# Multiple positive solutions for four-point boundary condition of fractional delay differential equations with $p$-Laplacian operator

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**Abstract:** In this work, a class of fractional delay differential equations with four-point boundary condition and $p$-Laplacian operator are discussed. Based on the Avery-Peterson theorem, the existence of at least triple positive solutions are derived. An simple example are given to show the validity of the conditions of our main theorem.

**Keywords:** Positive solution; Fractional differential equation; Mixed derivative; Delay; $p$-Laplacian operator

## 1 Introduction

Fractional calculus are generalized from the integer order and have been widely used in control system, aerodynamics, fluid flows and many other branches of engineering [1-7]. Recently, the fractional differential equation models has attracted great interest.

In the past several decades, fractional boundary value problems have obtained abundant theoretical achievements. There are many papers studying the existence of positive solutions under various boundary conditions by different methods, the approaches are mainly including the method of upper and lower solutions [8, 9], fixed point theorem on cones [10, 11], monotone iterative method [12, 13], Leray-Schauder degree [14, 15], Avery-Peterson theory [16-18], and so on. Especially, in [19], the existence multiple positive solutions for two classes of fractional differential equations with delay are analyzed with the help of Leggett-Williams theorem and a generalization of Leggett-Williams theorem.

At present, fractional differential equations with $p$-Laplacian operator have aroused the extensive attention of many scholars. There are many research results about the existence of solutions for fractional boundary value problems with $p$-Laplacian operator, one can refer to [8, 9, 12, 15, 18, 20-22] and references therein.

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Enlightened by the above literature, we discuss the following equation:

$$\begin{cases} D^{\beta}(\psi_p({}^cD^{\alpha}x(t))) = g(t, x_t, {}^c D^{\gamma}x(t)), & t \in (0,1), \\ x(t) = \xi(t), & t \in [-\tau, 0], \\ {}^cD^{\alpha}x(0) = x^{'}(0) = 0, \\ x(1) = cx(\lambda), \quad {}^cD^{\alpha}x(1) = d^cD^{\alpha}x(\zeta), \end{cases} \tag{1}$$

where $1 < \alpha, \beta \leq 2$, $0 < \gamma < 1$, $c \in [0, +\infty), d \in [0, +\infty), 0 < \lambda, \zeta < 1$ and $c < \lambda$. $D^{\beta}$ is the Riemann-Liouville fractional derivative, ${}^cD^{\alpha}$ and ${}^cD^{\gamma}$ are the Caputo fractional derivative. $\psi_q = \psi_p^{-1}$, $\psi_p$ is the $p$-Laplacian operator, $\psi_p(t) = |t|^{p-2}t$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. $g \in C([0,1] \times [0, +\infty) \times (-\infty, 0], [0, +\infty))$, $x_t(s) = x(t+s)$, for $t \in [0,1], s \in [-\tau, 0]$, $\tau > 0$. $\xi \in C_{\tau}(:= C([-\tau, 0], [0, +\infty))$, $C_{\tau}$ is a Banach space with $\|\xi\|_{[-\tau,0]} = \max_{s \in [-\tau, 0]} |\xi(s)|$.

In this paper, the sufficient conditions are obtained for the existence of at least three positive solutions for a class of four-point boundary value problems of fractional delay differential equations with $p$-Laplacian operator. This research method can be extended to many fractional boundary value problems.

For each $\xi \in C_{\tau}$ and $x \in C([0,1], R)$, we define and let

$$x_t(\varepsilon, \xi) = \begin{cases} x(t+\varepsilon), & t+\varepsilon \geq 0, \\ \xi(t+\varepsilon), & t+\varepsilon < 0, \varepsilon \in [-\tau, 0], \end{cases} \tag{2}$$

Obviously, $x_t(\cdot, \xi) \in C([0,1], R)$.

## 2 Preliminaries

This part introduce some useful definitions and important lemmas.

**Definition 2.1**([1]) The $\delta > 0$ order Riemann-Liouville fractional integral and derivative for a function $g(t)$ is defined, respectively

$$I^{\delta}g(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\theta)^{\delta-1} g(\theta) d\theta,$$

$$D^{\delta}g(t) = \frac{1}{\Gamma(n-\delta)} (\frac{d}{dt})^n \int_0^t (t-\theta)^{n-\delta-1} g(\theta) d\theta, \quad n = [\delta] + 1.$$

**Definition 2.2**([1]) The $\delta > 0$ order Caputo fractional derivative for a function $g(t)$ is defined as following

$${}^cD^{\delta}g(t) = \frac{1}{\Gamma(n-\delta)} \int_0^t (t-\theta)^{n-\delta-1} g^{(n)}(\theta) d\theta, \quad n = [\delta] + 1.$$

**Lemma 2.1**([1]) Assume that $\delta > 0$ and $n = [\delta] + 1$. If the function $g \in L[0,1] \cap C[0,1]$, then

$$I^{\delta} \, {}^cD^{\delta}g(t) = g(t) - a_1 - a_2 t - \cdots - a_n t^{n-1}, \quad a_i \in R, \; i = 1, 2, \cdots, n,$$

$$I^{\delta}D^{\delta}g(t)=g(t)-b_1t^{\delta-1}-b_2t^{\delta-2}-\cdots-b_nt^{\delta-n},\quad b_i\in R,\ i=1,2,\cdots,n.$$

**Lemma 2.2** If $g$ is continuous function and $1<\alpha,\beta\leq 2$, $0<\gamma<1$, then $x(t)$ is a solution of equation (1) when and only when, for certain $\xi(t)\in C_{\tau}$, x(t) is equivalent to the following integral form:

$$x(t)=a_1-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\eta)^{\alpha-1}k(\eta)d\eta, \tag{3}$$

where

$$a_1=\frac{\int_0^1(1-\eta)^{\alpha-1}k(\eta)d\eta-c\int_0^{\lambda}(\lambda-\eta)^{\alpha-1}k(\eta)d\eta}{(1-c)\Gamma(\alpha)}, \tag{4}$$

$$k(\eta)=\psi_q[-(\frac{1}{\Gamma(\beta)}\int_0^{\eta}(\eta-\theta)^{\beta-1}g(\theta,x_{\theta}(\cdot,\xi),{}^cD^{\gamma}x(\theta))d\theta+b_1\eta^{\beta-1})]. \tag{5}$$

Proof: From Lemmma 2.1, we get

$$\psi_p({}^cD^{\alpha}x(t))=\frac{1}{\Gamma(\beta)}\int_0^t(t-\theta)^{\beta-1}g(\theta,x_{\theta}(\cdot,\xi),{}^cD^{\gamma}x(\theta))d\theta+b_1t^{\beta-1}+b_2t^{\beta-2},$$

According to ${}^cD^{\alpha}x(0)=0$, ${}^cD^{\alpha}x(1)=d^cD^{\alpha}x(\zeta)$, we know that

$$b_2=0,\quad b_1=\frac{d^{p-1}\int_0^{\zeta}(\zeta-\theta)^{\beta-1}g(\theta,x_{\theta}(\cdot,\xi),{}^cD^{\gamma}x(\theta))d\theta-\int_0^1(1-\theta)^{\beta-1}g(\theta,x_{\theta}(\cdot,\xi),{}^cD^{\gamma}x(\theta))d\theta}{(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta)},$$

that is

$$\psi_p({}^cD^{\alpha}x(t))=\frac{1}{\Gamma(\beta)}\int_0^t(t-\theta)^{\beta-1}g(\theta,x_{\theta}(\cdot,\xi),{}^cD^{\gamma}x(\theta))d\theta+b_1t^{\beta-1}, \tag{6}$$

By (6), we have

$${}^cD^{\alpha}x(t)=\psi_q(\frac{1}{\Gamma(\beta)}\int_0^t(t-\theta)^{\beta-1}g(\theta,x_{\theta}(\cdot,\xi),{}^cD^{\gamma}x(\theta))d\theta+b_1t^{\beta-1})=-k(t).$$

From Lemmma 2.1, we obtain

$$x(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\eta)^{\alpha-1}k(\eta)d\eta+a_1+a_2t,$$

By use of $x'(0)=0$, $x(1)=cx(\lambda)$, one get

$$a_2=0,\quad a_1=\frac{\int_0^1(1-\eta)^{\alpha-1}k(\eta)d\eta-c\int_0^{\lambda}(\lambda-\eta)^{\alpha-1}k(\eta)d\eta}{(1-c)\Gamma(\alpha)},$$

Hence

$$x(t)=a_1-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\eta)^{\alpha-1}k(\eta)d\eta.$$

The conclusion has been proved.

**Lemma 2.3** If $g \in C([0,1] \times [0,+\infty) \times (-\infty,0],[0,+\infty))$, then $x(t) \geq 0$.

Proof: By (3)-(5), one has

$$
\begin{aligned}
k(\eta) =&\psi_q[\frac{\int_0^1 [\eta(1-\theta)]^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta - d^{p-1}\int_0^\zeta [\eta(\zeta-\theta)]^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta}{(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta)} \\
&- \frac{1}{\Gamma(\beta)}\int_0^\eta (\eta-\theta)^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta] \\
=&\psi_q[\frac{\int_0^1 [\eta(1-\theta)]^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta - \int_0^\eta (\eta-\theta)^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta}{\Gamma(\beta)} \\
&+ \frac{d^{p-1}\eta^{\beta-1}(\int_0^1 [\zeta(1-\theta)]^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta - \int_0^\zeta (\zeta-\theta)^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta)}{(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta)}] \\
\geq&\psi_q[\frac{\int_\eta^1 [\eta(1-\theta)]^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta}{\Gamma(\beta)} \\
&+ \frac{\int_0^\eta [[\eta(1-\theta)]^{\beta-1} - (\eta-\theta)^{\beta-1}] g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta}{\Gamma(\beta)}] \\
\geq&\psi_q[\frac{\int_\eta^1 [\eta(1-\theta)]^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta + \int_0^\eta [\eta(1-\theta)]^{\beta-1}\theta(1-\eta) g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta}{\Gamma(\beta)}] \\
\geq&\psi_q[\frac{\int_0^1 [\eta(1-\theta)]^{\beta-1}\theta(1-\eta) g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta}{\Gamma(\beta)}] \geq 0, \qquad (7)
\end{aligned}
$$

and

$$
\begin{aligned}
x(t) &= a_1 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1} k(\eta)d\eta \\
&= \frac{\int_0^1 (1-\eta)^{\alpha-1} k(\eta)d\eta - c\int_0^\lambda (\lambda-\eta)^{\alpha-1} k(\eta)d\eta}{(1-c)\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1} k(\eta)d\eta \\
&= \frac{\int_0^1 (1-\eta)^{\alpha-1} k(\eta)d\eta - \int_0^t (t-\eta)^{\alpha-1} k(\eta)d\eta}{\Gamma(\alpha)} \\
&\quad + \frac{c(\int_0^1 (1-\eta)^{\alpha-1} k(\eta)d\eta - \int_0^\lambda (\lambda-\eta)^{\alpha-1} k(\eta)d\eta)}{(1-c)\Gamma(\alpha)} \\
&\geq 0,
\end{aligned}
$$

Therefore, $x(t)$ is nonnegative.

**Lemma 2.4** Suppose that $x(t)$ is a solution of equation (1), then there exists $\varrho > 0$ such that

$$\max_{t\in[0,1]} |x(t)| \leq \varrho \max_{t\in[0,1]} |{}^c D^\gamma x(t)|, \quad \varrho = \frac{\Gamma(\alpha-\gamma)}{(1-c)\Gamma(\alpha)} > 0.$$

Proof: From (3), We calculate that

$$^cD^\gamma x(t) = -\frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-\eta)^{\alpha-\gamma-1}k(\eta)d\eta,$$

$$\max_{t\in[0,1]}|^cD^\gamma x(t)| \geq |^cD^\gamma x(1)| = \frac{1}{\Gamma(\alpha-\gamma)}\int_0^1 (1-\eta)^{\alpha-\gamma-1}k(\eta)d\eta,$$

Thus

$$\begin{aligned}\max_{t\in[0,1]}|x(t)| &\leq \frac{1}{(1-c)\Gamma(\alpha)}\int_0^1 (1-\eta)^{\alpha-1}k(\eta)d\eta \\ &\leq \frac{\Gamma(\alpha-\gamma)}{(1-c)\Gamma(\alpha)}\frac{1}{\Gamma(\alpha-\gamma)}\int_0^1 (1-\eta)^{\alpha-\gamma-1}k(\eta)d\eta \\ &\leq \varrho \max_{t\in[0,1]}|^cD^\gamma x(t)|.\end{aligned}$$

The conclusion has been proved.

**Lemma 2.5**([16]) Let $P$ is a cone of $U$, $\mu,\nu$ be nonnegative continuous convex functional on $P$, $\omega$ be a nonnegative continuous concave functional on $P$, and $\varpi$ be a nonnegative continuous functional on $P$. For $l,m,n,r>0$, define the following sets:

$P(\mu,r)=\{x\in P \mid \mu(x)<r\}$,

$P(\mu,\omega,m,r)=\{x\in P \mid \omega(x)\geq m, \mu(x)\leq r\}$,

$P(\mu,\nu,\omega,m,n,r)=\{x\in P \mid \omega(x)\geq m, \nu(x)\leq n, \mu(x)\leq r\}$,

and

$Q(\mu,\varpi,l,r)=\{x\in P \mid \varpi(x)\geq l, \mu(x)\leq r\}$.

If the functionals $\mu,\nu,\omega,\varpi$ satisfying $\varpi(\varepsilon z)\leq\varepsilon\varpi(x)$, $0\leq\varepsilon\leq 1$, such that for some $R,r>0$,

$$\omega(x)\leq\varpi(x), \quad \|x\|\leq R\mu(x),$$

for all $x\in\overline{P(\mu,r)}$. If $T:\overline{P(\mu,r)}\to\overline{P(\mu,r)}$ is completely continuous and there exist $l,m,n>0$ with $l<m$ such that

$(S_1)$ $\{x\in P(\mu,\nu,\omega,m,n,r):\omega(x)>m\}\neq\varnothing$ and $\omega(Tx)>m$ for $x\in P(\mu,\nu,\omega,m,n,r)$;

$(S_2)$ $\omega(Tx)>m$ for $x\in P(\mu,\omega,m,r)$ and $\nu(Tx)>n$;

$(S_3)$ $0\notin Q(\mu,\varpi,l,r)$ and $\varpi(Tx)<l$ for $x\in Q(\mu,\varpi,l,r)$ with $\varpi(x)=l$.

Then $T$ has at least exist three fixed point $x_1,x_2,x_3\in\overline{P(\mu,r)}$ as following

$$\mu(x_i)\leq r, \quad i=1,2,3;$$

$$m<\omega(x_1);$$

$$l<\varpi(x_2), \omega(x_2)<m;$$

and

$$\varpi(x_3)<l.$$

## 3 Main results

Next, the problem of positive solutions for equations (1) are studied. For convenience, some notations and hypotheses are presented as following

$$
\begin{aligned}
N_1 &= \Gamma(\alpha-\gamma+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1};\\
N_2 &= \frac{(1-c)\Gamma(\alpha)[\Gamma(\beta+2)]^{q-1}}{(1-\lambda)B(\alpha+q-1,\beta+q-1)};\\
N_3 &= (1-c)\Gamma(\alpha+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1};
\end{aligned}
$$

and

$(C_1)$ $\xi \in C([-\tau,0],[0,+\infty))$;

$(C_2)$ $g \in C([0,1]\times[0,+\infty)\times(-\infty,0],[0,+\infty))$ and $g(t,0,0)\neq 0$ for all $t\in[0,1]$;

$(C_3)$ $g(t,y,z) \le (rN_1)^{p-1}, \quad (t,y,z)\in[0,1]\times[0,\varrho r]\times[-r,r]$;

$(C_4)$ $g(t,y,z) > (mN_2)^{p-1}, \quad (t,y,z)\in[0,1]\times[m,n]\times[-r,r]$;

$(C_5)$ $g(t,y,z) < (lN_3)^{p-1}, \quad (t,y,z)\in[0,1]\times[0,l]\times[-r,r]$.

Let the Banach space $U=\{x\in C[0,1], {}^cD^\gamma x(t)\in C[0,1]\}$ with the norm

$$
\|x\| = max\{\max_{t\in[0,1]}|x(t)|, \max_{t\in[0,1]}|{}^cD^\gamma x(t)|\},
$$

and define the cone $P$ by

$$
P=\{x\in U|x\ge 0,\ \max_{t\in[0,1]}|x(t)|\le \varrho\max_{t\in[0,1]}|{}^cD^\gamma x(t)|,\ t\in[0,1]\}.
$$

**Lemma 3.1** Let the operator $T: P\to U$ is define as

$$
Tx(t)=a_1-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1}k(\eta)d\eta,
$$

Then $T: P\to P$ is completely continuous.

Proof: Obviously $T(P)\subseteq P$ and $T$ is continuous by using the continuity of $g(t,x_t,{}^cD^\gamma x(t))$. Assume that $H$ be a bounded subset in $P$, which is to say exists $M>0$ such that $\|x\|\le M$ for all $x\in H$, let

$$
L=\sup_{t\in[0,1],x\in[0,M]}|g(t,x_t,{}^cD^\gamma x(t))|+1,
$$

Then for $\forall x \in H$, we have

$$\begin{aligned}k(\eta) =&\psi_q[\frac{\int_0^1 [\eta(1-\theta)]^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta - d^{p-1}\int_0^\zeta [\eta(\zeta-\theta)]^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta}{(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta)} \\ &- \frac{1}{\Gamma(\beta)}\int_0^\eta (\eta-\theta)^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta] \\ \le&\psi_q(\frac{\int_0^1 [\eta(1-\theta)]^{\beta-1} g(\theta, x_\theta(\cdot,\xi), {}^c D^\gamma x(\theta))d\theta}{(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta)}) \\ \le&\psi_q(\frac{L}{(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)}) = (\frac{L}{(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)})^{q-1}, \end{aligned} \tag{8}$$

So one get

$$\begin{aligned}|Tx(t)| &= a_1 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1}k(\eta)d\eta \le \frac{\int_0^1 (1-\eta)^{\alpha-1}k(\eta)d\eta}{(1-c)\Gamma(\alpha)} \\ &\le \frac{L^{q-1}}{(1-c)\Gamma(\alpha+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1}},\end{aligned}$$

and

$$\begin{aligned}|{}^c D^\gamma (Tx(t))| &= \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-\eta)^{\alpha-\gamma-1}k(\eta)d\eta \\ &\le \frac{L^{q-1}}{\Gamma(\alpha-\gamma+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1}}.\end{aligned}$$

Hence, $T(H)$ is uniformly bounded.

For $\forall x \in H$ and $0 \le t_1 < t_2 \le 1$, one has

$$\begin{aligned}|Tx(t_2) - Tx(t_1)| &= |-\frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2-\eta)^{\alpha-1}k(\eta)d\eta + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-\eta)^{\alpha-1}k(\eta)d\eta| \\ &= \frac{1}{\Gamma(\alpha)}|\int_0^{t_2} (t_2-\eta)^{\alpha-1}k(\eta)d\eta - \int_0^{t_1} (t_1-\eta)^{\alpha-1}k(\eta)d\eta| \\ &\le \frac{L^{q-1}}{\Gamma(\alpha+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1}}(t_2^\alpha - t_1^\alpha),\end{aligned}$$

and

$$\begin{aligned}&|{}^c D^\gamma (Tx(t_2)) - {}^c D^\gamma (Tx(t_1))| \\ =& |\frac{1}{\Gamma(\alpha-\gamma)}\int_0^{t_2} (t_2-\eta)^{\alpha-\gamma-1}k(\eta)d\eta - \frac{1}{\Gamma(\alpha-\gamma)}\int_0^{t_1} (t_1-\eta)^{\alpha-\gamma-1}k(\eta)d\eta| \\ \le& \frac{L^{q-1}}{\Gamma(\alpha-\gamma+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1}}(t_2^{\alpha-\gamma} - t_1^{\alpha-\gamma}).\end{aligned}$$

Therefore, $T(H)$ is equicontinuous. We conclude that $T(H)$ is relatively compact on the basis of $Ascoli - Arzel\grave{a}$ theorem. The conclusion has been proved.

**Theorem 3.1** Assume that there exists constants $0 < l < m < n < r$ and $\frac{1-c}{1-\lambda}m \leq n$. If the assumptions $(C_1) - (C_5)$ holds, then equation (1) has at least three positive solutions $x_1$, $x_2$ and $x_3$ with

$$\mu(x_i) \leq r, \quad i = 1, 2, 3;$$

$$m < \omega(x_1);$$

$$l < \varpi(x_2), \omega(x_2) < m;$$

and

$$\varpi(x_3) < l.$$

Proof: Let

$$\omega(x) = \min_{t\in[0,\lambda]} |x(t)|, \quad \mu(x) = \max_{t\in[0,1]} |^c D^\gamma x(t)|, \quad \nu(x) = \varpi(x) = \max_{t\in[0,1]} |x(t)|,$$

Evidently, $\omega(x) \leq \varpi(x)$. By Lemmas 2.4, we know that

$$\|x\| \leq R\mu(x), \quad R = sup\{\varrho, 1\}.$$

For $x \in \overline{P(\mu, r)}$, in view of assumption $(C_3)$ and (8), we get

$$k(\eta) \leq \frac{rN_1}{[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1}},$$

then

$$\begin{aligned} \mu(x) &= \max_{t\in[0,1]} |^c D^\gamma (Tx(t))| = \max_{t\in[0,1]} | -\frac{1}{\Gamma(\alpha-\gamma)} \int_0^t (t-\eta)^{\alpha-\gamma-1} k(\eta) d\eta | \\ &\leq \frac{rN_1}{\Gamma(\alpha-\gamma+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1}} \leq r, \end{aligned}$$

Hence, $T : \overline{P(\mu, r)} \to \overline{P(\mu, r)}$.

Next, we prove conditions $(S_1) - (S_3)$ are ture for operator $T$. First, for constant function $x(t) = \frac{1-c}{1-\lambda}m \in P(\mu, \nu, \omega, m, n, r)$, we have $\mu(x) = 0 \leq r$, $\nu(x) = \frac{1-c}{1-\lambda}m \leq n$ and $\omega(x) = \frac{1-c}{1-\lambda}m > m$, which imply that $\{x \in P(\mu, \nu, \omega, m, n, r) : \omega(x) > m\} \neq \varnothing$. From assumption $(C_4)$ and (7), we have

$$k(\eta) > \psi_q[\frac{\eta^{\beta-1}(1-\eta)}{\Gamma(\beta+2)}(mN_2)^{p-1}] = \frac{\eta^{\beta+q-2}(1-\eta)^{q-1}}{[\Gamma(\beta+2)]^{q-1}} mN_2.$$

For $x \in P(\mu, \nu, \omega, m, n, r)$, one get

$$\begin{aligned}
\omega(Tx) &= \min_{t\in[0,\lambda]} |Tx(t)| = |T(x(\lambda))| \\
&= \frac{\int_0^1 (1-\eta)^{\alpha-1}k(\eta)d\eta - \int_0^\lambda (\lambda-\eta)^{\alpha-1}k(\eta)d\eta}{(1-c)\Gamma(\alpha)} \\
&= \frac{\int_\lambda^1 (1-\eta)^{\alpha-1}k(\eta)d\eta + \int_0^\lambda [(1-s)^{\alpha-1} - (\lambda-s)^{\alpha-1}]k(\eta)d\eta}{(1-c)\Gamma(\alpha)} \\
&\geq \frac{\int_\lambda^1 (1-\eta)^{\alpha-1}k(\eta)d\eta - \int_0^\lambda (1-s)^{\alpha-1}(1-\lambda)k(\eta)d\eta}{(1-c)\Gamma(\alpha)} \\
&> \frac{1-\lambda}{(1-c)\Gamma(\alpha)} \int_0^1 (1-\eta)^{\alpha-1}k(\eta)d\eta \\
&> \frac{mN_2(1-\lambda)}{(1-c)\Gamma(\alpha)[\Gamma(\beta+2)]^{q-1}} \int_0^1 \eta^{\beta+q-2}(1-\eta)^{\alpha+q-2}ds \\
&= \frac{mN_2(1-\lambda)B(\alpha+q-1, \beta+q-1)}{(1-c)\Gamma(\alpha)[\Gamma(\beta+2)]^{q-1}} = m,
\end{aligned}$$

These the condition $(S_1)$ is satisfied.

Secondly, if $z \in P(\mu, \omega, m, r)$ and $\nu(Tz) > n$, then

$$\begin{aligned}
\omega(Tx) &= \min_{t\in[0,\lambda]} |Tx(t)| = |T(x(\lambda))| \\
&\geq \frac{1-\lambda}{(1-c)\Gamma(\alpha)} \int_0^1 (1-\eta)^{\alpha-1}k(\eta)d\eta \\
&\geq \frac{1-\lambda}{1-c}\nu(Tx) > \frac{1-\lambda}{1-c}n \geq m,
\end{aligned}$$

Thus, condition $(S_2)$ also hold.

Finally, from assumption $(C_5)$ and (8), we have

$$k(\eta) < \frac{lN_3}{[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1}},$$

then

$$\begin{aligned}
\varpi(Tx) &= \max_{t\in[0,1]} |Tx(t)| \leq \frac{\int_0^1 (1-\eta)^{\alpha-1}k(\eta)d\eta}{(1-c)\Gamma(\alpha)} \\
&< \frac{lN_3}{(1-c)\Gamma(\alpha+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1}} = l,
\end{aligned}$$

Therefore, all conditions of Lemma 2.5 are holds. The conclusion has been proved.

## 4 Some Examples

**Example 4.1** For the equation (1), let $\alpha = \beta = \frac{3}{2}$, $\gamma = \frac{1}{2}$, $\lambda = \frac{2}{3}, \zeta = \frac{1}{4}$, $c = \frac{1}{3}, d = \frac{3}{2}$, $p = 2$, $\xi(t) \in C_\tau$, $\tau > 0$ and

$$g(t,y,z) = \begin{cases} \frac{t}{20} + \frac{1}{100} sin(\frac{z}{5000}), & 0 \le t \le 1, 0 \le y \le 1, \\ \frac{t}{20} + 535(y-1) + \frac{1}{100} sin(\frac{z}{5000}), & 0 \le t \le 1, 1 \le y \le 2, \\ \frac{t}{20} + 535 + 3(y-2) + \frac{1}{100} sin(\frac{z}{5000}), & 0 \le t \le 1, 2 \le y \le 6, \\ \frac{t}{20} + 547 + \frac{1}{100} sin(\frac{z}{5000}), & 0 \le t \le 1, y > 6. \end{cases}$$

By simple computation, we obtain $\varrho = \frac{3}{\sqrt{\pi}}$,
$N_1 = \Gamma(\alpha - \gamma + 1)[(1 - \zeta^{\beta-1} d^{p-1})\Gamma(\beta+1)]^{q-1} \approx 0.11045$,
$N_2 = \frac{(1-c)\Gamma(\alpha)[\Gamma(\beta+2)]^{q-1}}{(1-\lambda)B(\alpha+q-1,\beta+q-1)} \approx 265.868$,
$N_3 = (1-c)\Gamma(\alpha+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma(\beta+1)]^{q-1} \approx 0.098$.

In addition, if we take $l = 1, m = 2, n = 6$ and $r = 5000$, then $g(t,y,z)$ satisfies the following conditions:

$g(t,y,z) \le (rN_1)^{p-1} \approx 552.250, \quad (t,y,z) \in [0,1] \times [0, \frac{15000}{\sqrt{\pi}}] \times [-5000, 5000]$,
$g(t,y,z) > (mN_2)^{p-1} \approx 531.736, \quad (t,y,z) \in [0,1] \times [2,6] \times [-5000, 5000]$,
$g(t,y,z) < (lN_3)^{p-1} \approx 0.098, \quad (t,y,z) \in [0,1] \times [0,l] \times [-5000, 5000]$.

Then all conditions of Theorem 3.1 are satisfied. Thus problem (1) has at least exist three fixed point $x_1(t), x_2(t), x_3(t)$ such that

$$\mu(x_i) \le 5000, \quad i = 1,2,3;$$

$$2 < \omega(x_1);$$

$$1 < \varpi(x_2), \omega(x_2) < 2;$$

and

$$\varpi(x_3) < 1.$$

## 5 Conclusion

In this article, on the basis of the Avery-Peterson theorem, the sufficient conditions ensure that the existence at least three positive solutions are obtained. This research method can be extended to many fractional boundary value problems. It is worth noting that the equation involve time delay and $p$-Laplacian operator, compare to many pervious works, which has never been considered. In addition, our works is inspiring for future research as regards existence solutions of fractional langevin equations with delay.

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