TIME PERIODIC SOLUTIONS FOR THE FULL QUANTUM EULER EQUATION

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Abstract. In this paper, we establish the existence and uniqueness of a time periodic solution to the full compressible quantum Euler equations. First, we prove the existence of time periodic solutions under some smallness assumptions imposed on the external force in a periodic domain by a regularized approximation scheme and the Leray-Schauder degree theory. Then the result is generalized to $\mathbb{R}^3$ by adapting a limiting method and a diagonal argument. The uniqueness of the time periodic solutions is also given. Compared to classical Euler equations, the third-order quantum spatial derivatives are considered which need some elaborated treatments thereof in obtaining the highest-order energy estimates.

Keywords: Time periodic solutions, Full quantum Euler equations, Uniform estimates, Leray-Schauder degree theory.

1. Introduction

Quantum-type fluids are increasingly important with the development of the ultra-small electronic devices in nano-scale, where the quantum effects (particle tunneling through potential barriers and built-up in quantum wells) are present, such as the simulation of resonant tunneling diodes [10], superfluid [24] and superconductivity [9]. See also [3, 12] for more results and physics backgrounds. In contrast to the Kinetic theory, the advantage of the macroscopic quantum-type model relies on the fact that it provides the possibility of accurate and efficient numerical simulations by measuring the macroscopic variables (like electron density and the electron current density). Proceeding from the recently developed derivation for the macroscopic quantum-type fluid and related models, the compressible quantum hydrodynamical equations are perfectly demonstrated in [1, 10, 11, 18] by a moment expansion of the Wigner-Boltzmann equation [10] and the expansion of the thermal equilibrium Wigner distribution function [31]. In these models, the quantum effects terms can be viewed as the first quantum corrections $O(\hbar^2)$ of the classical hydrodynamic equations, where $\hbar$ is the Planck constant divided by 2π.

The main purpose of this paper is to consider the existence and uniqueness of a time periodic solution to the three dimensional full quantum Euler equations with damping and a time periodic external force

$$
\begin{cases}
\partial_t n + \operatorname{div}(nu) = 0, & (1.1a)\\
\partial_t (nu) + \operatorname{div}(nu\otimes u + P) = -\dfrac{nu}{\tau_m} + nf, & (1.1b)\\
\partial_t W + \operatorname{div}(uW + uP) + \operatorname{div} q = -\dfrac{1}{\tau_e}\left(W - \dfrac{3}{2}n(T-\bar{T})\right), & (1.1c)
\end{cases}
$$

where n, u, T are the electron density, the electron velocity and the temperature, respectively. $\bar{T}$ is the lattice temperature. $P = (P_{ij})_{3\times 3}$ is the quantum correction to the stress tensor which was derived by Ancona and Tiersten [2] and Ancona and Iafrate [1]. W is the quantum correction for energy density, which is first derived by Wigner [31]. q is the total heat flux consisted of third-order moment [10, 17] and the Chapman-Enskog expansion. Moreover, the constants τ_m, τ_e model the standard momentum and

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energy relaxation times approximations respectively. f is the given external time periodic force with a period $T^* > 0$. With quantum corrections, the above quantities q, P and W are defined by

$$q = -\kappa\nabla T - \frac{\hbar^2 n}{24}(\Delta u + 2\nabla \mathrm{div} u) + O(\hbar^4),$$

$$P = -nT\mathbb{I} - \frac{\hbar^2 n}{12}\nabla^2 \log n + O(\hbar^4),$$

and

$$W = \frac{3}{2}nT + \frac{1}{2}n|u|^2 - \frac{\hbar^2 n}{24}\Delta \log n + O(\hbar^4),$$

respectively, where $\mathbb{I}$ is the unit matrix in $R^{3\times 3}$. According to

$$\frac{\hbar^2}{12}\mathrm{div}\big(n(\nabla\otimes\nabla)\log n\big) = \frac{\hbar^2}{6}n\nabla\frac{\Delta\sqrt{n}}{\sqrt{n}},$$

the quantum mechanical effects can be interpreted as a force closely related to the quantum Bohm potential [4]

$$Q(n) = \frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}}.$$

We note that the majority of efforts of the quantum hydrodynamic equations were made in the well-posedness [10,17,18,25] and asymptotic behaviors [22,25,26]. For recent works, Pu and Guo [25] obtained the global existence of smooth solutions to the quantum hydrodynamic models with viscosity and established the semiclassical limit of solutions under small initial data. Later in [26], the initial boundary value problem of system (1.1) when $f = 0$ was studied. However, for the periodic solutions case, there is little mathematically result available. Therefore, the purpose of this paper is to study the existence and uniqueness of the time periodic solutions for the non-isentropic quantum Euler system around a constant state $(\bar{\psi}, 0, \bar{T})$ over the physical space $\mathbb{R}^3$.

There exists an extensive literature in past decades about the time periodic solutions for the classical compressible flow [5,7,8,13,15,20,21,23,29]. In the study of the Navier-Stokes equation, one can refer to [20,29] for the case of periodic domain and [16,19,21,23] for the whole space. See also [7,14] for the Navier-Stokes-Korteweg system and [5,6,13,28] for the magnetohydrodynamic equations. In addition to these results, one can see [8,15,30] and reference therein for the initial boundary value problem. Below we only review some of them related to our work. In [23], Ma, Ukai and Yang investigated the existence of time periodic solutions by the linear decay analysis and the contraction mapping theorem where the space dimension $n \geq 5$. Kagei and Tsuda [21] also considered the existence of time periodic solutions via the spectral properties on the case $n \geq 3$. Moreover, the existence of time periodic solutions of Navier-Stokes equations in the whole space $\mathbb{R}^3$ was present in Jin and Yang [19] by uniform estimates and the topological degree theory. Then the similar problem in the periodic domain $\Omega \subseteq \mathbb{R}^3$ was discussed in [20]. In a recent work, Cai et al. [7] studied the isentropic compressible Navier-Stokes-Korteweg System with friction, and pointed out that the friction term provides a good effect in the proof. However, when the viscous term is replaced by the damping term, the related study is very limited so far. Here, we only mention the work about the case $n = 3$ by Tan et al. [29], who proved the existence of time periodic solutions of the compressible Euler equation under some small and structure data assumptions in a bounded domain with periodic boundary. Our present work can be regarded as an extension of the previous work [29] in the sense that the quantum effects and thermal conduction are taken into account.

Even though we have employed the similar argument from [19,29], the so-called full quantum hydrodynamic model (1.1) carries some new feature such as the strong nonlinearity and coupling, which makes the study more mathematically challenging. We also remark that the case of full quantum Navier-Stokes equations is easier than the system under consideration since the high-order viscous term $\mu\Delta u + \lambda\nabla\mathrm{div}u$ can be used to control the quantum effects terms as [25,27]. Indeed, a full use of the energy equation is employed in this present paper. Precisely, there is only the low-order dissipation of damping in system

(1.1) when obtaining the highest-order energy estimates for the velocity field. And thus, we make a deep analysis of the structure for system (2.1) and introduce the elaborated energy estimates in some proper norm. The weighted energy norm we finally adopt is

$$
\begin{aligned}
\|\!|(\rho,u,\theta)\|\!| = &\sup_{t\in[0,T^*]}\Big(\|(u,\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t)\|_{H^3}^2+\|\nabla\rho(t)\|_{H^2}^2\Big)\\
&+\int_0^{T^*}\Big(\|\nabla\rho(t)\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t)\|_{H^3}^2\Big)dt\\
&+\varepsilon\int_0^{T^*}\|(\nabla\rho,\hbar\Delta\rho,\hbar^2\nabla\Delta\rho)(t)\|_{H^3}^2dt+\int_0^{T^*}\|(\hbar^2\nabla\mathrm{div}u,\hbar^3\nabla\Delta\rho)(t)\|_{H^3}^2\Big)dt.
\end{aligned}\tag{1.2}
$$

In addition, the structural assumptions of the external force is dispensable in this paper since we can obtain the basic L^2 estimates by the structure of (2.20) directly.

For convenience, we assume $\tau_m,\tau_e=1$ in system (1.1). By making use of the variable transformation $\psi^2=n$, system (1.1) can be reformulated as

$$
\begin{cases}
2\partial_t\psi+\psi\mathrm{div}u+2u\cdot\nabla\psi=0, & (1.3a)\\
\partial_t u+u\cdot\nabla u+\nabla T+2T\dfrac{\nabla\psi}{\psi}-\dfrac{\Delta\nabla\psi}{\psi}+\dfrac{\Delta\psi\nabla\psi}{\psi^2}+u=f, & (1.3b)\\
\partial_t T+\dfrac{2}{3}T\mathrm{div}u+u\cdot\nabla T-\dfrac{2\kappa}{3}\dfrac{\Delta T}{\psi^2}-\dfrac{\hbar^2}{18}\mathrm{div}\Delta u=\dfrac{\hbar^2}{9}\dfrac{\nabla\psi\cdot\nabla\mathrm{div}u}{\psi}-\dfrac{1}{3}|u|^2-(T-\bar T) & \\
\qquad+\dfrac{\hbar^2}{36}\Delta\log\psi^2, & (1.3c)\\
(\psi,u,T)(x,0)=(\psi_0,u_0,T_0). & (1.3d)
\end{cases}
$$

Our purpose of this present paper is to consider the time periodic solutions for system (1.3) around a constant state $(\bar\psi,0,\bar T)$. For this, letting $(\rho,u,\theta)=(\psi-\bar\psi,u,T-\bar T)$, system (1.3) can be written in the perturbation form

$$
\begin{cases}
2\partial_t\rho+\bar\psi\mathrm{div}u=-\rho\mathrm{div}u-2u\cdot\nabla\rho, & (1.4a)\\
\partial_t u+u\cdot\nabla u+\nabla\theta+2\gamma\nabla\rho-\dfrac{\hbar^2}{6\bar\psi}\nabla\Delta\rho+u=-\dfrac{\hbar^2}{6}\dfrac{\rho\nabla\Delta\rho}{\bar\psi(\rho+\bar\psi)}-2\dfrac{\theta\nabla\rho}{\rho+\bar\psi} & \\
\qquad+2\gamma\dfrac{\rho\nabla\rho}{\rho+\bar\psi}+\dfrac{\hbar^2}{6}\dfrac{\nabla\rho\Delta\rho}{(\rho+\bar\psi)^2}+f, & (1.4b)\\
\partial_t\theta+\dfrac{2}{3}\bar T\mathrm{div}u-\dfrac{2\kappa}{3\bar\psi^2}\Delta\theta-\dfrac{\hbar^2}{18}\mathrm{div}\Delta u=-\dfrac{2\kappa}{3}\dfrac{(\rho^2+2\rho\bar\psi)\Delta\theta}{\bar\psi^2(\rho+\bar\psi)^2}-u\cdot\nabla\theta & \\
\qquad-\dfrac{2}{3}\theta\mathrm{div}u+\dfrac{\hbar^2}{9}\dfrac{\nabla\rho\cdot\nabla\mathrm{div}u}{\rho+\bar\psi}-\dfrac{|u|^2}{3}-\theta+\dfrac{\hbar^2}{36}\Delta\log\psi^2, & (1.4c)\\
(\rho,u,\theta)(x,0)=(\rho_0,u_0,\theta_0), & (1.4d)
\end{cases}
$$

where $\gamma=\frac{\bar T}{\bar\psi}$.

Before starting, we introduce some notations which will be used frequently throughout this paper. We denote the solution spaces in the periodic domain Ω^R and in the whole space $\mathbb{R}^3$ respectively by

$$
X^R=\left(\begin{array}{cc}
 & (\rho,u,\theta)\in L^\infty(0,T^*;L^6(\Omega^R));\\
 & (u,\theta)\in L^\infty(0,T^*;L^2(\Omega^R))\cap L^2(0,T^*;L^2(\Omega^R));\\
 & (\nabla\rho,\nabla u)\in L^\infty(0,T^*;H^2(\Omega^R))\cap L^2(0,T^*;H^2(\Omega^R));\\
(\rho,u,\theta): & (\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\in L^\infty(0,T^*;H^3(\Omega^R))\cap L^2(0,T^*;H^3(\Omega^R));\\
 & (\nabla\theta,\hbar\mathrm{div}u,\hbar^2\nabla\mathrm{div}u,\hbar^3\nabla\Delta\rho)\in L^\infty(0,T^*;H^2(\Omega^R))\cap L^2(0,T^*;H^3(\Omega^R));\\
 & (\varepsilon^{1/2}\nabla\rho,\hbar\varepsilon^{1/2}\Delta\rho,\hbar^2\varepsilon^{1/2}\nabla\Delta\rho)\in L^\infty(0,T^*;H^2(\Omega^R))\cap L^2(0,T^*;H^3(\Omega^R));\\
 & \text{and } (\rho,u,\theta) \text{ is periodic in time and space that satisfies } \int_{\Omega^R}\rho dx=0,
\end{array}\right)\tag{1.5}
$$

and

$$X=\left(\begin{array}{c}(\rho,u,\theta):\end{array}\begin{array}{c}(\rho,u,\theta)\in L^\infty(0,T^*;L^6(\mathbb{R}^3));\\ (u,\theta)\in L^\infty(0,T^*;L^2(\mathbb{R}^3))\cap L^2(0,T^*;L^2(\mathbb{R}^3));\\ (\nabla\rho,\nabla u)\in L^\infty(0,T^*;H^2(\mathbb{R}^3))\cap L^2(0,T^*;H^2(\mathbb{R}^3);\\ (\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\in L^\infty(0,T^*;H^3(\mathbb{R}^3))\cap L^2(0,T^*;H^3(\mathbb{R}^3));\\ (\nabla\theta,\hbar\mathrm{div}u,\hbar^2\nabla\mathrm{div}u,\hbar^3\nabla\Delta\rho)\in L^\infty(0,T^*;H^2(\mathbb{R}^3))\cap L^2(0,T^*;H^3(\mathbb{R}^3));\\ (\varepsilon^{1/2}\nabla\rho,\hbar\varepsilon^{1/2}\Delta\rho,\hbar^2\varepsilon^{1/2}\nabla\Delta\rho)\in L^\infty(0,T^*;H^2(\mathbb{R}^3))\cap L^2(0,T^*;H^3(\mathbb{R}^3));\\ \text{and } (\rho,u,\theta) \text{ is periodic with the same time period as } f.\end{array}\right) \tag{1.6}$$

For some positive constant d, set

$$X_d^R=\{(\rho,u,\theta)\in X^R;\, \|\!|(\rho,u,\theta)|\!\|\le d^2\}, \tag{1.7}$$

and

$$X_d=\{(\rho,u,\theta)\in X;\, \|\!|(\rho,u,\theta)|\!\|\le d^2\}, \tag{1.8}$$

where the norm $\|\!|\cdot|\!\|$ is defined in (1.2).

Moreover, C denotes a generic constant independent of $\varepsilon, R, \hbar$, while C_ε is a constant dependent on ε. $[A,B]=AB-BA$ is the commutator of A and B, and the periodic domain is denoted by $\Omega^R=(-\pi,\pi)^3\subseteq\mathbb{R}^3$. In addition, $W^{k,p}(\Omega^R)$ is the usual Sobolev space, and we omit the domain Ω^R for simplicity if it does not cause any confusion. Let $C^{a,b}$ be the set of all functions, such that

$$\sup_{(x,t)\neq(y,s)}\frac{|f(x,t)-f(y,s)|}{|x-y|^a+|t-s|^b}<\infty.$$

Finally, for any multi-index α, we denote $\partial^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ by the partial differential derivatives. We sometimes abuse the notation $\partial^{\alpha\pm1}$ to mean $\partial^{\alpha\pm\beta}, |\beta|=1$, for some multi-index β.

1.1. **The main results of this paper.** The goal of this paper is to prove the following theorems

Theorem 1.1. *Let the time periodic force $f\in L^2(0,T^*;H^3(\mathbb{R}^3))$. If $\|f\|_{L^2(0,T^*;H^3(\mathbb{R}^3))}$ is sufficiently small, then the initial value problem (1.4) admits a solution $(\rho,u,\theta)\in X_{d_0}$, where X_{d_0} is defined in (1.8).*

Theorem 1.2. *Under the same assumption of Theorem 1.1, there exists a sufficiently small positive constant μ such that, provided that*

$$\sup_{t\in[0,T^*]}\left(\|\nabla\rho\|_{H^2}^2+\|(u,\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)\|_{H^3}^2\right)<\mu,$$

and the solution (ρ,u,θ) satisfies $(\rho_1,u_1,\theta_1)=(\rho_2,u_2,\theta_2)$ at infinite, system (1.4) admits a unique smooth solution $(\rho,u,\theta)\in X_{d_0}$.

The structure of this article is as follows. In Sect. 2, we introduce the operator χ and prove the existence of the time periodic solutions for the approximate problem (2.1) in a periodic domain by topological degree theory and the uniform (in $\varepsilon, R, \hbar$) energy estimates. It is the core of technical part of the proof. In Sect. 3, the uniform bounds and Arzela-Ascoli theorem allow us to get a limit function in $\mathbb{R}^3$. In addition, we prove the uniqueness of time periodic solutions. Finally, some preliminary inequalities are given in the Appendix.

2. THE EXISTENCE OF TIME PERIODIC SOLUTIONS IN A PERIODIC DOMAIN

To prove the existence of time periodic solutions in the periodic domain $\Omega^R = (-R,R)^3 \subseteq \mathbb{R}^3$, we introduce the following approximated system

$$
\begin{cases}
2\partial_t\rho + \bar{\psi}\mathrm{div}u - \varepsilon\Delta\rho = -\rho\mathrm{div}u - 2u\cdot\nabla\rho, & \text{(2.1a)}\\
\partial_t u + u\cdot\nabla u + \nabla\theta + 2\gamma\nabla\rho - \dfrac{\hbar^2}{6\bar{\psi}}\nabla\Delta\rho + u = -\dfrac{\hbar^2}{6}\dfrac{\rho\nabla\Delta\rho}{\bar{\psi}(\rho+\bar{\psi})} - 2\dfrac{\theta\nabla\rho}{\rho+\bar{\psi}} + 2\gamma\dfrac{\rho\nabla\rho}{\rho+\bar{\psi}} & \\
\quad + \dfrac{\hbar^2}{6}\dfrac{\nabla\rho\Delta\rho}{(\rho+\bar{\psi})^2} + f^R, & \text{(2.1b)}\\
\partial_t\theta + \dfrac{2}{3}\bar{T}\mathrm{div}u - \dfrac{2\kappa}{3\bar{\psi}^2}\Delta\theta - \dfrac{\hbar^2}{18}\mathrm{div}\Delta u + \theta - \dfrac{\hbar^2}{18\bar{\psi}}\Delta\rho = -\dfrac{2\kappa}{3}\dfrac{(\rho^2+2\rho\bar{\psi})\Delta\theta}{\bar{\psi}^2(\rho+\bar{\psi})^2} & \\
\quad - \dfrac{\hbar^2}{18}\dfrac{\rho\Delta\rho}{\bar{\psi}(\rho+\bar{\psi})} - u\cdot\nabla\theta - \dfrac{2}{3}\theta\mathrm{div}u + \dfrac{\hbar^2}{9}\dfrac{\nabla\rho\cdot\nabla\mathrm{div}u}{\rho+\bar{\psi}} - \dfrac{|u|^2}{3} - \dfrac{\hbar^2}{18}\dfrac{|\nabla\rho|^2}{(\rho+\bar{\psi})^2}, & \text{(2.1c)}
\end{cases}
$$

where f^R is a sufficiently smooth time periodic function with

$$
f^R \to f \text{ in } L^2(0,T^*;H^3(\mathbb{R}^3)) \text{ as } R\to\infty. \tag{2.2}
$$

Proposition 2.1. *Let f^R be a smooth function with periodic boundary and $f^R \in L^2(0,T^*;H^3(\Omega^R))$. There exists some suitably small positive constant λ and d_0, independent of R, $\hbar$ and ε, such that if*

$$
\|f^R\|_{L^2(0,T^*;H^3)} < \lambda,
$$

the initial value problem (2.1) admits a solution $(\rho,u,\theta)\in X^R_{d_0}$.

The rest of this section is devoting to the proof of Proposition 2.1. The result is obtained by the combination of the topological degree theory and the elaborate energy estimates. Notice that the assumption $\int_{\Omega^R}\rho dx = 0$ holds in general due to the conservation of mass. Different from the previous works [5, 7, 19, 20, 28, 29], the symmetry of the external force is unnecessary since we can obtain the L^2 estimates for the velocity and the temperature by the structure of (1.3) directly. Moreover, inspired by the idea in [29], we linearize the term $u\cdot\nabla u$ in the case of the quantum Euler system in order to guarantee the closed estimate in Lemma 2.2. Indeed, the main difficulty comes from the quantum Bohm potential term and the dispersive velocity $\hbar^2\Delta\mathrm{div}u$. For this, we fully utilize the linearized structure of system (2.6) and substitute the energy equation into the higher order energy estimate for $\Delta\theta$.

2.1. introduction of an operator χ.

To begin with, we work with the following linear system

$$
\begin{cases}
2\partial_t\rho + \bar{\psi}\mathrm{div}u - \varepsilon\Delta\rho = -\tau\tilde{\rho}\mathrm{div}\tilde{u} - 2\tau\tilde{u}\cdot\nabla\tilde{\rho}, & \text{(2.3a)}\\
\partial_t u + \tau\tilde{u}\cdot\nabla u + \nabla\theta + 2\gamma\nabla\rho - \dfrac{\hbar^2}{6\bar{\psi}}\nabla\Delta\rho + u = -\dfrac{\hbar^2\tau}{6}\dfrac{\tilde{\rho}\nabla\Delta\tilde{\rho}}{\tau\tilde{\rho}+\bar{\psi}} - 2\tau\dfrac{\tilde{\theta}\nabla\tilde{\rho}}{\tau\tilde{\rho}+\bar{\psi}} + 2\tau\gamma\dfrac{\tilde{\rho}\nabla\tilde{\rho}}{\tau\tilde{\rho}+\bar{\psi}} & \\
\quad + \dfrac{\hbar^2\tau}{6}\dfrac{\nabla\tilde{\rho}\Delta\tilde{\rho}}{(\tau\tilde{\rho}+\bar{\psi})^2} + \tau f^R, & \text{(2.3b)}\\
\partial_t\theta + \dfrac{2}{3}\bar{T}\mathrm{div}u - \dfrac{2\kappa}{3\bar{\psi}^2}\Delta\theta - \dfrac{\hbar^2}{18}\mathrm{div}\Delta u + \theta - \dfrac{\hbar^2}{18\bar{\psi}}\Delta\rho = -\tau\tilde{u}\cdot\nabla\tilde{\theta} - \dfrac{2}{3}\tau\tilde{\theta}\mathrm{div}\tilde{u} & \\
\quad + \dfrac{\hbar^2\tau}{9}\dfrac{\nabla\tilde{\rho}\cdot\nabla\mathrm{div}\tilde{u}}{\tau\tilde{\rho}+\bar{\psi}} - \dfrac{\tau}{2}|\tilde{u}|^2 - \dfrac{\hbar^2\tau}{18}\dfrac{|\nabla\tilde{\rho}|^2}{(\tau\tilde{\rho}+\bar{\psi})^2} - \dfrac{2\kappa}{3}\dfrac{(\tau^2\tilde{\rho}^2+2\tau\tilde{\rho}\bar{\psi})\Delta\tilde{\theta}}{\bar{\psi}^2(\tau\tilde{\rho}+\bar{\psi})^2} - \dfrac{\hbar^2\tau}{18}\dfrac{\tilde{\rho}\Delta\tilde{\rho}}{\bar{\psi}(\tau\tilde{\rho}+\bar{\psi})}, & \text{(2.3c)}
\end{cases}
$$

for any given $(\tilde{\rho},\tilde{u},\tilde{\theta})\in X^R_d$, for any $\tau\in[0,1]$. Next we define the operator χ

$$
\begin{aligned}
\chi : X^R_d \times [0,1] &\to X^R,\\
((\tilde{\rho},\tilde{u},\tilde{\theta}),\tau) &\to (\rho,u,\theta).
\end{aligned} \tag{2.4}
$$

In what follows, we focus on the properties of the operator χ.

Lemma 2.1. *For any $\tau \in [0,1]$, the operator χ is well defined.*

Proof. It follows from Sobolev embedding $H^2 \hookrightarrow L^\infty$ and the smallness of positive constant d that

$$\frac{\bar{n}}{2} \le \bar{n} + \tau\rho \le \frac{3\bar{n}}{2}, \ \frac{\bar{T}}{2} \le \bar{n} + \tau\theta \le \frac{3\bar{T}}{2}, \tag{2.5}$$

for any $(\rho, u, \theta) \in X_d^R$.

Consider the homogenous liner system

$$\begin{cases} 2\partial_t\rho + \bar{\psi}\mathrm{div}u - \varepsilon\Delta\rho = 0, & (2.6a) \\ \partial_t u + \tau\tilde{u}\cdot\nabla u + \nabla\theta + 2\gamma\nabla\rho - \dfrac{\hbar^2}{6\bar{\psi}}\nabla\Delta\rho + u = 0, & (2.6b) \\ \partial_t\theta + \dfrac{2}{3}\bar{T}\mathrm{div}u - \dfrac{2\kappa}{3\bar{\psi}^2}\Delta\theta - \dfrac{\hbar^2}{18}\mathrm{div}\Delta u + \theta - \dfrac{\hbar^2}{18\bar{\psi}}\Delta\rho = 0, & (2.6c) \end{cases}$$

with the initial value condition $(\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)$ that satisfies

$$\int_{\Omega^R} \rho_0 dx = 0. \tag{2.7}$$

Multiplying (2.6b) by u, and using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|u\|_{L^2}^2 = \frac{\tau}{2}\int_{\Omega^R}\mathrm{div}\tilde{u}|u|^2 - \int_{\Omega^R}\nabla\theta\cdot u - 2\gamma\int_{\Omega^R}\nabla\rho\cdot u + \frac{\hbar^2}{6\bar{\psi}}\int_{\Omega^R}\nabla\Delta\rho\cdot u. \tag{2.8}$$

For the first term of the RHS of (2.8), by Sobolev embedding $H^2 \hookrightarrow L^\infty$, we have

$$\frac{\tau}{2}\int_{\Omega^R}\mathrm{div}\tilde{u}|u|^2 \le C\tau\|\mathrm{div}\tilde{u}\|_{H^2}\|u\|_{L^2}^2.$$

For the second term of the RHS of (2.8), using integration by parts and (2.6c), we have

$$\begin{aligned} &-\int_{\Omega^R}\nabla\theta\cdot u \\ &= -\frac{3}{2\bar{T}}\int_{\Omega^R}\theta(\partial_t\theta - \frac{2\kappa}{3\bar{\psi}^2}\Delta\theta - \frac{\hbar^2}{18}\mathrm{div}\Delta u + \theta - \frac{\hbar^2}{18\bar{\psi}}\Delta\rho) \\ &= -\frac{3}{4\bar{T}}\frac{d}{dt}\int_{\Omega^R}|\theta|^2 - \frac{\kappa}{\bar{T}\bar{\psi}^2}\int_{\Omega^R}|\nabla\theta|^2 - \frac{\hbar^2}{12\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div}u - \frac{3}{2\bar{T}}\int_{\Omega^R}|\theta|^2 + \frac{\hbar^2}{12\bar{T}\bar{\psi}}\int_{\Omega^R}\Delta\rho\theta \\ &\le -\frac{3}{4\bar{T}}\frac{d}{dt}\int_{\Omega^R}|\theta|^2 - \frac{3}{2\bar{T}}\int_{\Omega^R}|\theta|^2 - \frac{\kappa}{\bar{T}\bar{\psi}^2}\int_{\Omega^R}|\nabla\theta|^2 - \frac{\hbar^2}{12\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div}u \\ &\quad + \delta\|\hbar\varepsilon^{1/2}\Delta\rho\|_{L^2}^2 + C_\varepsilon\hbar^2\|\theta\|_{L^2}^2, \end{aligned}$$

where δ is a sufficiently small positive constant. Again, (2.6a) gives

$$-2\gamma\int_{\Omega^R}\nabla\rho\cdot u = -\frac{2\gamma}{\bar{\psi}}\int_{\Omega^R}\rho(2\partial_t\rho - \varepsilon\Delta\rho) = -\frac{2\gamma}{\bar{\psi}}\frac{d}{dt}\int_{\Omega^R}|\rho|^2 - \frac{2\varepsilon\gamma}{\bar{\psi}}\int_{\Omega^R}|\nabla\rho|^2.$$

Similarly,

$$\frac{\hbar^2}{6\bar{\psi}}\int_{\Omega^R}\nabla\Delta\rho\cdot u = \frac{\hbar^2}{6\bar{\psi}^2}\int_{\Omega^R}\Delta\rho(2\partial_t\rho - \varepsilon\Delta\rho) = -\frac{\hbar^2}{6\bar{\psi}^2}\frac{d}{dt}\int_{\Omega^R}|\nabla\rho|^2 - \frac{\hbar^2\varepsilon}{6\bar{\psi}^2}\int_{\Omega^R}|\Delta\rho|^2.$$

Therefore, we have

$$\begin{aligned} &\frac{d}{dt}\|(u, \rho, \theta, \hbar\nabla\rho)\|_{L^2}^2 + \|\theta\|_{H^1}^2 + \|u\|_{L^2}^2 + \varepsilon\|\nabla\rho\|_{L^2}^2 + \hbar^2\varepsilon\|\Delta\rho\|_{L^2}^2 + \frac{\hbar^2}{12\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div}u \\ &\quad \le C\tau\|\mathrm{div}\tilde{u}\|_{H^2}\|u\|_{L^2}^2 + \delta\|\hbar\varepsilon^{1/2}\Delta\rho\|_{L^2}^2 + C_\varepsilon\hbar^2\|\theta\|_{L^2}^2. \end{aligned} \tag{2.9}$$

To close the estimate (2.9), we multiply (2.6b) by $-\frac{\hbar^2}{\bar{T}}\nabla \mathrm{div} u$ to obtain

$$
\begin{aligned}
\frac{d}{dt}\|\hbar \mathrm{div} u\|_{L^2}^2 + \frac{\hbar^2}{\bar{T}}\|\mathrm{div} u\|_{L^2}^2 =& \frac{\hbar^2\tau}{\bar{T}}\int_{\Omega^R} \tilde{u}\cdot\nabla u\cdot\nabla \mathrm{div} u - \frac{\hbar^4}{6\bar{\psi}\bar{T}}\int_{\Omega^R}\nabla\Delta\rho\cdot\nabla\mathrm{div} u \\
&+ \frac{2\gamma\hbar^2}{\bar{T}}\int_{\Omega^R}\nabla\rho\cdot\nabla\mathrm{div} u + \frac{\hbar^2}{\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div} u.
\end{aligned}
\tag{2.10}
$$

Now we will deal with the RHS of (2.10). For the first term, by integration by parts, we have

$$
\begin{aligned}
&\frac{\hbar^2\tau}{\bar{\psi}}\int_{\Omega^R}\tilde{u}\cdot\nabla u\cdot\nabla\mathrm{div} u \\
&= -\frac{\hbar^2\tau}{\bar{\psi}}\int_{\Omega^R}\mathrm{div}(\tilde{u}\cdot\nabla u)\mathrm{div} u \\
&= -\frac{\hbar^2\tau}{\bar{\psi}}\int_{\Omega^R}\nabla\tilde{u}:\nabla u\mathrm{div} u - \frac{\hbar^2\tau}{\bar{\psi}^2}\int_{\Omega^R}\tilde{u}\cdot\nabla\mathrm{div} u\mathrm{div} u \\
&= -\frac{\hbar^2\tau}{\bar{\psi}}\int_{\Omega^R}\nabla\tilde{u}:\nabla u\mathrm{div} u + \frac{\hbar^2\tau}{2\bar{\psi}^2}\int_{\Omega^R}\mathrm{div}\tilde{u}|\mathrm{div} u|^2 \\
&\le C\tau\|(\nabla\tilde{u},\mathrm{div}\tilde{u})\|_{L^\infty}\|\hbar\nabla u\|_{L^2}^2.
\end{aligned}
$$

For the second term on the RHS of (2.10), using (2.6a), we have

$$
\begin{aligned}
&-\frac{\hbar^4}{6\bar{\psi}\bar{T}}\int_{\Omega^R}\nabla\Delta\rho\cdot\nabla\mathrm{div} u \\
&= -\frac{\hbar^4}{6\bar{\psi}^2\bar{T}}\int_{\Omega^R}\Delta\rho\Delta(2\partial_t\rho - \varepsilon\Delta\rho) \\
&= -\frac{\hbar^4}{6\bar{\psi}^2\bar{T}}\frac{d}{dt}\int_{\Omega^R}|\Delta\rho|^2 - \frac{\hbar^4\varepsilon}{6\bar{\psi}^2\bar{T}}\int_{\Omega^R}|\nabla\Delta\rho|^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{2\gamma\hbar^2}{\bar{T}}\int_{\Omega^R}\nabla\rho\cdot\nabla\mathrm{div} u \\
&= -\frac{2\gamma\hbar^2}{\bar{\psi}\bar{T}}\int_{\Omega^R}\nabla\rho\cdot\nabla(2\partial_t\rho - \varepsilon\Delta\rho) \\
&= -\frac{2\gamma\hbar^2}{\bar{\psi}\bar{T}}\frac{d}{dt}\int_{\Omega^R}|\nabla\rho|^2 - \frac{2\gamma\hbar^2\varepsilon}{\bar{\psi}\bar{T}}\int_{\Omega^R}|\Delta\rho|^2.
\end{aligned}
$$

According to integration by parts and (2.6c), we decompose the last term on the RHS of (2.10) into

$$
\begin{aligned}
&\frac{\hbar^2}{\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div} u \\
&= \frac{\hbar^2}{12\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div} u - \frac{11\hbar^2}{12\bar{T}}\int_{\Omega^R}\Delta\theta\mathrm{div} u \\
&= \frac{\hbar^2}{12\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div} u - \frac{11\hbar^2\bar{\psi}^2}{8\bar{T}\kappa}\int_{\Omega^R}(\partial_t\theta + \frac{2}{3}\bar{T}\mathrm{div} u - \frac{\hbar^2}{18}\mathrm{div}\Delta u + \theta - \frac{\hbar^2}{18\bar{\psi}}\Delta\rho)\mathrm{div} u \\
&= \frac{\hbar^2}{12\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div} u - \frac{11\hbar^2\bar{\psi}^2}{12\kappa}\int_{\Omega^R}|\mathrm{div} u|^2 - \frac{11\hbar^4\bar{\psi}^2}{144\bar{T}\kappa}\int_{\Omega^R}|\nabla\mathrm{div} u|^2 - \frac{11\hbar^2\bar{\psi}^2}{8\bar{T}\kappa}\int_{\Omega^R}\partial_t\theta\mathrm{div} u \\
&\quad - \frac{11\hbar^2\bar{\psi}^2}{8\bar{T}\kappa}\int_{\Omega^R}(\theta - \frac{\hbar^2}{18\bar{\psi}}\Delta\rho)\mathrm{div} u.
\end{aligned}
$$

To make it easier to read, we separately deal with the above estimate. By integration by parts and (2.6b),

$$
\begin{aligned}
&-\frac{11\hbar^2\bar{\psi}^2}{8\bar{T}\kappa}\int_{\Omega^R}\partial_t\theta \mathrm{div}u \\
&=-\frac{11\hbar^2\bar{\psi}^2}{8\bar{T}\kappa}\frac{d}{dt}\int_{\Omega^R}\theta\mathrm{div}u-\frac{11\hbar^2\bar{\psi}^2}{8\bar{T}\kappa}\int_{\Omega^R}\nabla\theta\cdot\partial_t u \\
&=-\frac{11\hbar^2\bar{\psi}^2}{8\bar{T}\kappa}\frac{d}{dt}\int_{\Omega^R}\theta\mathrm{div}u+\frac{11\hbar^2\bar{\psi}^2}{8\bar{T}\kappa}\int_{\Omega^R}\nabla\theta\cdot(\tau\tilde{u}\cdot\nabla u+\nabla\theta+2\gamma\nabla\rho-\frac{\hbar^2}{6\bar{\psi}}\nabla\Delta\rho+u) \\
&\le-\frac{11\hbar^2\bar{\psi}^2}{8\bar{T}\kappa}\frac{d}{dt}\int_{\Omega^R}\theta\mathrm{div}u+C\tau\|\tilde{u}\|_{L^\infty}\|(\nabla\theta,\hbar\nabla u)\|_{L^2}^2+\delta\|(\varepsilon^{1/2}\nabla\rho,\hbar^2\varepsilon^{1/2}\nabla\Delta\rho)\|_{L^2}^2 \\
&\quad+C_\varepsilon\hbar^2\|(u,\nabla\theta)\|_{L^2}^2,
\end{aligned}
$$

where δ is a sufficient small positive constant. By Hölder's inequality and Young's inequality, we deduce

$$
-\frac{11\hbar^2}{8\bar{T}\bar{\psi}^2\kappa}\int_{\Omega^R}(\theta-\frac{\hbar^2}{18\bar{\psi}}\Delta\rho)\mathrm{div}u\le\delta\|\hbar\varepsilon^{1/2}\Delta\rho\|_{L^2}^2+C_\varepsilon\hbar\|(\hbar\mathrm{div}u,\theta)\|_{L^2}^2.
$$

Hence, we have

$$
\begin{aligned}
&\frac{\hbar^2}{\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div}u \\
&\le\frac{\hbar^2}{12\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div}u-\frac{11\hbar^2\bar{\psi}^2}{12\kappa}\int_{\Omega^R}|\mathrm{div}u|^2-\frac{11\hbar^4\bar{\psi}^2}{144\bar{T}\kappa}\int_{\Omega^R}|\nabla\mathrm{div}u|^2 \\
&\quad-\frac{11\hbar^2\bar{\psi}^2}{8\bar{T}\kappa}\frac{d}{dt}\int_{\Omega^R}\theta\mathrm{div}u+\delta\|(\varepsilon^{1/2}\nabla\rho,\hbar\varepsilon^{1/2}\Delta\rho,\hbar^2\varepsilon^{1/2}\nabla\Delta\rho)\|_{L^2}^2 \\
&\quad+C_\varepsilon\hbar\|(u,\nabla\theta,\hbar\mathrm{div}u,\theta)\|_{L^2}^2+C\tau\|\tilde{u}\|_{L^\infty}\|(\nabla\theta,\hbar\nabla u)\|_{L^2}^2.
\end{aligned}
$$

Summing up the above estimates, we conclude the desired inequality

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega^R}|(\hbar\mathrm{div}u,\hbar\nabla\rho,\hbar^2\Delta\rho)|^2+\hbar^2\|\mathrm{div}u\|_{L^2}^2+\hbar^4\|\nabla\mathrm{div}u\|_{L^2}^2+\hbar^2\varepsilon\|\Delta\rho\|_{L^2}^2+\hbar^4\varepsilon\|\nabla\Delta\rho\|_{L^2}^2 \\
&\quad-\frac{\hbar^2}{12\bar{T}}\int_{\Omega^R}\nabla\theta\cdot\nabla\mathrm{div}u+\frac{11\hbar^2}{8\bar{T}\bar{\psi}^2\kappa}\frac{d}{dt}\int_{\Omega^R}\theta\mathrm{div}u \\
&\le\delta\|(\varepsilon^{1/2}\nabla\rho,\hbar\varepsilon^{1/2}\Delta\rho,\hbar^2\varepsilon^{1/2}\nabla\Delta\rho)\|_{L^2}^2+C_\varepsilon\hbar\|(u,\nabla\theta,\hbar\mathrm{div}u,\theta)\|_{L^2}^2 \\
&\quad+C\tau\|(\tilde{u},\nabla\tilde{u},\mathrm{div}\tilde{u})\|_{L^\infty}\|(\nabla\theta,\hbar\nabla u)\|_{L^2}^2.
\end{aligned}
\tag{2.11}
$$

On the other hand, applying the operator $\nabla\times$ to (2.6b), and multiplying the result by $\hbar^2\nabla\times u$, we derive

$$
\frac{\hbar^2}{2}\frac{d}{dt}\int_{\Omega^R}|\nabla\times u|^2+\hbar^2\|\nabla\times u\|_{L^2}^2=-\tau\hbar^2\int_{\Omega^R}\nabla\times(\tilde{u}\cdot\nabla u)\nabla\times u.
$$

With the aid of (4.1), (4.5) and (4.7), we obtain

$$
\begin{aligned}
&-\tau\hbar^2\int_{\Omega^R}\nabla\times(\tilde{u}\cdot\nabla u)\nabla\times u \\
&=-\tau\hbar^2\int_{\Omega^R}\big(\nabla\times u\mathrm{div}\tilde{u}-\nabla\times u\cdot\nabla\tilde{u}-\nabla\times(\nabla\tilde{u}\cdot u)\big)\nabla\times u+\frac{\tau\hbar^2}{2}\int_{\Omega^R}\mathrm{div}\tilde{u}|\nabla\times u|^2 \\
&\le C\tau\|\nabla\tilde{u}\|_{H^2}\|\hbar\nabla\times u\|_{L^2}^2.
\end{aligned}
$$

Hence,

$$
\frac{\hbar^2}{2}\frac{d}{dt}\int_{\Omega^R}|\nabla\times u|^2+\hbar^2\|\nabla\times u\|_{L^2}^2\le C\tau\|\nabla\tilde{u}\|_{H^2}\|\hbar\nabla\times u\|_{L^2}^2.
\tag{2.12}
$$

Combining (2.9), (2.11) and (2.12), and using the curl-div decomposition for the gradient, we arrive at

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega^R}|(\rho,u,\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)|^2+\varepsilon\|\nabla\rho\|_{L^2}^2+\|\theta\|_{H^1}^2+\|u\|_{L^2}^2+\hbar^2\|\nabla u\|_{L^2}^2\\&\quad+\hbar^4\|\nabla\mathrm{div}u\|_{L^2}^2+\hbar^2\varepsilon\|\Delta\rho\|_{L^2}^2+\hbar^4\varepsilon\|\nabla\Delta\rho\|_{L^2}^2\le 0,\end{aligned}\tag{2.13}$$

for appropriately small positive constants d,δ and $\hbar$. Here, we don't need the uniform (in $R,\hbar,\varepsilon$) energy estimates. Based on the Gronwall inequality and Duhamel's principle, we have (ρ,u,θ) is the desired time periodic solution to system (2.3) since the period of $(\tilde\rho,\tilde u,\tilde\theta)$ is T^*.

For the uniqueness of the periodic solution (ρ,u,θ) to system (2.3) we assume there exists two different solutions (ρ_1,u_1,θ_1) and (ρ_2,u_2,θ_2) of system (2.3). Then, $(\rho_1,u_1,\theta_1)-(\rho_2,u_2,\theta_2)$ is the solution to the homogeneous system (2.6). Using the inequality (2.13) and integrating from 0 to T^*, we have

$$0\le\varepsilon\|\nabla\rho\|_{L^2}^2+\|\theta\|_{L^2}^2+\|u\|_{L^2}^2\le 0,$$

which leads $\rho\equiv u\equiv\theta\equiv 0$, thanks to poincáre inequality. Moreover, recalling the definition (2.4), we have,

$$\chi((\tilde\rho,\tilde u,\tilde\theta),0)=0.$$

Therefore, the operator χ is well defined. □

Lemma 2.2. *If d is appropriately small, the operator χ is compact and continuous.*

The proof is similar to Lemmas 2.2-2.3 in [29]. The only difference comes from the higher order quantum terms. Indeed, it is slightly easier than those of Section 2.2 since we don't need the uniform estimates w.r.t. $\hbar,\varepsilon,R$. Here, we only deal with the following quantum terms for simplicity. The other terms can be proved by "repeating" the proof of Lemma 2.1 in this paper and Lemmas 2.2-2.3 in [29]. The idea is still to apply ∂^α to (2.6b), and multiply the result by $\partial^\alpha(u-\frac{\hbar^2}{T}\nabla\mathrm{div}u)$ on both sides. Letting α be a multi-index with $0\le|\alpha|\le 3$, we obtain

$$\begin{aligned}I_1=&-\frac{\hbar^2}{6\bar\psi}\int_{\Omega^R}\partial^\alpha\nabla\Delta\rho\cdot\partial^\alpha u\\=&-\frac{\hbar^2}{6\bar\psi^2}\int_{\Omega^R}\partial^\alpha\Delta\rho\cdot\partial^\alpha(2\partial_t\rho-\varepsilon\Delta\rho+\tau\tilde\rho\mathrm{div}\tilde u+2\tau\tilde u\cdot\nabla\tilde\rho)\\=&\frac{\hbar^2}{6\bar\psi^2}\frac{d}{dt}\int_{\Omega^R}|\partial^\alpha\nabla\rho|^2+\frac{\hbar^2\varepsilon}{6\bar\psi^2}\int_{\Omega^R}|\partial^\alpha\Delta\rho|^2+\delta\hbar^2\varepsilon\|\Delta\rho\|_{H^3}^2+C_\varepsilon(\|\tilde\rho\|_{H^2}^2\|\hbar\mathrm{div}\tilde u\|_{H^3}^2+\|\tilde u\|_{H^3}^2\|\hbar\nabla\tilde\rho\|_{H^3}^2),\end{aligned}$$

where δ is a suitably small positive constant. By integration by parts and Lemmas 4.1 and 4.2, we deduce

$$\begin{aligned}I_2=&-\frac{\hbar^2\tau}{6}\int_{\Omega^R}\partial^\alpha(\frac{\tilde\rho\nabla\Delta\tilde\rho}{\bar\psi+\tau\tilde\rho})\cdot\partial^\alpha u\\=&\frac{\hbar^2\tau}{6}\int_{\Omega^R}\partial^{\alpha-1}(\frac{\tilde\rho\nabla\Delta\tilde\rho}{\bar\psi+\tau\tilde\rho})\cdot\partial^{\alpha+1}u\\\le&C\tau(\|\tilde\rho\|_{L^\infty}\|\hbar\partial^{\alpha-1}\nabla\Delta\tilde\rho\|_{L^2}+\|\partial^{\alpha-1}(\frac{\tilde\rho}{\bar\psi+\tau\tilde\rho})\|_{L^2}\|\hbar\nabla\Delta\tilde\rho\|_{L^\infty})\|\hbar\nabla u\|_{H^3}\\\le&\delta\|\hbar\nabla u\|_{H^3}^2+C\|\nabla\tilde\rho\|_{H^1}^2\|\hbar\nabla\tilde\rho\|_{H^3}^2.\end{aligned}$$

Using Lemma 4.2 again, we have

$$\begin{aligned}I_3=&\frac{\hbar^4\tau}{6\bar T}\int_{\Omega^R}\partial^\alpha(\frac{\tilde\rho\nabla\Delta\tilde\rho}{\bar\psi+\tau\tilde\rho})\cdot\partial^\alpha\nabla\mathrm{div}u\\\le&C\tau(\|\tilde\rho\|_{L^\infty}\|\hbar^2\partial^\alpha\nabla\Delta\tilde\rho\|_{L^2}+\|\partial^\alpha(\frac{\tilde\rho}{\bar\psi+\tau\tilde\rho})\|_{L^6}\|\hbar\nabla\Delta\tilde\rho\|_{L^3})\|\hbar^2\nabla\mathrm{div}u\|_{H^3}\\\le&\delta\|\hbar^2\nabla\mathrm{div}u\|_{H^3}^2+C\|\nabla\tilde\rho\|_{H^2}^2\|\hbar\nabla\tilde\rho\|_{H^3}^2+\delta\|\hbar^2\varepsilon^{1/2}\nabla\Delta\tilde\rho\|_{H^3}^2+C_\varepsilon\|\tilde\rho\|_{H^2}^2\|\hbar^2\nabla\mathrm{div}u\|_{H^3}^2.\end{aligned}$$

Combining the above estimates with the proof of Lemmas 2.2-2.3 in [29], and recalling (2.13), we finally obtain

$$\begin{aligned}&\frac{d}{dt}\|(\rho,u,\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}^2+\varepsilon\|\nabla\rho\|_{H^3}^2+\|\theta\|_{H^4}^2+\|u\|_{H^3}^2+\hbar^2\|\nabla u\|_{H^3}^2\\&\quad+\hbar^4\|\nabla\mathrm{div}u\|_{H^3}^2+\hbar^2\varepsilon\|\Delta\rho\|_{H^3}^2+\hbar^4\varepsilon\|\nabla\Delta\rho\|_{H^3}^2\\&\le C\big(\|\nabla\tilde\rho\|_{H^2}^2+\|(\tilde u,\tilde\theta,\hbar\nabla\tilde\rho,\hbar\nabla\tilde u,\hbar^2\Delta\tilde\rho)\|_{H^3}^2\big)\|(\tilde u,\tilde\theta,\nabla\tilde\theta,\hbar\nabla\tilde\rho,\hbar\nabla\tilde u,\hbar^2\Delta\tilde\rho,\hbar^2\nabla\mathrm{div}\tilde u)\|_{H^3}^2\\&\quad+\delta\|(\varepsilon^{1/2}\nabla\tilde\rho,\hbar\varepsilon^{1/2}\Delta\tilde\rho,\hbar^2\varepsilon^{1/2}\nabla\Delta\tilde\rho)\|_{H^3}^2+C\|f\|_{H^3}^2,\end{aligned}\tag{2.14}$$

thanks to the fact that $\delta,d,\hbar$ are enough small.

Integrating (2.14) over $[0,T^*]$, we arrive at

$$\begin{aligned}&\int_0^{T^*}\big(\varepsilon\|(\nabla\rho,\hbar\Delta\rho,\hbar^2\nabla\Delta\rho)\|_{H^3}^2+\|(\theta,u,\nabla\theta,\hbar\nabla u,\hbar^2\nabla\mathrm{div}u)\|_{H^3}^2\big)dt\\&\le C\sup_{t\in[0,T^*]}\big(\|\nabla\tilde\rho\|_{H^2}^2+\|(\tilde u,\tilde\theta,\hbar\nabla\tilde\rho,\hbar\nabla\tilde u,\hbar^2\Delta\tilde\rho)\|_{H^3}^2\big)\int_0^{T^*}\|(\tilde u,\tilde\theta,\nabla\tilde\theta,\hbar\nabla\tilde\rho,\hbar\nabla\tilde u,\hbar^2\Delta\tilde\rho,\hbar^2\nabla\mathrm{div}\tilde u)\|_{H^3}^2dt\\&\quad+\delta\int_0^{T^*}\|(\varepsilon^{1/2}\nabla\tilde\rho,\hbar\varepsilon^{1/2}\Delta\tilde\rho,\hbar^2\varepsilon^{1/2}\nabla\Delta\tilde\rho)\|_{H^3}^2dt+C\int_0^{T^*}\|f\|_{H^3}^2dt\triangleq M,\end{aligned}\tag{2.15}$$

which leads there exists $t_0\in(0,T^*)$ such that

$$\varepsilon\|(\nabla\rho,\hbar\Delta\rho,\hbar^2\nabla\Delta\rho)(t_0)\|_{H^3}^2+\|(\theta,u,\nabla\theta,\hbar\nabla u,\hbar^2\nabla\mathrm{div}u)(t_0)\|_{H^3}^2\le CM.\tag{2.16}$$

Then Poincare's inequality yields

$$\|(\rho,\theta,u,\nabla\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)(t_0)\|_{H^3}^2\le CM.\tag{2.17}$$

Integrating (2.14) again from t_0 to t with $t\in(t_0,T^*]$, we have

$$\|(\rho,\theta,u,\nabla\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)(t)\|_{H^3}^2\le CM.\tag{2.18}$$

Therefore, using the time periodicity, we have

$$\sup_{t\in[0,T^*]}\|(\rho,\theta,u,\nabla\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)(t)\|_{H^3}^2\le CM.\tag{2.19}$$

For the completeness of the proof, one can refer to [29] for details.

2.2. **Uniform energy estimates.** In this subsection, we will obtain some uniform (in R,ε) energy estimares for system (2.20). It is noted that the arguments are indeed independent of $\hbar$. To begin with, we introduce the nonlinear approximated system

$$\begin{cases}2\partial_t\rho+\bar\psi\mathrm{div}u-\varepsilon\Delta\rho=-\tau\rho\mathrm{div}u-2\tau u\cdot\nabla\rho, & \text{(2.20a)}\\ \partial_tu+\nabla\theta+2\dfrac{\bar T+\tau\theta}{\bar\psi+\tau\rho}\nabla\rho-\dfrac{\hbar^2}{6}\nabla(\dfrac{\Delta\rho}{\tau\rho+\bar\psi})+u=-\tau u\cdot\nabla u+\tau f^R, & \text{(2.20b)}\\ \partial_t\theta+\dfrac{2}{3}\bar T\mathrm{div}u-\dfrac{2\kappa}{3}\dfrac{\Delta\theta}{(\tau\rho+\bar\psi)^2}-\dfrac{\hbar^2}{18}\mathrm{div}\Delta u+\theta-\dfrac{\hbar^2}{18}\dfrac{\Delta\rho}{\tau\rho+\bar\psi}=-\tau u\cdot\nabla\theta-\dfrac{2\tau}{3}\theta\mathrm{div}u\\ \qquad-\tau\dfrac{|u|^2}{3}+\dfrac{\hbar^2\tau}{9}\dfrac{\nabla\rho\cdot\nabla\mathrm{div}u}{\tau\rho+\bar\psi}-\dfrac{\hbar^2\tau}{18}\dfrac{|\nabla\rho|^2}{(\tau\rho+\bar\psi)^2}, & \text{(2.20c)}\end{cases}$$

where $\tau\in(0,1]$. If $\tau=1$, we prove the existence of time periodic solutions to system (2.1).

Lemma 2.3. *Let X be a normal linear space, Ω be open and bounded subset of X, the map $I-F$ be completely continuous field in $\bar\Omega$, where I is the unit mapping. Provided that $(I-F)(\partial\Omega)\neq0$. Further, if deg$(I-F,\Omega,0)\neq0$, the equation $(I-F)(x)=0$ has a solution in Ω.*

The Leray-Schauder degree has the following property

Lemma 2.4. *Let $\chi : X_d^R \times [0,1] \to X^R$ be completed continuous. For any $\tau \in [0,1]$, if $(I-\chi)(\cdot,\tau) \neq 0$, we have $deg(I-\chi,\Omega,0)$ is dependent of τ.*

See [32] for more results about the topological degree theory.

Based on Lemmas 2.3-2.4, to show the existence of solutions to system (2.1), namely,

$$\chi(U,1)=U, \quad U=(\rho,u,\theta)\in X_d^R,$$

we need to prove that there exists some positive constant d_0, such that, for any $\tau \in (0,1]$,

$$(I-\chi(\cdot,\tau))(\partial \bar{B}_{d_0}(0)) \neq 0, \tag{2.21}$$

where we denote $\bar{B}_{d_0}(0)$ by a ball of radius d_0 centered at the origin.

Therefore, the main target is hereafter to obtain uniform (in $\varepsilon, R, \hbar$) bounds, which play a significant role in the proof of Proposition 2.1. At first, we give the zero, one, two and three order energy estimates for solutions to system (2.20).

Lemma 2.5. *Let $0<d_0<1$ be a suitably small constant, and $(\rho,u,\theta)\in\partial\hat{B}_{d_0}(0)$ be a solution of system (2.20). Then there holds*

$$\begin{aligned}
&\frac{d}{dt}\|(\rho,u,\theta,\hbar\nabla\rho)\|_{H^3}^2+\|u\|_{H^3}^2+\|\theta\|_{H^4}^2+\varepsilon\|\nabla\rho\|_{H^3}^2+\varepsilon\hbar^2\|\Delta\rho\|_{H^3}^2-\frac{\hbar^2}{12}\int_{\Omega^R}(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\Delta\hat{\theta}\mathrm{div}\hat{u}\\
&\quad\le C\tau d_0^2(\|\nabla\rho\|_{H^2}+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)\|_{H^3})+\delta\|\hbar^2\nabla\mathrm{div}u\|_{H^3}^2+C\hbar\|\hbar\nabla\rho\|_{H^3}^2\\
&\qquad+C\tau\|f^R\|_{H^3}^2.
\end{aligned}\tag{2.22}$$

Proof. Applying the operator ∂^α to system (2.20), we have

$$\begin{cases}
2\partial_t\hat{\rho}+(\bar{\psi}+\tau\rho)\mathrm{div}\hat{u}-\varepsilon\Delta\hat{\rho}=-2\tau u\cdot\nabla\hat{\rho}+h_1, & (2.23\text{a})\\
\partial_t\hat{u}+\hat{u}+\nabla\hat{\theta}+2\dfrac{\bar{T}+\tau\theta}{\bar{\psi}+\tau\rho}\nabla\hat{\rho}-\dfrac{\hbar^2}{6}\dfrac{\nabla\Delta\hat{\rho}}{\bar{\psi}+\tau\rho}=-\tau u\cdot\nabla\hat{u}+h_2+\tau\partial^\alpha f^R, & (2.23\text{b})\\
\partial_t\hat{\theta}+\dfrac{2}{3}(\bar{T}+\tau\theta)\mathrm{div}\hat{u}-\dfrac{2\kappa}{3}\dfrac{\Delta\hat{\theta}}{(\bar{\psi}+\tau\rho)^2}-\dfrac{\hbar^2}{18}\mathrm{div}\Delta\hat{u}+\hat{\theta}-\dfrac{\hbar^2}{18}\dfrac{\Delta\hat{\rho}}{\tau\rho+\bar{\psi}}=-\tau u\cdot\nabla\hat{\theta} & \\
\quad+\tau\partial^\alpha\Big(\dfrac{\hbar^2}{9}\dfrac{\nabla\rho\cdot\nabla\mathrm{div}u}{\bar{\psi}+\tau\rho}-\dfrac{|u|^2}{3}-\dfrac{\hbar^2}{18}\dfrac{|\nabla\rho|^2}{(\tau\rho+\bar{\psi})^2}\Big)+h_3, & (2.23\text{c})
\end{cases}$$

where

$$(\partial^\alpha\rho,\partial^\alpha u,\partial^\alpha\theta)=(\hat{\rho},\hat{u},\hat{\theta}),$$

and

$$\begin{aligned}
h_1=&-\tau[\partial^\alpha,\rho]\mathrm{div}u-2\tau[\partial^\alpha,u]\cdot\nabla\rho,\\
h_2=&-\tau[\partial^\alpha,u]\cdot\nabla u-2[\partial^\alpha,\frac{\bar{T}+\tau\theta}{\bar{\psi}+\tau\rho}]\nabla\rho+\frac{\hbar^2}{6}[\partial^\alpha\nabla,\frac{1}{\bar{\psi}+\tau\rho}]\Delta\rho,\\
h_3=&-\frac{2\tau}{3}[\partial^\alpha,\theta]\mathrm{div}u+\frac{2\kappa}{3}[\partial^\alpha,\frac{1}{(\bar{\psi}+\tau\rho)^2}]\Delta\theta+\frac{\hbar^2}{18}[\partial^\alpha,\frac{1}{\rho+\bar{\psi}}]\Delta\rho-\tau[\partial^\alpha,u]\cdot\nabla\theta.
\end{aligned}\tag{2.24}$$

According to Sobolev embedding and the Moser type calculus inequality (4.2), we get

$$\begin{aligned}
\|h_1\|_{L^{6/5}}\le&C\tau\|\nabla\rho\|_{L^3}\|\nabla u\|_{H^2}+C\tau\|\nabla\rho\|_{H^2}\|\nabla u\|_{L^3}\\
\le&C\tau d_0\|\nabla u\|_{H^2},
\end{aligned}\tag{2.25}$$

$$\|h_1\|_{H^k}\le C\tau d_0\|\nabla u\|_{H^{2+k}},\quad k=0,1,\tag{2.26}$$

$$\begin{aligned}\|h_2\|_{L^2} \leq & C\tau(\|\frac{\nabla\rho(\bar{T}+\tau\theta)}{(\bar{\psi}+\tau\rho)^2}+\frac{\nabla\theta}{\bar{\psi}+\tau\rho}\|_{L^\infty}\|\nabla\rho\|_{H^2}+\|\frac{\nabla\rho(\bar{T}+\tau\theta)}{(\bar{\psi}+\tau\rho)^2}+\frac{\nabla\theta}{\bar{\psi}+\tau\rho}\|_{H^2}\|\nabla\rho\|_{L^\infty}) \\ & +C\tau\|\nabla u\|_{L^\infty}\|\nabla u\|_{H^2}+C\tau(\|\nabla(\frac{1}{\bar{\psi}+\tau\rho})\|_{L^\infty}\|\hbar^2\Delta\rho\|_{H^3}+\|\hbar\nabla(\frac{1}{\bar{\psi}+\tau\rho})\|_{H^3}\|\hbar\Delta\rho\|_{L^\infty}) \\ \leq & C\tau d_0(\|(\nabla\rho,\nabla u)\|_{H^2}+\|(\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}),\end{aligned} \tag{2.27}$$

and

$$\|h_3\|_{L^2} \leq \tau d_0(\|(\nabla\theta,\nabla u)\|_{H^2}+\|(\nabla\theta,\hbar\nabla\rho)\|_{H^3}). \tag{2.28}$$

Multiplying system (2.23) by $A_0(\hat{\rho},\hat{u},\hat{\theta})^\top$, and integrating over the periodic domain Ω^R, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|A_0^{\frac{1}{2}}(\hat{\rho},\hat{u},\hat{\theta})\|_{L^2}^2+\|\hat{u}\|_{L^2}^2+\|\hat{\theta}\|_{H^1}^2+\varepsilon\|\nabla\hat{\rho}\|_{L^2}^2-\frac{\hbar^2}{12}\int_{\Omega^R}\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)^2\Delta\hat{\theta}\mathrm{div}\hat{u} \\ &\quad -\frac{\hbar^2}{6}\int_{\Omega^R}\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\nabla\Delta\hat{\rho}\cdot\hat{u}=\sum_{i=1}^2 R_{1,i},\end{aligned} \tag{2.29}$$

where

$$A_0=\begin{pmatrix}2 & 0 & 0\\ 0 & \frac{(\bar{\psi}+\tau\rho)^2}{\bar{T}+\tau\theta} & 0\\ 0 & 0 & \frac{3}{2}(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\end{pmatrix}, \tag{2.30}$$

$$R_{1,1}=\int_{\Omega^R}(h_1+h_2+h_3)A_0(\hat{\rho},\hat{u},\hat{\theta}), \tag{2.31}$$

and

$$\begin{aligned}R_{1,2}=&\frac{1}{2}\int_{\Omega^R}\partial_t A_0|(\hat{\rho},\hat{u},\hat{\theta})|^2+\frac{\tau}{2}\int_{\Omega^R}\mathrm{div}\Big(\frac{(\bar{\psi}+\tau\rho)^2u}{\bar{T}+\tau\theta}\Big)|\hat{u}|^2+\frac{3\tau}{4}\int_{\Omega^R}\mathrm{div}\Big(\frac{(\bar{\psi}+\tau\rho)^2u}{(\bar{T}+\tau\theta)^2}\Big)|\hat{\theta}|^2 \\ &+\tau\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^2}{\bar{T}+\tau\theta}\partial^\alpha f^R\cdot\hat{u}+\frac{3\tau}{2}\int_{\Omega^R}(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\partial^\alpha\Big(\frac{\hbar^2}{9}\frac{\nabla\rho\cdot\nabla\mathrm{div}u}{\bar{\psi}+\tau\rho}-\frac{|u|^2}{3}+\frac{\hbar^2}{18}\frac{|\nabla\rho|^2}{(\tau\rho+\bar{\psi})^2}\Big)\hat{\theta} \\ &+\frac{\hbar^2}{12}\int_{\Omega^R}\Delta(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\hat{\theta}\mathrm{div}\hat{u}+\frac{\hbar^2}{6}\int_{\Omega^R}\nabla(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\cdot\nabla\hat{\theta}\mathrm{div}\hat{u}-\kappa\int_{\Omega^R}\nabla(\frac{1}{\bar{T}+\tau\theta})^2\cdot\nabla\hat{\theta}\hat{\theta} \\ &+\int\nabla\Big(\frac{(\bar{\psi}+\tau\rho)^2}{\bar{T}+\tau\theta}\Big)\cdot\hat{u}\hat{\theta}+2\tau\int\nabla\rho\cdot\hat{u}\hat{\rho}-4\tau\int_{\Omega^R}u\cdot\nabla\hat{\rho}\hat{\rho}-\frac{\hbar^2}{12}\int_{\Omega^R}\nabla\Big(\frac{\bar{\psi}+\tau\rho}{(\bar{T}+\tau\theta)^2}\hat{\theta}\Big)\cdot\nabla\hat{\rho}.\end{aligned}$$

Here, we have used the equation

$$\begin{aligned}&-\frac{\hbar^2}{12}\int_{\Omega^R}\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)^2\mathrm{div}\Delta\hat{u}\hat{\theta} \\ &=\frac{\hbar^2}{12}\int_{\Omega^R}\nabla\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)^2\cdot\nabla\mathrm{div}\hat{u}\hat{\theta}+\frac{\hbar^2}{12}\int_{\Omega^R}\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)^2\nabla\mathrm{div}\hat{u}\cdot\nabla\hat{\theta} \\ &=-\frac{\hbar^2}{12}\int_{\Omega^R}\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)^2\Delta\hat{\theta}\mathrm{div}\hat{u}-\frac{\hbar^2}{12}\int_{\Omega^R}\Delta\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)^2\mathrm{div}\hat{u}\hat{\theta}-\frac{\hbar^2}{6}\int_{\Omega^R}\nabla\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)^2\cdot\nabla\hat{\theta}\mathrm{div}\hat{u}.\end{aligned} \tag{2.32}$$

In what follows, we focus on the following terms involving the quantum effects. For the other terms on (2.29), we can bound them as [29] by the continuity and momentum equations. More precisely, for the last term on the LHS of (2.29), we use integration by parts and (2.23a) to obtain

$$
\begin{aligned}
&-\frac{\hbar^2}{6}\int_{\Omega^R}\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\nabla\Delta\hat{\rho}\cdot\hat{u}\\
&=\frac{\hbar^2}{6}\int_{\Omega^R}\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Delta\hat{\rho}\mathrm{div}\hat{u}+\frac{\hbar^2}{6}\int_{\Omega^R}\nabla\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)\cdot\hat{u}\Delta\hat{\rho}\\
&=-\frac{\hbar^2}{6}\int_{\Omega^R}\frac{\Delta\hat{\rho}}{\bar{T}+\tau\theta}(2\partial_t\hat{\rho}-\varepsilon\Delta\hat{\rho}+2\tau u\cdot\nabla\hat{\rho}-h_1)+\frac{\hbar^2}{6}\int_{\Omega^R}\nabla\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)\cdot\hat{u}\Delta\hat{\rho}\\
&=\frac{\hbar^2}{6}\frac{d}{dt}\int_{\Omega^R}\frac{|\nabla\hat{\rho}|^2}{\bar{T}+\tau\theta}+\frac{\hbar^2\varepsilon}{6}\int_{\Omega^R}\frac{|\Delta\hat{\rho}|^2}{\bar{T}+\tau\theta}+\frac{\hbar^2}{3}\int_{\Omega^R}\nabla(\frac{1}{\bar{T}+\tau\theta})\cdot\nabla\hat{\rho}\partial_t\hat{\rho}\\
&\quad-\frac{\hbar^2}{6}\int_{\Omega^R}\partial_t(\frac{1}{\bar{T}+\tau\theta})|\nabla\hat{\rho}|^2+\frac{\hbar^2}{6}\int_{\Omega^R}\frac{\Delta\hat{\rho}h_1}{\bar{T}+\tau\theta}+\frac{\hbar^2\tau}{3}\int_{\Omega^R}\nabla(\frac{1}{\bar{T}+\tau\theta})\cdot\nabla\hat{\rho}u\cdot\nabla\hat{\rho}\\
&\quad-\frac{\hbar^2\tau}{6}\int_{\Omega^R}\mathrm{div}\Big(\frac{u}{\bar{T}+\tau\theta}\Big)|\nabla\hat{\rho}|^2+\frac{\hbar^2\tau}{3}\int_{\Omega^R}\frac{\nabla\hat{\rho}^\top\nabla u\nabla\hat{\rho}}{\bar{T}+\tau\theta}+\frac{\hbar^2}{6}\int_{\Omega^R}\nabla\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)\cdot\hat{u}\Delta\hat{\rho}\\
&\geq\frac{\hbar^2}{6}\frac{d}{dt}\int_{\Omega^R}\frac{|\nabla\hat{\rho}|^2}{\bar{T}+\tau\theta}+\frac{\hbar^2\varepsilon}{6}\int_{\Omega^R}\frac{|\Delta\hat{\rho}|^2}{\bar{T}+\tau\theta}-\delta\|(\hbar\varepsilon^{1/2}\Delta\rho,\hbar^2\nabla\mathrm{div}u)\|_{H^3}^2\\
&\quad-C\tau d_0^2\|(u,\nabla\theta,\hbar\mathrm{div}u,\hbar^2\Delta\rho)\|_{H^3},
\end{aligned}
\tag{2.33}
$$

thanks to Hölder's inequality, Young's inequality and the bound (2.5).

Estimates for the RHS of (2.29). For the first term $R_{1,1}$, combining Hölder's inequality, Gagliardo-Nirenberg inequality with the estimates (2.25), (2.27) and (2.28), we get

$$
\begin{aligned}
R_{1,1}\leq&\|A_0\|_\infty(\|h_1\|_{L^{6/5}}\|\hat{\rho}\|_{L^6}+\|(h_2,h_3)\|_{L^2}\|(\hat{u},\hat{\theta})\|_{L^2})\\
\leq&C\tau d_0^2(\|(\nabla\rho,\nabla u)\|_{H^2}+\|(\nabla\theta,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}).
\end{aligned}
\tag{2.34}
$$

It remains to estimate the last term on the RHS of (2.29). Combining Hölder inequality, Young's inequality with the estimates (2.38) and (2.40), we get

$$
\begin{aligned}
R_{1,2}\leq&C\tau d_0^2(\|\nabla\rho\|_{H^2}+\|(u,\theta,\nabla\theta,\hbar\mathrm{div}u)\|_{H^3})+\delta\|(u,\hbar^2\nabla\mathrm{div}u)\|_{H^3}^2\\
&+C\hbar\|(\nabla\theta,\hbar\nabla\rho)\|_{H^3}^2+C\tau\|f^R\|_{H^3}^2,
\end{aligned}
\tag{2.35}
$$

where δ is a sufficiently small positive constant. For simplify, we only deal with the following terms. By integration by parts, term $\frac{\hbar^2\tau}{6}\int_{\Omega^R}(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\partial^\alpha\big(\frac{\nabla\rho\cdot\nabla\mathrm{div}u}{\bar{\psi}+\tau\rho}\big)\hat{\theta}$ can be estimated as

$$
\begin{aligned}
&\frac{\hbar^2\tau}{6}\int_{\Omega^R}(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\partial^\alpha\Big(\frac{\nabla\rho\cdot\nabla\mathrm{div}u}{\bar{\psi}+\tau\rho}\Big)\hat{\theta}\\
&=-\frac{\hbar^2\tau}{6}\int_{\Omega^R}\partial\Big(\frac{(\bar{\psi}+\tau\rho)^2\hat{\theta}}{(\bar{T}+\tau\theta)^2}\Big)\partial^{\alpha-1}\Big(\frac{\nabla\rho\cdot\nabla\mathrm{div}u}{\bar{\psi}+\tau\rho}\Big)\\
&\leq C\hbar^2\tau(\|\frac{\nabla\rho}{\bar{\psi}+\tau\rho}\|_{L^\infty}\|\nabla\mathrm{div}u\|_{H^2}+\|\frac{\nabla\rho}{\bar{\psi}+\tau\rho}\|_{H^2}\|\nabla\mathrm{div}u\|_{L^\infty})(1+\|(\nabla\rho,\nabla\theta)\|_{H^2})\|\theta\|_{H^4}\\
&\leq C\tau d_0^2\|\theta\|_{H^4}.
\end{aligned}
\tag{2.36}
$$

Moreover, when $|\alpha|=0$,

$$
\begin{aligned}
&\frac{\hbar^2\tau}{6}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)\nabla\rho\cdot\nabla\mathrm{div}u\theta}{(\bar{T}+\tau\theta)^2}\\
&\quad\leq C\|\hbar\nabla\mathrm{div}u\|_{L^\infty}\|(\hbar\nabla\rho,\theta)\|_{L^2}^2\\
&\quad\leq C\tau d_0^2\|\hbar\nabla u\|_{L^2}.
\end{aligned}
\tag{2.37}
$$

Similarly, term $\int_{\Omega^R} u\cdot\nabla\rho\rho$ can be bounded by

$$\frac{\tau}{2}\int_{\Omega^R} u\cdot\nabla\rho\rho \le C\tau\|\rho\|_{L^6}\|\nabla\rho\|_{L^3}\|u\|_{L^2}\le C\tau d_0^2\|\nabla\rho\|_{H^1}.$$

Otherwise, when $|\alpha|>0$,

$$\frac{\tau}{2}\int_{\Omega^R} u\cdot\nabla\hat\rho\hat\rho = \frac{\tau}{4}\int_{\Omega^R}\mathrm{div}u|\hat\rho|^2\le C\tau d_0^2\|\nabla u\|_{H^2}.$$

Putting these estimates (2.33)-(2.35) for any multi-index $0\le|\alpha|\le 3$ together, recalling (2.5) and choosing $\delta,\hbar$ sufficiently small, we complete the proof of Lemma 2.5. □

From system (2.23) directly, we deduce the following estimates.

Lemma 2.6. *Under the same conditions in lemma 2.5, we derive*

$$\|\rho_t\|_{H^2}\le C\|(\nabla u,\varepsilon\Delta\rho)\|_{H^2}+C\tau d_0\|(\nabla u,\nabla\rho)\|_{H^2},\tag{2.38}$$

$$\|u_t\|_{H^2}\le C\|(\nabla\theta,\nabla\rho)\|_{H^2}+C\|(u,\hbar^2\Delta\rho)\|_{H^3}+C\tau d_0\|(\nabla u,\nabla\rho,\hbar\Delta\rho,\hbar^2\nabla\Delta\rho)\|_{H^2}+C\tau\|f^R\|_{H^2},\tag{2.39}$$

and

$$\|\theta_t\|_{H^2}\le C\|(\theta,\nabla u)\|_{H^2}+C\|(\nabla\theta,\hbar^2\nabla\mathrm{div}u)\|_{H^3}+C\tau d_0(\|\nabla\theta\|_{H^2}+\|(u,\nabla\theta,\hbar\mathrm{div}u)\|_{H^3}).\tag{2.40}$$

Note that it follows from Lemmas 2.5-2.6 that we need to deal with not just the estimates for $(\hbar\nabla u,\hbar^2\Delta\rho,\hbar^2\nabla\mathrm{div}u)$ but also the term $\frac{\hbar^2}{12}\int_{\Omega^R}(\frac{\bar\psi+\tau\rho}{\bar T+\tau\theta})^2\Delta\hat\theta\mathrm{div}\hat u$, which is not necessary in [7,25,29]. To do this, we employ the structure of equations (2.23a)-(2.23c) comprehensively and define a new weighted norm (1.2). For the higher order derivatives of ∇u, we use the curl-div decomposition of the gradient. Next, we give the estimates for the vorticity.

Lemma 2.7. *Under the same conditions in lemma 2.5, we have*

$$\frac{d}{dt}\|\hbar\nabla\times u\|_{H^3}^2+\hbar^2\|\nabla\times u\|_{H^3}^2\le C\tau d_0^2\|(\nabla\theta,\hbar\nabla\times u)\|_{H^3}+C\tau\|f^R\|_{H^3}^2.\tag{2.41}$$

Proof. Applying the operator $\partial^\alpha\nabla\times$ to (2.20b), and multiplying the result by $\hbar^2\nabla\times\hat u$, we derive

$$\begin{aligned}\frac{d}{dt}\|\hbar\nabla\times\hat u\|_{L^2}^2+\hbar^2\|\nabla\times\hat u\|_{L^2}^2 = {}&\frac{\hbar^2\tau}{2}\int_{\Omega^R}\mathrm{div}u|\nabla\times\hat u|^2-\hbar^2\tau\int_{\Omega^R}[\partial^\alpha,u]\cdot\nabla\nabla\times u\cdot\nabla\times\hat u\\ &-\hbar^2\int_{\Omega^R}\partial^\alpha\Big(2\nabla(\frac{\bar T+\tau\theta}{\bar\psi+\tau\rho})\times\nabla\rho+\tau\mathrm{div}u\nabla\times u-\tau\nabla\times u\cdot\nabla u-\tau f^R\Big)\cdot\nabla\times\hat u,\end{aligned}\tag{2.42}$$

thanks to Lemma 4.4. We proceed to deal with the RHS of (2.42). For the first term, by Hölder inequality, we have

$$\frac{\hbar^2\tau}{2}\int_{\Omega^R}\mathrm{div}u|\nabla\times u_\alpha|^2\le C\tau d_0^2\|\hbar\nabla\times u\|_{H^3}.\tag{2.43}$$

By Lemma 4.2, we have

$$\begin{aligned}&-\hbar^2\tau\int_{\Omega^R}[\partial^\alpha,u]\cdot\nabla\nabla\times u\cdot\nabla\times\hat u\\ &\le C\tau(\|\nabla u\|_{L^\infty}\|\hbar\nabla\times u\|_{H^3}+\|\nabla u\|_{H^2}\|\hbar\nabla\nabla\times u\|_{L^\infty})\|\hbar\nabla\times u\|_{H^3}\\ &\le C\tau d_0^2\|\hbar\nabla\times u\|_{H^3}.\end{aligned}\tag{2.44}$$

According to the estimates (4.1) and (4.3), we have

$$
\begin{aligned}
&-\frac{\hbar^2}{2}\int_{\Omega^R}\partial^\alpha\Big(\nabla(\frac{\bar{T}+\tau\theta}{\bar{\psi}+\tau\rho})\times\nabla\rho+\tau\mathrm{div}u\nabla\times u-\tau\nabla\times u\cdot\nabla u-\tau f^R\Big)\cdot\nabla\times\hat{u}\\
&\le C\|\nabla(\frac{\bar{T}+\tau\theta}{\bar{\psi}+\tau\rho})\|_{L^\infty}\|\hbar\nabla\rho\|_{H^3}\|\hbar\nabla\times u\|_{H^3}+C\|\hbar\nabla(\frac{\bar{T}+\tau\theta}{\bar{\psi}+\tau\rho})\|_{H^3}\|\nabla\rho\|_{L^\infty}\|\hbar\nabla\times u\|_{H^3}\\
&\quad+C\tau(\|(\mathrm{div}u,\nabla u)\|_{L^\infty}\|\hbar\nabla\times u\|_{H^3}+\|(\hbar\mathrm{div}u,\hbar\nabla u)\|_{H^3}\|\nabla\times u\|_{L^\infty})\|\hbar\nabla\times u\|_{H^3}\\
&\quad+C\tau\|f^R\|_{H^3}\|\hbar\nabla\times u\|_{H^3}\\
&\le\delta\|\hbar\nabla\times u\|_{H^3}^2+C\tau d_0^2\|(\nabla\theta,\hbar\nabla\times u)\|_{H^3}+C\tau\|f^R\|_{H^3}^2.
\end{aligned}
\tag{2.45}
$$

Putting all the above estimates for any multi-index $0\le|\alpha|\le3$ together, we completes the proof of Lemma 2.7. ☐

The next task is to obtain the higher order estimate for the divergence of the velocity field. To this end, we need to control the singular term $\frac{\hbar^2}{12}\int_{\Omega^R}(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\Delta\hat{\theta}\mathrm{div}\hat{u}$ in lemma 2.5 and derive some higher estimates for solutions to system (2.23). Indeed, by making full use of the energy equation (2.23c) and integration by parts, we obtain some "good term" which appears on the left as (2.48). This provides the possibility to close the inequality (2.52).

Lemma 2.8. *Under the same conditions in lemma 2.5, we have*

$$
\begin{aligned}
&\frac{d}{dt}\|(\hbar\mathrm{div}u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}^2+\frac{\hbar^2}{12}\int_{\Omega^R}(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\Delta\hat{\theta}\mathrm{div}\hat{u}+\frac{11\hbar^2}{8\kappa}\frac{d}{dt}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\hat{\theta}\mathrm{div}\hat{u}\\
&\quad+\|(\hbar\mathrm{div}u,\hbar^2\mathrm{div}u,\hbar\varepsilon^{1/2}\Delta\rho,\hbar^2\varepsilon\nabla\Delta\rho)\|_{H^3}^2\\
&\le\delta\|(\theta,\nabla\theta,\varepsilon^{\frac12}\nabla\rho,\hbar^3\nabla\Delta\rho)\|_{H^3}^2+C\hbar\|(u,\theta,\nabla\theta,\hbar\nabla u,\hbar\nabla\rho)\|_{H^3}^2+C\tau\|f^R\|_{H^3}^2\\
&\quad+C\tau d_0^2\|(u,\nabla\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}.
\end{aligned}
\tag{2.46}
$$

Proof. Multiplying (2.23b) by $-\frac{\hbar^2(\bar{\psi}+\tau\rho)^2\nabla\mathrm{div}\hat{u}}{(\bar{T}+\tau\theta)^2}$, integrating over the periodic domain Ω_R, we get

$$
\begin{aligned}
\frac{d}{dt}\|\hbar\mathrm{div}\hat{u}\|_{L^2}^2+\hbar^2\|\mathrm{div}\hat{u}\|_{L^2}^2=&-\frac{\hbar^4}{6}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)\nabla\Delta\hat{\rho}\cdot\nabla\mathrm{div}\hat{u}}{(\bar{T}+\tau\theta)^2}+2\hbar^2\int_{\Omega^R}\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\nabla\hat{\rho}\cdot\nabla\mathrm{div}\hat{u}\\
&+\hbar^2\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^2\nabla\hat{\theta}\cdot\nabla\mathrm{div}\hat{u}}{(\bar{T}+\tau\theta)^2}+\hbar^2\tau\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^2}{(\bar{T}+\tau\theta)^2}u\cdot\nabla\hat{u}\cdot\nabla\mathrm{div}\hat{u}\\
&-\hbar^2\int_{\Omega^R}\nabla\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)^2\mathrm{div}\hat{u}\hat{u}-\hbar^2\int_{\Omega^R}\nabla\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)^2\cdot\hat{u}_t\mathrm{div}\hat{u}\\
&+\frac{\hbar^2\tau}{2}\int_{\Omega^R}\partial_t\Big(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta}\Big)^2|\mathrm{div}\hat{u}|^2\\
&-\hbar^2\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^2(h_2+\tau\partial^\alpha f^R)\cdot\nabla\mathrm{div}\hat{u}}{(\bar{T}+\tau\theta)^2}\\
&\triangleq\sum_{i=1}^{8}R_{2,i}.
\end{aligned}
\tag{2.47}
$$

Estimates of the RHS of (2.47). For the first term $R_{2,1}$, by integration by parts, (2.23a), (2.26) and Lemmas 4.1-4.3, we derive

$$\begin{aligned}
R_{2,1} =& \frac{\hbar^4}{6}\int_{\Omega^R} \frac{(\bar\psi+\tau\rho)\Delta\hat\rho\Delta\mathrm{div}\hat u}{(\bar T+\tau\theta)^2} + \frac{\hbar^4}{6}\int_{\Omega^R}\nabla\Big(\frac{\bar\psi+\tau\rho}{(\bar T+\tau\theta)^2}\Big)\cdot\nabla\mathrm{div}\hat u\Delta\hat\rho \\
=& -\frac{\hbar^4}{6}\int_{\Omega^R}\frac{\Delta\hat\rho\Delta(2\partial_t\hat\rho-\varepsilon\Delta\hat\rho+2\tau u\cdot\nabla\hat\rho-h_1)}{(\bar T+\tau\theta)^2} - \frac{\hbar^4\tau}{6}\int_{\Omega^R}\frac{[\Delta,\rho]\mathrm{div}\hat u\Delta\hat\rho}{(\bar T+\tau\theta)^2} \\
&+\frac{\hbar^4}{6}\int_{\Omega^R}\nabla\Big(\frac{\bar\psi+\tau\rho}{(\bar T+\tau\theta)^2}\Big)\cdot\nabla\mathrm{div}\hat u\Delta\hat\rho \\
=& -\frac{\hbar^4}{6}\frac{d}{dt}\int_{\Omega^R}\frac{|\Delta\hat\rho|^2}{(\bar T+\tau\theta)^2} + \frac{\hbar^4}{6}\int_{\Omega^R}\partial_t\Big(\frac{1}{\bar T+\tau\theta}\Big)^2|\Delta\hat\rho|^2 - \frac{\hbar^4\varepsilon}{6}\int_{\Omega^R}\frac{|\nabla\Delta\hat\rho|^2}{(\bar T+\tau\theta)^2} \\
&+\frac{\hbar^4}{12}\int_{\Omega^R}\Delta\Big(\frac{1}{\bar T+\tau\theta}\Big)^2|\Delta\hat\rho|^2 + \frac{\hbar^4\tau}{6}\int_{\Omega^R}\mathrm{div}\Big(\frac{u}{(\bar T+\tau\theta^2)}\Big)|\Delta\hat\rho|^2 - \frac{\hbar^4\tau}{3}\int_{\Omega^R}\frac{\Delta\hat\rho[\Delta,u]\nabla\hat\rho}{(\bar T+\tau\theta)^2} \\
&-\frac{\hbar^4}{6}\int_{\Omega^R}\nabla\Big(\frac{\Delta\hat\rho}{(\bar T+\tau\theta)^2}\Big)\cdot\nabla h_1 - \frac{\hbar^4\tau}{6}\int_{\Omega^R}\frac{[\Delta,\rho]\mathrm{div}\hat u\Delta\hat\rho}{(\bar T+\tau\theta)^2} + \frac{\hbar^4}{6}\int_{\Omega^R}\nabla\Big(\frac{\bar\psi+\tau\rho}{(\bar T+\tau\theta)^2}\Big)\cdot\nabla\mathrm{div}\hat u\Delta\hat\rho \\
\le& -\frac{\hbar^4}{6}\frac{d}{dt}\int_{\Omega^R}\frac{|\Delta\hat\rho|^2}{(\bar T+\tau\theta)^2} - \frac{\hbar^4\varepsilon}{6}\int_{\Omega^R}\frac{|\nabla\Delta\hat\rho|^2}{(\bar T+\tau\theta)^2} + C\tau d_0^2\|(\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)\|_{H^3} \\
&+\delta\|(\hbar^2\nabla\mathrm{div}u,\hbar^3\nabla\Delta\rho)\|_{H^3}^2,
\end{aligned}$$

where δ is a sufficiently small positive constant. Similarly,

$$\begin{aligned}
R_{2,2}\le& -2\hbar^2\frac{d}{dt}\int_{\Omega^R}\frac{|\nabla\hat\rho|^2}{\bar T+\tau\theta} - 2\hbar^2\varepsilon\int_{\Omega^R}\frac{|\Delta\hat\rho|^2}{\bar T+\tau\theta} + \delta\|(\hbar\varepsilon^{1/2}\Delta\rho,\hbar^2\nabla\mathrm{div}u)\|_{H^3}^2 \\
&+C\tau d_0^2\|(\nabla\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}.
\end{aligned}$$

For the third term $R_{2,3}$ on the RHS of (2.47), there involves the five order derivatives of θ, which cannot be controlled as usual. To do this, we substitute the equation (2.23c) and apply integration by parts to divide it into

$$\begin{aligned}
R_{2,3} =& -\hbar^2\int_{\Omega^R}\frac{(\bar\psi+\tau\rho)^2\Delta\hat\theta\mathrm{div}\hat u}{(\bar T+\tau\theta)^2} - \hbar^2\int_{\Omega^R}\nabla\Big(\frac{\bar\psi+\tau\rho}{\bar T+\tau\theta}\Big)^2\cdot\nabla\hat\theta\mathrm{div}\hat u \\
=& -\frac{\hbar^2}{12}\int_{\Omega^R}\frac{(\bar\psi+\tau\rho)^2\Delta\hat\theta\mathrm{div}\hat u}{(\bar T+\tau\theta)^2} - \frac{11\hbar^2}{12}\int_{\Omega^R}\frac{(\bar\psi+\tau\rho)^2\Delta\hat\theta\mathrm{div}\hat u}{(\bar T+\tau\theta)^2} \\
&-\hbar^2\int_{\Omega^R}\nabla\Big(\frac{\bar\psi+\tau\rho}{\bar T+\tau\theta}\Big)^2\cdot\nabla\hat\theta\mathrm{div}\hat u \\
=& -\frac{\hbar^2}{12}\int_{\Omega^R}\frac{(\bar\psi+\tau\rho)^2\Delta\hat\theta\mathrm{div}\hat u}{(\bar T+\tau\theta)^2} - \frac{11\hbar^2}{12\kappa}\int_{\Omega^R}\frac{(\bar\psi+\tau\rho)^4|\mathrm{div}\hat u|^2}{\bar T+\tau\theta} - \frac{11\hbar^4}{144\kappa}\int_{\Omega^R}\frac{(\bar\psi+\tau\rho)^4|\nabla\mathrm{div}\hat u|^2}{(\bar T+\tau\theta)^2} \\
&-\frac{11\hbar^2}{8\kappa}\int_{\Omega^R}\frac{(\bar\psi+\tau\rho)^4\partial_t\hat\theta\mathrm{div}\hat u}{(\bar T+\tau\theta)^2} - \frac{11\hbar^4}{144\kappa}\int_{\Omega^R}\nabla\Big(\frac{(\bar\psi+\tau\rho)^4}{(\bar T+\tau\theta)^2}\Big)\nabla\mathrm{div}\hat u\mathrm{div}\hat u \\
&-\hbar^2\int_{\Omega^R}\nabla\Big(\frac{\bar\psi+\tau\rho}{\bar T+\tau\theta}\Big)^2\cdot\nabla\hat\theta\mathrm{div}\hat u - \frac{11\hbar^2}{8\kappa}\int_{\Omega^R}\frac{(\bar\psi+\tau\rho)^4}{(\bar T+\tau\theta)^2}\Big(\hat\theta-\frac{\hbar^2}{18}\frac{\Delta\hat\rho}{\tau\rho+\bar\psi}\Big)\mathrm{div}\hat u \\
&-\frac{11\hbar^2\tau}{8\kappa}\int_{\Omega^R}\frac{(\bar\psi+\tau\rho)^3}{(\bar T+\tau\theta)^2}\Big\{u\cdot\nabla\hat\theta - \partial^\alpha\Big(\frac{\hbar^2}{9}\frac{\nabla\rho\cdot\nabla\mathrm{div}u}{\bar\psi+\tau\rho} - \frac{|u|^2}{3} - \frac{\hbar^2}{18}\frac{|\nabla\rho|^2}{(\tau\rho+\bar\psi)^2}\Big) - h_3\Big\}\mathrm{div}\hat u.
\end{aligned} \tag{2.48}$$

We first deal with the fourth term of $R_{2,3}$, it requires more effort since there is higher order derivative of $\partial_t\theta$, which is not closed by Lemma 2.6. From integration by parts, (2.23b) and the estimates (2.27), (2.38) and (2.40), we have

$$
\begin{aligned}
&-\frac{11\hbar^2}{8\kappa}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\partial_t\hat{\theta}\mathrm{div}\hat{u}\\
&=-\frac{11\hbar^2}{8\kappa}\frac{d}{dt}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\hat{\theta}\mathrm{div}\hat{u}+\frac{11\hbar^2}{8\kappa}\int_{\Omega^R}\partial_t\Big(\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\Big)\hat{\theta}\mathrm{div}\hat{u}-\frac{11\hbar^2}{8\kappa}\int_{\Omega^R}\nabla\Big(\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\hat{\theta}\Big)\partial_t\hat{u}\\
&=-\frac{11\hbar^2}{8\kappa}\frac{d}{dt}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\hat{\theta}\mathrm{div}\hat{u}+\frac{11\hbar^2}{8\kappa}\int_{\Omega^R}\partial_t\Big(\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\Big)\hat{\theta}\mathrm{div}\hat{u}-\frac{11\hbar^4}{48\kappa}\int_{\Omega^R}\nabla\Big(\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\hat{\theta}\Big)\frac{\nabla\Delta\hat{\rho}}{\bar{\psi}+\tau\rho}\\
&\quad+\frac{11\hbar^2}{8\kappa}\int_{\Omega^R}\nabla\Big(\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\hat{\theta}\Big)(\hat{u}+\nabla\hat{\theta}+2\frac{\bar{T}+\tau\theta}{\bar{\psi}+\tau\rho}\nabla\hat{\rho}+\tau u\cdot\nabla\hat{u}-h_2-\tau\partial^\alpha f^R)\\
&\le-\frac{11\hbar^2}{8\kappa}\frac{d}{dt}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\hat{\theta}\mathrm{div}\hat{u}+\delta\|(\nabla\theta,\varepsilon^{1/2}\nabla\rho,\hbar^2\nabla\mathrm{div}u,\hbar^3\nabla\Delta\rho)\|_{H^3}^2\\
&\quad+C\tau d_0^2\|(u,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u)\|_{H^3}+C\tau\|f^R\|_{H^3}^2+C\hbar\|(u,\nabla\theta,\hbar\nabla\rho)\|_{H^3}^2,
\end{aligned}
$$

where δ is some sufficiently small positive constant. By using (2.28) and Young's inequality, the other terms in $R_{2,3}$ can be bounded by

$$
\delta\|(\theta,\hbar^2\Delta\rho,\hbar^2\nabla\mathrm{div}u)\|_{H^3}^2+C\tau d_0^2\|(\nabla\theta,\hbar\nabla\rho,\hbar\mathrm{div}u)\|_{H^3}^2+C\hbar\|\hbar\mathrm{div}u\|_{H^3}^2.
$$

Hence, combining all estimates for $R_{2,3}$, we have

$$
\begin{aligned}
R_{2,3}\le&-\frac{\hbar^2}{12}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^2\Delta\hat{\theta}\mathrm{div}\hat{u}}{(\bar{T}+\tau\theta)^2}-\frac{11\hbar^2}{12\kappa}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^4|\mathrm{div}\hat{u}|^2}{\bar{T}+\tau\theta}-\frac{11\hbar^4}{144\kappa}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^4|\nabla\mathrm{div}\hat{u}|^2}{(\bar{T}+\tau\theta)^2}\\
&-\frac{11\hbar^2}{8\kappa}\frac{d}{dt}\int_{\Omega^R}\frac{(\bar{\psi}+\tau\rho)^4}{(\bar{T}+\tau\theta)^2}\hat{\theta}\mathrm{div}\hat{u}+\delta\|(\theta,\nabla\theta,\varepsilon^{1/2}\nabla\rho,\hbar^2\Delta\rho,\hbar^2\nabla\mathrm{div}u,\hbar^3\nabla\Delta\rho)\|_{H^3}^2+C\tau\|f^R\|_{H^3}^2\\
&+C\tau d_0^2\|(u,\theta,\nabla\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}+C\hbar\|(u,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u)\|_{H^3}^2.
\end{aligned}
$$

For the fourth term $R_{2,4}$ on the RHS of (2.47), it already consists of the derivatives of the velocity at the five order. And thus we use integration by parts two times to obtain

$$
\begin{aligned}
R_{2,4}=&-\hbar^2\tau\int_{\Omega^R}u\cdot\nabla\hat{u}\cdot\nabla(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\mathrm{div}\hat{u}-\hbar^2\tau\int_{\Omega^R}(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\nabla u:\nabla\hat{u}\mathrm{div}\hat{u}\\
&-\hbar^2\tau\int_{\Omega^R}(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2u\cdot\nabla\mathrm{div}\hat{u}\mathrm{div}\hat{u}\\
=&-\hbar^2\tau\int_{\Omega^R}u\cdot\nabla\hat{u}\cdot\nabla(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\mathrm{div}\hat{u}-\hbar^2\tau\int_{\Omega^R}(\frac{\bar{\psi}+\tau\rho}{\bar{T}+\tau\theta})^2\nabla u:\nabla\hat{u}\mathrm{div}\hat{u}\\
&+\frac{\hbar^2\tau}{2}\int_{\Omega^R}\mathrm{div}\Big(\frac{(\bar{\psi}+\tau\rho)^2u}{(\bar{T}+\tau\theta)^2}\Big)|\mathrm{div}\hat{u}|^2\\
\le&C\tau d_0\|\hbar\nabla u\|_{H^3}^2.
\end{aligned}
$$

For the last four terms on the RHS of (2.47), by integration by parts, the Moser inequality (4.2), and the estimates (2.27) and (2.38)-(2.40) that

$$
\sum_{i=5}^{8}R_{2,i}\le C\tau d_0^2\|(\nabla\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}+\delta\|\hbar^2\nabla\mathrm{div}u\|_{H^3}^2+C\tau\|f^R\|_{H^3}^2.
$$

Hence, adding all above estimates together, choosing $\delta,\hbar$ enough small, we complete the proof of Lemma 2.8. $\square$

In the following, for constructing the closed estimate, we need some uniform bounds for the density.

Lemma 2.9. *Let $(\rho, u, \theta) \in \partial \hat{B}_{d_0}(0)$ be a solution to system (2.20). Then there exists a constant $0 < d_0 < 1$, independent ε, R, such that, for any positive constant $m < 1$,*

$$\begin{aligned} m^2\|\nabla\rho\|_{H^2}^2 + m^2\hbar^2\|\Delta\rho\|_{H^2}^2 + m^2\frac{d}{dt}\int_{\Omega^R} \partial^\beta u \partial^\beta \nabla\rho \le Cm^2\|(u, \mathrm{div}u, \nabla\theta, \varepsilon\Delta\rho)\|_{H^2}^2 + C\tau\|f^R\|_{H^2}^2 \\ + C\tau d_0^2(\|(\nabla\rho, \nabla u)\|_{H^2} + \|\hbar\nabla\rho\|_{H^3}), \end{aligned} \tag{2.49}$$

$$\begin{aligned} m^2\hbar^2\|\nabla\rho\|_{H^3}^2 + m^2\hbar^4\|\Delta\rho\|_{H^3}^2 + m^2\hbar^2\frac{d}{dt}\int_{\Omega_R} \hat{u}\cdot\hat{\rho} \le Cm^2\|(u, \nabla\theta, \hbar\nabla u, \hbar\varepsilon^{1/2}\Delta\rho)\|_{H^3}^2 \\ + C\tau d_0^2\|(\hbar\nabla\rho, \hbar\nabla u, \hbar^2\Delta\rho)\|_{H^3} + C\tau\|f^R\|_{H^3}^2, \end{aligned} \tag{2.50}$$

and

$$\begin{aligned} & m^2\hbar^4\|\Delta\rho\|_{H^3}^2 + m^2\hbar^6\|\nabla\Delta\rho\|_{H^3}^2 + m^2\hbar^4\frac{d}{dt}\int_{\Omega_R} \mathrm{div}\hat{u}\Delta\hat{\rho} \\ & \le Cm^2\|(u, \nabla\theta, \hbar\nabla u, \hbar\varepsilon^{1/2}\Delta\rho, \hbar^2\nabla\mathrm{div}u, \hbar^2\varepsilon\nabla\Delta\rho)\|_{H^3}^2 + C\tau d_0^2\|(\hbar\nabla\rho, \hbar\nabla u, \hbar^2\Delta\rho)\|_{H^3} + C\tau\|f^R\|_{H^3}^2. \end{aligned} \tag{2.51}$$

Proof. Applying the operator ∂^β ($|\beta| = 0, 1, 2$) to (2.20b) and multiplying the result by $\partial^\beta\nabla\rho$, we have

$$\begin{aligned} & 2\int_{\Omega^R} \frac{\bar{T}+\tau\theta}{\bar{\psi}+\tau\rho}|\partial^\beta\nabla\rho|^2 + \frac{\hbar^2}{6}\int_{\Omega^R}\frac{|\partial^\beta\Delta\rho|^2}{\bar{\psi}+\tau\rho} \\ & \le -\frac{d}{dt}\int_{\Omega^R}\partial^\beta u\partial^\beta\nabla\rho + C\|(u, \mathrm{div}u, \nabla\theta, \varepsilon\Delta\rho)\|_{H^2}^2 + C\tau\|f^R\|_{H^2}^2 \\ & \quad + C\tau d_0^2(\|(\nabla\rho, \nabla u)\|_{H^2} + \|\hbar\nabla\rho\|_{H^3}), \end{aligned}$$

thanks to Lemmas 4.1 and 4.2, Young's inequality and Hölder inequality. Multiplying a suitably small positive constant m^2 on both sides, and using (2.5), we obtain (2.49).

Again, we use the denotation $(\partial^\alpha\rho, \partial^\alpha u, \partial^\alpha\theta) = (\hat{\rho}, \hat{u}, \hat{\theta})$ in Lemma 2.5 for brevity. Multiplying (2.23b) by $\hbar^2\nabla\hat{\rho}$, integrating over the periodic domain Ω^R, we have

$$\begin{aligned} & 2\hbar^2\int_{\Omega^R}\frac{\bar{T}+\tau\theta}{\bar{\psi}+\tau\rho}|\nabla\hat{\rho}|^2 + \frac{\hbar^4}{6}\int_{\Omega^R}\frac{|\Delta\hat{\rho}|^2}{\bar{\psi}+\tau\rho} \\ & \le -\hbar^2\frac{d}{dt}\int_{\Omega_R}\hat{u}\cdot\nabla\hat{\rho} + \hbar^2\int_{\Omega_R}\mathrm{div}\hat{u}\big((\bar{\psi}+\tau\rho)\mathrm{div}\hat{u} - \varepsilon\Delta\hat{\rho} - h_1 + 2\tau u\cdot\nabla\hat{\rho}\big) + C\hbar^2\|(u, \nabla\theta)\|_{H^3}^2 \\ & \quad + C\tau d_0^2\|(\hbar\nabla\rho, \hbar\nabla u, \hbar^2\Delta\rho)\|_{H^3} + C\tau\|f^R\|_{H^3}^2 \\ & \le -\hbar^2\frac{d}{dt}\int_{\Omega_R}\hat{u}\cdot\nabla\hat{\rho} + C\|(u, \nabla\theta, \hbar\nabla u, \hbar\varepsilon^{1/2}\Delta\rho)\|_{H^3}^2 + C\tau d_0^2\|(\hbar\nabla\rho, \hbar\nabla u, \hbar^2\Delta\rho)\|_{H^3} + C\tau\|f^R\|_{H^3}^2, \end{aligned}$$

thanks to (4.4) and the estimate (2.27). Taking the summation for all $|\alpha| = 3, |\beta| = 2$ and combining (2.5), we prove (2.50).

Next, multiplying (2.23b) by $-\hbar^4\nabla\Delta\hat{\rho}$, and integrating over the periodic domain Ω^R, we get

$$\begin{aligned} & \hbar^4\|\Delta\rho\|_{H^3}^2 + \hbar^6\|\nabla\Delta\rho\|_{H^3}^2 \\ & \le -\hbar^4\frac{d}{dt}\int_{\Omega_R}\mathrm{div}\hat{u}\Delta\hat{\rho} - \hbar^4\int_{\Omega_R}\nabla\partial_t\hat{\rho}\nabla\mathrm{div}\hat{u} + C\|(u, \nabla\theta)\|_{H^3}^2 + C\tau d_0^2\|(\hbar\nabla\rho, \hbar\nabla u, \hbar^2\Delta\rho)\|_{H^3} \\ & \quad + C\tau\|f^R\|_{H^3}^2 \\ & \le -\hbar^4\frac{d}{dt}\int_{\Omega_R}\mathrm{div}\hat{u}\Delta\hat{\rho} + C\|(u, \nabla\theta, \hbar^2\nabla\mathrm{div}u, \hbar^2\varepsilon\nabla\Delta\rho)\|_{H^3}^2 + C\tau d_0^2\|(\hbar\nabla\rho, \hbar\nabla u, \hbar^2\Delta\rho)\|_{H^3} \\ & \quad + C\tau\|f^R\|_{H^3}^2, \end{aligned}$$

thanks to (2.23a). Then Lemma 2.9 is complete. □

By using these lemmas and the topological degree theory, we will establish the end of the proof of Proposition 2.1

Proof. Combining Lemma 2.5, Lemma 2.7, Lemma 2.8 and Lemma 2.9, and integrating the result from 0 to T^*, we obtain, for some suitably small positive constants m, ε and $\hbar$

$$\begin{aligned}
&\int_0^{T^*}\Big(\|\nabla\rho\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}^2+\|(\varepsilon^{\frac12}\nabla\rho,\hbar\varepsilon^{\frac12}\Delta\rho,\hbar^2\varepsilon\nabla\Delta\rho)\|_{H^3}^2\\
&\quad+\|(\hbar^2\nabla\mathrm{div}u,\hbar^3\nabla\Delta\rho)\|_{H^3}^2\Big)dt\\
&\le C\tau\|f^R\|_{L^2(0,T^*;H^3)}^2+C\tau d_0^2\Big(\int_0^{T^*}\|\nabla\rho\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}^2dt\Big)^{\frac12}\\
&\le C\tau\|f^R\|_{L^2(0,T^*;H^3)}^2+C\tau d_0^3,
\end{aligned}\tag{2.52}$$

where $i=0,1$. Here, we have used the curl-div decomposition of the gradient

$$\|\nabla u\|_{H^3}\le C\|\nabla u\|_{H^3}+C\|\mathrm{div}u\|_{H^3}.$$

From (2.52), there exists $t_0\in(0,T^*)$ such that

$$\begin{aligned}
&\|\nabla\rho(t_0)\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)(t_0)\|_{H^3}^2+\|(\varepsilon^{\frac12}\nabla\rho,\hbar\varepsilon^{\frac12}\Delta\rho,\hbar^2\varepsilon\nabla\Delta\rho)(t_0)\|_{H^3}^2\\
&\quad+\|(\hbar^2\nabla\mathrm{div}u,\hbar^3\nabla\Delta\rho)(t_0)\|_{H^3}^2\le\frac{C\tau}{T^*}\|f^R\|_{L^2(0,T^*;H^3)}^2+\frac{C\tau}{T^*}d_0^3.
\end{aligned}\tag{2.53}$$

Combining (2.22), (2.41) with (2.46), we have

$$\begin{aligned}
&\frac{d}{dt}\big(\|\nabla\rho\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)\|_{H^3}^2\big)\\
&\quad\le C\tau\|f^R\|_{H^3}^2+C\tau d_0^2\big(\|\nabla\rho\|_{H^2}+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)\|_{H^3}^2\big),
\end{aligned}\tag{2.54}$$

where we take $m,\delta,\varepsilon,\hbar$ suitably small (namely $0<\varepsilon\ll\hbar,\delta\ll m<1$) such that those terms without τ ahead disappear. Integrating (2.54) over t_0 to t ($t_0<t\le T^*$) and using (2.53), we deduce that

$$\begin{aligned}
&\|\nabla\rho(t)\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t)\|_{H^3}^2\\
&\quad\le\|\nabla\rho(t_0)\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t_0)\|_{H^3}^2+C\tau\|f^R\|_{L^2(0,T^*;H^3)}^2+C\tau d_0^3\\
&\quad\le C\tau\|f^R\|_{L^2(0,T^*;H^3)}^2+C\tau d_0^3.
\end{aligned}\tag{2.55}$$

Moreover, according to

$$\begin{aligned}
&\|\nabla\rho(0)\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(0)\|_{H^3}^2\\
&\quad=\|\nabla\rho(T^*)\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(T^*)\|_{H^3}^2\\
&\quad\le C\tau\|f^R\|_{L^2(0,T^*;H^3)}^2+C\tau d_0^3,
\end{aligned}\tag{2.56}$$

there holds

$$\sup_{t\in[0,T^*]}\big(\|\nabla\rho(t)\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t)\|_{H^3}^2\big)\le C\tau\|f^R\|_{L^2(0,T^*;H^3)}^2+C\tau d_0^3.\tag{2.57}$$

From (2.52) and (2.57), we have

$$\begin{aligned}
|\!|\!|(\rho,u,\theta)|\!|\!|^2\le&C\tau\|f^R\|_{L^2(0,T^*;H^3)}^2+C\tau d_0^3\\
\le&C\lambda^2+Cd_0^3\le\frac{d_0^2}{2},
\end{aligned}\tag{2.58}$$

when d_0 and λ are sufficiently small such that $C\lambda^2+Cd_0^3\le d_0^2/2$. Thus, we prove the condition (2.21) of the Leray-Schauder degree theory. Then based on Lemmas 2.3 and 2.4, we obtain the existence of time periodic solutions to the approximated system (2.1) in a sequence of bounded domain. Recalling the result $\chi((\tilde{\rho},\tilde{u},\tilde{\theta}),0)=0$ in the proof of Lemma 2.1, we have

$$\deg(I-\chi(\cdot,1),\hat{B}_{d_0}(0),0)=\deg(I-\chi(\cdot,0),\hat{B}_{d_0}(0),0)=\deg(I,\hat{B}_{d_0}(0),0)=1. \tag{2.59}$$

This completes the proof of Proposition 2.1. □

3. The existence and uniqueness of a time periodic solution in $\mathbb{R}^3$

3.1. **The proof of Theorem 1.1.** In this section, we focus on proving the existence of time periodic solutions in $\mathbb{R}^3$ by passing the limit in the approximated system (2.1).

Proof. To study the convergence as $\varepsilon\to 0,\ R\to\infty$, we denote the solution to the approximate system (2.1) by $(\rho_\varepsilon^R,u_\varepsilon^R,\theta_\varepsilon^R)$. From (2.58), we have

$$|||(\rho_\varepsilon^R,u_\varepsilon^R,\theta_\varepsilon^R)|||\le Cd_0,$$

where d_0 is independent of ε and R. Integrating (2.54) from t to $t+h$, and then integrating the result over $[0,T^*]$, we have

$$\begin{aligned}\int_0^{T^*}&\big\{\big(\|\nabla\rho(t+h)\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t+h)\|_{H^3}^2\big)\\&-\big(\|\nabla\rho(t)\|_{H^2}^2+\|(u,\theta,\nabla\theta,\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t)\|_{H^3}^2\big)\big\}dt\le C|d_0|.\end{aligned}$$

By the strong compactness of L^p, there exist the subsequence $\{(\rho_{\varepsilon_n}^R,u_{\varepsilon_n}^R,\theta_{\varepsilon_n}^R)\}_{n=1}^\infty$ and the solution $(\rho^R,u^R,\theta^R)\in X_{d_0}^R$, such that

$$(\rho_{\varepsilon_n}^R,u_{\varepsilon_n}^R,\theta_{\varepsilon_n}^R)\to(\rho^R,u^R,\theta^R)\qquad\text{strongly in } L^2(0,T^*;L^6(\Omega^R)),$$

and

$$(\nabla\theta_{\varepsilon_n}^R,\hbar\nabla\rho_{\varepsilon_n}^R,\hbar\nabla u_{\varepsilon_n}^R,\hbar^2\Delta\rho_{\varepsilon_n}^R)\to(\nabla\theta^R,\hbar\nabla\rho^R,\hbar\nabla u^R,\hbar^2\Delta\rho^R), \tag{3.1}$$

strong in $L^2(0,T^*;H^3(\Omega^R))$;

$$(\nabla u_{\varepsilon_n}^R,\nabla\rho_{\varepsilon_n}^R)\to(\nabla\rho^R,\nabla u^R), \tag{3.2}$$

strong in $L^2(0,T^*;H^2(\Omega^R))$.

Then, we claim

$$\|(\rho_\varepsilon^R,u_\varepsilon^R,\theta_\varepsilon^R)\|_{C^{\frac18,\frac12}\left((0,T^*)\times\Omega^R\right)}\le Cd_0.$$

With the aid of the fact $\rho_\varepsilon^R\in L^\infty(0,T^*,W^{1,6})$ and Sobolev embedding theorem, it is easy to see that $\rho_\varepsilon^R\in C^{\frac12}$, for any $t\in(0,T^*)$. Denote a ball centered at $x\in\Omega^R$ with radius $r=|t_2-t_1|^a$ $(0<t_1\le t_2<T^*)$ as B_r. From the condition $\partial_t\rho_\varepsilon^R\in L^2(0,T^*,L^2)$, there holds

$$\begin{aligned}&\int_{B_r}|\rho_\varepsilon^R(x,t_1)-\rho_\varepsilon^R(x,t_2)|dx\\&\le\int_{B_r}|\int_{t_1}^{t_2}\partial_t\rho_\varepsilon^R dt|dx\\&\le|(\int_{t_1}^{t_2}\int_{B_r}|\partial_t\rho_\varepsilon^R|^2dt)^{\frac12}(\int_{t_1}^{t_2}\int_{B_r}1^2dtdx)^{\frac12}|\\&\le C|t_1-t_2|^{\frac12+\frac{an}{2}},\end{aligned} \tag{3.3}$$

which leads, there exists a point $\tilde{x}\in B_r$, such that

$$|\rho_\varepsilon^R(\tilde{x},t_1)-\rho_\varepsilon^R(\tilde{x},t_2)|\le C|t_1-t_2|^{\frac12-\frac{an}{2}}.$$

Consequently, for any $x \in \Omega^R$,

$$\begin{aligned}|\rho_\varepsilon^R(x,t_1)-\rho_\varepsilon^R(x,t_2)| \leq & |\rho_\varepsilon^R(x,t_1)-\rho_\varepsilon^R(\tilde{x},t_1)|+|\rho_\varepsilon^R(\tilde{x},t_1)-\rho_\varepsilon^R(\tilde{x},t_2)|+|\rho_\varepsilon^R(\tilde{x},t_2)-\rho_\varepsilon^R(x,t_2)| \\ \leq & 2C|t_1-t_2|^{\frac{a}{2}}+C|t_1-t_2|^{\frac{1}{2}-\frac{an}{2}}.\end{aligned}$$

Setting $a=\frac{1}{4}$, we have

$$|\rho_\varepsilon^R(x,t_1)-\rho_\varepsilon^R(x,t_2)| \leq C|t_1-t_2|^{\frac{1}{8}}.$$

Thus, we deduce, for any $x,y \in \Omega^R$ and $t,s \in (0,T^*)$,

$$|\rho_\varepsilon^R(x,t)-\rho_\varepsilon^R(y,s)| \leq C|t-s|^{\frac{1}{8}}+C|x-y|^{\frac{1}{2}}.$$

The process to deal with the variables u_ε^R and θ_ε^R is similar. Therefore, by Arzela-Ascoli theorem,

$$(\rho_{\varepsilon_n}^R, u_{\varepsilon_n}^R, \theta_{\varepsilon_n}^R) \to (\rho^R, u^R, \theta^R) \qquad \text{uniformly.}$$

Thus as $\varepsilon \to 0$ ($n \to \infty$), we can conclude that the limit function (ρ^R, u^R, θ^R) satisfies system (1.4). This completes the proof for the existence of time periodic solutions of system (1.4) in the periodic domain Ω^R.

We are now in a position to construct a subsequence $(\rho^{R_k}, u^{R_k}, \theta^{R_k})$ such that $(\rho^{R_k}, u^{R_k}, \theta^{R_k})$ converges in Ω^{R_k}, and $(\rho^{R_{k+1}}, u^{R_{k+1}}, \theta^{R_{k+1}}) \subseteq (\rho^{R_k}, u^{R_k}, \theta^{R_k})$ converges in $\Omega^{R_{k+1}}$. Repeating the above process and combining the diagonal argument, we are able to choose a Cantor diagonal subsequence $(\rho^{R_n}, u^{R_n}, \theta^{R_n})$ such that $(\rho^{R_n}, u^{R_n}, \theta^{R_n}) \to (\rho, u, \theta)$ as $n \to \infty$ in $\mathbb{R}^3$. Consequently, we extend the time periodic classical solutions of system (1.4) to the whole space $\mathbb{R}^3$. This complete the proof. □

3.2. **The proof of Theorem 1.2.** Our next object is to prove the uniqueness of the time periodic solutions. Assume (ρ_1, u_1, θ_1) and (ρ_2, u_2, θ_2) are the time periodic solutions to system (1.4). Setting $q=\rho_1-\rho_2,\ v=u_1-u_2,\ \vartheta=\theta_1-\theta_2$, we have (q,v,ϑ) satisfies the following

$$\begin{cases} 2\partial_t q+\bar{\psi}\mathrm{div}v=-q\mathrm{div}u_1-\rho_2\mathrm{div}v-2v\cdot\nabla\rho_1-2u_2\cdot\nabla q, & (3.4a)\\ \partial_t v+\nabla\vartheta+2\gamma\nabla q-\dfrac{\hbar^2}{6}\dfrac{\nabla\Delta q}{\rho_2+\bar{\psi}}+v=-v\cdot\nabla u_1-u_2\cdot\nabla v-2\big(\tilde{g}(\rho_1,\theta_1)-\tilde{g}(\rho_2,\theta_2)\big)\nabla\rho_1 & \\ \quad -2\tilde{g}(\rho_2,\theta_2)\nabla q-\dfrac{\hbar^2}{6}\dfrac{q\nabla\Delta\rho_1}{(\rho_1+\bar{\psi})(\rho_2+\bar{\psi})}+\dfrac{\hbar^2}{6}(\bar{g}(\rho_1)-\bar{g}(\rho_2))\Delta\rho_1+\dfrac{\hbar^2}{6}\bar{g}(\rho_2))\Delta q, & (3.4b)\\ \partial_t\vartheta+\dfrac{2}{3}\bar{T}\mathrm{div}v-\dfrac{2\kappa}{3}\dfrac{\Delta\vartheta}{(\rho_2+\bar{\psi})^2}-\dfrac{\hbar^2}{18}\mathrm{div}\Delta u=-v\cdot\nabla\theta_1-u_2\cdot\nabla\vartheta-\dfrac{2}{3}\vartheta\mathrm{div}u_1-\dfrac{2}{3}\theta_2\mathrm{div}v & \\ \quad +\dfrac{\hbar^2}{9}\big((\rho_1+\bar{\psi})\bar{g}(\rho_1)-(\rho_2+\bar{\psi})\bar{g}(\rho_2)\big)\nabla\mathrm{div}u_1+\dfrac{\hbar^2}{9}(\rho_2+\bar{\psi})\bar{g}(\rho_2)\nabla\mathrm{div}v & \\ \quad -\dfrac{2\kappa}{3}\dfrac{(2\bar{\psi}+\rho_1+\rho_2)q\Delta\theta_1}{(\rho_1+\bar{\psi})^2(\rho_2+\bar{\psi})^2}, & (3.4c)\end{cases}$$

where

$$\tilde{g}(\theta,\rho)=\frac{\theta-\gamma\rho}{\bar{\psi}+\rho},\ \bar{g}(\rho)=\frac{\nabla\rho}{(\rho+\bar{\psi})^2}.$$

We can prove the uniqueness of system (1.4) by similar energy estimates as Sect. 2 when $f^R=0$. Thus, choosing a suitably small d_0, we can conclude

$$\int_0^{T^*}\Big(\|\nabla q\|_{H^2}^2+\|(v,\vartheta,\nabla\vartheta,\hbar\nabla q,\hbar\nabla v,\hbar^2\Delta q)\|_{H^3}^2\Big)dt \leq 0. \tag{3.5}$$

Combining the fact $(\rho_1,u_1,\theta_1)=(\rho_2,u_2,\theta_2)$ at infinite, we deduce $(q,v,\vartheta)=0$. This complete the proof of Theorem 1.2.

4. Appendix

For reader's convenience, we list some inequalities which will be frequently in the proof of Theorem 2.1.

Lemma 4.1. *Let $p, q, r \geq 1$ be some positive integers. Assuming $0 \leq a, m \leq l$, then we have, for some generic constant $c \in [0, 1]$,*

$$\|\nabla^a f\|_{L^p} \leq C\|\nabla^l f\|_{L^q}^c \|\nabla^m f\|_{L^r}^{1-c}, \tag{4.1}$$

where

$$\frac{a}{3} - \frac{1}{p} = (\frac{l}{3} - \frac{1}{q})c + (\frac{m}{3} - \frac{1}{r})(1 - c).$$

Provided that $c = 1$, we have $l - a \neq 3/q$.

Next, we give the Moser type calculus inequality

Lemma 4.2. *Let α be any multi-index with $|\alpha| = k$, $k \geq 1$. Then one has*

$$\|[\partial^\alpha, f]g\|_{L^p} \leq C\|\nabla f\|_{L^q}\|\partial^{\alpha-1} g\|_{L^r} + C\|\partial^\alpha f\|_{L^s}\|g\|_{L^t}, \tag{4.2}$$

and

$$\|\partial^\alpha (fg)\|_{L^p} \leq C\|f\|_{L^q}\|\partial^\alpha g\|_{L^r} + C\|\partial^\alpha f\|_{L^s}\|g\|_{L^t}, \tag{4.3}$$

where $f, g \in \mathbb{S}$, $p, r, s \in (1, \infty)$ that satisfies $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} = \frac{1}{s} + \frac{1}{t}$.

Lemma 4.3. *Let α be a multi-index and $f \in \mathbb{S}$, it holds*

$$\|\partial^\alpha \nabla^2 f\|_{L^2} \leq C\|\partial^\alpha \Delta f\|_{L^2}. \tag{4.4}$$

Proof. By the Riesz operator R_j, $\widehat{(R_j f)} = \frac{i\xi_j}{|\xi|}\widehat{f}$, where $R_i R_j$ is bounded from L^p to L^p with $1 < p < \infty$, we conclude

$$\|\partial^\alpha \nabla^2 f\|_{L^2} = \|\partial^\alpha \nabla \Delta^{-1} \nabla \Delta f\|_{L^2} = \|\partial^\alpha R_i R_j \Delta f\|_{L^2} \leq C\|\partial^\alpha \Delta f\|_{L^2}.$$

□

Lemma 4.4. *Assume that f, g be any vector functions. There holds*

$$f \cdot \nabla g = (\nabla \times g) \times f + (\nabla \times f) \times g + \nabla(f \cdot g) - \nabla f \cdot g, \tag{4.5}$$

$$\nabla \times (\nabla \times f) = \nabla(\mathrm{div} f) - \Delta f, \tag{4.6}$$

and

$$\nabla \times (f \times g) = f \mathrm{div} g - g \mathrm{div} f + (g \cdot \nabla) f - (f \cdot \nabla) g. \tag{4.7}$$

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