Average path length of a special class of hierarchical networks

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Abstract. Many of the behaviors observed in actual systems are comparable to scale-free and small-world structures in network research. In contrast to conventional hierarchical networks, the unusual fractal hierarchical network we created in this research has a pyramidal structure. The findings we get from this network are expanded to be applicable to arbitrary hierarchical networks. The average path length of unweighted and weighted hierarchical networks are the main topics of this paper. We demonstrate that, in the unweighted case, when the number of iterations z tends to infinity, the average path length is only related to the number of blocks of the hierarchical network. Additionally, in the weighted network, the average path length is related to the number of blocks r and the weighting factor w of the hierarchical network.

Keywords: Hierarchical network; Weighted networks; Average path length; Self-similar

1. Introduction

Complex networks presently play a significant role in scientific and social study because to the growth of fractal investigation, and many natural and social phenomena may be described by complex networks. Networks with random and dynamic properties have gained popularity during the last several years as a subject of study [1, 2]. The research has put a great deal of work into establishing their structure and studying the newly complicated features. It is well recognized that most real networks possess two distinctive characteristics of being workers, namely small world and scale-free [3], which also enhance the theory of complex networks. This is true despite the diversity of networks. Additionally, the majority of network systems have three distinguishing structural characteristics: clustering, degree distribution [4], and average path length [5]. The reader can referred to [6, 7], for more studies on structural properties in complex networks.

Additionally, complex networks show self-similarity, and their sharing dimension exhibits power-law properties. Recently, self-similarity fractals have been utilized to simulate how network iterations change over time [8]. Meanwhile, Zhang et al. [9, 10] employed Sierpinski gaskets to build evolutionary networks in a series of articles. Complex networks also have self-similar fractal models. For instance, Liu and Kong [11], Chen et al [12] examine Koch networks and Vicsek fractal-built networks, respectively. The average path length (APL) is an important indicator of small-world properties and represents the size of the network. The average path length is also the average of the shortest path length (SPL) between all two nodes in the network. Therefore, many formulas for calculating the average path length have been derived theoretically, which improves the efficiency when computing, but leads to a decrease in the efficiency of the computation as the size of the network grows. Accurate calculation of APL requires access to all SPLs, but this is impractical in social networks because information about all nodes is not available due to security and privacy protection constraints. In order to solve the problem of increasing computation due to the large-scale iteration of social networks, many scholars have introduced an alternative method of parameter estimation to simplify the computational process [13–15].

The preceding research concentrate on unweighted networks, but realistic application frequently shows that node-to-node connections may also be significant. As a consequence, many researchers have also made significant efforts to combine weighted networks with real-world issues. In order to assess its various

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properties and correlate them with their topological relatives, the authors of [16] examined Indian airports as a weighted hierarchical network. To better comprehend the topology and characteristics of global trade and financial networks, the authors of [17] investigate the model of international commerce and financial integration by integrating network analysis and weighting methods. Excellent research on weighted networks has also been done by several other writers [18–21].

Inspired by previous studies, in this paper, a class of three-dimensional triangular conical networks with scale-free and fractal structure is established. In Section 2, the network construction process is introduced. In Section 3, the definition and formula of the average path length in complex networks are first introduced, the average path length of hierarchical networks in the unweighted condition is calculated, and the relationship between the average path length and the iteration number in complex networks is explored. In Section 4, the weighed factor w is taken into consideration, and the average path length of weighted network is calculated at this stage. In section 5 provides the basic conclusions.

2. Construction of hierarchical network

This section focuses on the process of investigating the generation of fractal networks. First, we identified the networks after z iterations as $G_z^r, (r \geq 1, r, z \in N)$, where r denotes the number of blocks in the network. The construction procedures of the fractal network when $z = 1\ ,2\ ,3$ are given then to help the reader comprehend the iterative process. It is important to keep in mind that, regardless of how many iterations the network goes through, there is only one root node at the z-th generation, as indicated by the red, and the root node copied after the iterations are called vertexs as marked in green in Fig.1. In the graph, the blue marked points represent the bottom nodes in the network.

The following are the main construction steps:

Step 1. For $z = 0$, the network G_0^r is composed of a root node and three bottom nodes interconnected by lines in a triangular cone structure.

Step 2. For $z = 1$, the network G_0^r is duplicated with r times. With the network G_0^r at the root node as the center, $(r-1)$ networks G_0^r with similar structure are evenly scattered around, while the bottom nodes of the last layer are connected to the root node, thus the 1-th generation hierarchical network is obtained. Fig.1 illustrates the specific steps taken during the generation of the hierarchical network structure.

Step 3. Similarly, for $z \geq 2$, the network G_{z-1}^r is duplicated with r times. With the network G_{z-1}^r at the root node as the center, $(r-1)$ networks G_{z-1}^r with similar structure are evenly around, while the bottom nodes of the last layer are connected to the root node, thus the z-th generation network is generated, as Fig.2.

For the network model after z iterations, the total number of nodes is $N_z = 4r^z$, at this time there is only 1 largest root node and $(r-1)$ hub nodes, $3r^{z-1}$ vertices, the number of bottom nodes generated in the z generation that is the last layer of the bottom nodes is $3(r-1)^z$, where the hub nodes have connections with other bottom nodes, while the vertices have no links with other bottom nodes.

Given the rising complexity of G_{z+1}^r, this self-repeating pattern network can be viewed as a self-similar hierarchical fractal network. We simplify the network structure (as in Fig. 3) for the purpose of simplicity of calculating the average path length of the unweighted and weighted networks in the following section, and first we consider G_{z+1}^r as a network with r blocks. First, we consider G_{z+1}^r to be a network with r blocks, which are named $G_z^1, G_z^2, \ldots, G_z^r$.

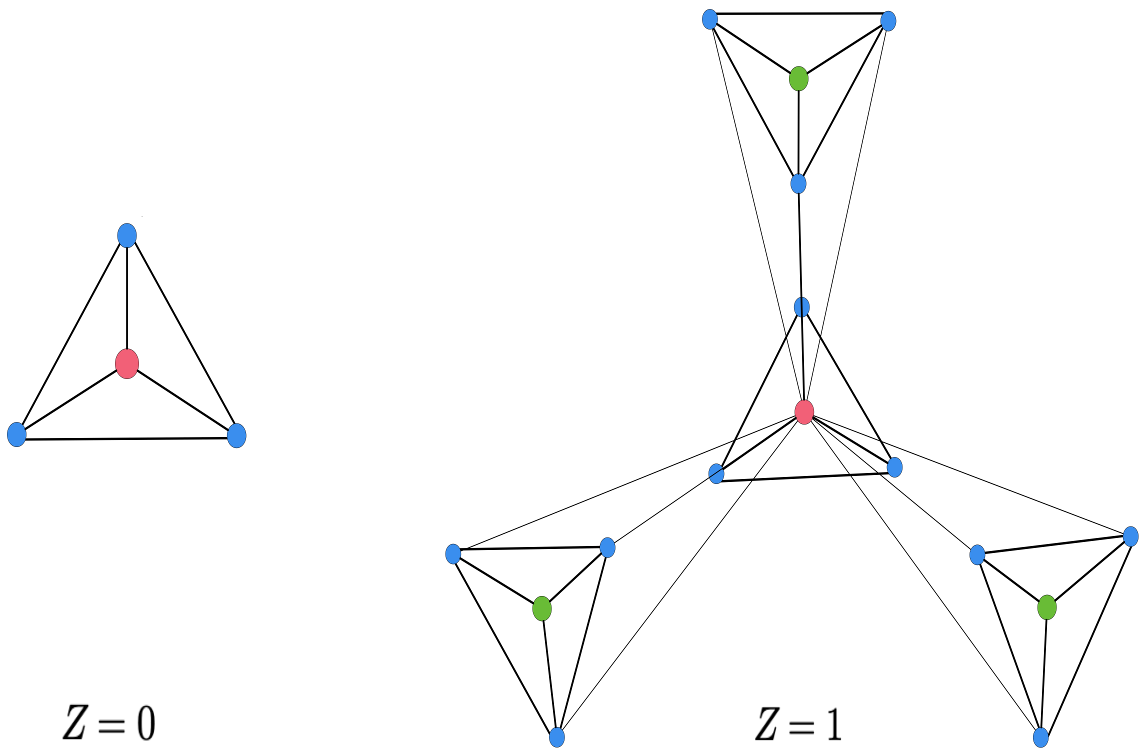

Figure 1: The network iteration process for $z = 0,\ 1$ with block number $r = 4$.

3. The average path length of unweighted networks

One of the most essential components of the network is the average path length, commonly referred to as the shortest average distance. The shortest distance d_{ij} between nodes i and j among all potential pairings of nodes can be used to compute it. Let D_z be the sum distance of the shortest path lengths between all two nodes in the fractal network given in Eq. (3.1) when iterating to the z-th time.

$$D_z = \frac{1}{2} \sum_{i \neq j \in G_z^r} d_{ij}(z). \tag{3.1}$$

The average path length indicated by $\overline{D_z}$ could then be determined.

$$\overline{D_z} = \frac{D_z}{N_z(N_z - 1)/2}, \tag{3.2}$$

where N_z represents the total number of nodes in the network.

Based on the fractal properties mentioned previous section, we can define the following network factors. Let M_z be the sum of the path lengths from the complex network's nodes other than the root node to the root node in the z-th generation, and L_z be the sum of the path lengths between the nodes in the z-th generation other than the bottom node. The network model shown in Fig.1 clearly shows that $M_0 = 3$ and $L_0 = 3$. Calculating the average path length of the network G_{z+1}^r generates the total path length between all node pairs in the network. The division is made with the root node and the underlying nodes for computational simplicity. M_{z+1} stands for the total of the root node distances at each node up to $(z+1)$ generations, with the root node as the center. Because the $(z+1)$-th generation network is composed of the z-th generation network replicated $(r-1)$ times, the analysis of M_{z+1} can be

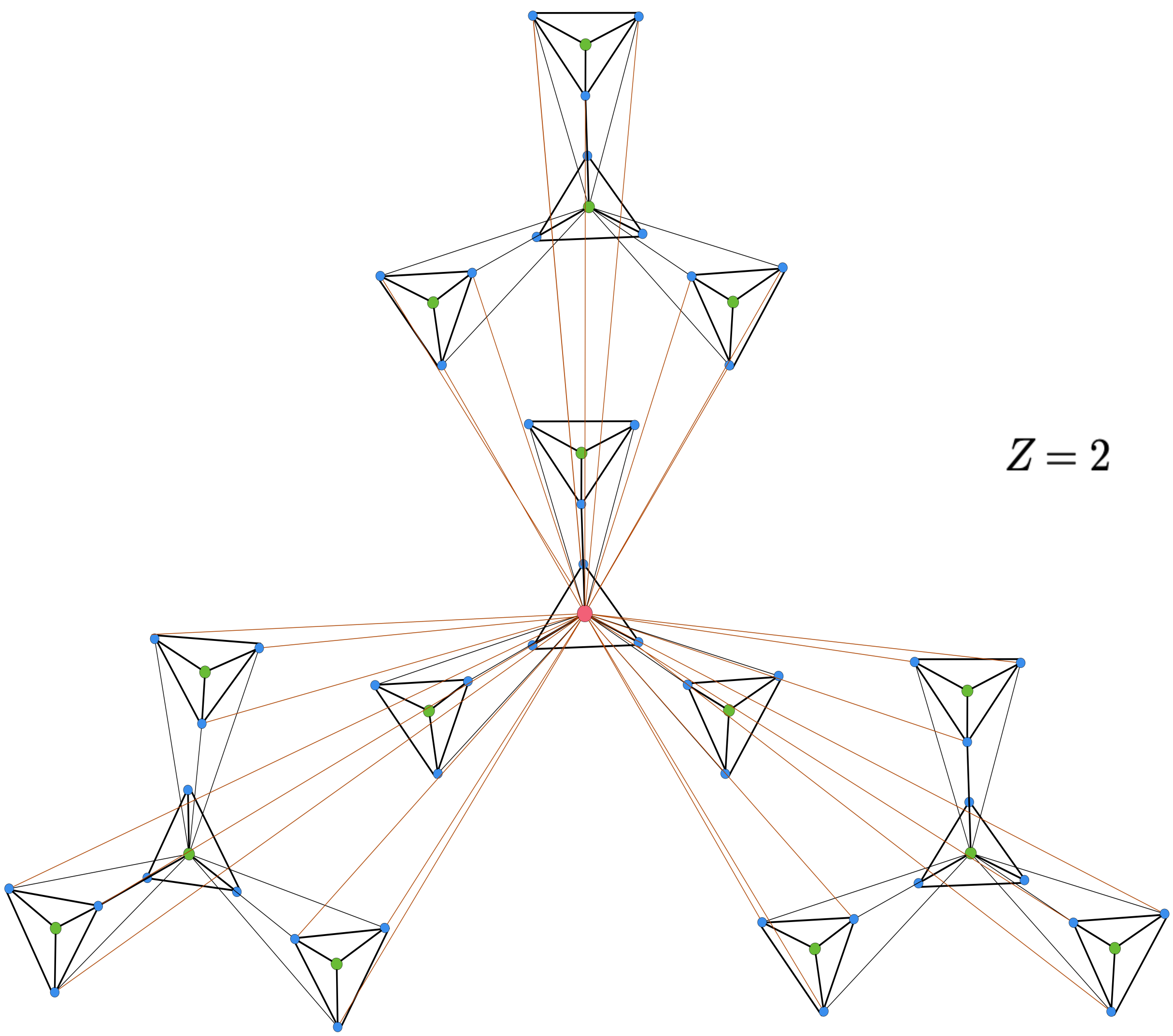

Figure 2: The network iteration process for $z = 2$ with block number $r = 4$.

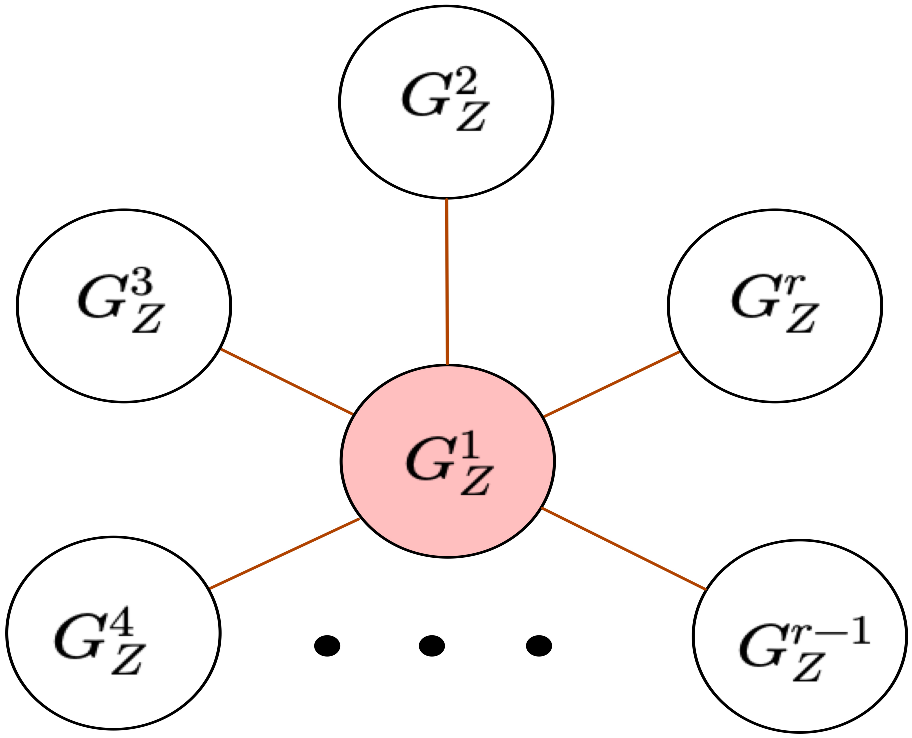

Figure 3: Structure of network model G^r_{z+1} for $(z+1)$-th generation.

split into two segments for the z-th and $(z+1)$-th generations, where the first segment could be referred to the total of distances from all nodes in the z-th generation network to the root node, and the second segment can be referred to the total of distances from the nodes generated in the $(z+1)$-th.

Correspondingly, the total amount of the distances from every node in the network to any bottom node in the $(z+1)$ generation, denoted by L_{z+1}, can be separated into two segments: the initial is the total of the distances from nodes produced in the $(z+1)$ generation with $(r-1)$ to their random bottom nodes, and the latter is the total of the distances from nodes produced in the z generation to the bottom nodes. The final portion is the total of the distances from the central node to the bottom node.

Therefore, according to the self-similar structure of the network, the total distance between different branches can be obtained using Eq. (3.3).

$$D_{z+1} = rD_z + \Delta_{z+1}, \tag{3.3}$$

where Δ_{z+1} is the sum of the lengths of all shortest paths between nodes in different branches G^r_z, $r \in N^+$.

Lemma 3.1. *In z-th generation, the network G^r_z has*

$$M_z = r^{z-2}(8z \cdot r - 8z - 8r + 11r^2),\ L_z = r^{z-2}\Big(8z(r-1) + 3r^2\Big).$$

Proof. First, in G^r_z, we can have any arbitrary node i. Let $m_i(z)$ be the smallest distance between i and the root node, the shortest path length between i and the last layer of bottom nodes is given by $l_i(z)$. In addition, M_z is represented as the total sum of all nodes in G^r_z, and L_z to be the sum of $l_i(z)$. As previously stated, we can deduce that the expression of M_{z+1} is

$$
\begin{aligned}
M_{z+1} &= \sum_{i\in G_z^1} m_i(z+1) + \sum_{i\in G_z^2} m_i(z+1) + \cdots + \sum_{i\in G_z^{r-1}} m_i(z+1) + \sum_{i\in G_z^r} m_i(z+1) \\
&=(r-1)\sum_{i\in G_z^r}(l_i(z)+1) + \sum_{i\in G_z^r} m_i(z) \\
&=(r-1)(L_z+N_z)+M_z.
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
L_{z+1} &= \sum_{i\in G_z^1} l_i(z+1) + \sum_{i\in G_z^2} l_i(z+1) + \cdots + \sum_{i\in G_z^{r-1}} l_i(z+1) + \sum_{i\in G_z^r} l_i(z+1) \\
&=(r-1)\sum_{i\in G_z^r} l_i(z) + \sum_{i\in G_z^r}(m_i(z)+1) \\
&=M_z+N_z+(r-1)L_z.
\end{aligned} \tag{3.5}
$$

On the other side, Eq. (3.4) can be rewritten as

$$
M_z = L_{z+1} - (r-1)L_z - N_z. \tag{3.6}
$$

The result from Eq. (3.6) is carried over to Eq. (3.5), one has

$$
L_{z+2} - rL_{z+1} = 8r^{z+1} - 8r^z.
$$

Then

$$
\begin{aligned}
r^{-(z+1)}L_{z+1} - r^{-z}L_z &= 8r^{-1} - 8r^{-2}, \\
&\vdots \\
r^{-2}L_2 - r^{-1}L_1 &= 8r^{-1} - 8r^{-2}, \\
r^{-1}L_1 - r^0L_0 &= 8r^{-1} - 8r^{-2}.
\end{aligned}
$$

Combining $L_0 = 3$, the expression for L_z can be obtained as

$$
\begin{aligned}
L_z &= (z+1)(8r^{-1} - 8r^{-2})r^{z+1} + 3r^{z+1} \\
&= r^{z-2}\Big(8z(r-1) + 3r^2\Big).
\end{aligned} \tag{3.7}
$$

Put the result of Eq. (3.7) into Eq. (3.6), we can derive

$$
M_z = r^{z-2}(8z\cdot r - 8z - 8r + 11r^2). \tag{3.8}
$$

□

Once the expressions for M_z and L_z have been solved, the next step is to compute Δ_{z+1}.

Lemma 3.2. *The network G_z^r with r blocks, then the total distance Δ_{z+1} can be defined as*

$$
\Delta_{z+1} = 4(r-1)r^{2z-1}\Big(7r^2 + 4(2z\cdot r - 2z - r - 2)\Big).
$$

Proof. Considering the self-similar structure of G_z^r, Δ_{z+1} could be written in the following way,

$$\begin{aligned}\Delta_{z+1} &= \Delta_{i\in G_z^1, j\in G_z^2} + \Delta_{i\in G_z^1, j\in G_z^3} + \cdots + \Delta_{i\in G_{z-1}^1, j\in G_z^2} \\ &\quad + \Delta_{i\in G_z^2, j\in G_z^3} + \Delta_{i\in G_z^2, j\in G_z^4} + \cdots + \Delta_{i\in G_z^{r-1}, j\in G_z^r} \\ &= (r-1)\Delta_{i\in G_z^1, j\in G_z^2} + \mathbf{C}_{r-1}^2 \Delta_{i\in G_z^2, j\in G_z^3}.\end{aligned} \tag{3.9}$$

From Eq. (3.9), the following assignment focuses on calculating the expressions for $\Delta_{i\in G_z^1, j\in G_z^2}$ and $\Delta_{i\in G_z^2, j\in G_z^3}$.

$$\begin{aligned}\Delta_{i\in G_z^1, j\in G_z^2} &= \sum_{i\in G_z^1, j\in G_z^2} d_{ij}(z+1) \\ &= \sum_{i\in G_z^1, j\in G_z^2} (m_i + 1 + l_i) \\ &= \sum_{i\in G_z^1} \sum_{j\in G_z^2} m_i(z) + \sum_{i\in G_z^1} \sum_{j\in G_z^2} (1 + l_i(z)) \\ &= N_z \cdot P_z + N_z^2 + N_z \cdot L_z \\ &= 8r^{2z-2}\Big(9r^2 - 8z + r(8z-4)\Big).\end{aligned} \tag{3.10}$$

Then we have

$$\begin{aligned}\Delta_{i\in G_z^2, j\in G_z^3} &= \sum_{i\in G_z^1, j\in G_z^2} d_{ij}(t+1) \\ &= \sum_{i\in G_z^1, j\in G_z^2} (l_i + 1 + 1 + l_j) \\ &= 2\sum_{i\in G_z^1} \sum_{j\in G_z^2} (l_i(z) + 1) \\ &= 2(N_z^2 + N_z \cdot L_z) \\ &= 8r^{2z-2}(8z\cdot r - 8z + 7r^2).\end{aligned} \tag{3.11}$$

Combining the results of Eqs. (3.9), (3.10), and (3.11), Δ_{z+1} can be given as

$$\begin{aligned}\Delta_{z+1} &= (r-1)\Delta_{i\in G_z^1, j\in G_z^2} + \mathbf{C}_{r-1}^2 \Delta_{i\in G_z^2, j\in G_z^3} \\ &= 4(r-1)r^{2z-3}\Big(7r^2 + 4(2z\cdot r - 2r - 2z - r)\Big).\end{aligned} \tag{3.12}$$

□

Theorem 3.3. *Let G_z^r be a network with r blocks iterated to the z-th time, then there exists*

$$D_z = 6r^z + 4r^{z-2}(8 + 12r - 7r^2 - 8r^z - 12r^{z+1} + 7r^{z+2} - 8z\cdot r^2 + 8z\cdot r^{z+1}). \tag{3.13}$$

Proof. Based on the previously obtained Eq. (3.3), we can obtain another expression for D_z differently as follows

$$D_{z+1} = r\cdot D_z + 4(r-1)r^{2z-3}\Big(7r^2 + 4(2z\cdot r - 2r - 2z - r)\Big).$$

Taking into account $D_z = 6$ in the initial condition, D_z can be converted into the iterative equation of Eq. (3.13)

$$\begin{aligned} D_z &= rD_{z-1} + \Delta_z \\ &= r^2 D_{z-2} + r\Delta_{z-1} + \Delta_z \\ &= r^z D_0 + r^{z-1}\Delta_1 + r^{z-2}\Delta_2 + \cdots + \Delta_z \\ &= r^z D_0 + \sum_{i=1}^{z} r^{z-i}\Delta_i. \end{aligned} \tag{3.14}$$

The crucial to solving Eq. (3.14) depends on the expression of $\sum_{i=1}^{z} r^{z-i}\Delta_i$

$$\sum_{i=1}^{z} r^{z-i}\Delta_i = 4z \cdot r^{2z-3}(r^2-1)(7r^2 - 8z + 8z \cdot r - 12r). \tag{3.15}$$

Therefore, Eq. (3.15) is put into Eq. (3.14), then the theorem holds and the proof is complete. □

Theorem 3.4. *Let G_z^r be a network with r blocks iterated to the z-th time, then there exists*

$$\lim_{z\to\infty} \frac{\overline{D_z}}{z} = \frac{4(r-1)}{r^2}.$$

Proof. In the network G_z^r, $N_z = 3$ and according to Eqs. (3.1) (3.2) (3.13), we have

$$\begin{aligned} \overline{D_z} &= \frac{D_z}{N_z(N_z-1)/2} \\ &= \frac{16 + 24r - 11r^2 - 16r^z - 24r^{z+1} + 14r^{z+2} - 16z \cdot r^z + 16z \cdot r^{z+1}}{r^2(4r^z - 1)}. \end{aligned} \tag{3.16}$$

According to the result of solving $\overline{D_z}$ and the method of solving the limit, then Theorem 3.4 can be proved.

□

As a result, when r is a specific value, there is a linear growth relationship between D_z and the number of iterations z, which is taken into account when $\overline{D_z} \propto \ln N_z$. Because of this, for large networks, the average distance increases logarithmically with network order. Furthermore, it can be seen that as r increases, As the iteration number z and the number of network blocks r increase, we notice that the growth trend of the average path lengthis almost purely influenced by the r, see Fig. 4.

Corollary 3.5. *For $r = 4$, the unweighted network G_z^r has the following propertie*

$$\overline{D_z} = \frac{\ln N_z}{\ln 4} + \frac{3}{4} + O(4^{-z}) \propto \ln N_z.$$

Proof. When the network is considered as 4 blocks, the total number of nodes in the z-th iteration at this time is $N_z = 4^{z+1}$, then $z = \log_4 N_z - 1$, combining Eqs. (3.14) and (3.16), we can obtain

$$\begin{aligned} \overline{D_z} &= \frac{D_z}{N_z(N_z-1)/2} \\ &= \frac{-4 + 7 \cdot 4^z + 4z(-4 + 4^z)}{4^{z+1} - 1} \\ &= \frac{N_z \log_4 N_z - 16\log_4 N_z - N_z + 16}{4(N_z - 1)}. \end{aligned}$$

We organize the above results to obtain the relationship between the average path length of network G_z^r and the total number of nodes N_z.

$$\overline{D_z} = \frac{\ln N_z}{\ln 4} + \frac{3}{4} + O(4^{-z}) \propto \ln N_z.$$

$\square$

When the unweighted hierarchical networks with 4 blocks, as a result, the average path length rises logarithmically with increasing network order in large networks. We compare the analytical simulation results and discover that they are nearly identical to the theoretical numerical results. Fig. 5 shows that the average path length growth trend of G_z^r is nearly consistent with $\ln N_z$.

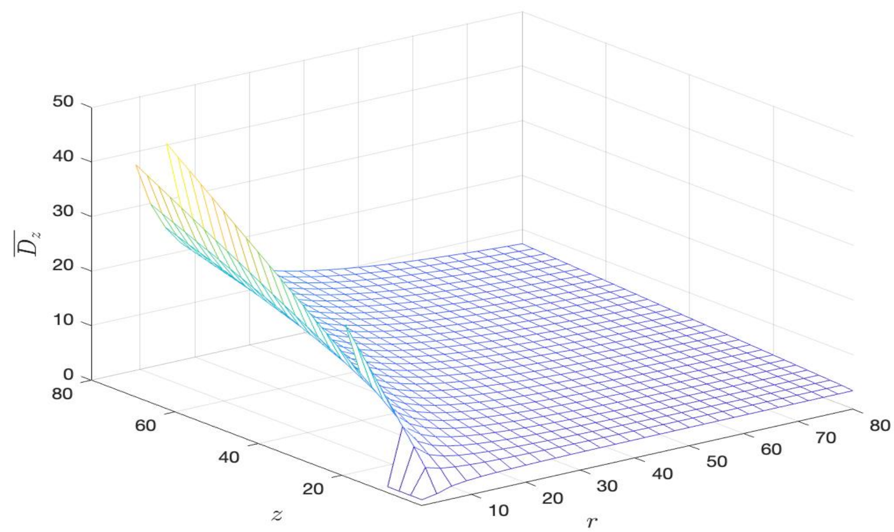

Figure 4: Relationship between the average path length $\overline{D_z}$ and z and r in unweighted network.

Remark 3.1. *The conclusions calculated in this paper about the unweighted hierarchical networks are basically similar to those from references [22, 23], which constructed a class of hierarchical networks connected by planar triangles. Therefore, it can be deduced that when the hierarchical networks are connected by triangles, whether they are planar triangular networks or spatial triangular iterative networks, thei In general, the conclusions are similar.*

4. The average path length of weighted networks

In real-world problems, there will always be some association between nodes and nodes in the network. In this section, the existence of a special meaning between paths is considered, and this association is given a special weight factor $w,\ w \in (0, 1)$, The weighting factor between nodes i and j is then denoted as w_{ij}, $(w_{ij} = w_{ji})$ is discussed in this article.

The unweighted network method described in the preceding section might be used to obtain the average path length of the weighted network. In the weighted hierarchical network, $\widehat{M_z}$ and $\widehat{L_z}$ denote, respectively, the total distance from the node to the root node and the distance to the bottom node in the z-th generation weighted network.

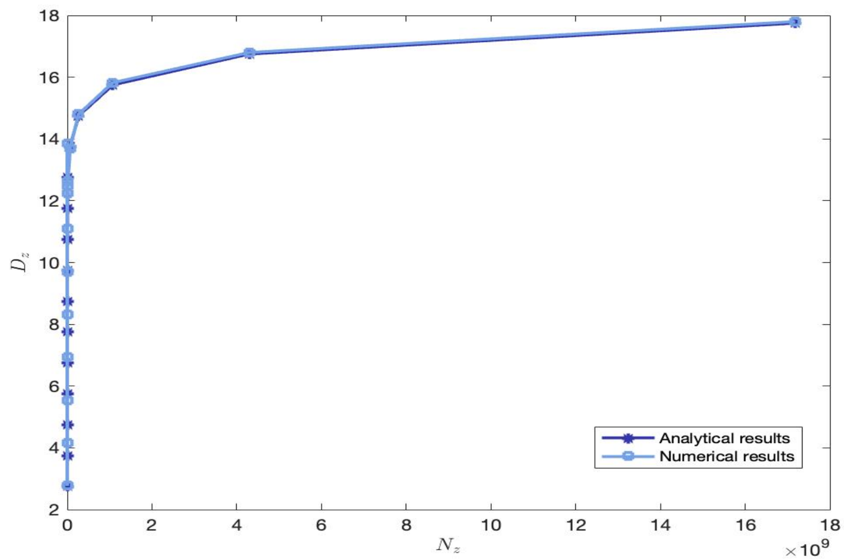

Figure 5: The numerical test for different network orders.

Lemma 4.1. *For a weighted network G_z^r with r blocks, at the z-th generation*

$$\begin{aligned}\widehat{M_z} &= \frac{r^{z-2}\Big(4-4w^z-4r(w^{z+1}-2w^z+1)-r^2(w-1)(4w^{z+1}-4w^z-3w)\Big)}{w(w-1)},\\ \widehat{L_z} &= \frac{r^{z-2}(4-4r-3r^2w+3r^2w^2-4w^z+4r\cdot w^z)}{w(w-1)}.\end{aligned} \tag{4.17}$$

Proof. Combining Eqs. (3.4) and (3.5), we can get

$$\begin{aligned}\widehat{M_{z+1}} &= \widehat{M_z}+(r-1)(\widehat{L_z}+4r^z\cdot w^z),\\ \widehat{L_{z+1}} &= (r-1)\widehat{L_z}+\widehat{M_z}+4w^z\cdot r^z.\end{aligned} \tag{4.18}$$

then

$$\widehat{M_z} = \widehat{L_{z+1}}-(r-1)\widehat{L_z}-4w^z\cdot r^z,$$

$$\begin{aligned}\widehat{M_{z+1}} \quad &= \widehat{L_{z+1}}-(r-1)\widehat{L_z}-4w^z\cdot r^z+(r-1)\widehat{L_z}+4(r-1)w^z\cdot r^z\\ &= \widehat{L_{z+1}}+4(r-2)w^z\cdot r^z.\end{aligned}$$

$\widehat{M_{z+1}}$ can be rewritten as follows

$$\widehat{M_{z+1}} = \widehat{L_{z+2}}-(r-1)\widehat{L_{z+1}}-4w^z\cdot r^z. \tag{4.19}$$

Based on the results obtained above, we can conclude that

$$\widehat{L_{z+2}}-r\widehat{L_{z+1}} = 4(r-1)w^z\cdot r^z. \tag{4.20}$$

By solving for Eq. (4.20), Lemma (4.1) can be proved.

$\square$

Theorem 4.2. *Let G_z^r be a weighted network with r blocks, at the z-th iterations*

$$\widehat{\Delta_{z+1}} = \frac{4(r-1)r^{2z-1}\Big(4+7r^2\cdot w(w-1)-4w^{z+1}-4r(1-w+w^2-2w^{z+1}+w^{z+2})\Big)}{w(w-1)}.$$

Proof. Combining the results of Eq. (3.9) with the initial values of M_z and L_z, we can obtain

$$\begin{aligned}\Delta_{i\in G_z^1}\widehat{\Delta}_{j\in G_z^2} &= -\frac{8r^{2z-2}\Big(4(w^z-1)+2r(w^{z+1}-3w^z+2)+r^2(w-1)(2w^{z+1}-2w^z-5w)\Big)}{w(w-1)},\\ \Delta_{i\in G_z^2}\widehat{\Delta}_{j\in G_z^3} &= \frac{8r^{2z-2}\Big(4+7r^2\cdot w(w-1)-4w^z+4r(w^z-1)\Big)}{w(w-1)}.\end{aligned} \tag{4.21}$$

The network's self-similar structure is maintained by weighted networks.

$$\begin{aligned}\widehat{\Delta_{z+1}} &= (r-1)\Delta_{i\in G_z^1}\widehat{\Delta}_{j\in G_z^2} + C_{r-1}^2\Delta_{i\in G_z^2}\widehat{\Delta}_{j\in G_z^3}\\ &= \frac{4(r-1)r^{2z-1}\Big(4+7r^2\cdot w(w-1)-4w^{z+1}-4r(1-w+w^2-2w^{z+1}+w^{z+2})\Big)}{w(w-1)}\end{aligned}$$

Then combining with the initial values of M_z and L_z, we can obtain $\widehat{\Delta_{z+1}}$. □

Theorem 4.3. *Let G_z^r be a weighted network with r blocks, at the z-th iterations*

$$\begin{aligned}\widehat{\overline{D_z}} = \Big(&8-8w-11r^3(w-1)w^2+14r^{z+3}(w-1)w^2-8r(1-3w+2w^2)+r^2\cdot w(8w^2+11w-19)\\ &+8r^z(w^{z+1}-1)-2r^{z+2}w(4w^{z+1})-8w^z+4w^2+3w-3)\\ &+8r^{z+1}(w^{z+2}-3w^{z+1}+w^2+1)\Big)\Big/\Big(wr^2(4r^z-1)(w-1)(rw-1)\Big).\end{aligned}$$

Proof. The proof procedure here can be referred to that of $\overline{D_z}$. □

Theorem 4.4. *Let G_z^r be a weighted network with r blocks, at the z-th iterations*

$$\lim_{z\to\infty}\frac{\widehat{\overline{D_z}}}{z} = \frac{4+7r^2(w-1)w+r(4+4w-4w^2)}{2r^3\cdot w(w-1)}. \tag{4.22}$$

Proof. The result of $\widehat{\overline{D_z}}$ derived from Theorem 4.3, the proof procedure is similar to that of Theorem 3.4. □

The results presented above show that as the number of network iterations increases, the average path length of the weighted network has always been associated with the weighting factor w and the number of network blocks r, showing that the average weighted path of the network is enclosed and the weighted network has a small-world property.

Then, in the weighted network, we consider a more special case, which leads to the results of Lemmas 4.4 and 4.5.

Lemma 4.5. *Assuming $w=\frac{1}{r}$, in the weighted network G_z^r, the average path length is*

$$\widehat{\overline{D_z}} = \frac{8-8r^z+22r^{z+1}+16z-r(19+16z)}{r(4r^z-1)}.$$

Proof. Adding $w = \frac{1}{r}$ to the formula of Theorem 4.3 completes the proof. $\square$

Lemma 4.6. *Assuming $w = \frac{1}{r}$, in the weighted network G_z^r, one has*

$$\lim_{z\to\infty} \frac{\widehat{D_z}}{z} = \frac{16(r-1)}{r}.$$

Proof. According to the result of Theorem 4.4, the above lemma can be proved by bringing $w = \frac{1}{r}$ to the limit formula of $\widehat{D_z}$ and z. $\square$

Remark 4.1. *A class of planar triangular connected hierarchical networks was investigated in reference [24]. The network is partitioned into m blocks using the parameter calculation method, and the weighting factor r is also introduced in order to streamline the calculation. The outcomes of lemma* 4.5 *and* 4.6 *in this paper are comparable to the theorem* 3.4 *that was reached. Therefore, we conclude that the average path length is solely dependent on the network parameters chosen, regardless of whether the hierarchical network is partitioned into triangles or other graphs, for the unweighted or weighted hierarchical network.*

5. Conclusions

In this paper, we analyzed the average path length of the unweighted and weighted hierarchical networks. We find that the average path length in the unweighted case has a logarithmic connection with the value of iterations z. Then, for the average path length, we find that it grows logarithmically as the network order increases. When a weighting factor (w) is applied, $\widehat{D_z}$ of the weighed network becomes a constant associated with the number of blocks r and the weighting factor w as the number of iterations z approaches infinity. Simultaneously, if w takes a special constant value ($w = \frac{1}{r}$), the result is $\frac{16(r-1)}{r}$ when the number of iterations z approaches infinity, ensuring that, in both unweighted and weighted hierarchical networks, the average path length of the network stays constrained with respect to the expansion of network iterations.

Author contributions

Jia-bao Liu: Writing-review and editing. Ya-qian Zheng: Writing-original draft; Writing-review and editing.

Declartion of competing interest

The authors did not report a potential conflict of interest.

Data availability statements

The data that support the finding of this study are availability within the article.

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