the behavior of minimizing sequences with the aid of concentration-compactness principles for fractional Sobolev spaces with variable exponents due to [26], which allows to recover compactness below some critical threshold.

Let $\mathcal{M}(\mathbb{R}^N)$ be the space of all signed finite Radon measures on $\mathbb{R}^N$ endowed with the total variation norm. Note that we may identify $\mathcal{M}(\mathbb{R}^N)$ with the dual of $C_0(\mathbb{R}^N)$, the completion of all continuous functions $u : \mathbb{R}^N \to \mathbb{R}$ whose support is compact relative to the supremum norm $\|\cdot\|_\infty$ (see, e.g., [22]).

Theorem 3.1. *Assume that $(\mathcal{P})$ and $(\mathcal{Q})$ hold. Let $\{u_n\}$ be a bounded sequence in $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ such that*

$$u_n \rightharpoonup u \quad in \quad W^{s,p(\cdot,\cdot)}(\mathbb{R}^N),$$

$$|u_n|^{\bar{p}} + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(s,y)}}\, dy \overset{*}{\rightharpoonup} \mu \quad in \quad \mathcal{M}(\mathbb{R}^N),$$

$$|u_n|^{q(x)} \overset{*}{\rightharpoonup} \nu \quad in \quad \mathcal{M}(\mathbb{R}^N).$$

Then, there exist sets $\{\mu_i\}_{i\in I} \subset (0,\infty)$, $\{\nu_i\}_{i\in I} \subset (0,\infty)$ and $\{x_i\}_{i\in I} \subset \mathcal{C}$, where I is an at most countable index set, such that

$$\mu \geq |u|^{\bar{p}} + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dy + \sum_{i\in I} \mu_i \delta_{x_i},$$

$$\nu = |u|^{q(x)} + \sum_{i \in I} \nu_i \delta_{x_i},$$

$$S_q \nu_i^{\frac{1}{\bar{p}_s^*}} \leq \mu_i^{\frac{1}{\bar{p}}}, \quad \forall i \in I.$$

For possible loss of mass at infinity, we have the following.

Theorem 3.2. *Assume that $(\mathcal{P})$, $(\mathcal{Q})$ and $(\mathcal{E}_\infty)$ hold. Let $\{u_n\}$ be a sequence in $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ as in Theorem 3.1. Set*

$$\nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{B_R^c} |u_n|^{q(x)}\, dx,$$

$$\mu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{B_R^c}\left[|u_n|^{\bar{p}} + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dy\right]\,dx.$$

Then

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{q(x)}\,dx = \nu(\mathbb{R}^N)+\nu_\infty,$$

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left[|u_n|^{\bar{p}} + \int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dy\right]\,dx = \mu(\mathbb{R}^N)+\mu_\infty.$$

Then

$$S_q\nu_\infty^{\frac{1}{q_\infty}} \le \mu_\infty^{\frac{1}{\bar{p}}}.$$

Now, we turn to prove $(PS)_c$ condition for $\mathcal{J}$. In order to apply Theorem 3.1 and 3.2, let us denote $G_1 = O(N)$ is the group of orthogonal linear transformations in $\mathbb{R}^N$. $E = W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$, $E_{G_1} = W^{s,p}_{O(N)} := \{u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) : gu(x) = u(g^{-1}x) = u(x), g \in O(N)\}$. $G_2 = \mathbb{Z}_2$, $Y = E \times E$, $X = Y_{G_1} = E_{G_1} \times E_{G_1}$. c denotes a positive constant and can be determined in concrete conditions.

To determine solutions to problem (1.1), we will apply Theorem 2.2 for Y endowed with the norm $\|(u,v)\|_{s,p} = \|u\|_{s,p} + \|v\|_{s,p}$. Consequently, by [3, 4, 28], we know that $(Y, \|\cdot\|_{s,p})$ is a reflexive Banach space. Let us consider the Euler-Lagrange functional associated to (1.1), defined by $\mathcal{J} : Y \to \mathbb{R}$

$$\begin{aligned}
\mathcal{J}(u,v) \;=\; & -\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy - \frac{1}{p(x)}\int_{\mathbb{R}^N}|u|^{p(x)}dx - \frac{1}{q(x)}\int_{\mathbb{R}^N}|u|^{q(x)}dx \\
& + \iint_{\mathbb{R}^{2N}}\frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy + \frac{1}{p(x)}\int_{\mathbb{R}^N}|v|^{p(x)}dx - \frac{1}{q(x)}\int_{\mathbb{R}^N}|v|^{q(x)}dx \\
& - \int_{\mathbb{R}^N}F(x,u,v)dx. \qquad (3.1)
\end{aligned}$$

It is clear that under the assumptions $(\mathcal{F})$, $\mathcal{J}$ is of class $C^1(Y,\mathbb{R})$. Moreover, for all $(u,v),(z_1,z_2) \in Y$, its Fréchet derivative is given by

$$\begin{aligned}
\langle \mathcal{J}'(u,v),(z_1,z_2)\rangle \;=\; & -[u,z_1] - \int_{\mathbb{R}^N}|u|^{p(x)-2}uz_1dx - \int_{\mathbb{R}^N}|u|^{q(x)-2}uz_1dx \\
& + [v,z_2] + \int_{\mathbb{R}^N}|v|^{p(x)-2}vz_2dx - \int_{\mathbb{R}^N}|v|^{q(x)-2}vz_2dx \\
& - \int_{\mathbb{R}^N}F_u(x,u,v)z_1dx - \int_{\mathbb{R}^N}F_v(x,u,v)z_2dx = 0,
\end{aligned}$$

where

$$[\zeta, z_i] := \iint_{\mathbb{R}^{2N}}\frac{|\zeta(x)-\zeta(y)|^{p(x,y)-2}(\zeta(x)-\zeta(y))(z_i(x)-z_i(y))}{|x-y|^{N+sp(x,y)}}dxdy \quad \text{for } i = 1,2.$$

It is easy to check that $\mathcal{J} \in C^1$ and the weak solutions for problem (1.1) coincide with the critical points of $\mathcal{J}$. By conditions $(\mathcal{P})$ and (F_4), it is immediate to see that $\mathcal{J}$ is $O(N)$-invariant. Then by the principle of symmetric criticality of Krawcewicz and Marzantowicz [30], we know that (u,v) is a critical point of $\mathcal{J}$ if and only if (u,v) is a critical point of $J = \mathcal{J}|_{X=E_{G_1}\times E_{G_1}}$. Therefore, it suffices to prove the existence of a sequence of critical points of $\mathcal{J}$ on Y.

Lemma 3.1. *Assume that* $(\mathcal{P})$, $(\mathcal{Q})$, $(\mathcal{E}_\infty)$ *and* $(\mathcal{F})$ *hold. Let* $\{(u_{n_k}, v_{n_k})\}$ *be a sequence such that* $\{(u_{n_k}, v_{n_k})\} \in X_{n_k}$,

$$J_{n_k}(u_{n_k}, v_{n_k}) \to c < \left(\frac{1}{\theta} - \frac{1}{q^-}\right) \min\left\{S_q^{\overline{p}\tau^+}, S_q^{\overline{p}\tau^-}\right\}, \quad dJ_{n_k}(u_{n_k}, v_{n_k}) \to 0, \quad \text{as } k \to \infty,$$

where $J_{n_k} = \mathcal{J}|_{X_{n_k}}$. *Then* $\{(u_{n_k}, v_{n_k})\}$ *contains a subsequence converging strongly in* X.

Proof. First, we show that $\{(u_{n_k}, v_{n_k})\}$ is bounded in X. If not, we may assume that $\|u_{n_k}\|_{s,p} > 1$ and $\|v_{n_k}\|_{s,p} > 1$ for any integer n. We have by condition (F_3),

$$\begin{aligned} o(1)\|u_{n_k}\|_{s,p} &\geq \langle -dJ_{n_k}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\rangle \\ &= \iint_{\mathbb{R}^{2N}} \frac{|u_{n_k}(x) - u_{n_k}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \int_{\mathbb{R}^N} |u_{n_k}|^{\overline{p}} dx \\ &\quad + \int_{\mathbb{R}^N} |u_{n_k}|^{q(x)} dx + \int_{\mathbb{R}^N} F_u(x, u_{n_k}, v_{n_k}) u_{n_k} dx \\ &\geq \iint_{\mathbb{R}^{2N}} \frac{|u_{n_k}(x) - u_{n_k}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \int_{\mathbb{R}^N} |u_{n_k}|^{\overline{p}} dx \\ &\geq \|u_{n_k}\|_{s,p}^{p^-}, \end{aligned} \tag{3.2}$$

Since $p^- > 1$, from (3.2), we know that $\{u_{n_k}\}$ is bounded. On the one hand, we have by condition (F_2),

$$\begin{aligned} c + o(1)\|v_{n_k}\|_{s,p} &= J_{n_k}(0, v_{n_k}) - \frac{1}{\theta}\langle dJ_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k})\rangle \\ &= \iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - v_{n_k}(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dxdy - \frac{1}{\theta} \iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - v_{n_k}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy \\ &\quad + \left(\frac{1}{\overline{p}} - \frac{1}{\theta}\right) \int_{\mathbb{R}^N} |v_{n_k}|^{\overline{p}} \mathrm{d}x + \left(\frac{1}{\theta} - \frac{1}{q(x)}\right) \int_{\mathbb{R}^N} |v_{n_k}|^{q(x)} \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} \left[F(x, 0, v_{n_k}) - \frac{1}{\theta} F_v(x, 0, v_{n_k}) v_{n_k}\right] dx \\ &\geq \left(\frac{1}{\overline{p}} - \frac{1}{\theta}\right) \iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - v_{n_k}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \left(\frac{1}{\overline{p}} - \frac{1}{\theta}\right) \int_{\mathbb{R}^N} |v_{n_k}|^{\overline{p}} \mathrm{d}x \\ &\geq \left(\frac{1}{\overline{p}} - \frac{1}{\theta}\right) \|v_{n_k}\|_{s,p}^{p^-}. \end{aligned}$$

This face implies that $\{v_{n_k}\}$ is bounded in E. Thus $\|u_{n_k}\|_{s,p} + \|v_{n_k}\|_{s,p}$ is bounded in X.

Next, we prove that $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence converging strongly in X.

On the one hand, we note that $\{u_{n_k}\}$ is bounded in E_{G_1}. Hence, up to a subsequence, $u_{n_k} \rightharpoonup u_0$ weakly in E_{G_1} and $u_{n_k}(x) \to u_0(x)$, a.e. in $\mathbb{R}^N$. We claim that $u_{n_k} \to u_0$ strongly in E_{G_1}. It follows from condition (F_3) that

$$\begin{aligned} 0 &\leftarrow \langle -dJ_{n_k}(u_{n_k} - u_0, v_{n_k}), (u_{n_k} - u_0, 0)\rangle \\ &= [u_{n_k} - u_0, u_{n_k} - u_0] + \int_{\mathbb{R}^N} |u_{n_k} - u_0|^{p(x)} \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |u_{n_k} - u_0|^{q(x)} dx + \int_{\mathbb{R}^N} F_u(x, u_{n_k} - u_0, v_{n_k})(u_{n_k} - u_0) dx \\ &\geq \|u_{n_k} - u_0\|_{s,p}^{p^-}. \end{aligned}$$

This fact imply that

$$u_{n_k} \to u_0 \quad \text{strongly in } E_{G_1}. \tag{3.3}$$

In the following we will prove that there exists $v \in E_{G_1}$ such that

$$v_{n_k} \to v_0 \quad \text{strongly in } E_{G_1}. \tag{3.4}$$

Since $\{v_{n_k}\}$ is also bounded in E. So we may assume that there exists v_0 and a subsequence, still denoted by $\{v_{n_k}\} \subset E$ such that

$$\begin{aligned}
& v_{n_k}(x) \to v_0(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^N, \\
& v_{n_k} \rightharpoonup v_0 \quad \text{in} \quad W^{s,p(\cdot,\cdot)}(\mathbb{R}^N), \\
& V_{n_k}(x) \rightharpoonup \mu \geq V_0(x) + \sum_{i\in I} \delta_{x_i}\mu_i \quad \text{weak*-sense of measures in } \mathcal{M}(\mathbb{R}^N), && (3.5) \\
& |v_{n_k}|^{q(x)} \rightharpoonup \nu = |v_0|^{q(x)} + \sum_{i\in I} \delta_{x_i}\nu_i \quad \text{weak*-sense of measures in } \mathcal{M}(\mathbb{R}^N), && (3.6) \\
& S_q \nu_i^{\frac{1}{p_s^*}} \leq \mu_i^{\frac{1}{\overline{p}}} \quad \text{for} \quad i \in I, && (3.7)
\end{aligned}$$

where

$$V_{n_k}(x) := |v_{n_k}(x)|^{\overline{p}} + \int_{\mathbb{R}^N} \frac{|v_{n_k}(x) - v_{n_k}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dy$$

and

$$V_0(x) := |v_0(x)|^{\overline{p}} + \int_{\mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dy$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. Moreover, we have

$$\begin{aligned}
& \limsup_{n\to\infty} \int_{\mathbb{R}^N} V_{n_k}(x)dx = \mu(\mathbb{R}^N) + \mu_\infty, && (3.8) \\
& \limsup_{n\to\infty} \int_{\mathbb{R}^N} |v_{n_k}|^{q(x)}dx = \nu(\mathbb{R}^N) + \nu_\infty, && (3.9) \\
& S_q \nu_\infty^{\frac{1}{q_\infty}} \leq \mu_\infty^{\frac{1}{\overline{p}}}. && (3.10)
\end{aligned}$$

First, we will prove that $I = \emptyset$. Now, we suppose on the contrary that $I \neq \emptyset$. Let $i \in I$ and we can construct a smooth cut-off function $\phi_{\epsilon,i}$ centered at z_i such that

$$0 \leq \phi_{\epsilon,i}(x) \leq 1,\ \phi_{\epsilon,i}(x) = 1 \text{ in } B\left(z_i, \frac{\epsilon}{2}\right),\ \phi_{\epsilon,i}(x) = 0 \text{ in } \mathbb{R}^N \setminus B(z_i,\epsilon),\ |\nabla\phi_{\epsilon,i}(x)| \leq \frac{4}{\epsilon},$$

for any $\epsilon > 0$ small. It is not difficult to see that $\{v_{n_k}\phi_{\epsilon,i}\}$ is a bounded sequence in E. From this, we can obtain that $\lim_{n\to\infty}\langle dJ_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k}\phi_{\epsilon,i})\rangle = 0$, that is, i.e.

$$\begin{aligned}
& -\iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - v_{n_k}(y)|^{p(x,y)-2}(v_{n_k}(x) - v_{n_k}(y))v_{n_k}(y)(\phi_{\epsilon,i}(x) - \phi_{\epsilon,i}(y))}{|x-y|^{N+sp(x,y)}} dxdy \\
& = \int_{\mathbb{R}^N} V_n(x)\phi_{\epsilon,i}dx - \int_{\mathbb{R}^N} F_v(x, u_{n_k}, v_{n_k})v_{n_k}\phi_{\epsilon,i}dx - \int_{\mathbb{R}^N} |v_{n_k}|^{q(x)}\phi_{\epsilon,i}dx + o_n(1). \qquad (3.11)
\end{aligned}$$

Note that the boundedness of $\{v_{n_k}\}$ in E implies the boundedness of $\{v_{n_k}\}$ in $L^{q(\cdot)}(\mathbb{R}^N)$ due to Theorem 3.3. Hence, from (F_1) and the Lebesgue dominated convergence theorem we have

$$\int_{\mathbb{R}^N} F_v(x, u_{n_k}, v_{n_k}) v_{n_k} \phi_{\epsilon,i}(x) \mathrm{d}x \to \int_{\mathbb{R}^N} F_v(x, u_0, v_0) v_0 \phi_{\epsilon,i}(x) \mathrm{d}x \quad \text{as } n \to \infty. \tag{3.12}$$

From the definition of $\phi_{\epsilon,i}(x)$, we obtain

$$\left| \int_{\mathbb{R}^N} F_v(x, u_0, v_0) v_0 \phi_{\epsilon,i} \mathrm{d}x \right| \le \int_{B_\epsilon(0)} |F_v(x, u_0, v_0) v_0| \mathrm{d}x \to 0 \quad \text{as } \epsilon \to 0. \tag{3.13}$$

On the other hand, let $\delta > 0$ be arbitrary and fixed. By the boundedness of $\{v_{n_k}\}$ in $L^{q(\cdot)}(\mathbb{R}^N)$ and the Young inequality, we have

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - v_{n_k}(y)|^{p(x,y)-2}(v_{n_k}(x) - v_{n_k}(y)) v_{n_k}(y)(\phi_{\epsilon,i}(x) - \phi_{\epsilon,i}(y))}{|x-y|^{N+sp(x,y)}} dxdy \right| \\
&\le \delta \iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - v_{n_k}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + C \iint_{\mathbb{R}^{2N}} |v_{n_k}(y)|^{p(x,y)} \frac{|\phi_{\epsilon,i}(x) - \phi_{\epsilon,i}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy \\
&\le C\delta + C \iint_{\mathbb{R}^{2N}} |v_{n_k}(y)|^{p(x,y)} \frac{|\phi_{\epsilon,i}(x) - \phi_{\epsilon,i}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx.
\end{aligned} \tag{3.14}$$

Taking limit superior in (3.17) as $n \to \infty$ then taking limit superior as $\epsilon \to 0$ with taking Lemma 4.4 into account in [29], we have

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \iint_{\mathbb{R}^{2N}} |v_{n_k}(y)|^{p(x,y)} \frac{|\phi_{\epsilon,i}(x) - \phi_{\epsilon,i}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx = 0. \tag{3.15}$$

Since $\delta > 0$ was chosen arbitrarily we obtain

$$\limsup_{\epsilon \to 0} \limsup_{n \to \infty} \iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - v_{n_k}(y)|^{p(x,y)-2}(v_{n_k}(x) - v_{n_k}(y)) v_{n_k}(y)(\phi_{\epsilon,i}(x) - \phi_{\epsilon,i}(y))}{|x-y|^{N+sp(x,y)}} dxdy = 0 \tag{3.16}$$

Since $\phi_{\epsilon,i}$ has compact support, letting $n \to \infty$ and $\epsilon \to 0$ in (3.11), we can deduce from (3.15) and (3.16) that

$$\mu_i \le \nu_i.$$

Inserting this into (3.7), we deduce

$$\nu_i \ge S^{\frac{q(z_i)\overline{p}}{q(z_i) - \overline{p}}} \ge \min \left\{ S_q^{\overline{p}\tau^+}, S_q^{\overline{p}\tau^-} \right\}. \tag{3.17}$$

It follows from (3.17) that

$$\begin{aligned}
c &= \lim_{n \to \infty} \left(J_{n_k}(0, v_{n_k}) - \frac{1}{\theta} \langle dJ_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k}) \rangle \right) \\
&\ge \lim_{n \to \infty} \left[\left(\frac{1}{\overline{p}} - \frac{1}{\theta} \right) \iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - v_{n_k}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \left(\frac{1}{\overline{p}} - \frac{1}{\theta} \right) \int_{\mathbb{R}^N} |v_{n_k}|^{\overline{p}} \mathrm{d}x \right] \\
&\quad + \left(\frac{1}{\theta} - \frac{1}{q(x)} \right) \int_{\mathbb{R}^N} |v_{n_k}|^{q(x)} \mathrm{d}x - \int_{\mathbb{R}^N} \left[F(x, 0, v_{n_k}) - \frac{1}{\theta} F_v(x, 0, v_{n_k}) v_{n_k} \right] dx \\
&\ge \int_{\mathbb{R}^N} \left(\frac{1}{\theta} - \frac{1}{q(x)} \right) |v_{n_k}|^{q(x)} dx \ge \left(\frac{1}{\theta} - \frac{1}{q^-} \right) \int_{\mathbb{R}^N} |v_{n_k}|^{q(x)} \phi_{\epsilon,i} dx \ge \left(\frac{1}{\theta} - \frac{1}{q^-} \right) \nu_i \\
&> \left(\frac{1}{\theta} - \frac{1}{q^-} \right) \min \left\{ S_q^{\overline{p}\tau^+}, S_q^{\overline{p}\tau^-} \right\}
\end{aligned}$$

as $\epsilon \to 0$, which is a contradiction. Hence $I = \emptyset$.

Next, we prove that $\nu_\infty = 0$. Suppose on the contrary that $\nu_\infty > 0$. To obtain the possible concentration of mass at infinity, we similarly define a cut off function $\phi_R \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_R(x) = 0$ on $|x| < R$ and $\phi_R(x) = 1$ on $|x| > R+1$. We can verify that $\{v_{n_k}\phi_R\}$ is bounded in E, hence $\langle dJ_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k}\phi_R)\rangle \to 0$ as $n \to \infty$, which implies

$$\begin{aligned}
&-\iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x)-v_{n_k}(y)|^{p(x,y)-2}(v_{n_k}(x)-v_{n_k}(y))v_{n_k}(y)(\phi_R(x)-\phi_R(y))}{|x-y|^{N+sp(x,y)}}dxdy \\
&= \int_{\mathbb{R}^N} V_n(x)\phi_R dx - \int_{\mathbb{R}^N} F_v(x,u_{n_k},v_{n_k})v_{n_k}\phi_R dx - \int_{\mathbb{R}^N} |v_{n_k}|^{q(x)}\phi_R dx + o_n(1). \qquad (3.18)
\end{aligned}$$

Note that $v_{n_k} \to v_0$ weakly in $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N} F_v(x,u,v)(v_{n_k}-v_0)\varphi_R dx \to 0$. As

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} (F_v(x,u_{n_k},v_{n_k}) - F_v(x,u_0,v_0))v_{n_k}\phi_R dx\right| &\le c|(F_v(x,u_{n_k},v_{n_k}) - F_v(x,u_0,v_0))\phi_R|_{(p_s^*)'}|v_{n_k}|_{p_s^*} \\
&\le c|(F_v(x,u_{n_k},v_{n_k}) - F_v(x,u_0,v_0))|_{(p_s^*)',\mathbb{R}^N\setminus B_R(0)},
\end{aligned}$$

by condition (F_1), for any $\varepsilon > 0$, there exists $R_1 > 0$ such that

$$|(F_v(x,u_{n_k},v_{n_k}) - F_v(x,u_0,v_0))|_{(p_s^*)',\mathbb{R}^N\setminus B_R(0)} < \varepsilon$$

as $R > R_1$ and $n \in \mathbb{N}$. Note that $\displaystyle\int_{\mathbb{R}^N} F(x,u_0,v_0)v_0\varphi_R dx \to 0$ as $R \to \infty$. Thus we obtain that

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N} F_v(x,u_{n_k},v_{n_k})v_{n_k}\varphi_R dx \\
&= \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N} [(F_v(x,u_{n_k},v_{n_k}) - F_v(x,u_0,v_0))v_{n_k}\varphi_R + F_v(x,u_0,v_0)(v_{n_k}-v) + F_v(x,u_0,v_0)v_0\varphi_R]dx \\
&= \lim_{R\to\infty}\left(\limsup_{n\to\infty}\int_{\mathbb{R}^N} (F_v(x,u_{n_k},v_{n_k}) - F_v(x,u_0,v_0))v_{n_k}\varphi_R dx + \int_{\mathbb{R}^N} F_v(x,u_0,v_0)v_0\varphi_R dx\right) \\
&= \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N} (F_v(x,u_{n_k},v_{n_k}) - F_v(x,u_0,v_0))v_{n_k}\varphi_R dx + \lim_{R\to\infty}\int_{\mathbb{R}^N} F_v(x,u_0,v_0)v_0\varphi_R dx \\
&= 0.
\end{aligned}$$

Moreover, we proceed as in (3.16) to get

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x)-v_{n_k}(y)|^{p(x,y)-2}(v_{n_k}(x)-v_{n_k}(y))v_{n_k}(y)(\phi_R(x)-\phi_R(y))}{|x-y|^{N+sp(x,y)}}dxdy\right| \\
&\le C\delta + C\iint_{\mathbb{R}^{2N}} |v_{n_k}(y)|^{p(x,y)}\frac{|\phi_R(x)-\phi_R(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx \qquad (3.19)
\end{aligned}$$

for each $\delta > 0$ arbitrary and fixed. Taking limit superior in the last estimate as $n \to \infty$ and then taking limit as $R \to \infty$, we obtain

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x)-v_{n_k}(y)|^{p(x,y)-2}(v_{n_k}(x)-v_{n_k}(y))v_{n_k}(y)(\phi_R(x)-\phi_R(y))}{|x-y|^{N+sp(x,y)}}dxdy = 0, \qquad (3.20)$$

since $\delta > 0$ can be taken arbitrarily. Letting $R \to \infty$ in (3.18), we can deduce from (3.19) and (3.20) that

$$\mu_\infty \leq \nu_\infty. \tag{3.21}$$

Combining (3.10) with (3.21) gives

$$\nu_\infty \geq S^{\frac{q_\infty \overline{p}}{q_\infty - \overline{p}}} \geq \min\left\{S_q^{\overline{p}\tau^+}, S_q^{\overline{p}\tau^-}\right\}. \tag{3.22}$$

If (3.22) holds. Thus

$$\begin{aligned} c &= \lim_{n\to\infty}\left(J_{n_k}(0, v_{n_k}) - \frac{1}{\theta}\langle dJ_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k})\rangle\right) \\ &\geq \int_{\mathbb{R}^N}\left(\frac{1}{\theta} - \frac{1}{q(x)}\right)|v_{n_k}|^{q(x)}dx \geq \left(\frac{1}{\theta} - \frac{1}{q^-}\right)\int_{\mathbb{R}^N}|v_{n_k}|^{q(x)}\phi_R dx \geq \left(\frac{1}{\theta} - \frac{1}{q^-}\right)\nu_\infty \\ &\geq \left(\frac{1}{\theta} - \frac{1}{q^-}\right)\min\left\{S_q^{\overline{p}\tau^+}, S_q^{\overline{p}\tau^-}\right\} \end{aligned} \tag{3.23}$$

as $R \to \infty$, which is a contradiction, we can prove that $\nu_\infty = 0$. Combining the facts that $I = \emptyset$ and $\nu_\infty = 0$, we obtain

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|v_{n_k}|^{q(x)}dx = \int_{\mathbb{R}^N}|v_0|^{q(x)}dx.$$

By a Brézis-Lieb Lemma type result for the Lebesgue spaces with variable exponents (see e.g., [[24], Lemma 3.9]), we have

$$\int_{\mathbb{R}^N}|v_{n_k} - v_0|^{q(x)}dx \to 0,$$

i.e., $v_{n_k} \to v_0$ in $L^{q(\cdot)}(\mathbb{R}^N)$. Consequently, we have

$$\int_{\mathbb{R}^N}|v_{n_k}|^{q(x)-2}v_{n_k}(v_{n_k} - v_0)dx \to 0, \tag{3.24}$$

by invoking Proposition 2.3 and the boundedness of $\{u_n\}_n$ in $L^{q(\cdot)}(\mathbb{R}^N)$. Also, we easily obtain

$$\int_{\mathbb{R}^N}F_v(x, u_{n_k}, v_{n_k})(v_{n_k} - v_0)dx \to 0. \tag{3.25}$$

Let us now introduce, for simplicity, for all $v \in E$ the linear functional $L(v)$ on E defined by

$$\langle L(v), w\rangle = \iint_{\mathbb{R}^{2N}}\frac{|v(x) - v(y)|^{p(x,y)-2}(v(x) - v(y))(w(x) - w(y))}{|x - y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N}|v|^{p(x)-2}vwdx$$

for all $w \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$. Obviously, from the Hölder inequality, we deduce that L is also continuous and satisfy

$$|\langle L(v), w\rangle| \leq \max\{\|v\|^{q^-}, \|v\|^{q^+}\}\|w\| \quad \text{for all } w \in E.$$

Hence, the weak convergence of $\{v_{n_k}\}$ in E gives that

$$\lim_{n\to\infty}\langle L(v_0), v_{n_k}-v_0\rangle = 0. \tag{3.26}$$

Clearly, $\langle L(v_{n_k}), v_{n_k}-v_0\rangle \to 0$ as $n\to\infty$. Hence, by (3.26), one has

$$\lim_{n\to\infty}\langle L(v_{n_k})-L(v_0), v_{n_k}-v_0\rangle = 0. \tag{3.27}$$

Let us now recall the well-known Simon inequalities:

$$|s-t|^p \le \begin{cases} C_p^{'}\left(|s|^{p-2}s-|t|^{p-2}t\right)\cdot(s-t) & \text{for} \quad p\ge 2\\ C_p{''}\left[\left(|s|^{p-2}s-|t|^{p-2}t\right)\cdot(s-t)\right]^{p/2}\left(|s|^p+|t|^p\right)^{(2-p)/2} & \text{for } 1<p<2,\end{cases} \tag{3.28}$$

for all $s,t\in\mathbb{R}^N$, where $C_p^{'}$ and $C_p{''}$ are positive constants depending only on p.

By (3.27) and (3.28) we get

$$\begin{aligned}&\lim_{n\to\infty}\left(\iint_{\mathbb{R}^{2N}}\frac{|v_{n_k}(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_{\mathbb{R}^N}|v_{n_k}-v_0|^{\overline{p}}dx\right)\\ &=\lim_{n\to\infty}\left(\langle L(v_{n_k}), v_{n_k}-v_0\rangle-\langle L(v_0), v_{n_k}-v_0\rangle\right)=0.\end{aligned}$$

This fact implies that $\{v_{n_k}\}$ strongly converges to v_0 in E.

In conclusion, we get $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence converging strongly in X. Hence the proof is complete. □

4 Proof of Theorem 1.1

Proof of Theorem 1.1 Now we shall verify the conditions of Theorem 2.2. Set

$$Y=U\oplus V,\quad U=E_{G_1}\times\{0\},\quad V=\{0\}\times E_{G_1},$$

$$Y_0=\{0\}\times E_{G_1}^{m\perp},\quad Y_1=\{0\}\times E_{G_1}^{(k)},$$

where m and k are to be determined. It is clear that Y_0, Y_1 are G-invariant and $\mathrm{codim}_V Y_0=m$, $\dim Y_1=k$. Obviously, (B_1), (B_2), (B_4) in Theorem 2.2 are satisfied. Set $V_j=E_{G_1}^{(j)}=\mathrm{span}\{e_1,e_2,\cdots,e_j\}$, then (B_3) is also satisfied. In the following we verify the conditions in (B_7). Since $\mathrm{Fix}(G)\cap V=0$, (a) of (B_7) holds. It remains to verify (b), (c) of (B_7). Next, we focus our attention on the case when $(u,v)\in X$ with $\|u\|_{s,p}<1$ and $\|v\|_{s,p}<1$.

(i) If $(0,v)\in Y_0\cap S_{\rho_m}$ (where ρ_m is to be determined) then by (f_1) and (f_3), for any $v\in E$ with $\|v\|_{s,p}<1$, we find that

$$\begin{aligned} J(0,v) &= \iint_{\mathbb{R}^{2N}}\frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy+\frac{1}{\overline{p}}\int_{\mathbb{R}^N}|v|^{\overline{p}}dx-\frac{1}{q(x)}\int_{\mathbb{R}^N}|v|^{q(x)}dx-\int_{\mathbb{R}^N}F(x,0,v)dx\\ &\ge \frac{1}{\overline{p}}\|v\|_{s,p}^{\overline{p}}-c\|v\|_{s,p}^{\overline{p}_s^*}-c\|v\|_{s,p}^{r^+},\end{aligned}$$

since $\overline{p}<r^+<\overline{p}_s^*$, there exists $\rho>0$ such that $J(0,v)\ge\alpha$ for every $\|v\|_{s,p}=\rho$, that is (b) of (B_7) holds.

(ii) From (H_1) we have

$$\begin{aligned} J(u,0) &= -\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy - \frac{1}{\overline{p}}\int_{\mathbb{R}^N}|u|^{\overline{p}}dx - \frac{1}{q(x)}\int_{\mathbb{R}^N}|u|^{q(x)}dx \\ &\quad -\int_{\mathbb{R}^N} F(x,u,0)dx \\ &\leq 0. \end{aligned}$$

Therefore, we can choose α such that

$$\alpha > \sup_{u\in E_{G_1}} J(u,0).$$

For each $(u,v) \in U \oplus Y_1$, we have

$$\begin{aligned} J(u,v) &= -\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy - \frac{1}{\overline{p}}\int_{\mathbb{R}^N}|u|^{\overline{p}}dx - \frac{1}{q(x)}\int_{\mathbb{R}^N}|u|^{q(x)}dx \\ &\quad +\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy + \frac{1}{\overline{p}}\int_{\mathbb{R}^N}|v|^{\overline{p}}dx - \frac{1}{q(x)}\int_{\mathbb{R}^N}|v|^{q(x)}dx \\ &\quad -\int_{\mathbb{R}^N} F(x,u,v)dx \qquad (4.1) \\ &\leq \frac{1}{\underline{p}}\|v\|^{p^-} - \frac{1}{\overline{q}}|v|^{q(\cdot)}_{q(\cdot)} + \alpha \end{aligned}$$

Because of the fact that on the finite-dimensional space Y_1 all norms are equivalent, we can choose $k > m$ and $\beta_k > \alpha_m$ such that

$$J_{U\oplus Y_1} \leq \beta_k,$$

so we get (c) in (B_7). By Lemma 3.1, for any $c \in [\alpha_m, \beta_k]$, $J(u,v)$ satisfies the condition of $(PS)^*_c$, then (B_6) in Theorem 2.2 holds. So according to Theorem 2.2,

$$c_j = \inf_{i^\infty(A)\geq j}\sup_{z\in A} J(u,v), \quad -k+1 \leq j \leq -m, \quad \alpha_m \leq c_j \leq \beta_k,$$

are critical values of J. Letting $m \to \infty$, we can get an unbounded sequence of critical values c_j. And because the functional J is even, we obtain two critical points $\pm u_j$ of J corresponding to c_j. □

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