A self-similar solution to time-fractional Stefan problem

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Abstract We derive the fractional version of one-phase one-dimensional Stefan model. We assume that the diffusive flux is given by the time-fractional Riemann-Liouville derivative, i.e. we impose the memory effect in the examined model. Furthermore, we find a special solution to this problem.

Key words: fractional derivatives, Stefan problem, self-similar solution.

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1 Introduction

The purpose of this paper is to study the process of changing the phase of medium, in which the diffusion exhibits non-local in time effects. We are motivated by the paper [2], where the authors represent the non-locality in time, assuming that the diffusive flux is given in the form of time-fractional Riemann-Liouville derivative of temperature gradient, i.e.

$$q^*(x,t) = -\partial^{1-\alpha} T_x(x,t). \tag{1}$$

Based on this assumption, the authors derived the sharp-interphase as well as the diffusive interphase fractional Stefan model. The sharp-interphase model obtained in [2] is characterized by the replacement of time derivative by fractional Caputo derivative. In last years several attempts to solve this problem have been done. In [4] the authors proved the existence of weak solutions in non-cylindrical domain with fixed boundary. In [10] under suitable regularity assumptions the Hopf lemma was proven. It is also worth to mention the paper [3] were the special solution to this problem was found. However, due to the lack of regularity results, the time-fractional Stefan problem with the Caputo derivative have not been solved.

It has been noticed already in [11] that the obtained diffusive-interphase model does not converge to the sharp one. This result encouraged the researchers to investigate the time-fractional Stefan model more deeply. In papers [1], [6] - [9] the authors discussed other possible formulations of time-fractional Stefan problem and compare the formulas for special solutions. In paper [5] there is shown that the time-fractional sharp-interphase model obtained in [2] is not a consequence of the assumption (1). Moreover, the authors obtained a new model based on (1). In this paper, we derive the sharp-interphase model with non-local flux given by (1) under mild regularity assumptions. We arrive at the similar model as in [5], however we obtain additional boundary condition. At last, we find a self-similar solution to this problem, which is the main result of this paper.

2 Formulation of the problem

In the paper we discuss one-dimensional domain $\Omega = (0, L)$ for a positive L. We assume that at the initial time $t = 0$ the domain Ω is divided onto two parts: $(0, x_0)$ - "liquid" and (x_0, L) - "solid". In particular, we admit the case where $x_0 = 0$. Following [2] we define the enthalpy function by $E = T + \phi$, where $T(x,t)$ is the temperature at point $x \in \Omega$ at time t and ϕ represents the latent heat. We consider the sharp-interface model, hence we assume that ϕ is given in the following form

$$\phi = \begin{cases} 1 & \text{in liquid,} \\ 0 & \text{in solid.} \end{cases} \tag{2}$$

We shall consider the one-phase model, i.e. we assume that $T \equiv 0$ in "solid" part. We denote by $q^*(x,t)$ the flux at $x \in \Omega$ at time t. In this setting, the principle of energy conservation takes the following form: for every $V = (a, b) \subseteq \Omega$

$$\frac{d}{dt} \int_V E(x,t)dx = q^*(a,t) - q^*(b,t). \tag{3}$$

We may easily see that if the model does not exhibit memory effects then identity (3) leads to classical one-phase Stefan problem. We state this result in the remark.

Remark 1. *If the flux is defined by the Fourier law $q^*(x,t) = -T_x(x,t)$, then (3) leads to the classical Stefan problem*

$$\frac{d}{dt}T(x,t) - T_{xx}(x,t) = 0 \quad \textit{for} \quad t > 0 \quad \textit{and} \quad x \in (0, L) \setminus \{s(t)\}, \tag{4}$$

$$\dot{s}(t) = -T_x^-(s(t),t) \quad \textit{for} \quad t > 0, \tag{5}$$

where $s(t)$ denotes an interface and

$$T_x^-(s(t),t) = \lim_{\varepsilon \to 0^+} T_x(s(t) - \varepsilon, t).$$

In order to study non-local model we recall the definitions of fractional operators. By I^{α}_{a} we denote the fractional integral given by

$$I^{\alpha}_{a} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{6}$$

We also introduce the Riemann-Liouville and Caputo fractional derivatives defined respectively by

$$\partial^{\alpha}_{a} f(t) = \frac{d}{dt} I^{1-\alpha}_{a} f(t), \quad D^{\alpha}_{a} f = \frac{d}{dt} I^{1-\alpha}_{a} [f(t) - f(a)].$$

If a subscript $a = 0$ we omit it in a notation. Following [2], we assume that the flux is given by the Riemann-Liouville fractional derivative with respect to the time variable, i.e.

$$q^{*}(x,t) = -\partial^{1-\alpha} T_x(x,t),$$

where

$$\partial^{1-\alpha} T_x(x,t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_{0}^{t} (t-\tau)^{\alpha-1} T_x(x,\tau) d\tau, \ \ \alpha \in (0,1).$$

We finish this section with a formal justification, why such a form of the flux seems to be reasonable in the model exhibiting memory effects.

Remark 2. *Let us denote by $s(t)$ the phase interface. We decompose the domain Ω on the solid and liquid parts.*

$$\Omega_l(t) = (0, s(t)) \ \text{ - liquid}, \quad \Omega_s(t) = (s(t), L) \ \text{ - solid}.$$

Let $V \subseteq \Omega$ be arbitrary. Then, if we assume that $V = (a,b)$ and denote

$$V_l(t) = \Omega_l(t) \cap V, \quad V_s(t) = \Omega_s(t) \cap V$$

then, (3) takes the form

$$\frac{d}{dt}\left[\int_{V_l(t)} (T(x,t)+1) dx\right] + \frac{d}{dt}\left[\int_{V_s(t)} T(x,t) dx\right] = \partial^{1-\alpha} T_x(b,t) - \partial^{1-\alpha} T_x(a,t). \tag{7}$$

Assuming that the temperature gradient is bounded with respect to time variable, after integrating with resect to time we arrive at

$$\int_{V_l(t)} (T(x,t)+1) dx + \int_{V_s(t)} T(x,t) dx = \int_{V_l(0)} (T(x,0)+1) dx + \int_{V_s(0)} T(x,0) dx$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} \left[T_x(b,\tau) - T_x(a,\tau)\right] d\tau, \tag{8}$$

i.e. the total enthalpy in V at time t is a sum of the initial enthalpy and the time-average of differences of local fluxes at the endpoints of V.

3 Main results

We derive the fractional Stefan model from the balance law (3) with the diffusive flux given by (1). In order to do it rigorously we have to impose certain regularity conditions on the phase interface s and the temperature function T. At first, we assume that t^* is positive and

$$\begin{gathered} s(t) \in AC[0,t^*], \;\; T_x(x,\cdot) \in L^\infty(U^x) \;\text{ for every } x \in \Omega, \\ T_x(\cdot,t) \in AC[0,s(t)-\varepsilon] \;\text{ for every } \varepsilon > 0 \;\text{ and every } t \in (0,t^*), \\ T_t(\cdot,t) \in L^1(0,s(t)) \quad \text{for each } t \in (0,t^*), \end{gathered} \tag{A1}$$

where we denote

$$\begin{gathered} Q_{s,t^*} = \{(x,t): \;\; 0 < x < s(t), \;\; t \in (0,t^*)\}, \\ U^x = \{t: \;\; (x,t) \in Q_{s,t^*}\}. \end{gathered}$$

Here and henceforth by AC we denote the space of absolutely continuous functions.
The standard setting of the initial-boundary condition for the Stefan problem is the following

$$T(x,0) = T_0(x) \geq 0 \;\text{ and }\; T(0,t) = T_D(t) \geq 0 \quad \text{or} \quad T_x(0,t) = T_N(t) \leq 0.$$

We expect that if T_0, $T_D \equiv 0$ or T_0, $T_N \equiv 0$, then $T \equiv 0$. Otherwise, we expect

$$\dot{s}(t) > 0, \tag{A2}$$

i.e. melting of solid. We note that since we consider one-phase Stefan problem the temperature in the solid vanishes. Therefore, the flux is nonzero only in the liquid part of the domain, i.e. in Q_{s,t^*} and it is given by the formula

$$q^*(x,t) = \begin{cases} -\partial^{1-\alpha}_{s^{-1}(x)} T_x(x,t) & \text{for} \quad (x,t) \in Q_{s,t^*}, \\ 0 & \text{for} \quad (x,t) \notin Q_{s,t^*}, \end{cases} \tag{9}$$

where

$$\partial^{1-\alpha}_{s^{-1}(x)} T_x(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{\alpha-1} T_x(x,\tau) d\tau & \text{for} \quad x \leq s(0), \\ \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_{s^{-1}(x)}^t (t-\tau)^{\alpha-1} T_x(x,\tau) d\tau & \text{for} \quad x > s(0). \end{cases} \tag{10}$$

This together with (2) leads to the following form of equality (3)

$$\frac{d}{dt}\left[\int_{V_l(t)} T(x,t) + 1 dx\right] = -q^*(b,t) + q^*(a,t). \tag{11}$$

The last of the regularity assumptions, that we will make advantage of, are

$$\dot{s}(t) \in L^\infty_{loc}((0,t^*]) \;\text{ and }\; D^\alpha_{s^{-1}(x)} T(\cdot,t) \in L^1(0,s(t)) \;\text{ for } t \in (0,t^*), \tag{A3}$$

Now we are ready to formulate the first result of this paper.

Theorem 1. *Let us discuss the sharp one-phase one-dimensional Stefan problem with the boundary condition $T(s(t),t)=0$. Then, under the assumptions (A1)-(A2), the conservation law (3) with the flux given by (1) leads to the following equation*

$$D^{\alpha}_{s^{-1}(x)}T(x,t)dx - T_{xx}(x,t) = \begin{cases} 0 & \textit{for} \quad x < s(0) \\ -\frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha} & \textit{for} \quad x \in (s(0), s(t)) \end{cases} \tag{12}$$

for a.a. $(x,t) \in Q_{s,t^*}$, *where*

$$D^{\alpha}_{s^{-1}(x)}T(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{d}{d\tau}T(x,\tau)d\tau & \textit{for} \quad x \le s(0) \\ \frac{1}{\Gamma(1-\alpha)}\int_{s^{-1}(x)}^t (t-\tau)^{-\alpha}\frac{d}{d\tau}T(x,\tau)d\tau & \textit{for} \quad x > s(0). \end{cases} \tag{13}$$

Moreover, functions T and s are related by the formula

$$\dot{s}(t) = -\frac{1}{\Gamma(\alpha)} \lim_{a \nearrow s(t)} \left[\frac{d}{dt} \int_{s^{-1}(a)}^t (t-\tau)^{\alpha-1} T_x(a,\tau)d\tau \right]. \tag{14}$$

Furthermore, if (A3) holds, then the additional boundary condition

$$T_x^-(s(t),t) = 0, \tag{15}$$

is satisfied, where T_x^- is defined as in Remark 1.

Remark 3. *We were informed by prof. Andrea N. Ceretani, that the equation (12) with the condition (14) have been already obtained in [5]. It is worth to mention that the fractional Stefan problem with the flux given by the Riemann-Liouville derivative were considered in [2]. However, the Authors obtained the following system of equations*

$$D^{\alpha}T(x,t)dx - T_{xx}(x,t) = 0, \tag{16}$$

$$D^{\alpha}s(t) = -T_x(s(t),t), \tag{17}$$

(see (17)and (18) in [2]). As pointed out in [5] and proved by careful calculations, the equations (16) and (17) are not the consequences of the assumptions imposed on the flux.

In this paper we present another derivation of (12) and (14), which leads to the additional boundary condition (15). Then, the following questions arise:

- *is the assumption (A3) to strong and this is way it implies "unexpected" boundary condition (15)?*
- *is there any relation between (14) and (15)?*

We partially answer to these questions. We show that, at least in the class of self-similar solutions, (A3) is satisfied and (14) implies (15).

Remark 4. *Passing formally with α to 1 in equations (12) and (14) we arrive at (4) - (5). Indeed, assuming that $T(s(t),t)=0$ we get*

$$D^{\alpha}_{s^{-1}(x)}T(x,t) + \frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha} = \partial^{\alpha}_{s^{-1}(x)}[T(x,t)+1] \to T_t(x,t) \quad \textit{as } \alpha \to 1.$$

Moreover, by (14)

$$\dot{s}(t) = -\lim_{a \to s(t)} \partial^{1-\alpha}_{s^{-1}(a)} T_x(a,t) \to -T_x^-(s(t),t) \quad \textit{as } \alpha \to 1.$$

Now we will present the second result of this paper. We will find a self-similar solution to the time-fractional Stefan problem in the domain

$$U=\{(x,t)\in\mathbb{R}\times(0,\infty):\ \ 0<x<s(t)\}, \tag{18}$$

where $(s(t),t)$ is the curve separating the phases. We impose a constant positive Dirichlet boundary condition on the left boundary and we assume that $s(0)=0$. In this case, the problem formulated in Theorem 1 takes the following form

$$D^{\alpha}_{s^{-1}(x)}u(x,t)=u_{xx}(x,t)-\frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\quad \text{in}\quad U, \tag{19}$$

$$u(s(t),t)=0, \tag{20}$$

$$u(0,t)=\gamma, \tag{21}$$

$$\dot{s}(t)=-\frac{1}{\Gamma(\alpha)}\lim_{a\nearrow s(t)}\frac{d}{dt}\left[\int_{s^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u_x(a,\tau)d\tau\right]. \tag{22}$$

Theorem 2. *For any $\gamma>0$ there exists a pair (u,s) which satisfies (19)-(22). Furthermore, the solution is given by*

$$s(t)=c_1t^{\frac{\alpha}{2}}, \tag{23}$$

$$u(x,t)=\int_{xt^{-\frac{\alpha}{2}}}^{c_1}H(p,xt^{-\frac{\alpha}{2}})G_{c_1}(p)dp\quad in\quad U, \tag{24}$$

where $c_1=c_1(\alpha,\gamma)>0$ and

$$G_{c_1}(y)=\frac{1}{\Gamma(1-\alpha)}\int_y^{c_1}(1-c_1^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}})^{-\alpha}d\mu\quad for\quad 0\le y\le c_1, \tag{25}$$

$$H(p,x)=1+\int_x^pN(p,y)dy\quad for\quad 0\le x\le p, \tag{26}$$

$$N(p,y)=\sum_{n=1}^{\infty}M_n(p,y)\quad for\quad 0\le y\le p, \tag{27}$$

where

$$M_1(p,y)=\frac{1}{\Gamma(1-\alpha)}\int_y^p(1-p^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}})^{-\alpha}d\mu\quad for\quad 0\le y\le p \tag{28}$$

and

$$M_n(p,y)=\int_y^pM_1(a,y)M_{n-1}(p,a)da\quad for\quad 0\le y\le p\quad and\quad n\ge 2. \tag{29}$$

For every $R>0$ the series (27) converges uniformly on $W_R=\{(p,y):\ \ 0\le y\le p\le R\}$. Functions M_n, N are positive on $\{(p,y):\ \ 0\le y<p\}$, hence u is positive in U. For every $a,\lambda>0$ function u satisfies the scaling property

$$u(x,t)=u(\lambda^ax,\lambda^{\frac{2a}{\alpha}}t) \tag{30}$$

and

$$u_x(s(t),t) = 0. \tag{31}$$

Furthermore, for every $t>0$ there hold $u_x(\cdot,t) \in C([0,s(t)])$, $u_{xx}(\cdot,t) \in L^1(0,s(t))$ and for every $x>0$ there holds $u_t(x,\cdot) \in C([s^{-1}(x),\infty))$. Finally, for every $x>0$ we have $u_x(x,\cdot) \in L^\infty(s^{-1}(x),\infty) \cap C([s^{-1}(x),\infty))$ and for every $t>0$ the hold $u_t(\cdot,t) \in L^1(0,s(t))$ and $D^\alpha_{s^{-1}(\cdot)} u(\cdot,t) \in L^1(0,s(t))$. In particular, the pair (u,s) satisfies the assumptions (A1) - (A3).

Corollary 1. *The Dirichlet condition (21) may be replaced by the Neumann condition*

$$u_x(0,t) = -\beta t^{-\frac{\alpha}{2}}, \;\; \beta > 0.$$

Then, Theorem 2 holds with $c_1 = c_1(\alpha,\beta) > 0$.

4 Derivation of the model

In this section we will prove Theorem 1.

Proof of Theorem 1. In order to derive the system of equations from (11), we apply the principle of energy conservation to an arbitrary subset V of the domain at time $t \in (0,t^*)$. We will consider two cases.

- If $V = (a,b) \subseteq (0,s(0))$, then from (A2) we have $V \subseteq (0,s(t))$ for each $t \in (0,t^*)$ and (11) gives

  $$\frac{d}{dt}\left[\int_V T(x,t) + 1 dx\right] = \partial^{1-\alpha} T_x(b,t) - \partial^{1-\alpha} T_x(a,t).$$

  Hence,

  $$\int_V \frac{d}{dt} T(x,t) dx = \partial^{1-\alpha} T_x(b,t) - \partial^{1-\alpha} T_x(a,t).$$

  We apply the fractional integral $I^{1-\alpha}$ with respect to the time variable to both sides of the identity and with a use of assumption (A1) we arrive at

  $$\int_V D^\alpha T(x,t) dx = T_x(b,t) - T_x(a,t).$$

  By the fundamental theorem of calculus we obtain

  $$\int_V [D^\alpha T(x,t) - T_{xx}(x,t)] dx = 0.$$

  Since $V \subseteq (0,s(0))$ is arbitrary, we get

  $$D^\alpha T(x,t) - T_{xx}(x,t) = 0 \qquad \text{for} \quad (x,t) \in (0,s(0)) \times (0,t^*). \tag{32}$$

- If $V=(a,b)$, where $s(0)<a<s(t)<b$, then (11) has the form

$$\frac{d}{dt}\left[\int_a^{s(t)} T(x,t)+1dx\right]=q^*(a,t)=-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_{s^{-1}(a)}^t (t-\tau)^{\alpha-1}T_x(a,\tau)d\tau.$$

Differentiating the integral on the left hand side leads to

$$\int_a^{s(t)}\frac{d}{dt}T(x,t)dx+\dot{s}(t)[T(s(t),t)+1]=-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_{s^{-1}(a)}^t (t-\tau)^{\alpha-1}T_x(a,\tau)d\tau.$$

Applaying $T(s(t),t)=0$, we get

$$\int_a^{s(t)}\frac{d}{dt}T(x,t)dx+\dot{s}(t)=-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_{s^{-1}(a)}^t (t-\tau)^{\alpha-1}T_x(a,\tau)d\tau. \tag{33}$$

If $a\nearrow s(t)$, then by the assumption $(A1)$ the first term vanishes and as a consequence we get (14). Next, if we apply the operator $I^{1-\alpha}_{s^{-1}(a)}$ (defined in (6)) to both sides of (33), then we obtain

$$\frac{1}{\Gamma(1-\alpha)}\int_{s^{-1}(a)}^t (t-\tau)^{-\alpha}\int_a^{s(\tau)}\frac{d}{d\tau}T(x,\tau)dxd\tau+\frac{1}{\Gamma(1-\alpha)}\int_{s^{-1}(a)}^t (t-\tau)^{-\alpha}\dot{s}(\tau)d\tau$$

$$=-\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(1-\alpha)}\int_{s^{-1}(a)}^t (t-\tau)^{-\alpha}\frac{d}{d\tau}\int_{s^{-1}(a)}^{\tau}(\tau-p)^{\alpha-1}T_x(a,p)dpd\tau. \tag{34}$$

We note that by the assumption (A1) we have $T_x(a,\cdot)\in L^\infty(U^a)$ hence, the right hand side of (34) may be written in the form

$$-I^{1-\alpha}_{s^{-1}(a)}\frac{d}{dt}\left[I^{\alpha}_{s^{-1}(a)}T_x(a,\cdot)(t)\right]=-\frac{d}{dt}\left[I^{1-\alpha}_{s^{-1}(a)}I^{\alpha}_{s^{-1}(a)}T_x(a,\cdot)(t)\right]=-T_x(a,t).$$

If we apply the Fubini theorem to the first term in (34), then we arrive at the identity

$$\int_a^{s(t)}D^{\alpha}_{s^{-1}(x)}T(x,t)dx+\frac{1}{\Gamma(1-\alpha)}\int_{s^{-1}(a)}^t (t-\tau)^{-\alpha}\dot{s}(\tau)d\tau=-T_x(a,t). \tag{35}$$

Applying the substitution $\tau=s^{-1}(x)$ we get

$$\frac{1}{\Gamma(1-\alpha)}\int_{s^{-1}(a)}^t (t-\tau)^{-\alpha}\dot{s}(\tau)d\tau=\frac{1}{\Gamma(1-\alpha)}\int_a^{s(t)}(t-s^{-1}(x))^{-\alpha}dx.$$

We expect that $T_x(\cdot,t)$ may admit singular behaviour near the phase change point. Thus, we proceed very carefully. We fix $\varepsilon>0$ such that $a<s(t)-\varepsilon$, then, by (A1) we have

$$-T_x(a,t)=\int_a^{s(t)-\varepsilon}T_{xx}(x,t)dx-T_x(s(t)-\varepsilon,t).$$

Making use of this identity in (35) we obtain

$$\int_a^{s(t)-\varepsilon}\left[D^{\alpha}_{s^{-1}(x)}T(x,t)dx - T_{xx}(x,t)+\frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx$$

$$= -\int_{s(t)-\varepsilon}^{s(t)}\left[D^{\alpha}_{s^{-1}(x)}T(x,t)dx+\frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx - T_x(s(t)-\varepsilon,t). \quad (36)$$

Let us choose arbitrary $\tilde{a}$ such that $s(0)<\tilde{a}<a$. Repeating the above calculations for $\tilde{a}$ instead of a, we obtain that

$$\int_{\tilde{a}}^{s(t)-\varepsilon}\left[D^{\alpha}_{s^{-1}(x)}T(x,t)dx - T_{xx}(x,t)+\frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx$$

$$= -\int_{s(t)-\varepsilon}^{s(t)}\left[D^{\alpha}_{s^{-1}(x)}T(x,t)dx+\frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx - T_x(s(t)-\varepsilon,t). \quad (37)$$

Subtracting the sides of (36) and (37) we arrive at

$$\int_{\tilde{a}}^{a}\left[D^{\alpha}_{s^{-1}(x)}T(x,t)dx - T_{xx}(x,t)+\frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx = 0 \quad (38)$$

for arbitrary $a,\tilde{a}\in(s(0),s(t)-\varepsilon)$ hence, we may deduce that

$$D^{\alpha}_{s^{-1}(x)}T(x,t)dx - T_{xx}(x,t)+\frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}=0 \quad \text{for} \quad x\in(s(0),s(t)), \quad (39)$$

i.e. (12) is proven.

It remains to show (15). From (37) and (39) we infer that

$$0 = -\int_{s(t)-\varepsilon}^{s(t)}\left[D^{\alpha}_{s^{-1}(x)}T(x,t)dx+\frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx - T_x(s(t)-\varepsilon,t).$$

In order to obtain additional information about $T_x(s(t),t)$, we employ further regularity assumptions. Applying (A3) we immediately get

$$\lim_{\varepsilon\to 0^+}\int_{s(t)-\varepsilon}^{s(t)}(t-s^{-1}(x))^{-\alpha}dx = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+}\int_{s(t)-\varepsilon}^{s(t)}D^{\alpha}_{s^{-1}(x)}T(x,t)dx = 0. \quad (40)$$

Making use of (40) we obtain

$$\lim_{\varepsilon\to 0^+}T_x(s(t)-\varepsilon,t)=0, \quad (41)$$

hence, we arrive at (15), which finishes the proof of Theorem 1.

□

5 Self-similar solution

This section is devoted to the proof of Theorem 2. The proof will be divided into a few steps. At first we will proceed with formal calculations that will lead us to appropriate scaling. We introduce parameters a, b, c and we define the function

$$u^\lambda(x,t) = \lambda^c u(\lambda^a x, \lambda^b t). \tag{42}$$

Our aim is to find a, b, c and the curve $(s(t), t)$ such that, if (u, s) is a solution to (19), then $u^\lambda = u$.
At first, we perform calculations. We note that $u_{xx}(x,t) = \lambda^{-c}\lambda^{-2a}u^\lambda_{xx}(\lambda^{-a}x, \lambda^{-b}t)$ and

$$\Gamma(1-\alpha)D^\alpha_{s^{-1}(x)}u(x,t) = \int_{s^{-1}(x)}^t (t-\tau)^{-\alpha}u_t(x,\tau)d\tau = \lambda^{-c}\lambda^{-b}\int_{s^{-1}(x)}^t (t-\tau)^{-\alpha}u^\lambda_t(\lambda^{-a}x, \lambda^{-b}\tau)d\tau$$

$$= \lambda^{-c}\int_{\lambda^{-b}s^{-1}(x)}^{t\lambda^{-b}} (t-\lambda^b p)^{-\alpha}u^\lambda_t(\lambda^{-a}x, p)dp = \lambda^{-c}\lambda^{-b\alpha}\int_{\lambda^{-b}s^{-1}(x)}^{t\lambda^{-b}} (t\lambda^{-b}-p)^{-\alpha}u^\lambda_t(\lambda^{-a}x, p)dp$$

$$= \lambda^{-c}\lambda^{-b\alpha}\Gamma(1-\alpha)D^\alpha_{\lambda^{-b}s^{-1}(x)}u^\lambda(\lambda^{-a}x, \lambda^{-b}t),$$

i.e.

$$D^\alpha_{s^{-1}(\lambda^a x)}u(\lambda^a x, \lambda^b x) = \lambda^{-c}\lambda^{-b\alpha}D^\alpha_{\lambda^{-b}s^{-1}(\lambda^a x)}u^\lambda(x,t).$$

Hence, if the pair (u, s) is a solution to (19), then

$$0 = D^\alpha_{s^{-1}(\lambda^a x)}u(\lambda^a x, \lambda^b t) - u_{xx}(\lambda^a x, \lambda^b t) + \frac{1}{\Gamma(1-\alpha)}(\lambda^b t - s^{-1}(\lambda^a x))^{-\alpha}$$

$$= \lambda^{-c}\lambda^{-b\alpha}D^\alpha_{\lambda^{-b}s^{-1}(\lambda^a x)}u^\lambda(x,t) - \lambda^{-c}\lambda^{-2a}u^\lambda_{xx}(x,t) + \frac{1}{\Gamma(1-\alpha)}\lambda^{-b\alpha}(t-\lambda^{-b}s^{-1}(\lambda^a x))^{-\alpha}.$$

Thus, if we set $c = 0$ and

$$b = \frac{2a}{\alpha}, \tag{43}$$

then we get

$$0 = D^\alpha_{\lambda^{-b}s^{-1}(\lambda^a x)}u^\lambda(x,t) - u^\lambda_{xx}(x,t) + \frac{1}{\Gamma(1-\alpha)}(t-\lambda^{-b}s^{-1}(\lambda^a x))^{-\alpha}.$$

We observe that, if $s(t)$ satisfies

$$s^{-1}(x) = \lambda^{-b}s^{-1}(\lambda^a x), \tag{44}$$

then u and u^λ are the solutions to the same equation. From the identity (43) we infer that $s^{-1}(x) = \lambda^{-\frac{2a}{\alpha}}s^{-1}(\lambda^a x)$. Hence, the function s^{-1} fulfills the functional equation $f(\lambda x) = \lambda^{\frac{2}{\alpha}}f(x)$. To solve this equation, it is enough to write

$$\frac{f(x)-f(\lambda x)}{x(1-\lambda)} = \frac{f(x)}{x}\frac{1-\lambda^{\frac{2}{\alpha}}}{1-\lambda}$$

and take the limit $\lambda \to 1$. Then we get that $f' = \frac{2}{\alpha}\frac{f}{x}$, i.e. $f(x) = cx^{\frac{2}{\alpha}}$. Thus, we obtained that, if there exists a self-similar solution, then the phase interface may have a form

$$s(t) = c_1 t^{\frac{\alpha}{2}} \tag{45}$$

for some positive c_1. If we denote

$$c_0 = c_1^{-\frac{2}{\alpha}}, \tag{46}$$

then we may write

$$s^{-1}(x) = c_0 x^{\frac{2}{\alpha}}. \tag{47}$$

Our aim is to find a special solution u to the system (19), (20), (22), when function s is given by (45). We will proceed as follows. At first, we will rewrite the equations (19), (22) in terms of a new self-similar solution. Subsequently, we will show that in this setting, assuming appropriate regularity of u, condition (22) implies $u_x(s(t), t) = 0$. Then, we will solve the problem

$$D^{\alpha}_{s^{-1}(x)} u(x,t) = u_{xx}(x,t) - \frac{1}{\Gamma(1-\alpha)}(t - s^{-1}(x))^{-\alpha} \quad \text{in} \quad U, \tag{48}$$

$$u(s(t),t) = 0, \;\; u_x(s(t),t) = 0 \quad \text{for} \quad t > 0,$$

with s given by (45). Then, we will show that the solution satisfies (22). In the final section, we will prove that obtained solution is positive and that for every $\gamma > 0$ we may find $c_1 > 0$ such that obtained solution satisfies Dirichlet boundary condition $u(0,t) = \gamma$.

5.1 Similarity variable

Let us begin with introducing a similarity variable

$$\xi = t x^{-\frac{2}{\alpha}}. \tag{49}$$

We define function f as follows

$$f(\xi) = f(t x^{-\frac{2}{\alpha}}) := u(x,t). \tag{50}$$

In the next proposition we establish how the expected regularity properties of u transforms to the properties of f. Furthermore, we will rewrite the conditions (48), (20), (22) in terms of f and prove that (22) implies vanishing of derivative of f in point c_0.

Proposition 1. *Let us assume that s is given by (45) and u satisfies (19), (20), (22). Suppose that u has following regularity. For some $k > 1$ and every $t > 0$ there hold $u_x(\cdot,t) \in L^1(0, s(t))$, $u_{xx}(\cdot,t) \in L^1(s(t)/k, s(t))$. Then, the function f defined by (50) satisfies $f' \in AC([c_0, k^{\frac{2}{\alpha}} c_0])$, $f \in C^2(c_0, k^{\frac{2}{\alpha}} c_0)$ and for $\xi \in (c_0, k^{\frac{2}{\alpha}} c_0)$ there hold*

$$\frac{1}{\Gamma(1-\alpha)} \int_{c_0}^{\xi} (\xi - p)^{-\alpha} f'(p) dp = \left(\frac{2}{\alpha}\right)^2 \xi^2 f''(\xi) + \left[\left(\frac{2}{\alpha}\right)^2 + \frac{2}{\alpha}\right] \xi f'(\xi) - \frac{(\xi - c_0)^{-\alpha}}{\Gamma(1-\alpha)}, \tag{51}$$

$$f(c_0)=0, \tag{52}$$

$$\left(\frac{\alpha}{2}\right)^2 c_0^{-2}\Gamma(\alpha)=\lim_{b\searrow c_0}\frac{d}{db}\left[\int_{c_0}^{b}(b-p)^{\alpha-1}f'(p)dp\right]. \tag{53}$$

The identity (51) together with regularity of f implies

$$\lim_{\xi\searrow c_0}(\xi-c_0)^{\alpha}f''(\xi)=\left(\frac{\alpha}{2}\right)^2\frac{c_0^{-2}}{\Gamma(1-\alpha)}, \tag{54}$$

while from (53) we deduce

$$f'(c_0)=0. \tag{55}$$

Proof. Let us begin with a simple calculation,

$$u_t(x,\tau)=f'(\tau x^{-\frac{2}{\alpha}})x^{-\frac{2}{\alpha}}, \tag{56}$$

$$u_x(x,t)=-\frac{2}{\alpha}f'(tx^{-\frac{2}{\alpha}})tx^{-\frac{2}{\alpha}-1}, \tag{57}$$

$$u_{xx}(x,t)=\left(\frac{2}{\alpha}\right)^2 f''(tx^{-\frac{2}{\alpha}})(tx^{-\frac{2}{\alpha}})^2x^{-2}+\frac{2}{\alpha}(\frac{2}{\alpha}+1)f'(tx^{-\frac{2}{\alpha}})(tx^{-\frac{2}{\alpha}})x^{-2}. \tag{58}$$

Applying the substitution $p=\tau x^{-\frac{2}{\alpha}}$ we get

$$D^{\alpha}_{s^{-1}(x)}u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_{c_0x^{\frac{2}{\alpha}}}^{t}(t-\tau)^{-\alpha}f'(\tau x^{-\frac{2}{\alpha}})x^{-\frac{2}{\alpha}}d\tau$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_{c_0}^{tx^{-\frac{2}{\alpha}}}(t-x^{\frac{2}{\alpha}}p)^{-\alpha}f'(p)dp=x^{-2}\frac{1}{\Gamma(1-\alpha)}\int_{c_0}^{tx^{-\frac{2}{\alpha}}}(tx^{-\frac{2}{\alpha}}-p)^{-\alpha}f'(p)dp.$$

Furthermore, we have

$$(t-c_0x^{\frac{2}{\alpha}})^{-\alpha}=x^{-2}(tx^{-\frac{2}{\alpha}}-c_0)^{-\alpha}.$$

Applying these results in equation (19) with s given by (45), we obtain (51). To show that (52) holds, it is enough to notice that, since the function u vanishes on the free boundary, we have

$$0=u(s(t),t)=u(c_1t^{\frac{\alpha}{2}},t)=f(c_0),$$

where we used (46). Now, we will prove the regularity results. By (57) we get

$$\infty>\int_0^{s(t)}|u_x(x,t)|dx=\frac{2}{\alpha}\int_0^{s(t)}|f'(tx^{-\frac{2}{\alpha}})|tx^{-\frac{2}{\alpha}-1}dx=\int_{c_0}^{\infty}|f'(\xi)|d\xi. \tag{59}$$

From (58) we obtain in the similar way that

$$\infty>\int_{s(t)/k}^{s(t)}|u_{xx}(x,t)|dx=\int_{s(t)/k}^{s(t)}\left|\left(\frac{2}{\alpha}\right)^2 f''(tx^{-\frac{2}{\alpha}})(tx^{-\frac{2}{\alpha}})^2x^{-2}+\frac{2}{\alpha}(\frac{2}{\alpha}+1)f'(tx^{-\frac{2}{\alpha}})(tx^{-\frac{2}{\alpha}})x^{-2}\right|dx$$

$$= \int_{c_0}^{k^{\frac{2}{\alpha}} c_0} \left| \frac{2}{\alpha} f''(\xi) \xi^{1+\frac{\alpha}{2}} t^{-\frac{\alpha}{2}} + (\frac{2}{\alpha}+1) f'(\xi) \xi^{\frac{\alpha}{2}} t^{-\frac{\alpha}{2}} \right| d\xi$$

$$\geq \frac{2}{\alpha} c_0^{1+\frac{\alpha}{2}} t^{-\frac{\alpha}{2}} \int_{c_0}^{k^{\frac{2}{\alpha}} c_0} |f''(\xi)|\, d\xi - (\frac{2}{\alpha}+1) k c_0^{\frac{\alpha}{2}} t^{-\frac{\alpha}{2}} \int_{c_0}^{\infty} |f'(\xi)| d\xi \quad \text{for every } t > 0$$

and as a consequence we obtain

$$\int_{c_0}^{k^{\frac{2}{\alpha}} c_0} |f''(\xi)|\, d\xi < \infty. \tag{60}$$

The estimates (59) and (60) lead to $f' \in AC([c_0, k^{\frac{2}{\alpha}} c_0])$. Making use of the absolute continuity of f' in identity (51) we deduce that $f \in C^2(c_0, k^{\frac{2}{\alpha}} c_0)$. Hence, we obtained postulated regularity results. Now, we shall rewrite the condition (22) in terms of the function f. We will show that it leads to (53). Let us fix $a \in (s(t)/k, s(t))$. Applying the substitution $p = a^{-\frac{2}{\alpha}} \tau$ we get that

$$A \equiv \frac{d}{dt} \left[\int_{s^{-1}(a)}^{t} (t-\tau)^{\alpha-1} u_x(a,\tau) d\tau \right] = -\frac{2}{\alpha} \frac{d}{dt} \left[\int_{c_0 a^{\frac{2}{\alpha}}}^{t} (t-\tau)^{\alpha-1} f'(\tau a^{-\frac{2}{\alpha}}) \tau a^{-\frac{2}{\alpha}-1} d\tau \right]$$

$$= -\frac{2}{\alpha} a^{\frac{2}{\alpha}-1} \frac{d}{dt} \left[\int_{c_0}^{t a^{-\frac{2}{\alpha}}} (t - a^{\frac{2}{\alpha}} p)^{\alpha-1} p f'(p) dp \right] = -\frac{2}{\alpha} a \frac{d}{dt} \left[\int_{c_0}^{t a^{-\frac{2}{\alpha}}} (t a^{-\frac{2}{\alpha}} - p)^{\alpha-1} p f'(p) dp \right].$$

After integrating by parts we obtain

$$A \equiv -\frac{2}{\alpha} a \frac{d}{dt} \left[\int_{c_0}^{t a^{-\frac{2}{\alpha}}} \frac{(t a^{-\frac{2}{\alpha}} - p)^{\alpha}}{\alpha} \left(f'(p) + p f''(p) \right) dp + \frac{(t a^{-\frac{2}{\alpha}} - c_0)^{\alpha}}{\alpha} c_0 f'(c_0) \right].$$

By the continuity of second derivatives of f in $(c_0, k^{\frac{2}{\alpha}} c_0)$ we obtain

$$\lim_{p \nearrow t a^{-\frac{2}{\alpha}}} \frac{(t a^{-\frac{2}{\alpha}} - p)^{\alpha}}{\alpha} \left(f'(p) + p f''(p) \right) = 0.$$

Therefore, we obtain

$$A = -\frac{2}{\alpha} a^{1-\frac{2}{\alpha}} \left[\int_{c_0}^{t a^{-\frac{2}{\alpha}}} (t a^{-\frac{2}{\alpha}} - p)^{\alpha-1} \left(f'(p) + p f''(p) \right) dp + (t a^{-\frac{2}{\alpha}} - c_0)^{\alpha-1} c_0 f'(c_0) \right].$$

Since $f' \in AC([c_0, k^{\frac{2}{\alpha}} c_0])$ we get

$$\lim_{a \nearrow s(t)} \int_{c_0}^{t a^{-\frac{2}{\alpha}}} (t a^{-\frac{2}{\alpha}} - p)^{\alpha-1} f'(p) dp = 0.$$

Applying these results together with (45) in (22) we obtain that

$$\frac{\alpha}{2}c_1 t^{\frac{\alpha}{2}-1} = \frac{1}{\Gamma(\alpha)}\frac{2}{\alpha}c_1^{1-\frac{2}{\alpha}}t^{\frac{\alpha}{2}-1}\lim_{a\nearrow s(t)}\left[\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}pf''(p)dp+(ta^{-\frac{2}{\alpha}}-c_0)^{\alpha-1}c_0f'(c_0)\right]. \tag{61}$$

We note that

$$\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}pf''(p)dp = -\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha}f''(p)dp+ta^{-\frac{2}{\alpha}}\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}f''(p)dp.$$

Moreover,

$$\lim_{a\nearrow s(t)}\left|\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha}f''(p)dp\right| \leq \lim_{a\nearrow s(t)}(ta^{-\frac{2}{\alpha}}-c_0)^{\alpha}\int_{c_0}^{ta^{-\frac{2}{\alpha}}}|f''(p)|\,dp = 0.$$

Making use of this convergence in (61), we obtain

$$\left(\frac{\alpha}{2}\right)^2 c_1^{\frac{2}{\alpha}}\Gamma(\alpha) = c_0\lim_{a\nearrow s(t)}\left[\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}f''(p)dp+(ta^{-\frac{2}{\alpha}}-c_0)^{\alpha-1}f'(c_0)\right],$$

i.e.

$$\left(\frac{\alpha}{2}\right)^2 c_0^{-2}\Gamma(\alpha) = \lim_{b\searrow c_0}\frac{d}{db}\left[\int_{c_0}^{b}(b-p)^{\alpha-1}f'(p)dp\right],$$

where we applied the equality

$$\int_{c_0}^{b}(b-p)^{\alpha-1}f''(p)dp = \frac{d}{db}\left[\int_{c_0}^{b}(b-p)^{\alpha-1}f'(p)dp\right]-(b-c_0)^{\alpha-1}f'(c_0). \tag{62}$$

Thus, we arrive at (53). To prove (54), we notice that from the equation (51) we get

$$\left(\frac{2}{\alpha}\right)^2(\xi-c_0)^{\alpha}\xi^2f''(\xi) = \frac{(\xi-c_0)^{\alpha}}{\Gamma(1-\alpha)}\int_{c_0}^{\xi}(\xi-p)^{-\alpha}f'(p)dp-\left[\left(\frac{2}{\alpha}\right)^2+\frac{2}{\alpha}\right](\xi-c_0)^{\alpha}\xi f'(\xi)+\frac{1}{\Gamma(1-\alpha)}.$$

The function f' is absolutely continuous on some neighborhood of c_0 thus, taking the limit at $\xi = c_0$ we obtain (54).
It remains to show that (53) implies $f'(c_0) = 0$. We note that

$$\frac{d}{db}\int_{c_0}^{b}(b-p)^{\alpha-1}f'(p)dp = \Gamma(\alpha)\partial_{c_0}^{1-\alpha}f'(b).$$

We fix $\varepsilon > 0$. Then, from (53), there exists $x_0 > c_0$ such that for every $x \in (c_0, x_0)$

$$\left(\frac{\alpha}{2c_0}\right)^2-\varepsilon \leq \partial_{c_0}^{1-\alpha}f'(x) \leq \left(\frac{\alpha}{2c_0}\right)^2+\varepsilon.$$

We note that, since f' is absolutely continuous we have $I^{1-\alpha}_{c_0}\partial^{1-\alpha}_{c_0} f' = f'$. Applying $I^{1-\alpha}_{c_0}$ to the above inequalities we obtain that for every $x \in (c_0, x_0)$

$$\left[\left(\frac{\alpha}{2c_0}\right)^2 - \varepsilon\right]\frac{(x-c_0)^{1-\alpha}}{\Gamma(2-\alpha)} \leq f'(x) \leq \left[\left(\frac{\alpha}{2c_0}\right)^2 + \varepsilon\right]\frac{(x-c_0)^{1-\alpha}}{\Gamma(2-\alpha)},$$

hence for every $x \in (c_0, x_0)$

$$\left(\frac{\alpha}{2c_0}\right)^2 - \varepsilon \leq f'(x)(x-c_0)^{\alpha-1}\Gamma(2-\alpha) \leq \left(\frac{\alpha}{2c_0}\right)^2 + \varepsilon.$$

The last pair of inequalities is equivalent with

$$\lim_{x\to c_0}\frac{f'(x)}{(x-c_0)^{1-\alpha}} = \left(\frac{\alpha}{2c_0}\right)^2\frac{1}{\Gamma(2-\alpha)}$$

and in particular $f'(c_0) = 0$. This way we finished the proof of Proposition 1. □

We note that, the converse statement also holds. Reverting the calculations, we obtain the following result.

Corollary 2. *Assume that $k > 1$ and function f is such that $f' \in AC([c_0, k^{\frac{2}{\alpha}}c_0])$, $f \in C^2(c_0, k^{\frac{2}{\alpha}}c_0)$ and for $\xi \in (c_0, k^{\frac{2}{\alpha}}c_0)$ the equality (51) holds. Then $u(x,t) := f(tx^{-\frac{2}{\alpha}})$ satisfies*

$$D^{\alpha}_{s^{-1}(x)}u(x,t) = u_{xx}(x,t) - \frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha} \quad \textit{for} \quad s(t)/k < x < s(t), \;\; 0 < t,$$

where $s(t)$ is given by (45). Furthermore, for every $t > 0$ there hold $u_x(\cdot,t)$, $u_{xx}(\cdot,t) \in L^1(s(t)/k, s(t))$ and for every $x > 0$ there holds $u_t(x,\cdot) \in AC([s^{-1}(x), s^{-1}(kx)])$. If in addition f satisfies (52), then $u(s(t),t) = 0$. Moreover, the condition (53) implies (22). As a consequence, (54) and (55) hold and then $u_x(s(t),t) = 0$.

5.2 Existence of solution

Now, we shall find the solution to the problem (51)-(53). As it was proven in the previous section, if the solution exists, then it also satisfies (55) so, it is convenient to consider the space

$$X_R := \{f \in C^1([c_0, R]) : \;\; f(c_0) = f'(c_0) = 0\},$$

for $R \in (c_0, \infty)$. Firstly, we transform the equation (51) into the weaker form and we obtain the existence of the solution to the transformed equation in the space X_R.

Let us apply the integral I_{c_0} to both sides of (51)

$$I^{2-\alpha}_{c_0} f'(\xi) = \left(\frac{2}{\alpha}\right)^2\int_{c_0}^{\xi}\tau^2 f''(\tau)d\tau + \left[\left(\frac{2}{\alpha}\right)^2 + \frac{2}{\alpha}\right]\int_{c_0}^{\xi}\tau f'(\tau)d\tau - \frac{(\xi-c_0)^{1-\alpha}}{\Gamma(2-\alpha)}.$$

If we integrate by parts and take into account that $f(c_0)=0$, $f'(c_0)=0$, then we obtain

$$I_{c_0}^{1-\alpha} f(\xi) = \left[\left(\frac{2}{\alpha}\right)^2 - \frac{2}{\alpha}\right]\int_{c_0}^{\xi} f(\tau)d\tau - \left[\left(\frac{2}{\alpha}\right)^2 - \frac{2}{\alpha}\right]\xi f(\xi) + \left(\frac{2}{\alpha}\right)^2 \xi^2 f'(\xi) - \frac{(\xi-c_0)^{1-\alpha}}{\Gamma(2-\alpha)}.$$

We apply again I_{c_0} to both sides and integrate by parts to get

$$I_{c_0}^{2-\alpha} f(\xi) = \left[\left(\frac{2}{\alpha}\right)^2 - \frac{2}{\alpha}\right] I_{c_0}^2 f(\xi) - \left[3\left(\frac{2}{\alpha}\right)^2 - \frac{2}{\alpha}\right]\int_{c_0}^{\xi} \tau f(\tau)d\tau + \left(\frac{2}{\alpha}\right)^2 \xi^2 f(\xi) - \frac{(\xi-c_0)^{2-\alpha}}{\Gamma(3-\alpha)}.$$

The above equality has the following form

$$f(\xi) = Kf(\xi) + g(\xi), \tag{63}$$

where

$$Kf(\xi) = \left(\frac{\alpha}{2}\right)^2 \xi^{-2} I_{c_0}^{2-\alpha} f(\xi) + \left[\frac{\alpha}{2} - 1\right]\xi^{-2} I_{c_0}^2 f(\xi) + \left[3 - \frac{\alpha}{2}\right]\xi^{-2}\int_{c_0}^{\xi} \tau f(\tau)d\tau$$

and

$$g(\xi) = \left(\frac{\alpha}{2}\right)^2 \xi^{-2}\frac{(\xi-c_0)^{2-\alpha}}{\Gamma(3-\alpha)}.$$

Proposition 2. *Assume that $R \in (c_0, \infty)$. Then there exists the unique $f \in X_R$ solution to (63). Furthermore, the obtained solution belongs to $C^2((c_0,R))$ and it satisfies (51) on (c_0,R).*

Proof. At first, we note that $g \in X_R$ and the operator K is linear and bounded on X_R. After applying Arzeli-Ascoli theorem we deduce that K is compact operator in X_R hence, by Fredholm alternative the equation (63) has the unique solution provided, the homogeneous equation has only one solution. Indeed, from the estimate

$$|Kf(\xi)| \le \left[\left(\frac{\alpha}{2}\right)^2 c_0^{-2}\frac{(\xi-c_0)^{1-\alpha}}{\Gamma(2-\alpha)} + (1-\frac{\alpha}{2})c_0^{-2}(\xi-c_0) + (3-\frac{\alpha}{2})c_0^{-1}\right]\int_{c_0}^{\xi}|f(\tau)|d\tau$$

and Gronwall lemma we deduce that the only solution in X_R of $f - Kf = 0$ is $f \equiv 0$. Since the right hand side of (63) belongs to $C^2((c_0,R))$, then so does f. Hence, we may invert the calculations leading to identity (63) and we obtain that f satisfies (51) on (c_0,R). □

Proposition 3. *For every $R > 0$ there exists exactly one f belonging to $C^1([c_0,R]) \cap C^2(c_0,R)$ which satisfies the system (51) - (55).*

Proof. It remains to show that the solution obtained in Proposition 2 satisfies (53) and (54). We note that (54) is a simple consequence of (51) and continuity of f'. Let us show (53). We fix $\varepsilon > 0$. Then, by (54) there exists $\xi_0 > c_0$ such that for every $c_0 < \xi < \xi_0$

$$\left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} - \varepsilon \le (\xi-c_0)^{\alpha} f''(\xi) \le \left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} + \varepsilon.$$

Hence, for every $c_0 < \xi < \xi_0$

$$\left(\left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} - \varepsilon\right)(\xi - c_0)^{-\alpha} \leq f''(\xi) \leq \left(\left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} + \varepsilon\right)(\xi - c_0)^{-\alpha}.$$

Applying $\frac{1}{\Gamma(1-\alpha)} I^{\alpha}_{c_0}$ to both these inequalities we obtain that for every $c_0 < \xi < \xi_0$

$$\left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} - \varepsilon \leq \frac{1}{\Gamma(1-\alpha)} I^{\alpha} f''(\xi) \leq \left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} + \varepsilon.$$

Hence,

$$I^{\alpha} f''(\xi) \to \left(\frac{\alpha}{2}\right)^2 c_0^{-2} \quad \text{as } \xi \to c_0.$$

If we recall that $f'(c_0) = 0$, then from (62) we have

$$\lim_{\xi \to c_0} \frac{d}{d\xi} \int_{c_0}^{\xi} (\xi - p)^{\alpha - 1} f'(p) dp = \lim_{\xi \to c_0} \Gamma(\alpha) I^{\alpha} f''(\xi) = \Gamma(\alpha) \left(\frac{\alpha}{2}\right)^2 c_0^{-2}$$

and we arrive at (53). $\square$

From Corollary 2 and Proposition 3 we deduce the following result.

Corollary 3. *Let f be the solution to (51)-(55) given by Proposition 3. Then, for every $k \in (1, \infty)$ function $u(x,t) := f(tx^{-\frac{2}{\alpha}})$ satisfies*

$$D^{\alpha}_{s^{-1}(x)} u(x,t) = u_{xx}(x,t) - \frac{1}{\Gamma(1-\alpha)} (t - s^{-1}(x))^{-\alpha} \quad \textit{for} \quad s(t)/k < x < s(t), \;\; t > 0,$$

$$u(s(t), t) = 0,$$

$$\dot{s}(t) = -\frac{1}{\Gamma(\alpha)} \lim_{a \nearrow s(t)} \frac{d}{dt} \left[\int_{s^{-1}(a)}^{t} (t - \tau)^{\alpha - 1} u_x(a, \tau) d\tau \right],$$

$$u_x(s(t), t) = 0,$$

where $s(t)$ is given by (45). Furthermore, for every $t > 0$ there hold $u_x(\cdot, t)$, $u_{xx}(\cdot, t) \in L^1(s(t)/k, s(t))$ and for every $x > 0$ there holds $u_t(x, \cdot) \in AC([s^{-1}(x), s^{-1}(kx)])$.

Now, we shall examine the positivity of u given in the above corollary. By (54) and the equality

$$f(\xi) = \int_{c_0}^{\xi} (\xi - \tau) f''(\tau) d\tau$$

we deduce that there exists $\xi_0 \in (c_0, R)$ such that $f(\xi) > 0$ for $\xi \in (c_0, \xi_0)$. In the next subsection we shall show that $f(\xi) > 0$ for each $\xi > c_0$ and we determine the limit at infinity.

5.3 Positivity of solution

Proposition 4. *The solution f given by Proposition 3 is positive on (c_0,∞). Furthermore,*

$$f(tx^{-\frac{2}{\alpha}})=\int_{xt^{-\frac{\alpha}{2}}}^{c_1}\sum_{n=0}^{\infty}(L^nG(y))dy,\quad \text{if}\ \ tx^{-\frac{2}{\alpha}}\in[0,c_1], \tag{64}$$

where

$$(Lh)(x):=\frac{1}{\Gamma(1-\alpha)}\int_x^{c_1}\int_\mu^{c_1}(1-p^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}})^{-\alpha}h(p)dpd\mu, \tag{65}$$

$$G(x)=\frac{1}{\Gamma(1-\alpha)}\int_x^{c_1}(1-c_0\mu^{\frac{2}{\alpha}})^{-\alpha}d\mu \tag{66}$$

and the series converges uniformly on $[0,c_1]$. Moreover, if $F(\mu):=f(\mu^{-\frac{2}{\alpha}})$, then $F\in C^1([0,c_1])$ and $F''\in L^1(0,c_1)$.

Proof. In order to prove the positivity of f on (c_0,∞) we have to transform the equation (51). We introduce $\mu:=\xi^{-\frac{\alpha}{2}}$ and

$$F(\mu):=f(\mu^{-\frac{2}{\alpha}})=f(\xi). \tag{67}$$

We note that if $\xi\in(c_0,\infty)$, then $\mu\in(0,c_1)$ and $f(c_0)=f'(c_0)=0$ implies $F(c_1)=F'(c_1)=0$. We will rewrite the identity (51) in terms of function F. We note that

$$F'(\mu)=-\frac{2}{\alpha}\mu^{-\frac{2}{\alpha}-1}f'(\mu^{-\frac{2}{\alpha}}) \tag{68}$$

and

$$F''(\mu)=\frac{2}{\alpha}(\frac{2}{\alpha}+1)\mu^{-\frac{2}{\alpha}-2}f'(\mu^{-\frac{2}{\alpha}})+(\frac{2}{\alpha})^2\mu^{-\frac{2}{\alpha}-1}\mu^{-\frac{2}{\alpha}-1}f''(\mu^{-\frac{2}{\alpha}}).$$

Hence,

$$\mu^2F''(\mu)=\left[\left(\frac{2}{\alpha}\right)^2+\frac{2}{\alpha}\right]\xi f'(\xi)+\left(\frac{2}{\alpha}\right)^2\xi^2f''(\xi).$$

Furthermore,

$$\int_\mu^{c_1}(\mu^{-\frac{2}{\alpha}}-p^{-\frac{2}{\alpha}})^{-\alpha}F'(p)dp=-\frac{2}{\alpha}\int_\mu^{c_1}(\mu^{-\frac{2}{\alpha}}-p^{-\frac{2}{\alpha}})^{-\alpha}p^{-\frac{2}{\alpha}-1}f'(p^{-\frac{2}{\alpha}})dp.$$

Applying the substitution $p^{-\frac{2}{\alpha}}=w$ we get

$$\int_\mu^{c_1}(\mu^{-\frac{2}{\alpha}}-p^{-\frac{2}{\alpha}})^{-\alpha}F'(p)dp=-\int_{c_0}^{\mu^{-\frac{2}{\alpha}}}(\mu^{-\frac{2}{\alpha}}-w)^{-\alpha}f'(w)dw=-\int_{c_0}^{\xi}(\xi-w)^{-\alpha}f'(w)dw.$$

Using this calculations in (51) we get that function F satisfies

$$F''(\mu)=-\frac{1}{\Gamma(1-\alpha)}\mu^{-2}\int_\mu^{c_1}(\mu^{-\frac{2}{\alpha}}-p^{-\frac{2}{\alpha}})^{-\alpha}F'(p)dp+\frac{1}{\Gamma(1-\alpha)}\mu^{-2}(\mu^{-\frac{2}{\alpha}}-c_0)^{-\alpha},$$

which is equivalent with

$$F''(\mu) = -\frac{1}{\Gamma(1-\alpha)}\int_\mu^{c_1}(1-p^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}})^{-\alpha}F'(p)dp + \frac{1}{\Gamma(1-\alpha)}(1-c_0\mu^{\frac{2}{\alpha}})^{-\alpha}. \tag{69}$$

Integrating this equality from x to c_1 and recalling that $F'(c_1)=0$ we get

$$F'(x) = \frac{1}{\Gamma(1-\alpha)}\int_x^{c_1}\int_\mu^{c_1}(1-p^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}})^{-\alpha}F'(p)dpd\mu - \frac{1}{\Gamma(1-\alpha)}\int_x^{c_1}(1-c_0\mu^{\frac{2}{\alpha}})^{-\alpha}d\mu. \tag{70}$$

We are going to obtain an explicit formula for F and we will show that F is positive in $[0,c_1]$. Since f' is continuous in $[c_0,\infty)$ from (68) we may deduce that $F' \in C(0,c_1]$.

Then, identity (70) may be written as

$$F'(x) = (LF')(x) - G(x) \tag{71}$$

where the operator L and function G are defined by (65) and (66), respectively. We apply L to both sides of (71) and we deduce that

$$F'(x) = (L^2F')(x) - (G(x) + LG(x)).$$

Iterating this procedure we obtain that for every $n \in \mathbb{N}$ and every $x \in (0,c_1)$ there holds

$$F'(x) = (L^nF')(x) - \sum_{k=0}^{n}(L^kG)(x). \tag{72}$$

Let us show that for every fixed $x_0 \in (0,c_1)$

$$\lim_{n\to\infty}\max_{x\in[x_0,c_1]}|(L^nF')(x)| = 0. \tag{73}$$

At first we note that for any $x_0 \in [0,c_1]$ and $h \in C([x_0,c_1])$ there holds

$$\|L^nh\|_{C([x_0,c_1])} \le \|h\|_{C([x_0,c_1])}\,\|L^n1\|_{C([x_0,c_1])}. \tag{74}$$

Let us focus on the estimate of L^n1. By the Fubini theorem we have

$$\frac{1}{\Gamma(1-\alpha)}\int_x^{c_1}\int_\mu^{c_1}(1-p^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}})^{-\alpha}dpd\mu = \frac{1}{\Gamma(1-\alpha)}\int_x^{c_1}\int_x^{p}(1-p^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}})^{-\alpha}d\mu dp.$$

We note that

$$\int_x^p(1-p^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}})^{-\alpha}d\mu \le \frac{\alpha}{2}B(\frac{\alpha}{2},1-\alpha)p, \tag{75}$$

where we applied the substitution $w := p^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}}$. Hence, we obtain

$$0 < L1(x) \le \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}c_1(I_x1)(c_1) \quad \text{for} \quad x \in [0,c_1). \tag{76}$$

We shall show by induction that

$$0 < L^n 1(x) \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1\right]^n (I_x^n 1)(c_1) \quad \text{for} \quad x \in [0, c_1) \tag{77}$$

for each $n \in \mathbb{N}$. Indeed, suppose that (77) holds for $n = k-1$ and then we have

$$L^k 1(x) = LL^{k-1}1(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^{c_1} \int_x^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu (L^{k-1}1)(p) dp$$

$$\leq \frac{1}{\Gamma(1-\alpha)} \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1\right]^{k-1} \int_x^{c_1} \int_x^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu (I_p^{k-1} 1)(c_1) dp$$

$$\leq \frac{1}{\Gamma(1-\alpha)} \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1\right]^{k-1} \int_x^{c_1} \frac{\alpha}{2} B(\frac{\alpha}{2}, 1-\alpha) p (I_p^{k-1} 1)(c_1) dp,$$

where in the last inequality we used (75). Thus, we have

$$L^k 1(x) \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1\right]^k \int_x^{c_1} (I_p^{k-1} 1)(c_1) dp = \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1\right]^k I_x^k 1(c_1)$$

and (77) is proven. We note that

$$(I_x^n 1)(c_1) = \frac{1}{\Gamma(n)} \int_x^{c_1} (c_1 - \tau)^{n-1} d\tau = \frac{(c_1 - x)^n}{n!} \tag{78}$$

hence, by (77) we get

$$0 < L^n 1(x) \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1^2\right]^n \frac{1}{n!}. \tag{79}$$

Applying the estimate (79) in (74) we obtain that

$$\max_{x \in [x_0, c_1]} |(L^n F')(x)| \leq \max_{x \in [x_0, c_1]} |F'(x)| \max_{x \in [x_0, c_1]} |L^n 1(x)|$$

$$\leq \max_{x \in [x_0, c_1]} |F'(x)| \left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1^2\right)^n \frac{1}{n!}$$

and due to the presence of factorial function in the denominator the convergence (73) holds. We will show that the series $\sum_{k=0}^{\infty} (L^k G)(x)$ is uniformly convergent on $[0, c_1]$. Indeed, applying the substitution $w := c_0 \mu^{\frac{2}{\alpha}}$ in the definition of G we obtain that

$$G(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\alpha}{2} c_1 \int_{c_0 x^{\frac{2}{\alpha}}}^1 (1-w)^{-\alpha} w^{\frac{\alpha}{2}-1} dw.$$

Thus,

$$\max_{x \in [0, c_1]} |G(x)| \leq \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1.$$

Applying estimates (74) and (79) for $x_0 = 0$ we arrive at

$$\max_{x\in[0,c_1]} |L^n G(x)| \le \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1 \left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1^2\right)^n \frac{1}{n!} =: a_n.$$

We note that

$$\frac{a_{n+1}}{a_n} = \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1^2 \frac{1}{n+1} \to 0 \quad \text{as } n \to \infty.$$

Hence, by comparison criterion and d'Alembert criterion for convergence of the series we obtain that $\sum_{k=0}^{\infty}(L^k G)(x)$ is uniformly convergent on $[0, c_1]$. Finally, we may pass to the limit in (72) to obtain

$$F'(x) = -\sum_{n=0}^{\infty}(L^n G)(x) \quad \text{for every } x \in [0, c_1], \tag{80}$$

where the right hand side converges uniformly. As a consequence, $F \in C^1([0, c_1])$ and by (69) we get $F'' \in L^1(0, c_1)$.

We note that $L^n G(x) > 0$ for $[0, c_1)$ thus,

$$F' < 0 \quad \text{on } [0, c_1). \tag{81}$$

Applying the fundamental theorem of calculus, we may write

$$F(x) = -\int_x^{c_1} F'(y)dy = \int_x^{c_1} \sum_{n=0}^{\infty}(L^n G)(y)dy \quad \text{for every } x \in [0, c_1]. \tag{82}$$

Thus, we have obtained that F is positive on $[0, c_1)$. We recall that the functions f and F are related by the equality (67) therefore, we proved the claim. □

From Corollary 3 and Proposition 4 we arrive at the following conclusion.

Corollary 4. *Let $c_1 > 0$ and $s(t) = c_1 t^{\frac{\alpha}{2}}$. Let us define*

$$u(x,t) := \int_{xt^{-\frac{\alpha}{2}}}^{c_1} \sum_{n=0}^{\infty}(L^n G(y))dy \quad \textit{for} \quad x \in [0, s(t)], \ \ t > 0,$$

where L and G are given by (65) and (66), respectively. Then, the above series converges uniformly and for every $n \in \mathbb{N}$ there holds $L^n G(y) > 0$ for every $y \in [0, c_1)$. Moreover, $u(x,t)$ satisfies

$$D^{\alpha}_{s^{-1}(x)} u(x,t) = u_{xx}(x,t) - \frac{1}{\Gamma(1-\alpha)}(t - s^{-1}(x))^{-\alpha} \quad \textit{for} \quad 0 < x < s(t),$$

$$u(s(t),t) = 0,$$

$$\dot{s}(t) = -\frac{1}{\Gamma(\alpha)} \lim_{a \nearrow s(t)} \frac{d}{dt}\left[\int_{s^{-1}(a)}^{t} (t-\tau)^{\alpha-1} u_x(a,\tau)d\tau\right],$$

$$u_x(s(t),t) = 0,$$

for $t > 0$. Finally, by equality $u(x,t) = F(xt^{-\frac{\alpha}{2}})$ we get: for every $t > 0$ there hold $u_x(\cdot,t) \in C([0,s(t)])$, $u_{xx}(\cdot,t) \in L^1(0,s(t))$ and for every $x > 0$ there holds $u_t(x,\cdot) \in C([s^{-1}(x),\infty))$.

Corollary 5. *Functions u and s defined in Corollary 4 satisfy additionally $u_x < 0$, $u_t > 0$ in $\{(x,t) \in \mathbb{R} \times (0,\infty) : \ 0 < x < s(t)\}$,*

$$\forall \ x > 0 \ \ u_x(x,\cdot) \in L^\infty(s^{-1}(x),\infty) \cap C([s^{-1}(x),\infty)) \tag{83}$$

and

$$\forall \ t > 0 \ \ u_t(\cdot,t) \in L^1(0,s(t)) \quad \text{and} \ \ D^\alpha_{s^{-1}(\cdot)} u(\cdot,t) \in L^1(0,s(t)). \tag{84}$$

In particular, the pair (u,s) satisfies the assumptions (A1) - (A3).

Proof. At first, we recall that

$$u_x(x,t) = t^{-\frac{\alpha}{2}} F'(\mu), \quad u_t(x,t) = -\frac{\alpha}{2} x t^{-\frac{\alpha}{2}-1} F'(\mu),$$

where $\mu = xt^{-\frac{\alpha}{2}}$. Hence, by (81) we infer $u_x < 0$, $u_t > 0$. Since, $F' \in C([0,c_1])$ and for fixed $x > 0$ μ is continuous and bounded function of t on $[s^{-1}(x),\infty)$, we obtain (83). To prove (84), we note that for every $t > 0$

$$\|u(\cdot,t)\|_{L^1(0,s(t))} = \int_0^{s(t)} u_t(x,t)dx = -\frac{\alpha}{2}\int_0^{c_1 t^{\frac{\alpha}{2}}} x t^{-\frac{\alpha}{2}-1} F'(xt^{-\frac{\alpha}{2}})dx = -\frac{\alpha}{2} t^{\frac{\alpha}{2}-1} \int_0^{c_1} pF'(p)dp < \infty,$$

because $F' \in C([0,c_1])$. Using this results we obtain further,

$$\int_0^{s(t)} \left| D^\alpha_{s^{-1}(x)} u(x,t) \right| dx = \frac{1}{\Gamma(1-\alpha)} \int_0^{s(t)} \int_{s^{-1}(x)}^t (t-\tau)^{-\alpha} u_t(x,\tau) d\tau dx$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \int_0^{s(\tau)} u_t(x,\tau)dxd\tau = -\frac{1}{\Gamma(1-\alpha)} \frac{\alpha}{2} \int_0^{c_1} pF'(p)dp \int_0^t (t-\tau)^{-\alpha} \tau^{\frac{\alpha}{2}-1} d\tau$$

$$= -\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} \int_0^{c_1} pF'(p)dp < \infty.$$

Corollary 4 together with (83) and (84) implies that the pair (u,s) satisfies the assumptions (A1) - (A3). □

5.4 Boundary condition

By Corollary 4, for each $c_1 > 0$ we have obtained self-similar solution of time-fractional Stefan problem $(u,s)_{c_1}$ such that

$$u(0,t) = \int_0^{c_1} \sum_{n=0}^{\infty} (L^n G(y))dy. \tag{85}$$

Now, we address to Dirichlet boundary condition (21). We investigate whether for given $\gamma > 0$ it is possible to find $c_1 > 0$ such that $(u,s)_{c_1}$ satisfy (19)-(22).

For this purpose we write explicitly the dependence of solution on c_1. Recall, that from (65), (66) and the Fubini theorem we have

$$(L_{c_1}h)(y) = \frac{1}{\Gamma(1-\alpha)} \int_y^{c_1} \int_y^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu h(p) dp, \tag{86}$$

$$G_{c_1}(y) = \frac{1}{\Gamma(1-\alpha)} \int_y^{c_1} (1 - c_1^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu. \tag{87}$$

The next proposition provides the representation (24) of the self-similar solution.

Proposition 5. *If c_1 is positive and $s(t) = c_1 t^{\frac{\alpha}{2}}$, then for $t > 0$ and $x \in [0, s(t)]$ there holds*

$$\int_{xt^{-\frac{\alpha}{2}}}^{c_1} \sum_{n=0}^{\infty} (L_{c_1}^n G_{c_1}(y)) dy = \int_{xt^{-\frac{\alpha}{2}}}^{c_1} H(p, xt^{-\frac{\alpha}{2}}) G_{c_1}(p) dp, \tag{88}$$

where the function H is defined by (26)-(29). Furthermore, $H - 1$ is positive on the set $W := \{(p, x) : \; 0 \leq x < p\}$ and H is continuous on $\overline{W}$.

Proof. We will find another recursive formula for $L_{c_1}^n G_{c_1}$. For $0 \leq y \leq p < \infty$ we denote

$$M_1(p, y) := \frac{1}{\Gamma(1-\alpha)} \int_y^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu. \tag{89}$$

Then, we may write

$$(L_{c_1}h)(y) = \int_y^{c_1} M_1(p, y) h(p) dp.$$

Further, we obtain

$$(L_{c_1}^2 G_{c_1})(y) = \int_y^{c_1} M_1(p, y)(L_{c_1} G_{c_1})(p) dp = \int_y^{c_1} \int_y^r M_1(p, y) M_1(r, p) dp G_{c_1}(r) dr.$$

Thus, if we denote

$$M_2(r, y) := \int_y^r M_1(p, y) M_1(r, p) dp \tag{90}$$

then,

$$(L_{c_1}^2 G_{c_1})(y) = \int_y^{c_1} M_2(p, y) G_{c_1}(p) dp.$$

By induction we obtain

$$(L_{c_1}^n G_{c_1})(y) = \int_y^{c_1} M_n(p, y) G_{c_1}(p) dp \qquad \text{for} \quad n \geq 1 \tag{91}$$

where we set

$$M_n(p, y) := \int_y^p M_1(a, y) M_{n-1}(p, a) da \qquad \text{for} \quad n \geq 2. \tag{92}$$

Now, we shall obtain the estimate for M_n. By (75) we get

$$M_1(p,y) \le \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} p. \tag{93}$$

Then,

$$M_2(p,y) \le \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\right]^2 p \int_y^p a da \le \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} p\right]^2 (I_y 1)(p).$$

We prove by induction that

$$M_n(p,y) \le \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} p\right]^n (I_y^{n-1} 1)(p), \quad n \ge 2. \tag{94}$$

Indeed, if

$$M_k(p,y) \le \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} p\right]^k (I_y^{k-1} 1)(p),$$

then by (93) we obtain

$$M_{k+1}(p,y) \le \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} p\right]^k \int_y^p \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} a (I_a^{k-1} 1)(p) da$$

$$\le \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} p\right]^{k+1} \int_y^p (I_a^{k-1} 1)(p) da = \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} p\right]^{k+1} (I_y^k 1)(p)$$

hence, we arrive at (94). Applying (78) in (94) we get the following estimate

$$M_n(p,y) \le \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} p\right]^n \frac{p^{n-1}}{(n-1)!} \quad \text{for} \quad n \ge 2. \tag{95}$$

Let us define

$$N(p,y) := \sum_{n=1}^{\infty} M_n(p,y), \;\; 0 \le y \le p < \infty. \tag{96}$$

If $R > 0$, then by (95) the series converges uniformly on the set

$$W_R = \{(p,y) : \;\; 0 \le y \le p \le R\}. \tag{97}$$

In particular, N is continuous, non-negative and bounded on W_R for each R positive.

If we sum over n the sides of (91), then we get

$$\sum_{n=1}^{\infty} L_{c_1}^n G_{c_1}(y) = \int_y^{c_1} N(p,y) G_{c_1}(p) dp. \tag{98}$$

Therefore, we have

$$\int_{xt^{-\frac{\alpha}{2}}}^{c_1} \sum_{n=0}^{\infty} (L_{c_1}^n G_{c_1}(y)) dy = \int_{xt^{-\frac{\alpha}{2}}}^{c_1} G_{c_1}(y) dy + \int_{xt^{-\frac{\alpha}{2}}}^{c_1} \int_y^{c_1} N(p,y) G_{c_1}(p) dp dy.$$

If we denote

$$H(p,x) := 1 + \int_x^p N(p,y)dy \quad \text{for} \quad 0 \le x \le p, \tag{99}$$

then after applying Fubini theorem we obtain (88). $\square$

Now, we shall investigate the dependence of the self-similar solution obtained in Corollary 4 from the parameter c_1. For this purpose we apply the representation given by Proposition 5 and we denote

$$F_{c_1}(x) = \int_x^{c_1} H(p,x) G_{c_1}(p) dp, \tag{100}$$

Recall, that the function H is continuous and bounded. We examine the continuity of the mapping

$$c_1 \mapsto F_{c_1}(x) = \int_x^{c_1} H(p,x) G_{c_1}(p) dp. \tag{101}$$

The precise formulation is stated below.

Proposition 6. *Assume that c_1 is positive. Then for every $x \in [0, c_1]$*

$$\lim_{\bar{c}_1 \searrow c_1} F_{\bar{c}_1}(x) = F_{c_1}(x). \tag{102}$$

Moreover, we have

$$\lim_{c_1 \searrow 0} F_{c_1}(0) = 0 \tag{103}$$

and

$$\lim_{c_1 \nearrow \infty} F_{c_1}(0) = \infty. \tag{104}$$

Furthermore, if $\gamma > 0$, then there exists positive c_1 such that

$$F_{c_1}(0) = \int_0^{c_1} H(p,0) G_{c_1}(p) dp = \gamma. \tag{105}$$

Proof. Let us fix $x \in [0, c_1]$ and assume that $\bar{c}_1 > c_1$. Then by formula (100) we get

$$F_{\bar{c}_1}(x) - F_{c_1}(x) = \int_{c_1}^{\bar{c}_1} H(p,x) G_{\bar{c}_1}(p) dp + \int_x^{c_1} H(p,x)[G_{\bar{c}_1}(p) - G_{c_1}(p)] dp.$$

We note that H is bounded on $\{(p,x) : \ 0 \le x \le p \le \bar{c}_1\}$ and

$$|G_{\bar{c}_1}(p)| \le \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} \bar{c}_1$$

hence, the first integral converges to zero, if $\bar{c}_1 \searrow c_1$. Next, we write

$$G_{\bar{c}_1}(p) - G_{c_1}(p) = \frac{\alpha}{2\Gamma(1-\alpha)} \left[(\bar{c}_1 - c_1) \int_{\bar{c}_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}}^{1} (1-w)^{-\alpha} w^{\frac{\alpha}{2}-1} dw + c_1 \int_{\bar{c}_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}}^{c_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}} (1-w)^{-\alpha} w^{\frac{\alpha}{2}-1} dw \right].$$

The first integral is uniformly bounded by $B(1-\alpha,\frac{\alpha}{2})$ hence, the first term converges to zero, if $\bar{c}_1 \searrow c_1$. The second integral also converges to zero because

$$\int_{\bar{c}_1^{-\frac{2}{\alpha}}p^{\frac{2}{\alpha}}}^{c_1^{-\frac{2}{\alpha}}p^{\frac{2}{\alpha}}}(1-w)^{-\alpha}w^{\frac{\alpha}{2}-1}dw \le \sup_{W\subset[0,1],|W|\le(\frac{\bar{c}_1}{c_1})^{\frac{2}{\alpha}}-1}\int_W (1-w)^{-\alpha}w^{\frac{\alpha}{2}-1}dw \to 0,$$

if $\bar{c}_1 \searrow c_1$. Therefore, we obtained (102).

To get (103) we note that

$$F_{c_1}(0)=\int_0^{c_1}H(p,0)G_{c_1}(p)dp \le \|H\|_{L^\infty(W_{c_1})}\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}c_1 \to 0,$$

if $c_1 \searrow 0$.

Recall that N is non-negative, thus we have

$$F_{c_1}(0)\ge\int_0^{c_1}G_{c_1}(p)dp=\frac{\alpha}{2\Gamma(1-\alpha)}c_1\int_0^{c_1}\int_{c_1^{-\frac{2}{\alpha}}p^{\frac{2}{\alpha}}}^1(1-w)^{-\alpha}w^{\frac{\alpha}{2}-1}dwdp$$

$$\ge\frac{\alpha}{2\Gamma(1-\alpha)}c_1\int_0^{c_1}\int_{c_1^{-\frac{2}{\alpha}}p^{\frac{2}{\alpha}}}^1(1-w)^{-\alpha}dwdp=\frac{\alpha}{2\Gamma(2-\alpha)}c_1\int_0^{c_1}(1-c_1^{-\frac{2}{\alpha}}p^{\frac{2}{\alpha}})^{1-\alpha}dp$$

$$=\frac{\left(\frac{\alpha}{2}\right)^2c_1^2}{\Gamma(2-\alpha)}B(2-\alpha,\frac{\alpha}{2})=\frac{\alpha\Gamma(1+\frac{\alpha}{2})}{2\Gamma(2-\frac{\alpha}{2})}c_1^2\to\infty \text{ as } c_1\to\infty$$

and we proved (104)

Finally, it remains to prove that for each $\gamma\in(0,\infty)$ there exists $c_1\in(0,\infty)$ such that

$$F_{c_1}(0)=\gamma.$$

From (102) we deduce the continuity of $(0,\infty)\ni c_1\mapsto F_{c_1}(0)$. Applying the Darboux property together with (103), (104) we deduce that this map is onto $(0,\infty)$. □

To prove Theorem 2, it remains to collect the obtained results.

Proof of Theorem 2. The result is a direct consequence of Corollary 4, Corollary 5, Proposition 5 and Proposition 6. □

Proof of Corollary 1. We note that Corollary 1 is a simple consequence of Theorem 2. Indeed, from the formula (24) we obtain that

$$u_x(0,t)=-t^{-\frac{\alpha}{2}}\left[c_1\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}+\int_0^{c_1}N(p,0)G_{c_1}(p)dp\right]=:-t^{-\frac{\alpha}{2}}g(c_1).$$

Since N is continuous and bounded on W_R for every $R>0$ and G_{c_1} is continuous with respect to c_1, we obtain that g is continuous as well. Furthermore, $g(0)=0$ and $\lim_{c_1\to\infty}g(c_1)=\infty$. Thus, Corollary 1 follows from the Darboux property. □

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