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Symmetry Analysis and Conservation Laws of a Family of Non-linear Viscoelastic Wave Equation.

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Summary

This work considers a non-linear viscoelastic wave equation with non-linear damping and source terms. We analyse the partial differential equation from the point of view of Lie symmetries. Firstly, we apply Lie's method to obtain new symmetries. Hence, we transform the partial differential equation into an ordinary differential equation, by using the symmetries. Moreover, new solutions are derived from the ordinary differential equations. Finally, by using the direct method of multipliers, we construct low-order conservation laws depending on the form of the damping and source terms.

KEYWORDS:
viscoelastic wave equation, Lie symmetries, conversation laws

1 | INTRODUCTION

Recently, several viscoelastic wave equations have been studied. The single viscoelastic wave equation of the form

$$u_{tt} - \Delta u + \int_0^1 h(t-s)\Delta u(x,s)ds + f(u_t) = g(u)$$

in $\Omega \times (0,\infty)$, where Ω is a bounded domain of R^N $(N \geq 1)$, with initial and boundary conditions, has been extensively studied. Many results concerning non-existence and blow-up in finite time have been proved[1,2,3,4,5,6]. For instance, for $p > m \geq 2$, Messaoudi[7] proved a blow-up result for solutions with negative initial energy and a global result for $2 \leq p \leq m$. This result has been later improved to accommodate certain solutions with positive initial energy. Messaoudi and Said-Houari[8] proved a global non-existence result of certain solutions with positive initial energy of a system of viscoelastic wave equations with non-linear damping and source terms acting in both equations. Nevertheless, in the absence of the source term ($g = 0$), it is well-known that the damping term $f(u_t)$ assures global existence and decay of the solution energy for arbitrary initial data. Also, in the absence of the damping term, the source term causes finite time blow-up of solutions with a large initial data (negative initial energy).

Furthermore, the non-linear viscoelastic wave equation with damping and source terms

$$u_{tt} - \Delta u + f(u_t) = g(u), \qquad x \in \Omega,\ t > 0, \tag{1}$$

has also been very studied obtaining similar results. As in the single viscoelastic wave equation, in the absence of the source term ($g = 0$), it is well-known that the damping term $f(u_t)$ assures global existence and decay of the solution energy for arbitrary initial data. In the same way, in the absence of the damping term, the source term causes finite time blow-up of solutions with a large initial data (negative initial energy). Here, the interaction between the damping term and the source term makes the problem more interesting. Moreover, in the linear damping case $f(u_t) = au_t$ and $g(u) = b|u|^{p-2}u$ a polynomial source term, Levine[9,10] showed that solutions with negative initial energy blow-up in finite time. The main tool used was the "concavity method".

[0]**Abbreviations:** ANA, anti-nuclear antibodies; APC, antigen-presenting cells; IRF, interferon regulatory factor

In addition, Messaoudi[11] considered the non-linear viscoelastic wave equation with damping and source terms

$$u_{tt} - \Delta u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \qquad x \in \Omega,\ t > 0.$$

For this model, Georgiev and Todorova[12] showed that solutions with negative energy continue to exist globally "in time" if $m \geq p \geq 2$, and blow-up in finite time if $p > m \geq 2$ and the initial energy is sufficiently negative. The main tool used was also the "concavity method". This latter result has been pushed by Messaoudi[13] to the situation where the initial energy $E(0) < 0$.

Nevertheless, the resolution of non-linear partial differential equations (PDEs) is a very important field of research in applied mathematics. Symmetry reductions and exact solutions have several applications in the context of differential equations. For instance, exact solutions arising from symmetry methods can be used to study properties such as asymptotic and "blow-up". A large amount of literature has been published about the application of Lie transformation group theory to construct solutions of non-linear PDEs[14,15,16,17,18,19,20,21,22]. The symmetries leaving invariant the equation can reduce the number of independent variables, transforming the PDEs into ordinary differential equations (ODEs), generally easier to solve.

Conservation laws analyse which physical properties of a PDE do not change in the course of time. In particular, local conservation laws are continuity equations providing conserved quantities of physical importance for all solutions of a particular equation. For any PDE, a complete classification of conservation laws can be determined by the multiplier method[23,24].

To sum up, the aim of this work is to do a complete Lie group classification of Eq. (1). Afterwards, we present the reductions obtained from the different symmetries, transforming the PDE into an ODE. Moreover, we obtain travelling wave solutions by the comparison between Eq. (1) and similar equations studied previously[25,26,27]. Finally, we give a complete classification of the conservation laws admitted by Eq. (1).

The structure of the paper is as follows: In Sec. 2 we study the Lie symmetries of equation (1) obtaining the symmetry reductions, the symmetry variables and the reduced ODEs. Then, in Sec. 4 we present the classification of the conservation laws and the multipliers of Eq. (1). Finally, in Sec. 5 we give some conclusions of the work.

2 | LIE SYMMETRIES CLASSIFICATION AND REDUCTIONS

The idea of Lie's theory of symmetry analysis of differential equations relies on the invariance of the latter under a transformation of independent and dependent variables. This transformation constitutes a local group of point transformations providing a diffeomorphism on the space of independent and dependent variables, mapping solutions of the equations to other solutions. Any transformation of the independent and dependent variables leads to a transformation of the derivatives[28].

The application of Lie's theory to differential equations is completely algorithmic. However, it usually involves a lot of tedious calculations. Nevertheless, we make use of powerful softwares like Maple® and the needed calculations are done rapidly.

In this section, let us briefly describe the classical Lie method and its application to Eq. (1), obtaining the symmetry reductions, the symmetry variables and the reduced equations.

It is considered the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u) + \mathcal{O}(\epsilon^2), \\ t^* &= t + \epsilon\tau(x,t,u) + \mathcal{O}(\epsilon^2), \\ u^* &= u + \epsilon\eta(x,t,u) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{2}$$

where ϵ is the group parameter. These transformations leave invariant the set of solutions of Eq. (1). The associated Lie algebra of infinitesimal symmetries is given by the infinitesimal generator

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \tag{3}$$

Each infinitesimal generator (3) generates a transformation obtained by solving the system of ODEs

$$\frac{\partial \hat{x}}{\partial \epsilon} = \xi(\hat{x},\hat{t},\hat{u}), \qquad \frac{\partial \hat{t}}{\partial \epsilon} = \tau(\hat{x},\hat{t},\hat{u}), \qquad \frac{\partial \hat{u}}{\partial \epsilon} = \eta(\hat{x},\hat{t},\hat{u}),$$

satisfying the initial conditions

$$\hat{x}|_{\epsilon=0} = x, \qquad \hat{t}|_{\epsilon=0} = t, \qquad \hat{u}|_{\epsilon=0} = u,$$

with ϵ the group parameter.

The symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi(x,t,u)u_x + \tau(x,t,u)u_t - \eta(x,t,u) = 0.$$

For Eq. (1), a PDE with two independent variables, a single group reduction transforms the PDE into ODEs, easier to solve than the original equation.

We require that the transformation (2) leaves invariant the set of solutions of Eq. (1). This leads to an overdetermined linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$, generated by requiring that

$$\mathrm{pr}^{(2)}\mathbf{v}\,(\Delta)\,|_{\Delta=0}=0,$$

where $\mathrm{pr}^{(2)}\mathbf{v}$ is the 2-th order prolongation of the vector field $\mathbf{v}$ defined by

$$\mathrm{pr}^{(2)}\mathbf{v}=\mathbf{v}+\eta_x\frac{\partial}{\partial u_x}+\eta_t\frac{\partial}{\partial u_t}+\eta_{xx}\frac{\partial}{\partial u_{xx}}+\eta_{xt}\frac{\partial}{\partial u_{xt}}+\eta_{tt}\frac{\partial}{\partial u_{tt}},$$

with the coefficients

$$\begin{aligned}
\eta_x &= D_x\eta-u_tD_x\tau-u_xD_x\xi,\\
\eta_t &= D_t\eta-u_tD_t\tau-u_xD_t\xi,\\
\eta_{xx} &= D_x(\eta_x)-u_{xt}D_x\tau-u_{xx}D_x\xi,\\
\eta_{xt} &= D_t(\eta_x)-u_{xt}D_x\tau-u_{xx}D_t\xi,\\
\eta_{tt} &= D_t(\eta_t)-u_{tt}D_t\tau-u_{xt}D_t\xi,
\end{aligned}$$

where D_x and D_t are the total derivatives of x and t, respectively.

Applying the previous condition to Eq. (1), we get a system of equations for the infinitesimals. Then, by solving the system, we obtain the following classification:

For $f(u_t)$ and $g(u)$ arbitrary functions, the symmetries admitted by Eq. (1) are given by the infinitesimal generators

$$\mathbf{v}_1=\partial_x,\quad \mathbf{v}_2=\partial_t. \tag{4}$$

The symmetries (4) yield to the one-parameter symmetry transformation groups

$$\begin{aligned}
(\hat{x},\hat{t},\hat{u})_1 &= (x+\epsilon,t,u),\ \text{space translation},\\
(\hat{x},\hat{t},\hat{u})_2 &= (x,t+\epsilon,u),\ \text{time translation}.
\end{aligned}$$

• For the generator $\lambda\mathbf{v}_1+\mathbf{v}_2$, we obtain travelling wave reductions,

$$z=x-\lambda t,\quad u(x,t)=h(z), \tag{5}$$

where $h(z)$ satisfies

$$(\lambda^2-1)h''+f(-\lambda\,h')-g(h)=0. \tag{6}$$

From Eq. (1), according to the expression of f, we can distinguish different cases for which Eq. (1) has extra symmetries. Since Eq. (1) has additional symmetries and the reductions that correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ have already been derived, we determine the similarity variables and similarity solutions corresponding to the additional generators.

Case 1: For $f(u_t)=-\mathrm{e}^{-n}{u_t}^n+f_1$ and $g(u)=\left(\left(g_0\,u+g_1\right)\left(\frac{2}{n}-1\right)\right)^{\frac{n}{2-n}}-f_1$, the symmetries are

$$\mathbf{v}_1,\quad \mathbf{v}_2,\quad \mathbf{v}_3^1=g_0\left(\frac{n-1}{n-2}\right)x\partial_x+g_0\left(\frac{n-1}{n-2}\right)t\partial_t+(g_0u+g_1)\partial_u. \tag{7}$$

The symmetry (7) yields to the one-parameter symmetry transformation group

$$(\hat{x},\hat{t},\hat{u})_3^1=(x\mathrm{e}^{\frac{g_0\,(n-1)\epsilon}{n-2}},t\mathrm{e}^{\frac{g0\,(n-1)\epsilon}{n-2}},\mathrm{e}^{g_0\,\epsilon}u+\mathrm{e}^{g_0\,\epsilon}\int_0^\epsilon g_1\,\mathrm{e}^{-g_0\,z_1}\,\mathrm{d}z_1),\ \text{scaling and shift}.$$

Then, we obtain the symmetry reduction

$$z=\tfrac{x}{t},\quad u=x^{\frac{n-2}{n-1}}h(z)-\tfrac{g_1}{g_0},$$

where $h(z)$ satisfies

$$\begin{aligned}
&z^{\frac{3n^2+2n}{n^2-3n+2}}\left(-z^{\frac{1}{n-1}}\right)^{3n}\Bigg(\left(g_0{}^{\frac{n}{n-2}}\,(2-n)^{\frac{n}{n-2}}\,(n-1)^2\,e^{f_2\,n}\,z^4-g_0{}^{\frac{n}{n-2}}\,(2-n)^{\frac{n}{n-2}}\,(n-1)^2\,e^{f_2\,n}\,z^2\right)h^{\frac{n}{n-2}+\frac{2}{n-2}}\,h''\\
&+\left(2\,g_0{}^{\frac{n}{n-2}}\,(2-n)^{\frac{n}{n-2}}\,(n-1)^2\,e^{f_2\,n}\,z^3-2\,g_0{}^{\frac{n}{n-2}}\,(2-n)^{\frac{n}{n-2}}\,(n-2)\,(n-1)\,e^{f_2\,n}\,z\right)h^{\frac{n}{n-2}+\frac{2}{n-2}}\,h'\\
&+\left(e^{f_2\,n}\left(n^{\frac{n}{n-2}}\,\left(n^2-2\,n+1\right)\right)h^{\frac{2}{n-2}}+\left(g_0{}^{\frac{n}{n-2}}\,(2-n)^{\frac{n}{n-2}}\,n-2\,g_0{}^{\frac{n}{n-2}}\,(2-n)^{\frac{n}{n-2}}\right)h^{\frac{2n}{n-2}}\right)\Bigg)\\
&-\left(g_0{}^{\frac{n}{n-2}}\,(2-n)^{\frac{n}{n-2}}\,(n-1)^2\,z^{\frac{2n^3}{n^2-3n+2}}\right)h^{\frac{n}{n-2}+\frac{2}{n-2}}\,\left(h'\right)^n=0.
\end{aligned}$$

Case 2: For $f(u_t) = au_t^2$ and $g(u) = k\, e^{cu}$, besides $\mathbf{v}_1$ and $\mathbf{v}_2$, the equation has an extra symmetry,

$$\mathbf{v}_1, \quad \mathbf{v}_2, \quad \mathbf{v}_3^2 = -\frac{1}{2}cx\partial_x - \frac{1}{2}ct\partial_t + \partial_u. \tag{8}$$

The symmetry (8) yields to the one-parameter symmetry transformation group

$$(\hat{x}, \hat{t}, \hat{u})_3^2 = (x\mathrm{e}^{-1/2\,c\epsilon}, t\mathrm{e}^{-1/2\,c\epsilon}, \epsilon + u), \text{ scaling in } t \text{ and } x \text{ combined with translation in } u.$$

The similarity variables and similarity solutions are

$$z = \tfrac{x}{t}, \quad u = -\tfrac{2}{c}\ln t + h(z),$$

where $h(z)$ satisfies

$$\left(c^2z^2 - c^2\right) h'' + \left(a\,z^2c^2\right)(h')^2 + \left(4\,z\,c\,a + 2\,z\,c^2\right) h' + k\,\mathrm{e}^{hc}c^2 + 4\,a + 2\,c = 0.$$

Case 3: For $f(u_t) = k$ and $g(u) = (au + b)^n - k$, the symmetries admitted by the equation are

$$\mathbf{v}_1, \quad \mathbf{v}_2, \quad \mathbf{v}_3^3 = \frac{1}{2}a(n-1)x\partial_x + \frac{1}{2}a(n-1)t\partial_t + (au + b)\partial_u. \tag{9}$$

The symmetry (9) yields to the one-parameter symmetry transformation group

$$(\hat{x}, \hat{t}, \hat{u})_3^3 = (\mathrm{e}^{1/2\,ak_3\,(n-1)\epsilon}x + \int_0^\epsilon k_1\,\mathrm{e}^{-1/2\,ak_3\,(n-1)z_1}\,\mathrm{d}z_1\mathrm{e}^{1/2\,ak_3\,(n-1)\epsilon}, \mathrm{e}^{1/2\,ak_3\,(n-1)\epsilon}t + \int_0^\epsilon k_2\,\mathrm{e}^{-1/2\,ak_3\,(n-1)z_1}\,\mathrm{d}z_1\mathrm{e}^{1/2\,ak_3\,(n-1)\epsilon},$$
$$\mathrm{e}^{-ak_3\,\epsilon}u + \int_0^\epsilon -bk_3\,\mathrm{e}^{ak_3\,z_1}\,\mathrm{d}z_1\mathrm{e}^{-ak_3\,\epsilon}), \text{scaling and shift.}$$

Substituting the infinitesimals into the invariant surface condition, the invariant solutions obtained are

$$z = \frac{x}{t}, \qquad u = \frac{1}{c}\left(t^{-\frac{2}{n-1}}h(z) - b\right).$$

These variables reduce the equation into the following ODE:

$$\begin{aligned}&\left(n^2z^2 - 2nz^2 - n^2 + z^2 + 2n\right) h'' - h''h + \left(2n^2z - 2za\right) h'\\ &+ \left(an^2 - 2an + a\right)(h^n) + (2n+2)\,h = 0.\end{aligned}$$

Case 4: For $f(u_t) = au_t^{\frac{4}{3}}$ and $g(u) = (cu + k)^2$, the symmetries obtained are

$$\mathbf{v}_1, \quad \mathbf{v}_2, \quad \mathbf{v}_3^4 = -\frac{1}{2}c\,x\partial_x - \frac{1}{2}c\,t\partial_t + (c\,u + k)\partial_u. \tag{10}$$

The symmetry (10) yields to the one-parameter symmetry transformation group

$$(\hat{x}, \hat{t}, \hat{u})_3^4 = (x\mathrm{e}^{-1/2\,c\epsilon}, t\mathrm{e}^{-1/2\,c\epsilon}, \mathrm{e}^{c\epsilon}u + \int_0^\epsilon k\mathrm{e}^{-cz_1}\,\mathrm{d}z_1\mathrm{e}^{c\epsilon}), \text{ scaling and shift.}$$

Substituting the infinitesimals into the invariant surface condition, the invariant solutions obtained are

$$z = \frac{x}{t}, \qquad u = \frac{1}{c}\left(t^{-2}h(z) - k\right).$$

These variables reduce the equation into the following ODE:

$$\left(z^2 - 1\right) h'' + \left(-\sqrt[3]{\frac{-h'z - 2\,h}{c}}az + 6z\right) h' + c\,h^2 + \left(-2\,\sqrt[3]{\frac{-h'z - 2\,h}{c}}a + 6\right) h = 0.$$

3 | TRAVELLING WAVES

In this section we are studying Equation (6) in order to find travelling wave solutions of Equation (1). The other ordinary differential equations obtained are not considered because they are non-autonomous differential equations.
Let us consider the second-order Equation (6)

$$h'' = \frac{1}{1-\lambda^2}f(-\lambda h') + \frac{1}{1-\lambda^2}g(h). \tag{11}$$

We can meet Equation (6) in studying of other mathematical models. For instance, the general solution of a second-order ODE of the form

$$h'' = \frac{1}{\lambda}\left(\mu h' + \frac{1}{2}h^2 - \omega h - c_0\right), \tag{12}$$

with c_0 an arbitrary constant and λ, μ, and ω satisfying $\omega = \frac{6\mu^2}{25\lambda}$, was obtained by Kudryashov[25]. The general solution is given in terms of the Weierstrass elliptic function, with invariants $g_2 = 0$ and $g_3 = c_1$,

$$h(z) = \omega_k + \frac{6\alpha^2}{25\beta} - \exp\left\{\frac{2z\alpha}{5\beta}\right\}\mathcal{P}\left(c_2 - \frac{5\beta}{\alpha\sqrt{12\beta}}\exp\left\{\frac{z\alpha}{5\beta}\right\}, 0, c_1\right), \tag{13}$$

where c_1 and c_2 are arbitrary constants.
Comparison between Equation (11) and Equation (12) shows that these equations are the same if

$$\begin{aligned} f(-\lambda h') &= \tfrac{1-\lambda^2}{\lambda}\,\mu h', \\ g(h) &= \tfrac{1}{\lambda}\left(\tfrac{1}{2}h^2 - \omega h - c_0\right). \end{aligned}$$

Hence, the solutions of Equation (11) and Equation (12) are equal.
In addition, by the same procedure, the authors[26] obtained the general solution of a second-order ODE of the form

$$h'' = \frac{2b(\beta q - p) - \alpha}{b(\beta^2+1)}h' + \frac{m_1 n_1}{b(\beta^2+1)}h^2 - \frac{b(p^2+q^2)+a}{b(\beta^2+1)}h. \tag{14}$$

Thus, we can derive the general solution for this equation for

$$\begin{aligned} f(-\lambda h') &= \tfrac{2b(\beta q - p) - \alpha}{b(\beta^2+1)}(1-\lambda^2)h', \\ g(h) &= \tfrac{m_1 n_1}{b(\beta^2+1)}(1-\lambda^2)h^2 - \tfrac{b(p^2+q^2)+a}{b(\beta^2+1)}(1-\lambda^2)h. \end{aligned}$$

Finally, by undoing the change of variables (5), a exact solution of the non-linear viscoelastic wave Equation (1) is

$$u(x,t) = \omega + \frac{6\alpha^2}{25\beta} - \exp\left\{\frac{2(x-\lambda\, t)\alpha}{5\beta}\right\}\mathcal{P}\left(c_2 - \frac{5\beta}{\alpha\sqrt{12\beta}}\exp\left\{\frac{(x-\lambda\, t)\alpha}{5\beta}\right\}, 0, c_1\right). \tag{15}$$

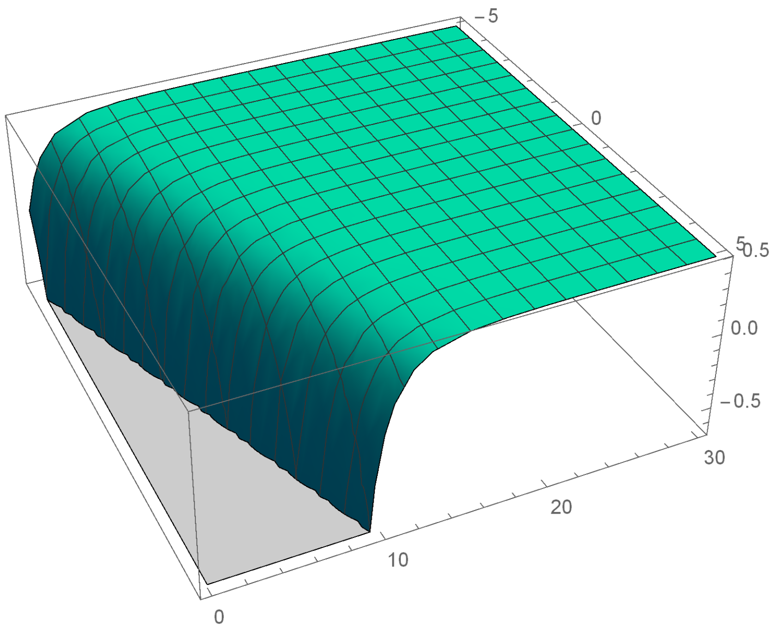

FIGURE 1 Solution (15) for $\lambda = \alpha = \beta = c_1 = c_2 = 1$.

Solution (15) is a soliton (see Figure 1). Furthermore, we can find another solution by using a different procedure.
Let us assume that Equation (6) has a solution of the form

$$h = \alpha\, H^{\beta}(z),$$

where α and β are parameters to be determined later. Here, $H(z)$ is a solution of the Jacobi equation

$$(H')^2 = r + p\,H^2 + q\,H^4, \tag{16}$$

with r, p, and q constants.
H has the expression of an exponential or polynomial function. If H is a solution of Equation (16), we can distinguish three cases: (i) H is the Jacobi elliptic sine function $sn(z,m)$, (ii) H is the Jacobi elliptic cosine function, $cn(z,m)$, (iii) H is the Jacobi elliptic function of the third kind $dn(z,m)$. However, we focus on the first case.
If $H(z) = sn(z,m)$,

$$h(z) = p\, sn^q(z|m) \tag{17}$$

is a solution of (6). Substituting (17) into (6), we obtain the expressions of $f(-\lambda h')$, $g(h)$, and the parameters that make (17) a solution of (6).
This procedure was applied by BruzÃşn and Gandarias[27] to a similar equation, obtaining an exact solution. In the same way, an exact solution of Equation (1) is

$$u(x,t) = p\, sn^q(x - \lambda\, t|m). \tag{18}$$

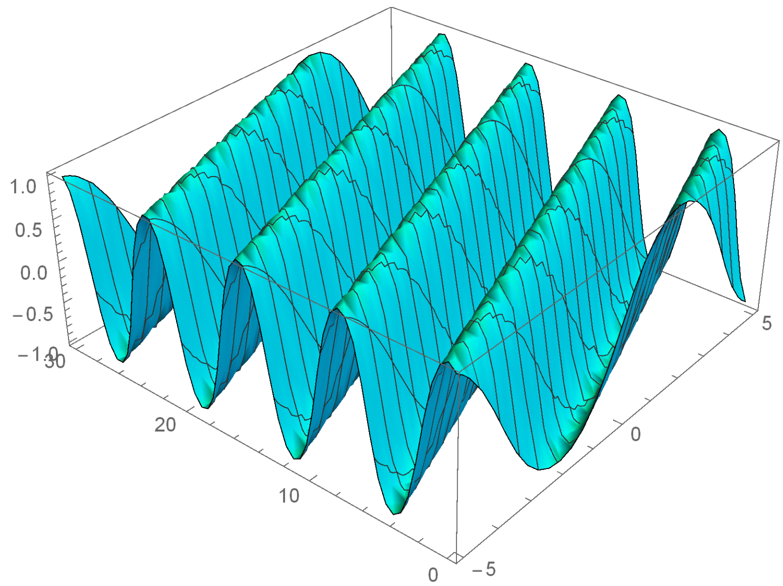

FIGURE 2 Solution (18) for $\lambda = p = q = 1$ and $m = 0.5$.

Solution (18) is shown in Figure 2 . Particularly, the solution is a stable non-linear non-harmonic oscillatory periodic wave.

4 | CONSERVATION LAWS

The concept of a conservation law, which is a mathematical formulation of the familiar physical laws of conservation of energy, conservation of momentum and so on, plays an important role in the analysis of basic properties of the solutions. For instance, invariance of a variational principle under a group of time translations implies the conservation of energy for the solutions of the associated Euler-Lagrange equations, and invariance under a group of spatial translations implies conservation of momentum[18].
Anco and Bluman presented a direct conservation law method for PDEs expressed in normal form. A PDE is in normal form if it can be expressed in a solved form for some leading derivative of u such that all the other terms in the equation contain neither the leading derivative nor its differential consequences[23].
A conservation law admitted by Eq. (1) satisfies the divergence identity

$$D_tT + D_xX = (u_{tt} - u_{xx} + f(u_t) = g(u))Q,$$

called the characteristic equation for the conserved density T and the conserved flux T.
However, the general form for low-order multipliers Q in terms of u and derivatives of u is given by those variables that can be differentiated to obtain a leading derivative of the equation. Clearly, u_{tt} can be obtained by differentiation of u_t with respect to t, and u_{xx} can be obtained by differentiation of u_x with respect to x.
This determines

$$Q(t, x, u, u_t, u_x)$$

as the general form for a low-order multiplier for Eq. (1).
All low-order multipliers can be found by solving the determining equation

$$E_u((u_{tt} - u_{xx} + f(u_t) - g(u))Q) = 0, \tag{19}$$

where E_u represents the Euler operator with respect to u[18], that is

$$E_u = \partial_u - D_x \partial_{u_x} - D_t \partial_{u_t} + D_x D_t \partial_{u_{xt}} + D_x^2 \partial_{u_{xx}} + \cdots .$$

Hence, we write and split the determining equation (19) with respect to u_{xx}, u_{tt}, u_{tx}, yielding an overdetermined system in Q, $f(u_t)$, $g(u)$. The multipliers are found by solving the system with the same algorithmic method used for the determining equation for infinitesimal symmetries. Thus, we obtain a complete classification of multipliers and conservation laws.

TABLE 1 Multipliers admitted by Eq. (1), with $f(u_t) \neq 0$.

$f(u_t)$	$g(u)$	Q
$f_0 u_t + f_1$	arbitrary	$u_x\, e^{f_0 t}$
f_0	arbitrary	u_x, u_t
f_0	$g_1\, e^{g_0 u} - f_0$	u_t, u_x, $tu_t + xu_x + \frac{2}{g_0}$
$-g_0 - \frac{1}{f_0 u_t + f_1}$	g_0	$f_0 u_t u_x + f_1 u_x$
$-\frac{4 f_0}{u_t + f_1} + f_2$	$\frac{4 f_0}{f_1} - f_2$	$u_t + f_1$

The solutions with $f(u_t) \neq 0$ are shown in Table (1).

4.1 | Conservation Laws

The non-trivial low-order conservation laws of the viscoelasticity model $u_{tt} - u_{xx} + f(u_t) = g(u)$, with $f(u_t) \neq 0$, are the following:

• For $f(u_t) = f_0$, $g(u)$ arbitrary function and $Q = u_t$, we obtain the following conservation law:

$$\begin{aligned} T &= \frac{1}{2} {u_x}^2 + \frac{1}{2} {u_t}^2 + \int g\left(u\right) + f_0 \,\mathrm{d}u, \\ X &= -u_t u_x. \end{aligned} \tag{20}$$

• For $f(u_t) = f_0$, $g(u)$ arbitrary function and $Q = u_x$, we obtain the following conservation law:

$$\begin{aligned} T &= u_t u_x, \\ X &= -\frac{1}{2} {u_x}^2 - \frac{1}{2} {u_t}^2 + \int g\left(u\right) + f_0 \,\mathrm{d}u. \end{aligned} \tag{21}$$

• For $f(u_t) = f_0$, $g(u) = g_1\, e^{g_0 u} - f_0$ and $Q = tu_t + xu_x + \frac{2}{g_0}$, we obtain the following conservation law:

$$\begin{aligned} T &= \frac{1}{2 g_0} 2\, t e^{u g_0} g_1 + \left(t {u_t}^2 + t {u_x}^2 + 2\, u_x x u_t\right) g_0 + 4\, u_t, \\ X &= \frac{1}{2 g_0} 2\, x e^{u g_0} g_1 + \left(-2\, t u_t u_x - x {u_t}^2 - {u_x}^2 x\right) g_0 - 4\, u_x. \end{aligned} \tag{22}$$

• For $f(u_t) = f_0 u_t + f_1$, $g(u)$ an arbitrary function and $Q = u_x\, e^{f_0 t}$, we obtain the following conservation law:

$$\begin{aligned} T &= u_x \mathrm{e}^{f_0 t} u_t, \\ X &= \int \mathrm{e}^{f_0 t} \left(g\left(u\right) + f_1\right)\, \mathrm{d}u + \frac{1}{2}\left(-{u_t}^2 - {u_x}^2\right) \mathrm{e}^{f_0 t}. \end{aligned} \tag{23}$$

• For $f(u_t) = -g_0 - \frac{1}{u_t f_0 + f_1}$, $g(u) = g_0$ and $Q = f_0 u_t u_x + f_1 u_x$, we obtain the following conservation law:

$$\begin{aligned} T &= \frac{1}{6} f_0\, {u_x}^3 + \frac{1}{2} f_0\, {u_t}^2 u_x + f_1\, u_x u_t, \\ X &= -\frac{1}{2} f_0\, u_t {u_x}^2 - \frac{1}{2} {u_x}^2 f_1 - \frac{1}{6} f_0\, {u_t}^3 - u - \frac{1}{2} {u_t}^2 f_1. \end{aligned} \tag{24}$$

• For $f(u_t) = -\frac{4 f_0}{u_t + f_1} + f_2$, $g(u) = \frac{4 f_0}{f_1} - f_2$ and $Q = u_t + f_1$, we obtain the following conservation law:

$$\begin{aligned} T &= \frac{1}{2} {u_x}^2 + \frac{1}{2} {u_t}^2 + f_1\, u_t + 4\, \frac{f_0\, u}{f_1}, \\ X &= \left(-u_t - f_1\right) u_x. \end{aligned} \tag{25}$$

Next, we study the meaning of some of these conservation laws. Every conservation law yields a corresponding conserved integral

$$C[u] = \int_{\Omega} T\, dx,$$

where Ω is the domain of solutions $u(t, x, y)$.

For Equation (1) with $f(u_t)$ constant and $g(u)$ nonlinear function, conservation law (20) yields conservation of an energy quantity

$$\mathcal{E}[u] = \int_{\Omega} \frac{1}{2} {u_x}^2 + \frac{1}{2} {u_t}^2 + \int g\left(u\right) + f_0\, \mathrm{d}u\, dx,$$

which represents the total energy for solution $u(x, t)$.

Conservation law (21) yields the conserved quantity

$$\mathcal{M}[u] = \int_{\Omega} u_t u_x\, dx,$$

which is momentum quantities.

5 | CONCLUSIONS

In this paper, we have obtained a complete Lie group classification for the viscoelastic wave equation (1) in the presence of damping and source terms, for different expressions of the functions f and g. Then, we have constructed the corresponding reduced equations. These reductions make easier the resolution of the viscoelastic wave equation (1) in order to obtain solutions of physical interest such as solitons. Moreover, we have obtained these travelling wave solutions from the reduced equations by the comparison between equation (1) and comparable equations studied before by other authors. Furthermore, classical Lie symmetries are not the only ones that can be studied. Another symmetries such as non-classical or potential symmetries can be studied in the future. Finally, we have derived the non-trivial low-order conservation laws by using the multiplier method developed by Anco and Bluman.

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Author contributions

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Conflict of interest

The authors declare no potential conflict of interests.

References

1. Kafini M, Messaoudi SA. A blow up result for a viscoelastic system in R^N. *Electron. J. Differential Equations.* 2006;113:1–7.
2. Kafini M, Messaoudi SA. A blow up result in a Cauchy viscoelastic problem. *Appl. Math. Lett.*. 2008;21(6):549–553.
3. Wang Y. A global nonexistence theorem for viscoelastic equations with arbitrary positive initial energy. *Appl. Math. Lett.*. 2009;22:1394–1400.
4. Kalantarov VK, Ladyzhenskaya OA. The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type. *J. Soviet Math.*. 1978;10:53–70.
5. Haraux A, Zuazua E. Decay estimates for some semilinear damped hyperbolic problems. *Arch. Ration. Mech. Anal.*. 1988;150:191–206.
6. Ball J. Remarks on blow up and nonexistence theorems for nonlinear evolutions equations. *Quart. J. Math. Oxford.* 1977;28(2):473–486.
7. Messaoudi SA. Blow up of solutions with positive initial energy in a nonlinear viscoelastic wave equations. *J. Math. Anal. Appl.*. 2006;320:902–915.
8. Messaoudi SA, Said-Houari B. Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms. *J. Math. Anal. Appl.*. 2010;365:277–287.
9. Levine HA. Instability and nonexistence of global solutions to nonlinear wave equations of the form. *Trans. Amer. Math. Soc.*. 1974;192:1–21.
10. Levine HA. Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. *J. Math. Anal.*. 1974;5:138–146.
11. Messaoudi SA. Blow up and global existence in nonlinear viscoelastic wave equations. *Math. Nachr.*. 2003;260:58–66.
12. Georgiev V, Todorova G. Existence of a solution of the wave equation with nonlinear damping and source term. *J. Differential Equations.* 1994;109:295–308.
13. Messaoudi SA. Blow up in a nonlinearly damped wave equation. *Math. Nachr.*. 2001;231:1–7.
14. Abramowitz M, Stegun IA. *Handbook of Mathematical Functions*. New York: Dover; 1972.
15. Bluman GW, Cole JD. The general similarity solution of the heat equation. *J. Math. Mech.*. 1969;18:1025–1042.
16. Bluman GW, Kumei S. *Symmetries and differential equations*. Verlag: Springer; 1989.
17. Champagne B, Hereman W, Winternitz P. The computer calculation of Lie point symmetries of large systems of differential equations. *Com. Phys. Comm.*. 1991;66:319–340.
18. Olver PJ. *Applications of Lie groups to differential equations*. Verlag: Springer; 1986.
19. Ovsiannikov LV. *Group Analysis of Differential Equations*. New York: Academic; 1982.

20. Rosa M, Camacho JC, Bruzón MS, Gandarias ML. Classical and potential symmetries for a generalized Fisher equation. *Journal of Computational and Applied Mathematics.* 2017;318:181–188.

21. Khalique CM, Adem KR. Exact solutions of the (2+1)-dimensional ZakharovâĂŞKuznetsov modified equal width equation using Lie group analysis. *Mathematical and Computer Modelling.* 2011;54:184–189.

22. TracinÃă R, BruzÃşn MS, Gandarias ML, Torrisi M. Nonlinear self-adjointness, conservation laws, exact solutions of a system of dispersive evolution equations. *Communications in Nonlinear Science and Numerical Simulation.* 2014;19:3036–3043.

23. Anco S. Symmetry properties of conservation laws. *International Journal of Modern Physics B.* 2016;30(28–29):1640003–1640015.

24. Anco S. Generalization of Noether's theorem in modern form to non-variational partial differential equations, in Recent progress and Modern Challenges in Applied Mathematics. *Modeling and Computational Science, Fields Institute Communications.* 2016;:1–51.

25. Kudryashov N. On "new travelling wave solutions" of the KdV and the KdV-Burgers equations. *Commun Nonlinear Sci Numer Simulat.* 2009;:1891–1900.

26. Bruzón MS, Garrido TM, Recio E, Rosa R. Lie symmetries and travelling wave solutions of the nonlinear waves in the inhomogeneous Fisher-Kolmogorov equation. *Math Meth Appl Sci..* 2019;2:1–9.

27. Bruzón MS, Gandarias ML. Travelling wave solutions for a generalized double dispersion equation. *Nonlinear Analysis.* 2009;71:e2109–e2117.

28. Oliveri F. Lie Symmetries of Differential Equations: Classical Results and Recent Contributions. *Symmetry.* 2010;2:658–706.

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