BEST PROXIMITY POINT RESULTS FOR MIXED MULTIVALUED MAPPINGS WITH APPLICATION TO HOMOTOPY THEORY

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ABSTRACT. In this paper, first, we introduce a concept of mixed multivalued contraction mapping. Then, we present some best proximity point results for such mappings on 0-complete partial metric spaces. Hence, we extend and generalize some famous and nice results existing in the literature such as Abkar and Gabeleh [2], Gabeleh [11] and Aydi et al. [5]. Also, we provide some nontrivial illustrative examples to support our results and to compare with the results mentioned before. Finally, the first time, we give some applications to homotopy theory via new best proximity point results. Hence, we obtain some best proximity point results for homotopic mappings

1. INTRODUCTION AND PRELIMINARIES

In 1922, Banach [8] proved a very important result, known as Banach contraction principle, for fixed point theory on metric spaces. Because of its applicable in various fields of nonlinear analysis and applied mathematical analysis, this principle has been generalized and extended in different ways [13, 19]. One of the interesting and famous generalizations was proved by Nadler [17] for multivalued mappings on metric spaces as follows:

Theorem 1 ([17]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. If there exists $k \in [0, 1)$ such that*

$$H_d(T\xi, T\eta) \leq kd(\xi, \eta)$$

for all $\xi, \eta \in X$, where $CB(X)$ is the family of all nonempty closed and bounded subsets of X and H_d is a Hausdorff metric with respect to d on $CB(X)$. Then, T has a fixed point in X.

Later, a number of interesting fixed point theorems for multivalued mappings have been obtained. In this sense, Mizoguchi and Takahashi [16] proved a famous and nice generalization of Nadler's fixed point theorem which is a partial answer of Problem 9 in Reich [18]:

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Theorem 2 ([16]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Assume that*

$$H_d(T\xi, T\eta) \leq \mu(d(\xi, \eta))d(\xi, \eta)$$

for all $\xi, \eta \in X$, where $\mu : [0, \infty) \to [0, 1)$ is a function satisfying $\limsup_{s\to\gamma^+} \mu(s) < 1$ for all $\gamma \geq 0$. Then, T has a fixed point in X.

A function $\mu : [0, \infty) \to [0, 1)$ satisfying $\limsup_{s\to\gamma^+} \mu(s) < 1$ for all $\gamma \geq 0$ is called MT-function in the literature. The set of all MT-function μ will be denoted by Φ. Note that, each nondecreasing (nonincreasing) function is a MT-function and so the class of MT-functions is considerable.

Another generalization of Banach contraction principle was obtained by Matthews [15] introducing a new concept so called partial metric to study of denotational semantics of dataflow networks. Then, many fixed point results for single valued and multivalued mappings in the settings of partial metric spaces have been obtained in various ways[1, 10, 20]..

Now, we recall definition of the partial metric space and its topological properties.

Definition 1 ([15]). *Let X be a nonempty set and $\sigma : X \times X \to \mathbb{R}^+$ (nonnegative real numbers) be a function. Then, σ is said to be a partial metric if the following conditions hold:*

p1) $\sigma(\xi, \xi) = \sigma(\xi, \eta) = \sigma(\eta, \eta)$ *if and only if* $\xi = \eta$
p2) $\sigma(\xi, \xi) \leq \sigma(\xi, \eta)$
p3) $\sigma(\xi, \eta) = \sigma(\eta, \xi)$
p4) $\sigma(\xi, \zeta) \leq \sigma(\xi, \eta) + \sigma(\eta, \zeta) - \sigma(\eta, \eta)$

for all $\xi, \eta, \zeta \in X$. The pair (X, σ) is called partial metric space.

It is clear that every metric space is a partial metric space, but the converse may not be true. A well-known example of partial metric spaces shows this fact. Indeed, let $X = \mathbb{R}^+$ and $\sigma : X \times X \to \mathbb{R}^+$ be a function defined as $\sigma(\xi, \eta) = \max\{\xi, \eta\}$ for all $\xi, \eta \in X$. Then, (X, σ) is a partial metric space, but it is not a metric space. For other examples of partial metric spaces, we refer to [3, 4, 14].

Each partial metric σ on X generates T_0 topology τ_σ which has as a base the family open σ-balls $\{B_\sigma(\xi, \varepsilon) : \xi \in X,\ \varepsilon > 0\}$ where

$$B_\sigma(\xi, \varepsilon) = \{\eta \in X : \sigma(\xi, \eta) < \sigma(\xi, \xi) + \varepsilon\}$$

for all $\xi \in X$ and $\varepsilon > 0$.

Let (X, σ) be a partial metric space, $\{\xi_n\}$ be a sequence in X and $\xi \in X$. Then, the sequence $\{\xi_n\}$ converges to ξ with respect to τ_σ iff $\lim_{n\to\infty} \sigma(\xi_n, \xi) = \sigma(\xi, \xi)$. Further, (X, σ) is said to be complete if every Cauchy sequence $\{\xi_n\}$ in X converges with respect to τ_σ to a point $\xi \in X$ such that $\sigma(\xi, \xi) = \lim_{m,n\to\infty} \sigma(\xi_n, \xi_m)$. Recall that the sequence $\{\xi_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} \sigma(\xi_n, \xi_m)$ exists and is finite.

Romaguera [20] introduced the concept of 0-complete partial metric space. Hence, a weaker form of completeness in partial metric spaces has been obtained.

Definition 2 ([20]). *Let (X, σ) be a partial metric space and $\{\xi_n\}$ be a sequence in X.*

(i) *$\{\xi_n\}$ is called 0-Cauchy sequence if*

$$\lim_{n,m\to\infty} \sigma(\xi_n, \xi_m) = 0.$$

(ii) *(X, σ) is 0-complete partial metric space if every 0-Cauchy sequence converges to a point ξ in X with respect to τ_σ such that*

$$\lim_{n,m\to\infty} \sigma(\xi_n, \xi_m) = \sigma(\xi, \xi) = 0.$$

It is clear that every 0-Cauchy sequence in X is a Cauchy sequence and so every complete partial metric space is 0-complete. But, the converse may not be true. Indeed, let us consider the set $X = \mathbb{Q} \cap [0, \infty)$ is endowed with the partial metric defined as $\sigma(\xi, \eta) = \max\{\xi, \eta\}$ for all $\xi, \eta \in X$. Then, (X, σ) is a 0-complete partial metric space which is not a complete.

If (X, σ) is a partial metric space, then the function $d_\sigma : X \times X \to \mathbb{R}^+$ defined by

$$d_\sigma(\xi, \eta) = 2\sigma(\xi, \eta) - \sigma(\xi, \xi) - \sigma(\eta, \eta)$$

is a metric on X.

Now, we give the relations between partial metric space (X, σ) and corresponding metric space (X, d_σ) which are important for our main results.

Lemma 1 ([15]). *Let (X, σ) be a partial metric space.*

(i) *$\{\xi_n\}$ is a Cauchy sequence in (X, σ) if and only if $\{\xi_n\}$ is a Cauchy sequence in (X, d_σ)*

(ii) *(X, σ) is a complete partial metric space if and only if (X, d_σ) is a complete metric space*

(iii) *Given a sequence $\{\xi_n\}$ in X and $\xi \in X$. Then, we have*

$$\lim_{n\to\infty} d_\sigma(\xi_n, \xi) = 0 \Longleftrightarrow \sigma(\xi, \xi) = \lim_{n\to\infty} \sigma(\xi_n, \xi) = \lim_{n,m\to\infty} \sigma(\xi_n, \xi_m).$$

Using Lemma 1 (iii), it can be easily seen that

$$\lim_{n\to\infty} d_\sigma(\xi_n, \xi) = 0 \text{ and } \lim_{n\to\infty} d_\sigma(\eta_n, \eta) = 0 \Longrightarrow \lim_{n\to\infty} \sigma(\xi_n, \eta_n) = \sigma(\xi, \eta).$$

The following lemma is very useful in our main results

Lemma 2. *Let (X, σ) be a partial metric space and $\emptyset \neq \Omega \subseteq \dot{X}$. Then, we have*

$$\xi \in \overline{\Omega} \Longleftrightarrow \sigma(\xi, \Omega) = \sigma(\xi, \xi)$$

where $\overline{\Omega}$ is closure of Ω with respect to τ_σ.

In the rest of paper, the set of all nonempty closed and bounded subsets of (X,σ) will be denoted by $CB^{\sigma}(X)$.

Aydi et al. [5] defined partial Hausdorff metric of (X,σ) on $CB^{\sigma}(X)$. The function

$$H_{\sigma}: CB^{\sigma}(X)\times CB^{\sigma}(X)\to\mathbb{R}^{+}$$

defined by

$$H_{\sigma}(\Omega,\Psi)=\max\left\{\sup_{a\in\Omega}\sigma(a,\Psi),\sup_{b\in\Psi}\sigma(b,\Omega)\right\}$$

for all $\Omega,\Psi\in CB^{\sigma}(X)$ where $\sigma(\xi,\Omega)=\inf\{\sigma(\xi,\eta):\eta\in\Omega\}$ is said to be the partial Hausdorff metric of (X,σ).

The properties of partial Hausdorff metric were given in [5] as follows:

Lemma 3. *Let (X,σ) be a partial metric space. For all $\Omega,\Psi\in CB^{\sigma}(X)$, we have*

(i) $H_{\sigma}(\Omega,\Omega)\leq H_{\sigma}(\Omega,\Psi)$,
(ii) $H_{\sigma}(\Omega,\Psi)=H_{\sigma}(\Psi,\Omega)$,
(iii) $H_{\sigma}(\Omega,\Psi)\leq H_{\sigma}(\Omega,C)+H_{\sigma}(C,\Psi)-inf_{c\in C}\sigma(c,c)$,
(iv) $H_{\sigma}(\Omega,\Psi)=0$ *implies* $\Omega=\Psi$.

Then, they obtained the following fixed point theorem for multivalued mapping on complete partial metric space and so generalized Nadler's fixed point theorem [17]

Theorem 3 ([5])**.** *Let (X,σ) be a complete partial metric space and $T: X\to CB^{\sigma}(X)$ be a multivalued mapping. If there exists $k\in(0,1)$ such that*

$$H_{\sigma}(T\xi,T\eta)\leq k\sigma(\xi,\eta)$$

for all $\xi,\eta\in X$, then T has a fixed point in X.

Note that, since τ_{σ} may not be a T_1-space, Theorem 3 is not a generalization for single-valued mappings on partial metric spaces unlike in the settings of metric spaces. Moreover, Romaguera [21] showed that $CB^{\sigma}(X)$ may be empty in the following example.

Example 1. *Let $X=[0,\infty)$ and $\sigma: X\times X\to\mathbb{R}^{+}$ be a function defined as $\sigma(\xi,\eta)=\max\{\xi,\eta\}$ for all $\xi,\eta\in X$. Then, (X,σ) is a partial metric space. Now, consider the closed subset Ω of (X,σ). In this case, $\Omega=[\xi,\infty)$ for some $\xi\in X$. However, it can be easily seen that the set Ω is unbounded. Hence, each closed subset Ω of X is unbounded, that is, $CB^{\sigma}(X)=\emptyset$.*

To remedy this big problem, Romaguera [21] introduced the notion of a mixed multivalued mapping $T: X\to X\cup CB^{\sigma}(X)$ on a partial metric space (X,σ). According to this new notion, $T\xi$ is singleton ($|T\xi|=1$) or $T\xi\in CB^{\sigma}(X)$ for all $\xi\in X$. For any subset Ω of X, image of Ω under the mixed multivalued mapping T is defined as

$$T(\Omega)=\bigcup_{\xi\in\Omega}T\xi.$$

On the other hand, recently, taking into account nonself mappings, Basha and Veeramani [7] introduced the concept of best proximity point. Let (X, d) be a metric space, $\Omega, \Psi \subseteq X$ and $T : \Omega \to \Psi$ be a mapping. If $\Omega \cap \Psi = \emptyset$, then T cannot have a fixed point which leads to think existence of an approximate solution ξ for the minimization problem $\min_{\xi\in\Omega} d(\xi, T\xi)$. By this motivation, the best approximation theory and the best proximity point theory have been arisen. Although best approximation theorems do not guarantee the existence $\xi \in X$ such that $d(\xi, T\xi)$ is minimum, best proximity point theorems ensure an optimal solution of the problem mentioned before and so an approximate solution of the equation $T\xi = \xi$. Recall a point ξ is called best proximity point of T if $d(\xi, T\xi) = d(\Omega, \Psi)$. It is clear that every best proximity point is a fixed point whenever $\Omega = \Psi = X$. Since best proximity point theorems are natural extension of fixed point results, the best proximity point theory have been studied in various ways on generalized metric spaces by many authors [6, 22]. Now, we remember the basic concepts and notations of best proximity point theory in the settings of partial metric spaces.

Let (X, σ) be a partial metric space, Ω, Ψ be nonempty subsets of X and $T : \Omega \to \Psi$ be a mapping. We regard the following subsets of Ω and Ψ, respectively:

$$\Omega_0 = \{\xi \in \Omega : \sigma(\xi, \eta) = \sigma(\Omega, \Psi) \text{ for some } \eta \in \Psi\}$$

and

$$\Psi_0 = \{\eta \in \Psi : \sigma(\xi, \eta) = \sigma(\Omega, \Psi) \text{ for some } \xi \in \Omega\}$$

where $\sigma(\Omega, \Psi) = \inf \{\sigma(\xi, \eta) : \xi \in \Omega \text{ and } \eta \in \Psi\}$.

Definition 3 ([12]). *Let (X, σ) be a partial metric space, $\Omega, \Psi \subseteq X$, and $\Omega_0 \neq \emptyset$. Then the pair (Ω, Ψ) is said to have the weak σ-Property if and only if*

$$\left.\begin{array}{l} \sigma(\xi_1, \eta_1) = \sigma(\Omega, \Psi) \\ \sigma(\xi_2, \eta_2) = \sigma(\Omega, \Psi) \end{array}\right\} \Longrightarrow \sigma(\xi_1, \xi_2) \leq \sigma(\eta_1, \eta_2)$$

for all $\xi_1, \xi_2 \in \Omega_0$ and $\eta_1, \eta_2 \in \Psi_0$.

In this paper, we first introduce a concept of mixed multivalued contraction mapping. Then, we present some best proximity point results for such mappings on 0-complete partial metric spaces. Hence, we extend and generalize some famous and nice results existing in the literature such as Abkar and Gabeleh [2], Gabeleh [11] and Aydi et al. [5]. Also, we provide some nontrivial illustrative examples to support our results and to compare with the results mentioned before. Finally, the first time, we give some applications to homotopy theory via new best proximity point results. Hence, we obtain some best proximity point results for homotopic mappings

2. Main Results

We begin to the following very important lemmas to obtain best proximity point results for multivalued mappings on partial metric spaces.

Lemma 4 ([5]). *Let (X,σ) be a partial metric space and $\Omega,\Psi \in CB^{\sigma}(X)$. Then for each $a \in \Omega$ and $\varepsilon > 0$, there exists $b \in \Psi$ such that*

$$\sigma(a,b) \leq H_{\sigma}(\Omega,\Psi) + \varepsilon.$$

Lemma 5 ([9]). *Let $\mu : [0,\infty) \to [0,1)$ be an MT-function, then the function $\beta : [0,\infty) \to [0,1)$ defined as $\beta(\gamma) = \frac{1+\mu(\gamma)}{2}$ is also an MT-function.*

Now, we introduce a concept of mixed multivalued contraction mapping.

Definition 4. *Let (X,σ) be a partial metric space, Ω,Ψ be subsets of X and $T : \Omega \to \Psi \cup CB^{\sigma}(\Psi)$ be a mixed multivalued mapping. The mapping T is said to be a mixed multivalued contraction mapping if there exists $\mu \in \Phi$ such that*

$$H_{\sigma}(T\xi, T\eta) \leq \mu(\sigma(\xi,\eta))\sigma(\xi,\eta) \tag{2.1}$$

for all $\xi,\eta \in \Omega$.

Theorem 4. *Let (X,σ) be a 0-complete partial metric space, $\Omega,\Psi \subseteq X$ be two nonempty closed subsets of X and $\Omega_0 \neq \emptyset$. Assume that $T : \Omega \to \Psi \cup CB^{\sigma}(\Psi)$ is a mixed multivalued contraction mapping satisfying $T(\Omega_0) \subseteq \Psi_0$ and the pair (Ω,Ψ) has the weak σ-Property. Then, T has a best proximity point ξ^* in Ω. Moreover, $\sigma(\xi^*,\xi^*) = 0$.*

Proof. First, we define $\beta : [0,\infty) \to [0,1)$ by

$$\beta(\gamma) = \frac{\mu(\gamma)+1}{2}$$

for all $\gamma \geq 0$. Then, from Lemma 5, β is also an MT-function. Now, by taking an arbitrary point $\xi_0 \in \Omega_0$ we consider the following two cases:

Case 1. Let $|T\xi_0| = 1$. Then, there exists $\eta_0 \in \Psi$ such that $\eta_0 = T\xi_0$. Since $\eta_0 = T\xi_0 \in T(\Omega_0) \subseteq \Psi_0$, there exists $\xi_1 \in \Omega_0$ such that

$$\sigma(\xi_1,\eta_0) = \sigma(\Omega,\Psi). \tag{2.2}$$

If $\xi_0 = \xi_1$, then ξ_0 is a best proximity point of T. Moreover, since

$$\begin{aligned} \sigma(\xi_0,\xi_0) &\leq \sigma(\eta_0,\eta_0) \\ &= H_{\sigma}(T\xi_0, T\xi_0) \\ &\leq \mu(\sigma(\xi_0,\xi_0))\sigma(\xi_0,\xi_0), \end{aligned}$$

we get $\sigma(\xi_0,\xi_0) = 0$. Assume $\xi_0 \neq \xi_1$, then $\sigma(\xi_0,\xi_1) > 0$. In this case, we claim that there exists $\xi_2 \in \Omega_0$ such that

$$\sigma(\xi_1,\xi_2) \leq \beta(\sigma(\xi_0,\xi_1))\sigma(\xi_0,\xi_1). \tag{2.3}$$

Indeed, if $|T\xi_1| = 1$, then there exists $\eta_1 \in \Psi$ such that $\eta_1 = T\xi_1$. Since $\eta_1 = T\xi_1 \in T(\Omega_0) \subseteq \Psi_0$, there exists $\xi_2 \in \Omega_0$ such that

$$\sigma(\xi_2,\eta_1) = \sigma(\Omega,\Psi).$$

Since (Ω,Ψ) has the weak σ-Property, using contractivity of T, we have

$$\begin{aligned}\sigma(\xi_1,\xi_2) &\leq \sigma(\eta_0,\eta_1)\\ &= H_\sigma(T\xi_0,T\xi_1)\\ &\leq \mu(\sigma(\xi_0,\xi_1))\sigma(\xi_0,\xi_1)\\ &\leq \beta(\sigma(\xi_0,\xi_1))\sigma(\xi_0,\xi_1).\end{aligned}$$

Hence, (2.3) holds. Now, if $|T\xi_1|>1$, then, from Lemma 4, there exists $\eta_1\in T\xi_1$ such that

$$\sigma(\eta_0,\eta_1)\leq H_\sigma(T\xi_0,T\xi_1)+\frac{1-\mu(\sigma(\xi_0,\xi_1))}{2}\sigma(\xi_0,\xi_1).$$

Using contractivity of T, we have

$$\begin{aligned}\sigma(\eta_0,\eta_1) &\leq H_\sigma(T\xi_0,T\xi_1)+\frac{1-\mu(\sigma(\xi_0,\xi_1))}{2}\sigma(\xi_0,\xi_1) \qquad (2.4)\\ &\leq \mu(\sigma(\xi_0,\xi_1))\sigma(\xi_0,\xi_1)+\frac{1-\mu(\sigma(\xi_0,\xi_1))}{2}\sigma(\xi_0,\xi_1)\\ &= \beta(\sigma(\xi_0,\xi_1))\sigma(\xi_0,\xi_1).\end{aligned}$$

Since $\eta_1\in T\xi_1\subseteq T(\Omega_0)\subseteq\Psi_0$, there exists $\xi_2\in\Omega_0$ such that

$$\sigma(\xi_2,\eta_1)=\sigma(\Omega,\Psi).$$

Because of the fact that (Ω,Ψ) has the weak σ-Property, from (2.2)and (2.4), we have

$$\sigma(\xi_1,\xi_2)\leq\sigma(\eta_0,\eta_1)\leq\beta(\sigma(\xi_0,\xi_1))\sigma(\xi_0,\xi_1)$$

and hence, (2.3) holds.

Case 2. Let $|T\xi_0|>1$. Then, there exists $\eta_0\in\Psi$ such that $\eta_0\in T\xi_0$. Since $\eta_0\in T\xi_0\subseteq T(\Omega_0)\subseteq\Psi_0$, there exists $\xi_1\in\Omega_0$ such that

$$\sigma(\xi_1,\eta_0)=\sigma(\Omega,\Psi). \qquad (2.5a)$$

If $\xi_0=\xi_1$, then ξ_0 is a best proximity point of T. Similar to Case 1, it can be easily seen that $\sigma(\xi_0,\xi_0)=0$. Assume $\xi_0\neq\xi_1$, then $\sigma(\xi_0,\xi_1)>0$. In this case, we claim that there exists $\xi_2\in\Omega_0$ satisfying (2.3). Indeed, if $|T\xi_1|=1$, then there exists $\eta_1\in\Psi$ such that $\eta_1=T\xi_1$. Since $\eta_1=T\xi_1\in T(\Omega_0)\subseteq\Psi_0$, there exists $\xi_2\in\Omega_0$ such that

$$\sigma(\xi_2,\eta_1)=\sigma(\Omega,\Psi).$$

Since (Ω,Ψ) has the weak σ-Property, using contractivity of T, we have

$$\begin{aligned}\sigma(\xi_1,\xi_2) &\leq \sigma(\eta_0,\eta_1)\\ &\leq H_\sigma(T\xi_0,T\xi_1)\\ &\leq \mu(\sigma(\xi_0,\xi_1))\sigma(\xi_0,\xi_1)\\ &\leq \beta(\sigma(\xi_0,\xi_1))\sigma(\xi_0,\xi_1),\end{aligned}$$

that is, (2.3) holds. If $|T\xi_1| > 1$, then, from Lemma 4, there exists $\eta_1 \in T\xi_1$ such that

$$\sigma(\eta_0, \eta_1) \leq H_\sigma(T\xi_0, T\xi_1) + \frac{1 - \mu(\sigma(\xi_0, \xi_1))}{2}\sigma(\xi_0, \xi_1).$$

Using contractivity of T, we have

$$\begin{aligned} \sigma(\eta_0, \eta_1) &\leq H_\sigma(T\xi_0, T\xi_1) + \frac{1 - \mu(\sigma(\xi_0, \xi_1))}{2}\sigma(\xi_0, \xi_1) \\ &\leq \mu(\sigma(\xi_0, \xi_1))\sigma(\xi_0, \xi_1) + \frac{1 - \mu(\sigma(\xi_0, \xi_1))}{2}\sigma(\xi_0, \xi_1) \\ &= \beta(\sigma(\xi_0, \xi_1))\sigma(\xi_0, \xi_1). \end{aligned} \tag{2.6}$$

Since $\eta_1 \in T\xi_1 \subseteq T(\Omega_0) \subseteq \Psi_0$, there exists $\xi_2 \in \Omega_0$ such that

$$\sigma(\xi_2, \eta_1) = \sigma(\Omega, \Psi).$$

Because of the fact that (Ω, Ψ) has the weak σ-Property, from (2.5a)and (2.6), we have

$$\sigma(\xi_1, \xi_2) \leq \sigma(\eta_0, \eta_1) \leq \beta(\sigma(\xi_0, \xi_1))\sigma(\xi_0, \xi_1)$$

and hence (2.3) holds.

Continuing this process, we can construct two sequences $\{\xi_n\}$ in Ω and $\{\eta_n\}$ in Ψ such that $\xi_n \in \Omega_0$, $\eta_n \in T\xi_n$ (we can suppose consecutive terms of $\{\xi_n\}$ are different, otherwise the proof is complete) and

$$\sigma(\xi_{n+1}, \eta_n) = \sigma(\Omega, \Psi) \tag{2.7}$$

$$\sigma(\xi_n, \xi_{n+1}) \leq \sigma(\eta_{n-1}, \eta_n)$$

$$\sigma(\eta_{n-1}, \eta_n) \leq \beta(\sigma(\xi_{n-1}, \xi_n))\sigma(\xi_{n-1}, \xi_n)$$

for all $n \in \mathbb{N}$. Since $\beta(t) < 1$ for all $t \in [0, \infty)$, we have

$$\begin{aligned} \sigma(\xi_n, \xi_{n+1}) &\leq \sigma(\eta_{n-1}, \eta_n) \\ &\leq \beta(\sigma(\xi_{n-1}, \xi_n))\sigma(\xi_{n-1}, \xi_n) \\ &< \sigma(\xi_{n-1}, \xi_n) \end{aligned} \tag{2.8}$$

Hence, $\{\sigma(\xi_n, \xi_{n+1})\}$ is a decreasing sequence in $\mathbb{R}$ and so it converges to a point $\gamma \geq 0$. Since $\limsup_{r \to \gamma^+} \beta(r) < 1$ and $\beta(\gamma) < 1$, there exist $r \in [0, 1)$ and $\varepsilon > 0$ such that $\beta(s) \leq r$ for all $s \in [\gamma, \gamma + \varepsilon]$. Because of the fact that $\lim_{n\to\infty} \sigma(\xi_n, \xi_{n+1}) = \gamma$, there exists $n_0 \in \mathbb{N}$ such that $\gamma \leq \sigma(\xi_n, \xi_{n+1}) \leq \gamma + \varepsilon$ for all $n \geq n_0$. Then, we have

$$\begin{aligned} \sigma(\xi_n, \xi_{n+1}) &\leq \beta(\sigma(\xi_{n-1}, \xi_n))\sigma(\xi_{n-1}, \xi_n) \\ &\leq r\sigma(\xi_{n-1}, \xi_n) \\ &\;\;\vdots \\ &\leq r^{n-n_0}\sigma(\xi_{n_0}, \xi_{n_0+1}) \end{aligned}$$

for all $n \geq n_0$. Let $m, n \in \mathbb{N}$ with $m > n \geq n_0$. Then, we get

$$\begin{aligned}
\sigma(\xi_n, \xi_m) &\leq \sigma(\xi_n, \xi_{n+1}) + \sigma(\xi_{n+1}, \xi_{n+2}) + \cdots + \sigma(\xi_{m-1}, \xi_m) \\
&\leq r^{n-n_0}\sigma(\xi_{n_0}, \xi_{n_0+1}) + r^{n-n_0+1}\sigma(\xi_{n_0}, \xi_{n_0+1}) \\
&\quad + \cdots + r^{m-n_0-1}\sigma(\xi_{n_0}, \xi_{n_0+1}) \\
&= r^{n-n_0}\sigma(\xi_{n_0}, \xi_{n_0+1})(1 + r + \cdots + r^{m-n-1}) \\
&= r^{n-n_0}\sigma(\xi_{n_0}, \xi_{n_0+1})\frac{1 - r^{m-n}}{1 - r} \\
&\leq \frac{r^{n-n_0}}{1 - r}\sigma(\xi_{n_0}, \xi_{n_0+1}).
\end{aligned}$$

Hence, we have $\lim_{n,m\to\infty} \sigma(\xi_n, \xi_m) = 0$. Then, $\{\xi_n\}$ is a 0-Cauchy sequence in Ω. From (2.8), $\{\eta_n\}$ is a 0-Cauchy sequence in Ψ. Since (ξ, σ) is a 0-complete partial metric and Ω, Ψ are closed subsets of X , there exist $\xi^* \in \Omega$ and $\eta^* \in \Psi$ such that

$$\sigma(\xi^*, \xi^*) = \lim_{n\to\infty} \sigma(\xi_n, \xi^*) = \lim_{n,m\to\infty} \sigma(\xi_n, \xi_m) = 0 \tag{2.9}$$

and

$$\sigma(\eta^*, \eta^*) = \lim_{n\to\infty} \sigma(\eta_n, \eta^*) = \lim_{n,m\to\infty} \sigma(\eta_n, \eta_m) = 0.$$

From (2.7) and Lemma 1 (iii), we have

$$\sigma(\xi^*, \eta^*) = \sigma(\Omega, \Psi). \tag{2.10}$$

Also, from (2.1) and (2.9), we get

$$\lim_{n\to\infty} H_\sigma(T\xi_n, T\xi^*) = 0. \tag{2.11}$$

Since $\eta_n \in T\xi_n$, from (2.7) and (2.10), we have

$$\begin{aligned}
\sigma(\Omega, \Psi) &\leq \sigma(\xi_{n+1}, T\xi^*) \\
&\leq \sigma(\xi_{n+1}, \eta_n) + \sigma(\eta_n, T\xi^*) \\
&\leq \sigma(\xi_{n+1}, \eta_n) + H_\sigma(T\xi_n, T\xi^*) \\
&= \sigma(\Omega, \Psi) + H_\sigma(T\xi_n, T\xi^*).
\end{aligned}$$

Taking limit $n \to \infty$ in last inequality, we obtain

$$\lim_{n\to\infty} \sigma(\xi_{n+1}, T\xi^*) = \sigma(\Omega, \Psi).$$

On the other hand, we get

$$\begin{aligned}
\sigma(\Omega, \Psi) &\leq \sigma(\xi^*, T\xi^*) \\
&\leq \sigma(\xi^*, \xi_{n+1}) + \sigma(\xi_{n+1}, T\xi^*)
\end{aligned}$$

and so taking limit $n \to \infty$ in last inequalit, we have $\sigma(\xi^*, T\xi^*) = \sigma(\Omega, \Psi)$. Hence, T has a best proximity point ξ^* in Ω. Finally, from (2.9), we have $\sigma(\xi^*, \xi^*) = 0$. □

Example 2. *Let $X = \{\upsilon_0, \upsilon_1, \upsilon_2, \cdots, \upsilon_n, \cdots\}$ be a countable set. Let define a bounded sequence $\{\omega_n\}$ of positive real numbers satisfying $0 < \omega_0 = \omega_1$ and $0 < \omega_n < \omega_{n+1}$ for all $n \geq 1$ and the function $\sigma : X \times X \to \mathbb{R}^+$ defined by*

$$\sigma(\upsilon_n, \upsilon_m) = \begin{cases} 0 & , \quad n = m \in \{0,1\} \\ \frac{\omega_n}{2} & , \quad n = m \geq 2 \\ \omega_n + \omega_m & , \quad \textit{otherwise} \end{cases}$$

Denote by $\omega_\infty := \lim_{n\to\infty} \omega_n$. Then, (X, σ) is a partial metric space. Further, (X, σ) is 0-complete. Since there is no 0-Cauchy sequence other than $\xi_n = \upsilon_0$ and $\xi_n = \upsilon_1$ for all $n \in \mathbb{N}$, (X, σ) is a 0-complete partial metric space. Let $\Omega = \{\upsilon_0, \upsilon_2, \upsilon_4, \cdots \upsilon_{2n} \cdots\}$, $\Psi = \{\upsilon_1, \upsilon_3, \cdots, \upsilon_{2n+1}, \cdots\}$ be closed subset of X, then we have $\Omega_0 = \{\upsilon_0\}$ and $\Psi_0 = \{\upsilon_1\}$. Moreover, we get $d(\Omega, \Psi) = \omega_0 + \omega_1$ and the pair (Ω, Ψ) has the weak σ-Property. Define the mapping $T : \Omega \to \Psi \cup CB^\sigma(\Psi)$ as

$$T\upsilon_{2n} = \begin{cases} \{\upsilon_1\} & , \quad n = 0 \\ X_{2n-1} \cap \Psi & , \quad n \geq 1 \end{cases}$$

where $X_n = \{\upsilon_n, \upsilon_{n+1}, \upsilon_{n+2}, \cdots\}$ for all $n \in \mathbb{N}$. Then, we have $T(\Omega_0) \subseteq \Psi_0$. Now, we take $\mu : [0, \infty) \to [0, 1)$ defined by

$$\mu(\gamma) = \begin{cases} \frac{\omega_{n-1}+\omega_{m-1}}{\omega_n+\omega_m} & , \quad \textit{if } \gamma = \omega_n + \omega_m \textit{ for some } n, m \geq 1 \\ 0 & , \quad \textit{otherwise} \end{cases}$$

Since $\limsup_{r\to\gamma^+} \mu(r) = 0 < 1$, μ is a MT-function. We shall show that T is a mixed multivalued contraction mapping. Then, we consider the following cases:

Case 1. Let $n = m = 0$. Then, we have

$$H_\sigma(T\upsilon_0, T\upsilon_0) = 0 = \mu(\sigma(\upsilon_0, \upsilon_0))\sigma(\upsilon_0, \upsilon_0).$$

Case 2. Let $n = 0$ and $m \geq 1$. Then, since $\omega_0 = \omega_1$, we have

$$\begin{aligned} H_\sigma(T\upsilon_0, T\upsilon_{2m}) &= H_\sigma(\{\upsilon_1\}, X_{2m-1} \cap \Psi) \\ &= \omega_1 + \omega_{2m-1} \\ &= \mu(\sigma(\upsilon_0, \upsilon_{2m}))\sigma(\upsilon_0, \upsilon_{2m}). \end{aligned}$$

Case 3 : Let $n, m \geq 1$. Then, we have

$$\begin{aligned} H_\sigma(T\upsilon_{2n}, T\upsilon_{2m}) &= H_\sigma(X_{2n-1} \cap \Psi, X_{2m-1} \cap \Psi) \\ &= \omega_{2n-1} + \omega_{2m-1} \\ &= \mu(\sigma(\upsilon_{2n}, \upsilon_{2m}))\sigma(\upsilon_{2n}, \upsilon_{2m}). \end{aligned}$$

Hence, all hypotheses of Theorem 4 are satisfied, and so T has a best proximity point in Ω which is υ_0. Moreover $\sigma(\upsilon_0, \upsilon_0) = 0$. Note that, there is no $k \in [0, 1)$ satisfying

$$H_\sigma(T\xi, T\eta) \leq k\sigma(\xi, \eta) \tag{2.12}$$

for all $\xi, \eta \in \Omega$. Assume that there exists $k \in [0,1)$ satisfying (2.12). In this case, for all $n, m \geq 1$, we have

$$H_\sigma(T\upsilon_{2n}, T\upsilon_{2m}) = \omega_{2n-1} + \omega_{2m-1} \leq k(\omega_{2n} + \omega_{2m}) = k\sigma(\upsilon_{2n}, \upsilon_{2m}).$$

and so

$$\frac{\omega_{2n-1} + \omega_{2m-1}}{\omega_{2n} + \omega_{2m}} \leq k.$$

Taking limit $n, m \to \infty$ in last inequality, we have $1 \leq k$ which is a contradiction.

Taking $\mu(\gamma) = k \in [0,1)$ for all $\gamma \in [0, \infty)$ in Theorem 4, we obtain the following result.

Corollary 1. *Let (X, σ) be a 0-complete partial metric space, $\Omega, \Psi \subseteq X$ be two nonempty closed subsets of X and $\Omega_0 \neq \emptyset$. Assume that $T : \Omega \to \Psi \cup CB^\sigma(\Psi)$ is a mixed multivalued mapping satisfying $T(\Omega_0) \subseteq \Psi_0$ and (Ω, Ψ) has the weak σ-Property. If there exists $k \in [0,1)$ such that*

$$H_\sigma(T\xi, T\eta) \leq k\sigma(\xi, \eta)$$

for all $\xi, \eta \in \Omega$, then T has a best proximity point ξ^* in Ω. Moreover, $\sigma(\xi^*, \xi^*) = 0$.

Taking $\Omega = \Psi = X$ in Theorem 4 and Corollary 1, we have the following fixed point results which is more general than the main result of [5].

Corollary 2. *Let (X, σ) be a 0-complete partial metric space,and $T : X \to X \cup CB^\sigma(X)$ be a mixed multivalued mapping. If there exists $\mu \in \phi$ such that*

$$H_\sigma(T\xi, T\eta) \leq \mu(\sigma(\xi, \eta))\sigma(\xi, \eta)$$

for all $\xi, \eta \in X$, then T has a fixed point ξ^* in X. Moreover, $\sigma(\xi^*, \xi^*) = 0$.

Proof. If we take $\Omega = \Psi = X$ in Theorem 4, then there exists $\xi^* \in X$ such that

$$\sigma(\xi^*, T\xi^*) = \sigma(X, X) \text{ and } \sigma(\xi^*, \xi^*) = 0.$$

Also, since $\sigma(X, X) \leq \sigma(\xi^*, \xi^*)$ and $\sigma(X, X) \leq \sigma(T\xi^*, T\xi^*)$, then we have $\sigma(\xi^*, \xi^*) = \sigma(\xi^*, T\xi^*) = \sigma(T\xi^*, T\xi^*)$. Therefore, we get $\xi^* \in T\xi^*$, that is, ξ^* is a fixed point of T in X. □

Corollary 3. *Let (X, σ) be a 0-complete partial metric space,and $T : X \to X \cup CB^\sigma(X)$ be a mixed multivalued mapping. If there exists $k \in [0,1)$ such that*

$$H_\sigma(T\xi, T\eta) \leq k\sigma(\xi, \eta)$$

for all $\xi, \eta \in X$, then T has a fixed point ξ^* in X. Moreover, $\sigma(\xi^*, \xi^*) = 0$.

3. Application

In this section, we obtain homotopy results by applying Theorem 4 and Corollary 2. Hence, the first time in literature, it will be obtained applications to homotopy theory via best proximity point theorems. Now, we recall the definition of homotopy.

Definition 5. *Let X, Y be topological spaces and $T, S : X \to Y$ be continuous mappings. Then, a homotopy from T to S is a continuous function $\hat{H} : X \times [0,1] \to Y$ such that $\hat{H}(\xi, 0) = T\xi$ and $\hat{H}(\xi, 1) = S\xi$ for all $\xi \in X$. Also, T and S are called homotopic mappings.*

Here, the family of all functions μ in Φ satisfying the following implication will be denoted by Φ_H

$$\lim_{i\to\infty} (1 - \mu(s_i))\, s_i = 0 \implies \lim_{i\to\infty} s_i = 0 \tag{3.1}$$

for every sequence $\{s_i\} \subseteq [0, \infty)$.

It is clear that $\mu(\gamma) = k \in [0,1)$ for all $\gamma \in [0,\infty)$ belongs to Φ_H, that is, $\Phi_H \neq \emptyset$. Moreover, Φ_H is a proper subset of Φ. Indeed, let $\mu : [0,\infty) \to [0,1)$ be a function defined by

$$\mu(\gamma) = \begin{cases} \gamma & , \quad [0,1) \\ 0 & \quad \gamma \geq 1 \end{cases}$$

for all $\gamma \in [0, \infty)$. It can be easily seen that $\mu \in \Phi$. If we take $s_i = \frac{i}{i+1}$ for all $i \in \mathbb{N}$, then we have $\lim_{i\to\infty} (1 - \mu(s_i))\, s_i = 0$. However, $\lim_{i\to\infty} s_i = 1$. Hence, we have $\mu \notin \Phi_H$.

Theorem 5. *Let (X, σ) be a 0-complete partial metric space, Ω, Ψ be closed subsets of X, $\Omega_0 \neq \emptyset$ and $\emptyset \neq U \subseteq \Omega$. Assume that $F : \Omega \times [0,1] \to \Psi \cup CB^\sigma(\Psi)$ be a mixed multivalued mapping and the pair (Ω, Ψ) has the weak σ-Property.*

(i) $\sigma(\xi, F(\xi, u)) > \sigma(\Omega, \Psi)$ for all $\xi \in \Omega \backslash U$ and $u \in [0,1]$,

(ii) there exists $\mu \in \Phi_H$ such that

$$H_\sigma(F(\xi, u), F(\eta, u)) \leq \mu(\sigma(\xi, \eta))\sigma(\xi, \eta) \tag{3.2}$$

for all $\xi, \eta \in \Omega$ and $u \in [0,1]$,

(iii) there exists a continuous function $\varphi : [0,1] \to \mathbb{R}$ such that

$$H_\sigma\left(F(\xi, u), F(\xi, \upsilon)\right) \leq \mu(\sigma(\xi, \xi))\, |\varphi(u) - \varphi(\upsilon)|$$

for all $u, \upsilon \in [0,1]$ and each $\xi \in \Omega$,

(iv) for all $u \in [0,1]$ satisfying $\sigma(\xi, F(\xi, u)) = \sigma(\Omega, \Psi)$ for some $\xi \in U$, there exists $\varepsilon_u > 0$ such that $F(\Omega_0, u^*) \subseteq \Psi_0$ for all $u^* \in (u - \varepsilon_u, u + \varepsilon_u)$.

(v) if $\sigma(\xi, F(\xi, u)) = \sigma(\Omega, \Psi)$ for some $\xi \in \Omega$ and $u \in [0,1]$, then $F(\xi, u)$ is singleton.

If $F(\cdot, 0)$ has a best proximity point in Ω, then $F(\cdot, 1)$ has a best proximity point in Ω.

Proof. Define the set

$$K = \{u \in [0,1] : \sigma(\xi, F(\xi, u)) = \sigma(\Omega, \Psi) \text{ for some } \xi \in U\}.$$

Since $F(.,0)$ has a best proximity point in Ω and (i) holds, then $0 \in K$. Hence, K is a nonempty set. We will show that K is both open and closed in $[0,1]$ and hence by connectedness of $[0,1]$, we have that $K = [0,1]$. We first show that K is closed. Let $\{u_n\}$ be sequence in K with $u_n \to u^* \in [0,1]$ as $n \to \infty$. Using definition of K, there exists $\xi_n \in U$ such that

$$\sigma(\xi_n, F(\xi_n, u_n)) = \sigma(\Omega, \Psi) \tag{3.3}$$

for all $n \in \mathbb{N}$ and so, from (v), $F(\xi_n, u_n)$ is singleton for all $n \in \mathbb{N}$. Then, since (Ω, Ψ) has the weak σ-Property, from (ii), we have

$$\begin{aligned}
\sigma(\xi_n, \xi_m) &\leq \sigma(F(\xi_n, u_n), F(\xi_m, u_m)) \\
&= H_\sigma(F(\xi_n, u_n), F(\xi_m, u_m)) \\
&\leq H_\sigma(F(\xi_n, u_n), F(\xi_n, u_m)) + H_\sigma(F(\xi_n, u_m), F(\xi_m, u_m)) \\
&\leq \mu(\sigma(\xi_n, \xi_n))\, |\varphi(u_n) - \varphi(u_m)| + \mu(\sigma(\xi_n, \xi_m))\sigma(\xi_n, \xi_m) \\
&< |\varphi(u_n) - \varphi(u_m)| + \mu(\sigma(\xi_n, \xi_m))\sigma(\xi_n, \xi_m)
\end{aligned}$$

for all $n, m \in \mathbb{N}$. Since φ is a continuous function and the sequence $\{u_n\}$ is convergent, we have

$$\lim_{n,m\to\infty} (1 - \mu(\sigma(\xi_n, \xi_m))\, \sigma(\xi_n, \xi_m) = 0$$

and so, from (3.1), we get $\lim_{n,m\to\to\infty} \sigma(\xi_n, \xi_m) = 0$. Hence, $\{\xi_n\}$ is a $0-$Cauchy sequence. Since (X, σ) is 0-complete and Ω is a closed subset of X, there exists $\xi^* \in \Omega$ such that

$$\lim_{n,m\to\infty} \sigma(\xi_n, \xi_m) = \lim_{n\to\infty} \sigma(\xi_n, \xi^*) = \sigma(\xi^*, \xi^*) = 0.$$

Then, from (3.3), we have

$$\begin{aligned}
\sigma(\Omega, \Psi) &\leq \sigma(\xi_n, F(\xi^*, u^*)) \\
&\leq \sigma(\xi_n, F(\xi_n, u_n)) + \sigma(F(\xi_n, u_n), F(\xi^*, u^*)) \\
&= \sigma(\Omega, \Psi) + \sigma(F(\xi_n, u_n), F(\xi^*, u^*)) \\
&\leq \sigma(\Omega, \Psi) + H_\sigma(F(\xi_n, u_n), F(\xi^*, u^*)) \\
&\leq \sigma(\Omega, \Psi) + H_\sigma(F(\xi_n, u_n), F(\xi_n, u^*)) + H_\sigma(F(\xi_n, u^*), F(\xi^*, u^*)) \\
&\leq \sigma(\Omega, \Psi) + \mu(\sigma(\xi_n, \xi_n)\, |\varphi(u_n) - \varphi(u^*)| + \mu(\sigma(\xi_n, \xi^*))\sigma(\xi_n, \xi^*) \\
&< \sigma(\Omega, \Psi) + |\varphi(u_n) - \varphi(u^*)| + \sigma(\xi_n, \xi^*)
\end{aligned}$$

and so we get

$$\sigma(\xi^*, F(\xi^*, u^*)) = \lim_{n\to\infty} \sigma(\xi_n, F(\xi^*, u^*)) = \sigma(\Omega, \Psi).$$

From definition of K, we have $u^* \in K$ and so K is closed in $[0,1]$.

Now, we will show that K is open. Let $u_0 \in K$. Then, there exists $\xi_0 \in U$ such that $\sigma(\xi_0, F(\xi_0, u_0)) = \sigma(\Omega, \Psi)$. From (iv), for $u_0 \in [0,1]$, there exists $\varepsilon_{u_0} > 0$ such that $F(\Omega_0, u^*) \subseteq \Psi_0$ for all $u^* \in (u_0 - \varepsilon_{u_0}, u_0 + \varepsilon_{u_0})$. If we consider the mapping $F(\cdot, u^*) : \Omega \to \Psi \cup CB^\sigma(\Psi)$ for all $u^* \in (u_0 - \varepsilon_{u_0}, u_0 + \varepsilon_{u_0})$, then, from (ii), $F(\cdot, u^*)$ is a mixed multivalued contraction mapping. Therefore, all hypotheses of Theorem 4 are satisfied. Hence, for all $u^* \in (u_0 - \varepsilon_{u_0}, u_0 + \varepsilon_{u_0})$, $F(\cdot, u^*)$ has a best proximity point $\xi^*_{u^*}$ in Ω. From (i), we have $\xi^*_{u^*} \in U$ for all $u^* \in (u_0 - \varepsilon_{u_0}, u_0 + \varepsilon_{u_0})$. Hence, from definition of K, $u_0 \in (u_0 - \varepsilon_{u_0}, u_0 + \varepsilon_{u_0}) \subseteq K$, that is, K is open in $[0, 1]$. □

If we take $\mu : [0, \infty) \to [0, 1)$ defined by $\mu(\gamma) = k \in [0, 1)$ for all $\gamma \in [0, \infty)$ in Theorem 5, we obtain the following result.

Corollary 4. *Let (X, σ) be a 0-complete partial metric space, Ω, Ψ be closed subsets of X, $\Omega_0 \neq \emptyset$ and $\emptyset \neq U \subseteq \Omega$. Assume that $F : \Omega \times [0, 1] \to \Psi \cup CB^\sigma(\Psi)$ be a mixed multivalued mapping and the pair (Ω, Ψ) has the weak σ-Property.*

(i) $\sigma(\xi, F(\xi, u)) > \sigma(\Omega, \Psi)$ for all $\xi \in \Omega \backslash U$ and $u \in [0, 1]$,

(ii) there exists $k \in [0, 1)$ such that

$$H_\sigma(F(\xi, u), F(\eta, u)) \leq k\sigma(\xi, \eta)$$

for all $\xi, \eta \in \Omega$ and $u \in [0, 1]$,

(iii) there exists a continuous function $\varphi : [0, 1] \to \mathbb{R}$ such that

$$H_\sigma \left(F(\xi, u), F(\xi, \upsilon)\right) \leq k \left|\varphi(u) - \varphi(\upsilon)\right|$$

for all $u, \upsilon \in [0, 1]$ and each $\xi \in \Omega$,

(iv) for all $u \in [0, 1]$ satisfying $\sigma(\xi, F(\xi, u)) = \sigma(\Omega, \Psi)$ for some $\xi \in U$, there exists $\varepsilon_u > 0$ such that $F(\Omega_0, u^*) \subseteq \Psi_0$ for all $u^* \in (u - \varepsilon_u, u + \varepsilon_u)$.

(v) if $\sigma(\xi, F(\xi, u)) = \sigma(\Omega, \Psi)$ for some $\xi \in \Omega$ and $u \in [0, 1]$, then $F(\xi, u)$ is singleton.

If $F(\cdot, 0)$ has a best proximity point in Ω, then $F(\cdot, 1)$ has a best proximity point in Ω.

Taking $\Psi = X$ in Theorem 5 and Corollary 4, we can present the following fixed point results.

Corollary 5. *Let (X, σ) be a 0-complete partial metric space, Ω be a closed subset of X and $\emptyset \neq U \subseteq \Omega$. Assume that $F : \Omega \times [0, 1] \to X \cup CB^\sigma(X)$ be a mixed multivalued contraction mapping*

(i) $\xi \notin F(\xi, u))$ for all $\xi \in \Omega \backslash U$ and $u \in [0, 1]$,

(ii) for all $\xi, \eta \in \Omega$ and $u \in [0, 1]$, there exists $\mu \in \Phi_H$ such that

$$H_\sigma(F(\xi, u), F(\eta, u)) \leq \mu(\sigma(\xi, \eta))\sigma(\xi, \eta)$$

(iii) there exists a continuous function $\varphi : [0, 1] \to \mathbb{R}$ such that

$$H_\sigma(F(\xi, u), F(\xi, \upsilon)) \leq \mu(\sigma(\xi, \xi)) \left|\varphi(u) - \varphi(\upsilon)\right|$$

for all $u, \upsilon \in [0, 1]$ and each $\xi \in \Omega$,

(iv) for all $u \in [0, 1]$ satisfying $\xi \in F(\xi, u)$ for some $\xi \in U$, there exists $\varepsilon_u > 0$ such that $F(\Omega, u^*) \subseteq \Omega$ for all $u^* \in (u - \varepsilon_u, u + \varepsilon_u)$.

(v) if $\xi \in F(\xi, u)$ for some $\xi \in \Omega$ and $u \in [0, 1]$, then $F(\xi, u) = \{\xi\}$.

If $F(\cdot, 0)$ has a fixed point in Ω, then $F(\cdot, 1)$ has a fixed point in Ω.

Proof. Assume that $\xi \in F(\xi, 0)$ for some $\xi \in \Omega$. From (ii) and (v), it is clear that $\sigma(\xi, \xi) = 0$ and so, we have $\sigma(\Omega, X) = 0$. Hence, the conditions (i), (ii) and (iii) of Theorem 5 are satisfied. Further, we have $\Omega_0 = \Psi_0 = \Omega$ whenever $\Psi = X$, and so the condition (iv) is also hold. Therefore, there exists $\xi^* \in \Omega$ such that

$$\sigma(\xi^*, F(\xi^*, 1)) = \sigma(\Omega, X) = 0$$

Hence, $\xi^*$is a fixed point of $F(\cdot, 1)$. □

Corollary 6. *Let (X, σ) be a 0-complete partial metric space, Ω be a closed subset of X and $\emptyset \neq U \subseteq \Omega$. Assume that $F : \Omega \times [0, 1] \to X \cup CB^\sigma(X)$ be a mixed multivalued contraction mapping*

(i) $\xi \notin F(\xi, u))$ for all $\xi \in \Omega \backslash U$ and $u \in [0, 1]$,

(ii) for all $\xi, \eta \in \Omega$ and $u \in [0, 1]$, there exists $k \in [0, 1)$ such that

$$H_\sigma(F(\xi, u), F(\eta, u)) \leq k\sigma(\xi, \eta)$$

(iii there exists a continuous function $\varphi : [0, 1] \to R$ such that

$$H_\sigma(F(\xi, u), F(\xi, \upsilon) \leq k\,|\varphi(u) - \varphi(\upsilon)|$$

for all $u, \upsilon \in [0, 1]$ and each $\xi \in \Omega$,

(iv) for all $u \in [0, 1]$ satisfying $\xi \in F(\xi, u)$ for some $\xi \in U$, there exists $\varepsilon_u > 0$ such that $F(\Omega, u^*) \subseteq \Omega$ for all $u^* \in (u - \varepsilon_u, u + \varepsilon_u)$.

(v)if $\xi \in F(\xi, u)$ for some $\xi \in \Omega$ and $u \in [0, 1]$, then $F(\xi, u) = \{\xi\}$.

If $F(\cdot, 0)$ has a fixed point in Ω, then $F(\cdot, 1)$ has a fixed point in Ω.

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