EXISTENCE OF SOLUTIONS FOR FRACTIONAL m-POINT BOUNDARY-VALUE PROBLEMS AT RESONANCE WITH p-LAPLACIAN OPERATOR

YANQIANG WU

ABSTRACT. In this paper, we considered a class of m-point boundary-value problem of fractional differential equations at resonance with p-Laplacian operator in the following:

$$\begin{cases} D^{\beta}_{0+}\varphi_p(D^{\alpha}_{0+}u(t)) = f(t,u(t),D^{\alpha-2}_{0+}u(t),D^{\alpha-1}_{0+}u(t),D^{\alpha}_{0+}u(t)), \quad t\in(0,1),\\ u(0)=u'(0)=D^{\alpha}_{0+}u(0)=0, \quad D^{\alpha-2}_{0+}u(1)=\sum_{i=1}^{m-2}a_iD^{\alpha-2}_{0+}u(\eta_i), \end{cases}$$

where $2<\alpha\le 3$, $\eta_1<\eta_2<\cdots<\eta_{m-2}$, $0<\beta\le 1$, $3<\alpha+\beta\le 4$, $\sum_{i=1}^{m-2}a_i\eta_i=1$, D^{α}_{0+} denote the Riemann-Liouville fractional derivative, $\varphi_p(s)=|s|^{p-2}s$ is p-Laplacian operator. The existence of solutions to above problem is obtained by using the extension of Mawhin's continuation theorem. It is note that our method dropped a usual condition in the process of investigating above problem. So, in some sense, we got a new result under weaker condition than previous ones[8].

1. Introduction

In the present paper, we investigated a fractional m-point boundary value problem with p-Laplacian operator in the following

$$\begin{cases} D^{\beta}_{0+}\varphi_p(D^{\alpha}_{0+}u(t)) = f(t,u(t),D^{\alpha-2}_{0+}u(t),D^{\alpha-1}_{0+}u(t),D^{\alpha}_{0+}u(t)), \ t\in(0,1),\\ u(0)=u'(0)=D^{\alpha}_{0+}u(0)=0, \quad D^{\alpha-2}_{0+}u(1)=\sum_{i=1}^{m-2}a_iD^{\alpha-2}_{0+}u(\eta_i), \end{cases} \tag{1.1}$$

where $2<\alpha\le 3$, $0<\beta\le 1$, $3<\alpha+\beta\le 4$, $0<\eta_1<\cdots<\eta_i<\cdots<\eta_{m-2}<1$, $a_i\in\mathbb{R}$, $\sum_{i=1}^{m-2}a_i\eta_i=1$, $\varphi_p(s)=|s|^{p-2}s$, $1<p, 1/p+1/q=1$, φ_p is invertible and φ_q is its inverse operator, D^{α}_{0+} is Riemann-Liouville fractional derivative, $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ is continuous. Moreover, FBVP (1.1) happens to be at resonance because the following problem

$$\begin{cases} D^{\beta}_{0+}\varphi_p(D^{\alpha}_{0+}u(t)) = 0, \quad t\in(0,1),\\ u(0)=u'(0)=D^{\alpha}_{0+}u(0)=0, \quad D^{\alpha-2}_{0+}u(1)=\sum_{i=1}^{m-2}a_iD^{\alpha-2}_{0+}u(\eta_i) \end{cases}$$

has a solution $x(t)=ct^{\alpha-1}$, where $c\in\mathbb{R}$. In the passing decades, fractional calculus became a very important method for many fields such as control theory, biology,etc(see [1, 3, 10, 11]). Many scholars have paid more attention to it and gained a few achievement.

In addition, the turbulent flow in a porous medium is a very important mechanics problem. Leibenson [4] firstly introduced the p-Laplacian equation which is

$$(\phi_p(x'(t)))' = f(t,x(t),x'(t)), \tag{1.2}$$

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where $\phi_p(s) = |s|^{p-2}s,\ p > 1$.

Furthermore, there are a few articles which consider fractional differential equation at resonance with p-Laplacian. For example, in [8], Shen and Liu considered the following problem

$$\begin{cases} D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(t)) = f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t)),\ t\in(0,1), \\ u(0) = u'(0) = D_{0+}^{\alpha}u(0) = 0, \quad D_{0+}^{\alpha-1}u(1) = \sum_{i=1}^{m}\sigma_i D_{0+}^{\alpha-1}u(\eta_i), \end{cases} \tag{1.3}$$

where $2 < \alpha \leq 3,\ 0 < \beta \leq 1,\ 3 < \alpha+\beta \leq 4,\ \eta_i \in (0,1),\ \sigma_i \in \mathbb{R},\ \sum_{i=1}^{m}\sigma_i = 1$, $1 < m, m \in N$. The author[[8]] used the condition

$$\Delta = \frac{1}{\Gamma(\beta+1)^{q-1}(q\beta-\beta+1)}(1-\sum_{i=1}^{m}\sigma_i\eta_i^{q\beta-\beta+1}) \neq 0.$$

It is the purpose of this paper to show that the assumption like above condition 1is not necessary for this class of differential equations. For the sake of better illustrating the conclusion, we take the following boundary value problem (1.1) which differ from (4.1). So, in some sense, our paper generalize some results(see[8]).

2. Preliminaries

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. We call that a operator

$$M|_{\operatorname{dom} M\cap X} : X\cap \operatorname{dom} M \to Y,$$

is quasi-linear if

(i) $\operatorname{Im} M$ is a closed subset of Y,
(ii) $\ker M := \{u \in X\cap\operatorname{dom} M : Mu = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n\in\mathbb{N}$.

Let $X_1 = \ker M$, X_2 be the complement space of X_1 in X, i.e., $X = X_1 \oplus X_2$. Similarly, suppose Y_1 be a subspace of Y, and Y_2 a complement space of Y_1 in Y. Suppose $P : X \to X_1$ be a projector, $Q : Y \to Y_1$ be a semi-projector.

Let $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is a continuous operator. Let $\Sigma_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}$. N_λ is said to be M-compact in $\overline{\Omega}$ if there is a $Y_1 \subset Y$ with $\dim Y_1 = \dim X_1$ and an operator $R : \overline{\Omega}\times[0,1] \to X$ continuous and compact such that for $\lambda \in [0,1]$,

$$(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y, \tag{2.1}$$

$$QN_\lambda x = \theta, \lambda\in(0,1) \Leftrightarrow QNx = \theta, \tag{2.2}$$

$$R(\cdot,\lambda)\mid_{\Sigma_\lambda} = (I-P)\mid_{\Sigma_\lambda} \tag{2.3}$$

and $R(\cdot,0)$ is the zero operator,

$$M[P+R(\cdot,\lambda)] = (I-Q)N_\lambda. \tag{2.4}$$

Lemma 2.1. [6] *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two Banach spaces, and $\Omega\subset X$ an open and bounded nonempty set. Suppose $M : X\cap\operatorname{dom} M \to Y$ be a quasi-linear operator and $N_\lambda : \overline{\Omega}\to Y$, $\lambda\in[0,1]$ be M-compact in $\overline{\Omega}$. Moreover, if the following conditions hold*

(i) *$Mu \neq N_\lambda u$ for all $(u,\lambda) \in (\operatorname{dom} M\cap\partial\Omega)\times(0,1)$,*
(ii) *$QNu \neq 0$ for all $u\in\partial\Omega\cap\ker M$,*
(iii) *$\deg(JQN, \ker M\cap\Omega, 0) \neq 0$, where $J : \operatorname{Im} Q \to \ker M$ is a homeomorphism with $J(\theta) = \theta$ and $N = N_1$,*

then the equation $Mu = Nu$ has at least one solution in $\operatorname{dom} M\cap\overline{\Omega}$.

Definition 2.2. [6] *Let X be a Banach space and $X_1 \subset X$ is a subspace. A mapping $Q : X \to X_1$ is a semi-projector, if Q satisfies*
$(i) Q^2 x = Qx,\ \forall x \in X,$
$(ii) Q(\mu x) = \mu Qx,\ \forall x \in X,\ \mu \in \mathbf{R}.$

Definition 2.3. [10] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function u is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right side integral is point-wise defined on $(0, +\infty)$.

Definition 2.4. [10] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function u is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$, here n is the smallest integer greater than or equal to α.

Lemma 2.5. [10] *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where

$$c_i = \frac{\left(I_{0+}^{n-\alpha} x(t)\right)^{(n-i)}\big|_{t=0}}{\Gamma(\alpha - i + 1)},$$

here n is the smallest integer greater than or equal to α.

Lemma 2.6. [10] *Suppose $u(t) \in C[0,1]$ and $0 \leq \beta \leq \alpha$, then $D_{0+}^{\beta} I_{0+}^{\alpha} u(t) = I_{0+}^{\alpha-\beta} u(t)$.*

Lemma 2.7. [10] *Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then the differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

has a solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, where $n - 1 < \alpha < n$.

Lemma 2.8. *There exists $k \in \{0, 1, \ldots, m-2\}$ satisfies $\sum_{i=1}^{m-2} a_i \eta_i^{c+k(q-1)} \neq 1$ for $\forall\ c > 0$ and $q > 1$.*

Proof. Suppose the conclusion is not true, firstly, we have

$$\begin{pmatrix} \eta_1^c & \eta_2^c & \cdots & \eta_{m-2}^c \\ \eta_1^{c+(q-1)} & \eta_2^{c+(q-1)} & \cdots & \eta_{m-2}^{c+(q-1)} \\ \vdots & \vdots & & \vdots \\ \eta_1^{c+(m-2)(q-1)} & \eta_2^{c+(m-2)(q-1)} & \cdots & \eta_{m-2}^{c+(m-2)(q-1)} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{m-2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

It is equal to

$$\begin{pmatrix} \eta_1^c & \eta_2^c & \cdots & \eta_{m-2}^c & 1 \\ \eta_1^{c+(q-1)} & \eta_2^{c+(q-1)} & \cdots & \eta_{m-2}^{c+(q-1)} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \eta_1^{c+(m-3)(q-1)} & \eta_2^{c+(m-3)(q-1)} & \cdots & \eta_{m-2}^{c+(m-3)(q-1)} & 1 \\ \eta_1^{c+(m-2)(q-1)} & \eta_2^{c+(m-2)(q-1)} & \cdots & \eta_{m-2}^{c+(m-2)(q-1)} & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{m-2} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \tag{2.5}$$

Secondly,

$$\begin{vmatrix} \eta_1^c & \eta_2^c & \cdots & \eta_{m-2}^c & 1 \\ \eta_1^{c+(q-1)} & \eta_2^{c+(q-1)} & \cdots & \eta_{m-2}^{c+(q-1)} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \eta_1^{c+(m-3)(q-1)} & \eta_2^{c+(m-3)(q-1)} & \cdots & \eta_{m-2}^{c+(m-3)(q-1)} & 1 \\ \eta_1^{c+(m-2)(q-1)} & \eta_2^{c+(m-2)(q-1)} & \cdots & \eta_{m-2}^{c+(m-2)(q-1)} & 1 \end{vmatrix}.$$

Then, one has

$$\begin{vmatrix} \eta_1^c & \eta_2^c & \cdots & \eta_{m-2}^c & 1 \\ \eta_1^{c+(q-1)} & \eta_2^{c+(q-1)} & \cdots & \eta_{m-2}^{c+(q-1)} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \eta_1^{c+(m-3)(q-1)} & \eta_2^{c+(m-3)(q-1)} & \cdots & \eta_{m-2}^{c+(m-3)(q-1)} & 1 \\ \eta_1^{c+(m-2)(q-1)} & \eta_2^{c+(m-2)(q-1)} & \cdots & \eta_{m-2}^{c+(m-2)(q-1)} & 1 \end{vmatrix}$$

$$= \eta_1^c \eta_2^c \ldots \eta_{m-2}^c \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \eta_1^{q-1} & \eta_2^{q-1} & \cdots & \eta_{m-2}^{q-1} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \eta_1^{(m-3)(q-1)} & \eta_2^{(m-3)(q-1)} & \cdots & \eta_{m-2}^{(m-3)(q-1)} & 1 \\ \eta_1^{(m-2)(q-1)} & \eta_2^{(m-2)(q-1)} & \cdots & \eta_{m-2}^{(m-2)(q-1)} & 1 \end{vmatrix}.$$

So,

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \eta_1^{q-1} & \eta_2^{q-1} & \cdots & \eta_{m-2}^{q-1} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \eta_1^{(m-3)(q-1)} & \eta_2^{(m-3)(q-1)} & \cdots & \eta_{m-2}^{(m-3)(q-1)} & 1 \\ \eta_1^{(m-2)(q-1)} & \eta_2^{(m-2)(q-1)} & \cdots & \eta_{m-2}^{(m-2)(q-1)} & 1 \end{vmatrix}$$

is Vandermonde Determinant and $0 < \eta_1^{q-1} < \eta_1^{q-1} < \cdots < \eta_{m-2}^{q-1} < 1$, it is well known that the Vandermonde Determinant is not equal to zero, then

$$\begin{vmatrix} \eta_1^c & \eta_2^c & \cdots & \eta_{m-2}^c & 1 \\ \eta_1^{c+(q-1)} & \eta_2^{c+(q-1)} & \cdots & \eta_{m-2}^{c+(q-1)} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \eta_1^{c+(m-3)(q-1)} & \eta_2^{c+(m-3)(q-1)} & \cdots & \eta_{m-2}^{c+(m-3)(q-1)} & 1 \\ \eta_1^{c+(m-2)(q-1)} & \eta_2^{c+(m-2)(q-1)} & \cdots & \eta_{m-2}^{c+(m-2)(q-1)} & 1 \end{vmatrix} \neq 0$$

by a similar way. This is a contradiction with (2.5), so we get the conclusion. $\square$

Remark. When $q = 2$, we can see that Lemma 3.2 in [9] is a particular result of Lemma 2.8.

In the following, let $X = \{u|u, D_{0+}^{\alpha-2}u, D_{0+}^{\alpha-1}u, D_{0+}^{\alpha}u \in C[0,1]\}$ with the usual norm $\|u\|_X = \max\{\|u\|_\infty, \|D_{0+}^{\alpha-2}u\|_\infty \|D_{0+}^{\alpha-1}u\|_\infty, \|D_{0+}^{\alpha}u\|_\infty\}$, where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$, and $Y = C[0,1]$ with the usual norm $\|y\|_Y = \|y\|_\infty$.

Define the operator $M : \operatorname{dom} M \subset X \to Y$ by

$$Mu = D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(t)), \tag{2.6}$$

$$\begin{aligned}\operatorname{dom} M = \Big\{u \in X : D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u) \in Y,\ u(0) = u'(0) = D_{0+}^{\alpha}u(0) = 0,\\ D_{0+}^{\alpha-2}u(1) = \sum_{i=1}^{m-2} a_i D_{0+}^{\alpha-2}u(\eta_i)\Big\}.\end{aligned} \tag{2.7}$$

Define the operator $N_\lambda : X \to Y$, $\lambda \in [0,1]$,

$$N_\lambda u(t) = \lambda f(t, u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha}u(t)), t \in [0,1],$$

so, FBVP(1.1) is equivalent to the abstract equation $Mu = Nu$, where $N = N_1$.

3. Main result

In this section, we give the main results of this paper. First of all, we list the following hypotheses.

(H1) There exist nonnegative functions $a, b, c, d, e \in Y$ satisfying

$$|f(t, u, v, w, z))| \le a(t) + b(t)|u|^{p-1} + c(t)|v|^{p-1} + d(t)|w|^{p-1} + e(t)|z|^{p-1},$$

for all $t \in [0,1]$, $(u, v, w, z) \in \mathbb{R}^4$.

(H2) There exists a constant $A > 0$ satisfying

$$\begin{aligned}&\int_0^1 (1-s)\varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, u, v, w, z)d\tau\Big)ds\\ &-\sum_{i=1}^{m} a_i \int_0^{\eta_i} (\eta_i - s)\varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, u, v, w, z)d\tau)ds \neq 0,\end{aligned}$$

for all $t \in [0,1]$, $(u, v, w, z) \in \mathbb{R}^4$, $|v| + |w| > A$.

(H3) There exists a constant $B > 0$ satisfying

$$c(1-\lambda)\frac{C(f(t, ct^{\alpha-1}, c\Gamma(\alpha)t, c\Gamma(\alpha), 0)}{C_0} < 0, \tag{3.1}$$

or

$$c(1-\lambda)\frac{C(f(t, ct^{\alpha-1}, c\Gamma(\alpha)t, c\Gamma(\alpha), 0)}{C_0} > 0, \tag{3.2}$$

for all $|c| > B$, $c \in \mathbb{R}$. Where $C(y)$ and C_0 are defined in (3.7).

We give the main result of this paper.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous and the condition* (H1)–(H3) *hold, then BVP* (1.1) *has at least one solution provided that*

$$\frac{1}{\Gamma(\beta+1)}\Big(\frac{(\alpha+1)^{p-1}}{\Gamma(\alpha+1)^{p-1}}D\|b\|_\infty + D\|c\|_\infty + D\|d\|_\infty + \|e\|_\infty\Big) < 1. \tag{3.3}$$

To get the conclusion, we need the following Lemmas.

Lemma 3.2. *The operator $M : \operatorname{dom} M \cap X \to Y$ is a quasi-linear, and*

$$\ker M = \{u \in X : u(t) = ct^{\alpha-1},\ \forall t \in [0,1],\ c \in \mathbb{R}\}, \tag{3.4}$$

$$\begin{aligned}\operatorname{Im} M = \Big\{ y \in Y : &\int_0^1 (1-s)\varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} y(\tau)d\tau\Big)ds \\ &- \sum_{i=1}^{m-2} a_i \int_0^{\eta_i} (\eta_i - s)\varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} y(\tau)d\tau\Big)ds = 0\Big\}.\end{aligned} \tag{3.5}$$

Proof. By Lemma 2.5 and $D^{\beta}_{0+}\varphi_p(D^{\alpha}_{0+}u(t)) = 0$, then

$$D^{\alpha}_{0+}u(t) = \varphi_q(c_0 t^{\beta-1}).$$

so $c_0 = 0$. Thus,

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

Combining with $u(0) = u'(0) = 0$, we have $c_2 = c_3 = 0$. So, $u(t) = c_1 t^{\alpha-1}$, $c_1 \in \mathbb{R}$. i.e., (3.4) is satisfied.

Suppose $y \in \operatorname{Im} M$, then there exists $u \in \operatorname{dom} M$ satisfying

$$y(t) = D^{\beta}_{0+}\varphi_p(D^{\alpha}_{0+}u(t)).$$

Again, by Lemma 2.5, one has

$$u(t) = I^{\alpha}_{0+}\varphi_q(I^{\beta}_{0+}y(s)) + c_1 t^{\alpha-1}$$

and

$$D^{\alpha-2}_{0+}u(t) = D^{\alpha-2}_{0+}I^{\alpha}_{0+}\varphi_q(I^{\beta}_{0+}y(s)) + c_1\Gamma(\alpha)t.$$

Combining with $\sum_{i=1}^{m-2} a_i\eta_i = 1$, one has

$$\begin{aligned}&\int_0^1 (1-s)\varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} y(\tau)d\tau\Big)ds \\ &- \sum_{i=1}^{m-2} a_i \int_0^{\eta_i} (\eta_i - s)\varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} y(\tau)d\tau\Big)ds = 0.\end{aligned} \tag{3.6}$$

On the other hand, suppose $y \in Y$ and satisfies (3.6). Let $u(t) = I^{\alpha}_{0+}\varphi_q(I^{\beta}_{0+}y(t))$, we have $u \in \operatorname{dom} M$ and $Mu(t) = D^{\beta}_{0+}\varphi_p(D^{\alpha}_{0+}u(t)) = y(t)$. So $y \in \operatorname{Im} M$, i.e. (3.5) holds. From above statements, we know M is a quasi-linear operator. □

Lemma 3.3. *Suppose $\Omega \subset X$ be an open and bounded set, then N_λ is M-compact in $\overline{\Omega}$.*

Proof. Define the projectors $P : X \to X_1$ and $Q : Y \to Y_1$ respectively by

$$\begin{aligned}Pu(t) &= \tfrac{1}{\Gamma(\alpha)} D^{\alpha-1}_{0+}u(0)t^{\alpha-1}, \quad t \in [0,1],\\ Qy(t) &= \tfrac{C(y)}{C_0}t^k, \quad t \in [0,1],\end{aligned}$$

where $X_1 = \ker M, Y_1 = \{c\,t^k, c \in \mathbb{R}\}$,
and we define a functional

$$C(y) = \varphi_p\Big(\int_0^1 (1-s)\varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} y(\tau)d\tau\Big)ds$$

$$- \sum_{i=1}^{m-2} a_i \int_0^{\eta_i} (\eta_i - s)\varphi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} y(\tau)d\tau)ds)\Big) \quad (3.7)$$

and a constant

$$\begin{aligned} C_0 &= \varphi_p\Big(\int_0^1 (1-s)\varphi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1}\tau^k d\tau)ds \\ &\quad - \sum_{i=1}^{m-2} a_i \int_0^{\eta_i} (\eta_i - s)\varphi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1}\tau^k d\tau)ds)\Big) \\ &= \frac{k!}{\Gamma(\beta+k+1)} \cdot \frac{\Big(1 - \sum_{i=1}^{m-2} a_i \eta_i^{\beta(q-1)+2+k(q-1)}\Big)^{p-1}}{((\beta+k)(q-1)+2)^{p-1}((\beta+k)(q-1)+1)^{p-1}}. \end{aligned}$$

In fact, $C_0 = C(t^k)$. Here $k \in \{0, 1, \dots, m-2\}$ satisfies $\sum_{i=1}^{m-2} a_i \eta_i^{\beta(q-1)+2+k(q-1)} \neq 1$ which be as in lemma 2.8. Obviously, $X_1 = \ker M = \operatorname{Im} P$ and $Y_1 = \operatorname{Im} Q$. Thus, we have $\dim Y_1 = \dim X_1 = 1$. For any $y \in Y$, we get

$$Q^2 y = \frac{C(Qy)}{C_0} t^k = \frac{C(y)}{C_0^2} C(t^k) t^k = \frac{C(y)}{C_0} t^k = Qy.$$

Hence, $Q^2 = Q$, Q is a semi-projector and $\ker Q = \operatorname{Im} M$.

Let $\Omega \subset X$ be an open and bounded set with $\theta \in \Omega$. For each $u \in \overline{\Omega}$, we have $Q[(I-Q)N_\lambda(u)] = 0$. Thus, $(I-Q)N_\lambda(u) \in \operatorname{Im} M = \ker Q$. Next, taking any $y \in \mathrm{ImM}$ and noting $Qy = 0$, one get $y \in (I-Q)Y$. So, (2.2) holds.

Define $R : \overline{\Omega} \times [0,1] \to X_2$ by

$$R(u,\lambda)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q\Big(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1}((I-Q)N_\lambda u(\tau))d\tau\Big)ds.$$

Firstly, We know $R(u,\lambda)$ is continuous on $\overline{\Omega} \times [0,1]$. Moreover, for all $u \in \overline{\Omega}$, there exists a constant $L > 0$ such that $|I_{0+}^\beta (I-Q)N_\lambda u(\tau))| \leq L$, so $R(\overline{\Omega}, \lambda)$, $D_{0+}^{\alpha-2} R(\overline{\Omega}, \lambda)$, $D_{0+}^{\alpha-1} R(\overline{\Omega}, \lambda)$ and $D_{0+}^{\alpha} R(\overline{\Omega}, \lambda)$ are equicontinuous and uniformly bounded. Thus, $R : \overline{\Omega} \times [0,1] \to X_2$ is compact.
In fact, for $u \in \overline{\Omega}$, $0 < t_1 < t_2 \leq 1$, $2 < \alpha \leq 3$, $0 < \beta \leq 1$, $3 < \alpha + \beta \leq 4$, we have

$$\begin{aligned} & |R(u,\lambda)(t_2) - R(u,\lambda)(t_1)| \\ = \;& \frac{1}{\Gamma(\alpha)} \Big| \int_0^{t_2} (t_2 - s)^{\alpha-1} \varphi_q(I_{0+}^\beta((I-Q)N_\lambda u(\tau)))ds \\ - \;& \int_0^{t_1} (t_1 - s)^{\alpha-1} \varphi_q(I_{0+}^\beta((I-Q)N_\lambda u(\tau)))ds \Big| \\ \leq \;& \frac{\varphi_q(L)}{\Gamma(\alpha)} \Big(\int_0^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds\Big) \\ = \;& \frac{\varphi_q(L)}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha), \end{aligned}$$

$$\Big| D_{0+}^{\alpha-2} R(u,\lambda)(t_2) - D_{0+}^{\alpha-2} R(u,\lambda)(t_1) \Big|$$

$$
\begin{aligned}
&= |\int_0^{t_2}(t-s)\varphi_q(I_{0+}^{\beta}((I-Q)N_\lambda u(\tau)))ds - \int_0^{t_1}(t-s)\varphi_q(I_{0+}^{\beta}((I-Q)N_\lambda u(\tau)))ds| \\
&\le \varphi_q(L)(\int_0^{t_1}(t_2-s)-(t_1-s)ds + \int_{t_1}^{t_2}(t_2-s)ds) \\
&= \frac{\varphi_q(L)}{2}(t_2^2-t_1^2)
\end{aligned}
$$

and

$$
\begin{aligned}
&|D_{0+}^{\alpha-1}R(u,\lambda)(t_2) - D_{0+}^{\alpha-1}R(u,\lambda)(t_1)| \\
= & |\int_0^{t_2}\varphi_q(I_{0+}^{\beta}((I-Q)N_\lambda u(\tau)))ds - \int_0^{t_1}\varphi_q(I_{0+}^{\beta}((I-Q)N_\lambda u(\tau)))ds| \\
\le & \varphi_q(L)(t_2-t_1).
\end{aligned}
$$

Since t^α is uniformly continuous on $[0,1]$, so $R(\overline{\Omega},\lambda)$, $D_{0+}^{\alpha-2}R(\overline{\Omega},\lambda)$ and $D_{0+}^{\alpha-1}R(\overline{\Omega},\lambda)$ are equicontinuous. Similarly, $I_{0+}^{\beta}((I-Q)N_\lambda u(\tau)) \subset C[0,1]$ is equicontinuous too. Because $\varphi_q(s)$ is uniformly continuous on $[-T,T]$, we have $D_{0+}^{\alpha}R(\overline{\Omega},\lambda) = I_{0+}^{\beta}((I-Q)N_\lambda(\overline{\Omega}))$ is equicontinuous. Thus, $R:\overline{\Omega}\times[0,1]\to X_2$ is compact.

For each $u\in\Sigma_\lambda$, we have $D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(t)) = N_\lambda(u(t)) \in \operatorname{Im} M$. Thus,

$$
\begin{aligned}
R(u,\lambda)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}((I-Q)N_\lambda u(\tau))d\tau)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(\tau))d\tau)ds,
\end{aligned}
$$

Furthermore, one has

$$
R(u,\lambda)(t) = u(t) - \frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1} = (I-P)u(t).
$$

Because $R(u,0)(t)$ is zero operator, so (2.3) holds. Moreover, for any $u\in\overline{\Omega}$,

$$
\begin{aligned}
&M[Pu+R(u,\lambda)](t) \\
&= M[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}((I-Q)N_\lambda u(\tau))d\tau)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1}] \\
&= (I-Q)N_\lambda u(t),
\end{aligned}
$$

which implies (2.4). So N_λ is M-compact in $\overline{\Omega}$. □

Lemma 3.4. *Suppose* (H1), (H2) *hold, then the set*

$$
\Omega_1 = \{u\in \operatorname{dom} M\setminus\ker M : Mu = \lambda Nu,\ \lambda\in(0,1)\}
$$

is bounded.

Proof. By lemma 2.5, one has

$$
u(t) = I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3},
$$

where

$$
c_i = \frac{\left(I_{0+}^{n-\alpha}x(t)\right)^{(n-i)}\big|_{t=0}}{\Gamma(\alpha-i+1)}, i=1,2,3.
$$

Combining this with $u(0) = u'(0) = 0$, we get $c_1 = c_2 = 0$ and

$$c_1 = \frac{D_{0^+}^{\alpha-1}x(0)}{\Gamma(\alpha)}.$$

$$\begin{aligned}
\|u\|_\infty &= \|I_{0^+}^\alpha D_{0^+}^\alpha u + \frac{D_{0^+}^{\alpha-1}x(0)}{\Gamma(\alpha)} t^{\alpha-1}\|_\infty \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \| \|D_{0^+}^\alpha u\|_\infty + \frac{|D_{0^+}^{\alpha-1}x(0)|}{\Gamma(\alpha)} \\
&\le \frac{1}{\Gamma(\alpha)} \|D_{0^+}^{\alpha-1}u\|_\infty + \frac{1}{\Gamma(\alpha+1)} \|D_{0^+}^\alpha u\|_\infty.
\end{aligned}$$

Take any $u \in \Omega_1$, then $Nu \in \mathrm{Im} M = \ker Q$ and $QNu = 0$ for all $t \in [0,1]$. By using (H2), there exists $t_0 \in [0,1]$ such that $|D_{0^+}^{\alpha-2}u(t_0)| + |D_{0^+}^{\alpha-1}u(t_0)| \le A$. Thus

$$D_{0^+}^{\alpha-1}u(t) = D_{0^+}^{\alpha-1}u(t_0) + \int_{t_0}^t D_{0^+}^\alpha u(t) dt, \tag{3.8}$$

$$D_{0^+}^{\alpha-2}u(t) = D_{0^+}^{\alpha-2}u(t_0) + \int_{t_0}^t D_{0^+}^{\alpha-1}u(t) dt, \tag{3.9}$$

$$\|D_{0^+}^{\alpha-1}u\|_\infty \le A + \|D_{0^+}^\alpha u\|_\infty, \tag{3.10}$$

$$\|D_{0^+}^{\alpha-2}u\|_\infty \le A + \|D_{0^+}^{\alpha-1}u\|_\infty \le 2A + \|D_{0^+}^\alpha u\|_\infty, \tag{3.11}$$

$$\|u\|_\infty \le \frac{1}{\Gamma(\alpha+1)} (\alpha+1) \|D_{0^+}^\alpha u\|_\infty + \bar{C}, \tag{3.12}$$

where $\overline{C} = \frac{A}{\Gamma(\alpha)}$. Combining with $Mu = \lambda Nu$ and $D_{0^+}^\alpha u(0) = 0$, we obtain

$$\varphi_p(D_{0^+}^\alpha u(t)) = \lambda I_{0^+}^\beta Nu(t).$$

From (H1) and $\lambda \in (0,1)$, one has

$$\begin{aligned}
|\varphi_p(D_{0^+}^\alpha u(t))| \le & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |f(s, u(s), D_{0^+}^{\alpha-2}u(s), D_{0^+}^{\alpha-1}u(s), D_{0^+}^\alpha u(s))| ds \\
& + d(s)|D_{0^+}^{\alpha-1}u(s)|^{p-1} + e(s)|D_{0^+}^\alpha u(s)|^{p-1}) ds \\
\le & \frac{1}{\Gamma(\beta+1)} (\|a\|_\infty + \|b\|_\infty \|u\|_\infty^{p-1} + \|c\|_\infty \|D_{0^+}^{\alpha-2}u\|_\infty^{p-1} \\
& + \|d\|_\infty \|D_{0^+}^{\alpha-1}u\|_\infty^{p-1} + \|e\|_\infty \|D_{0^+}^\alpha u\|_\infty^{p-1}), \quad \forall t \in [0,1].
\end{aligned}$$

By the virtue of $|\varphi_p(D_{0^+}^\alpha u(t))| = |D_{0^+}^\alpha u(t)|^{p-1}$ and the inequality $(|a| + |b|)^p \le \overline{B}(|a|^p + |b|^p)$, where $\overline{B} = 2^{p-1}$ when $p > 2$ and $\overline{B} = 1$ when $1 < p < 2$, $a, b \in \mathbb{R}$. One has

$$\begin{aligned}
\|D_{0^+}^\alpha u\|_\infty^{p-1} \le & \frac{1}{\Gamma(\beta+1)} (\|a\|_\infty + \|b\|_\infty \overline{B} \frac{(\alpha+1)^{p-1}}{\Gamma(\alpha+1)^{p-1}} \|D_{0^+}^\alpha u\|_\infty^{p-1} + C_1 \\
& + \|c\|_\infty \overline{B}(C_2 + \|D_{0^+}^\alpha u\|_\infty^{p-1}) + \|d\|_\infty \overline{B}(C_3 + \|D_{0^+}^\alpha u\|_\infty^{p-1}) \\
& + \|e\|_\infty \|D_{0^+}^\alpha u\|_\infty^{p-1}),
\end{aligned}$$

where C_1, C_2, C_3 are some constants. From (3.3), there exists a constant $M_1 > 0$ satisfying

$$\|D_{0^+}^\alpha u\|_\infty < M_1. \tag{3.13}$$

Then, Ω_1 is bounded. $\square$

Lemma 3.5. *Suppose* (H2) *holds, then the set* $\Omega_2 = \{u \in \ker M : Nu \in \operatorname{Im} M\}$ *is bounded.*

Proof. For each $u \in \Omega_2$, we have $u(t) = ct^{\alpha-1}$ for all $c \in \mathbb{R}$ and $QNu = 0$. By using (H2), there exists a $t_0 \in [0,1]$ satisfying $|D_{0^+}^{\alpha-1}u(t_0)| + |D_{0^+}^{\alpha-2}u(t_0)| \le A$, which implies $|c| \le \frac{A}{\Gamma(\alpha)(1+t_0)}$. Therefore, Ω_2 is bounded. $\square$

Define the isomorphism $J : \operatorname{Im} Q \to \ker M$ by $J(ct^k) = ct^{\alpha-1}, c \in \mathbb{R}$, for all $t \in [0,1]$.

Lemma 3.6. *Suppose (3.1) holds, then*

$$\Omega_3 = \{u \in \ker M : -\lambda u + (1-\lambda)JQNu = 0,\ \lambda \in [0,1]\}$$

is bounded.

Proof. Suppose (3.1) holds, for $u \in \Omega_3$, we have $u(t) = ct^{\alpha-1}$ for $c \in \mathbb{R}$. Then

$$\lambda ct^{\alpha-1} = (1-\lambda)\frac{C(f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha)\tau, c\Gamma(\alpha), 0)}{C_0}t^{\alpha-1}, \tag{3.14}$$

where $C(y)$ is defined in (3.7). If $\lambda = 1$, then $c = 0$. If $\lambda \neq 1$, in view of (3.1), one has

$$c(1-\lambda)\frac{C(f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha)\tau, c\Gamma(\alpha), 0)}{C_0} < 0, \tag{3.15}$$

which contradicts to $\lambda c^2 \ge 0$. i.e., Ω_3 is bounded.

Suppose (3.2) holds, it is similar to proof

$$\Omega_3 = \{u \in \ker M : \lambda u + (1-\lambda)JQNu = 0,\ \lambda \in [0,1]\}$$

is bounded. $\square$

Proof of Theorem 3.1. Assume that Ω is a bounded open set of X with $\cup_{i=1}^{3}\overline{\Omega}_i \subset \Omega$. By Lemma 3.3, Lemmas 3.4 and 3.5, we have N is M-compact on $\overline{\Omega}$,

(i) $Mx \neq N_\lambda x$ for each $(u,\lambda) \in (\operatorname{dom} M \cap \partial\Omega) \times (0,1)$,
(ii) $QNu \neq 0$, for all $u \in \partial\Omega \cap \ker M$.

Let $H(u,\lambda) = \lambda u + (1-\lambda)JQN(u)$. By Lemma 3.6 we know $H(u,\lambda) \neq 0$ for each $u \in \partial\Omega \cap \ker M$. Thus, we have

$$\begin{aligned}
\deg(JQN|_{\ker M}, \Omega \cap \ker M, 0) &= \deg(H(\cdot,0), \Omega \cap \ker M, 0) \\
&= \deg(H(\cdot,1), \Omega \cap \ker M, 0) \\
&= \deg(I, \Omega \cap \ker M, 0) \neq 0,
\end{aligned}$$

then (iii) of Lemma 2.1 is holds. Consequently, FBVP (1.1) have at least one solution. $\square$

4. Conclusion

There are some articles which consider the boundary value problems of fractional differential equation at resonance with p-Laplacian. For example, in [8], Shen and Liu studied the following problem

$$\begin{cases} D_{0^+}^{\beta}\varphi_p(D_{0^+}^{\alpha}u(t)) = f(t,u(t),D_{0^+}^{\alpha-2}u(t),D_{0^+}^{\alpha-1}u(t),D_{0^+}^{\alpha}u(t)),\ t\in(0,1), \\ u(0)=u'(0)=D_{0^+}^{\alpha}u(0)=0, \quad D_{0^+}^{\alpha-1}u(1)=\sum_{i=1}^{m}\sigma_i D_{0^+}^{\alpha-1}u(\eta_i), \end{cases} \tag{4.1}$$

where $2<\alpha\le 3$, $0<\beta\le 1$, $3<\alpha+\beta\le 4$, $\eta_i\in(0,1)$, $\sigma_i\in\mathbb{R}$, $\sum_{i=1}^{m}\sigma_i=1$, $1<m, m\in N$, $\varphi_p(s)=|s|^{p-2}s$, $1<p, 1/p+1/q=1$. In this paper, the author used the condition

$$\Delta = \frac{1}{\Gamma(\beta+1)^{q-1}(q\beta-\beta+1)}(1-\sum_{i=1}^{m}\sigma_i\eta_i^{q\beta-\beta+1}) \neq 0.$$

Moreover, in other papers, the authors need the same assumption which similar to above condition. In this paper, we considered the similar problems which do not need the assumption like above condition. So, in some sense, our paper generalize some results(see[8]).

5. Declarations

Competing interests.

The author declare that they have no competing interests.

Authors' contributions.

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Yanqiang Wu

School of Mathematics, China University of Mining and Technology, Xuzhou 221008, China

E-mail address: wyq1976819@126.com