A NEW BLOW UP CRITERION FOR THE 3D MAGNETO-MICROPOLAR FLUID FLOWS WITHOUT MAGNETIC DIFFUSION

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Abstract. This note obtains a new regularity criterion for the three-dimensional magneto-micropolar fluid flows in terms of one velocity component and the gradient field of the magnetic field, i.e. the weak solution (u, ω, b) to the magneto-micropolar fluid flows can be extended beyond time $t = T$, provided if $u_3 \in L^\beta(0, T; L^\alpha(\mathbf{R}^3))$ with $\frac{2}{\beta} + \frac{3}{\alpha} \leq \frac{3}{4} + \frac{1}{2\alpha}, \alpha > \frac{10}{3}$ and $\nabla b \in L^{\frac{4p}{3(p-2)}}(0, T; \dot{M}_{p,q}(\mathbf{R}^3))$ with $1 < q \leq p < \infty$ and $p \geq 3$.

Keywords: Magneto-micropolar fluid flow; Regularity criterion; weak solution; Morrey-Campanato spaces

Mathematics Subject Classification: 35Q35; 76D03

1. Introduction

The aim of this paper is to understand the regularity criterion for the following three dimensional magneto-micropolar fluid flows without magnetic diffusion

$$
\begin{cases}
\partial_t u - (\mu + \chi)\Delta u + \nabla \pi = -u \cdot \nabla u + b \cdot \nabla b - \chi \nabla \times \omega, & (x,t) \in \mathbf{R}^3 \times (0,T), \\
\partial_t \omega - \gamma \Delta \omega - \kappa \nabla \mathrm{div} \omega + 2\chi \omega = -u \cdot \nabla \omega + \chi \nabla \times u, & (x,t) \in \mathbf{R}^3 \times (0,T), \\
\partial_t b + u \cdot \nabla b = b \cdot \nabla u, & (x,t) \in \mathbf{R}^3 \times (0,T), \\
\nabla \cdot u = \nabla \cdot b = 0, & (x,t) \in \mathbf{R}^3 \times (0,T), \\
u|_{t=0} = u_0, \omega|_{t=0} = \omega_0, b|_{t=0} = b_0, & x \in \mathbf{R}^3.
\end{cases}
\tag{1.1}
$$

This system is a special case of the classical three dimensional magneto-micropolar fluid flows

$$
(1.2)\qquad \begin{cases} \partial_t u - (\mu+\chi)\Delta u + \nabla\pi = -u\cdot\nabla u + b\cdot\nabla b - \chi\nabla\times\omega, & (x,t)\in\mathbf{R}^3\times(0,T),\\ \partial_t\omega - \gamma\Delta\omega - \kappa\nabla\mathrm{div}\omega + 2\chi\omega = -u\cdot\nabla\omega + \chi\nabla\times u, & (x,t)\in\mathbf{R}^3\times(0,T),\\ \partial_t b + u\cdot\nabla b - \nu\Delta b = b\cdot\nabla u, & (x,t)\in\mathbf{R}^3\times(0,T),\\ \mathrm{div}u = \mathrm{div}b = 0, & (x,t)\in\mathbf{R}^3\times(0,T),\\ u|_{t=0} = u_0, \omega|_{t=0} = \omega_0, b|_{t=0} = b_0, & x\in\mathbf{R}^3, \end{cases}
$$

where $u = (u_1, u_2, u_3), \omega = (\omega_1, \omega_2, \omega_3), b = (b_1, b_2, b_3)$ and π denote the unknown velocity field, the micro-rotational velocity, the magnetic field and the unknown scalar pressure at the point $(x,t)\in\mathbf{R}^3\times(0,T)$, respectively. While u_0, ω_0, b_0 are the prescribed initial data and $\mathrm{div}u = \mathrm{div}b = 0$ in the sense of distributions. The constants $\mu, \chi, \kappa, \gamma$ are positive numbers associated to properties of the material: μ is the kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities (more details see [11]).

The megneto-micropolar fluid system (1.2) was first proposed by Galdi and Rionero [3]. The existence of global-in-time weak solutions were then established by Rojas-Medar and Boldrini [14], while the local strong solutions and global strong solutions in bounded domain for the small initial data were considered, respectively, by Rojas-Medar [13] and Ortega-Torres and Rojas-Medar [12]. However, whether the local strong solutions can exist globally or the global weak solution is regular and unique is an outstanding open problem. Hence there are many regularity criteria to ensure the smoothness of solutions. Gala [2] proved that if the velocity field satisfies $u\in L^{\frac{2}{1-r}}(0,T;\dot{M}_{p,\frac{3}{r}}(\mathbf{R}^3))$ or the gradient of velocity field satisfies $\nabla u\in L^{\frac{2}{2-r}}(0,T;\dot{M}_{p,\frac{3}{r}}(\mathbf{R}^3))$, then the local smooth solution (u,ω,b) can be extended beyond $t = T$. Zhang and Yao [18] demonstrated that if $\nabla u\in L^p(0,T;\dot{F}^0_{q,\frac{2q}{3}}(\mathbf{R}^3))$ with $\frac{2}{p}+\frac{3}{q} = 2, \frac{3}{2} < q \le \infty$, then the weak solution (u,ω,b) is smooth on $[0,T]$.

When the micro-rotational velocity $\omega = 0$ and $\chi = 0$,the equation (1.2) becomes the standard magneto-hydrodynamic equations. In recent years, the problem of regularity criteria involving one components have been investigated for the MHD equation (see e. g. [4], [5], [6], [8], [9]). In 2016, Yamazaki [15] proved that if

$$u_3\in L^p(0,T;L^q(\mathbf{R}^3))$$

with $\frac{2}{p}+\frac{3}{q}\le\frac{1}{3}+\frac{1}{2q}, \frac{15}{2} < q\le\infty$ and

$$j_3\in L^{p'}(0,T;L^{q'}(\mathbf{R}^3))$$

with $\frac{2}{p'}+\frac{3}{q'}\le 2, \frac{13}{2} < q'\le\infty$, then the weak solution (u,b) is regular, where j_3 is the third component of $\nabla\times b = (j_1,j_2,j_3)$. Later Zhang [17] refined the result of Yamazaki's. He proved that if

$$u_3\in L^p(0,T;L^q(\mathbf{R}^3))$$

with $\frac{2}{p}+\frac{3}{q}\le\frac{4}{9}+\frac{1}{3q}, \frac{15}{2} < q\le\infty$ and

$$j_3\in L^{p'}(0,T;L^{q'}(\mathbf{R}^3))$$

with $\frac{2}{p'}+\frac{3}{q'}\le 2, \frac{3}{2} < q'\le\infty$, then the weak solution (u,b) is regular.

We further assume that $\omega = \chi = 0$, the system (1.1) is usually named MHD equation without magnetic diffusion. In order to present our motivation, we list some information on regularity criteria for 2D-MHD equation without magnetic diffusion. In 2011, Zhou and Fan [16] proved if $\nabla b\in L^1(0,T;BMO(\mathbf{R}^2))$, then the local strong solution (u,b) is regular. Gala, Ragusa and Ye [7] improved Zhou and Fan's result. They showed that if $\nabla b \in L^{\frac{p}{p-1}}(0,T;\dot{M}_{p,q}(\mathbf{R}^2))$ with $p \geq q > 1$, the local strong solution (u,b) to the MHD equation with magnetic diffusion is regular.

Motivated by [2], [15], and [7], we will investigate the regularity criterion on the weak solution to the magneto-micropolar flows involving one velocity component and the gradient of magnetic field satisfying (1.3) and (1.4). Our result is stated as follows.

Theorem 1.1. *Let $(u_0,b_0) \in H^1(\mathbf{R}^3)$ and $\omega_0 \in H^1(\mathbf{R}^3)$, with the initial data $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Assume that (u,ω,b) be the weak solution to the equations* (1.1) *defined on* $[0,T)$ *for some $0 < T < \infty$. If (u,b) satisfies*

$$u_3 \in L^\beta(0,T;L^\alpha(\mathbf{R}^3)), \frac{2}{\beta} + \frac{3}{\alpha} \leq \frac{3}{4} + \frac{1}{2\alpha}, \alpha > \frac{10}{3}, \tag{1.3}$$

and

$$\nabla b \in L^{\frac{4p}{3(p-2)}}(0,T;\dot{M}_{p,q}(\mathbf{R}^3)), 1 < q \leq p < \infty, p \geq 3, \tag{1.4}$$

then the solution (u,ω,b) to (1.1) *can be smoothly extended beyond $t = T$.*

Remark 1.2. *In our knowledge, this is the first regularity criterion result is concerned with weak solution to the 3D incompressible MHD equations without magnetic diffusion in Morrey Campanato space. The most difficulties that arising is to handle the nonlinear term $\int_{\mathbf{R}^3} u \cdot \nabla u \cdot \Delta u dx$. For the two dimension case, due to $\int_{\mathbf{R}^2} u \cdot \nabla u \cdot \Delta u = 0$, the condition $\nabla b \in L^{\frac{2p}{p-2}}(0,T;\dot{M}_{p,q}(\mathbf{R}^2))$ is sufficient (see* [7]*). Compared with the result in* [2]*, due to the magneto micropolor fluid flows discussed in Theorem 1.1 are lack of magnetic diffusion, it increases the difficulties to deal with the nonlinear terms in H^1-energy estimtes, especially for the term $\int_{\mathbf{R}^3} u \cdot \nabla w \cdot \Delta w dx$. Fortunately, L^3-energy estimate for ω helps us to overcome these problems.*

When $\omega = 0, \chi = 0$, the magneto micropolar equations (1.1) becomes the classical MHD equations without magnetic diffusion. Theorem 1.1 converts into the following corollary.

Corollary 1.3. *Let $(u_0,b_0) \in H^1(\mathbf{R}^3)$ with the initial data $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Assume that (u,b) be the weak solution to the incompressible MHD equations defined on $[0,T)$ for some $0 < T < \infty$. If (u,b) satisfies*

$$u_3 \in L^\beta(0,T;L^\alpha(\mathbf{R}^3)), \frac{2}{\beta} + \frac{3}{\alpha} \leq \frac{3}{4} + \frac{1}{2\alpha}, \alpha > \frac{10}{3}$$

and

$$\nabla b \in L^{\frac{4p}{3(p-2)}}(0,T;\dot{M}_{p,q}(\mathbf{R}^3)), 1 < q \leq p < \infty, p \geq 3$$

then the local strong solution (u,b) can be smoothly extended beyond $t = T$.

Remark 1.4. *Comparing with the results either in* [15] *or in* [17]*, though the MHD equations discussed in Corollary 1.3 are lack of magnetic diffusion and the spatial space of magnetic field is enlightened, it is hard to say our results have refined the one* [15] *or that in* [17]*.*

The difficulties and strategy are listed as follows:

• The first big tiger is to estimate the nonlinear term $\int_{\mathbf{R}^3} u \cdot \nabla u \cdot \Delta u dx$. Since $\int_{\mathbf{R}^3} u \cdot \nabla u \cdot \Delta u dx \neq 0$, it is impossible to handle it as in the second dimension. We have to build the horizontal energy estimate $\|\nabla_h u\|_{L^2}, \|\nabla_h b\|_{L^2}$.

• The second thorn is the nonlinear term $\int_{\mathbf{R}^3} u \cdot \nabla w \cdot \Delta w dx$. Integrating by part gives us $\int_{\mathbf{R}^3} \Delta u \cdot \nabla \omega \cdot \omega dx$ and $\int_{\mathbf{R}^3} \nabla u \cdot \nabla^2 \omega \cdot \omega dx$. As usual, if we use the Hölder inequality and Young inequality, we get

$$\begin{aligned}|\int_{\mathbf{R}^3} u \cdot \nabla \omega \cdot \Delta \omega dx| &\le \|u\|_{L^6} \|\nabla \omega\|_{L^3} \|\Delta \omega\|_{L^2} \le \|\nabla u\|_{L^2} \|\nabla \omega\|_{L^2}^{\frac{1}{2}} \|\Delta \omega\|_{L^2}^{\frac{3}{2}} \\ &\le \frac{1}{4} \|\Delta \omega\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|\nabla \omega\|_{L^2}^2,\end{aligned}$$

which doesn't work. Thanks to L^3-norm estimate of ω and some suitable interpolating inequality, which can be referred in section 3, we can overcome these difficulties. More precisely, These methods helps us to handle $\int_{\mathbf{R}^3} \Delta u \cdot \nabla \omega \cdot \omega dx$ and $\int_{\mathbf{R}^3} \nabla u \cdot \nabla^2 \omega \cdot \omega dx$ successfully.

The rest of this paper is organized as follow. The definition of some functional spaces and some useful lemmas are presented in Section 2. The L^3-norm of ω is given in Section 3. The proof of Theorem 1.1 is provided in Section 4.

2. Preliminaries

In this section, we will present some information on the Morrey Campanato space and introduce the definition of weak solution to the magneto microploar equation (1.1).

Definition 2.1. *(see [10]) (1)For $1 < q \le p \le \infty$,the homogeneous Morrey-Campanato space $\dot{M}_{p,q}$ is defined by*

$$\dot{M}_{p,q} = \{f \in L^q_{loc}(\mathbf{R}^3) : \|f\|_{\dot{M}_{p,q}(\mathbf{R}^3)} = \sup_{x \in \mathbf{R}^3} \sup_{R>0} R^{\frac{3}{p}-\frac{3}{q}} \|f\|_{L^q(B(x,R))} < \infty\}$$

where $B(x,R)$ denotes the closed ball in $\mathbf{R}^3$ with center x and radius R.

(2)Let $1 < p' \le q' < \infty$, the following homogeneous space $Z_{p',q'}$ is defined by:

$$Z_{p',q'} = \{f \in L^{p'} | f = \sum_{k \in N} g_k, where (g_k) \subset L^{q'}_{comp}(\mathbf{R}^3),$$

$$\sum_{k \in N} d_k^{3(\frac{1}{p'}-\frac{1}{q'})} \|g_k\|_{L^{q'}} < \infty, where \forall\ k, d_k = diam(supp g_k) < \infty\}$$

The following lemma plays a crucial role in proving the regularity criterion for the magneto-micropolar fluid flows (1.1).

Lemma 2.2. *(see [10]) (1)Let $1 < p' \le q' < \infty$ and p, q such that $\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$,then $\dot{M}_{p,q}$ is the dual space of $Z_{p',q'}$.*

(2)Let $1 < p' \le q' < 2, \frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$ and $r = \frac{3}{p}$, there exists $C > 0$ such that $\forall f \in L^2(\mathbf{R}^3)$ and $\forall g \in \dot{H}^r(\mathbf{R}^3), h \in \dot{M}_{p,q}(\mathbf{R}^3)$ satisfies

$$\int_{\mathbf{R}^3} |f(x)g(x)h(x)| dx \le C \|h\|_{\dot{M}_{p,q}} \|fg\|_{Z_{p',q'}} \le C \|h\|_{\dot{M}_{p,q}} \|f\|_{L^2} \|g\|_{\dot{H}^r}.$$

The Sobolev-Ladyzhenskaya inequality in the whole space $\mathbf{R}^3$ reads as follows.

Lemma 2.3. *(see [1]) There exists a constant $C > 0$ such that*

$$\|\phi\|_{L^p} \le C \|\phi\|_{L^2}^{\frac{6-p}{2p}} \|\partial_1 \phi\|_{L^2}^{\frac{p-2}{2p}} \|\partial_2 \phi\|_{L^2}^{\frac{p-2}{2p}} \|\partial_3 \phi\|_{L^2}^{\frac{p-2}{2p}}, \tag{2.1}$$

for every $\phi \in H^1(\mathbf{R}^3)$ and every $p \in [2,6]$, where C is a constant depending only on p.

The definition of weak solution to the magnetic micropolar equation is provided in the following.

Definition 2.4. *Let* $(u_0, b_0) \in L^2_\sigma(\mathbf{R}^3), \omega \in L^2(\mathbf{R}^3)$ *and* $T > 0$. *A measurable function* (u, b, ω) *is said to be a weak solution to* (1.1) *on* $(0, T)$ *if*

(i)

$$\begin{aligned}(u, b) &\in L^\infty(0, T; L^2_\sigma(\mathbf{R}^3)) \cap L^2(0, T; H^1(\mathbf{R}^3)),\\ \textit{and} \quad \omega &\in L^\infty(0, T; L^2(\mathbf{R}^3)) \cap L^2(0, T; H^1(\mathbf{R}^3));\end{aligned}$$

(ii) For every $\phi, \varphi \in H^1(0, T; H^1_\sigma(\mathbf{R}^3))$ *and* $\psi \in H^1(0, T; H^1(\mathbf{R}^3))$ *with* $\phi(T) = \varphi(T) = \psi(T) = 0$,

$$\begin{aligned}&\int_0^T \langle -u, \partial_t \phi\rangle + \langle u \cdot \nabla u, \phi\rangle + (\mu + \chi)\langle \nabla u, \nabla \phi\rangle dt\\ &\quad - \int_0^T \langle b \cdot \nabla b, \phi\rangle + \chi \langle \nabla \times \omega, \phi\rangle dt = -\langle u_0, \phi_0\rangle,\end{aligned}$$

$$\begin{aligned}&\int_0^T \langle -\omega, \partial_t \varphi\rangle + \gamma\langle \omega, \nabla \varphi\rangle + \kappa \langle \nabla \cdot u, \nabla \cdot \varphi\rangle dt\\ &\quad + \int_0^T \langle u \cdot \nabla \omega, \varphi\rangle + 2\chi\langle \omega, \varphi\rangle - 2\chi \langle \nabla \times u, \varphi\rangle dt = -\langle \omega_0, \phi_0\rangle,\end{aligned}$$

and

$$\int_0^T \langle -b, \partial_t \psi\rangle + \langle u \cdot \nabla b, \psi\rangle - \langle b \cdot \nabla u, \psi\rangle dt = -\langle b_0, \psi_0\rangle.$$

where $L^2_\sigma = \{u | u \in L^2, \nabla \cdot u = 0\}$.

3. Some useful Lemmas

The following L^3-energy estimate is significant to prove our result.

Lemma 3.1. *Let* (u, ω, b) *be the weak solution to the magneto-micropolar equation* (1.1). *Then*

$$\|\omega\|^3_{L^3} + \int_0^t \|\nabla |\omega|^{\frac{3}{2}}\|^2_{L^2} d\tau \le C(\|\omega_0\|_{L^3}, \|\omega_0\|_{L^2}, \|u_0\|_{L^2}, \|b_0\|_{L^2}, T), \tag{3.1}$$

for any $t \in [0, T)$.

Proof. Multiplying $(1.1)_2$ by $|\omega|\omega$ and integrating over $\mathbf{R}^3$, we have

$$\begin{aligned}&\frac{1}{3}\frac{d}{dt}\|\omega\|^3_{L^3} + 2\|\omega\|^3_{L^3} + \frac{4}{9}\|\nabla|\omega|^{\frac{3}{2}}\|^2_{L^2} + \frac{1}{2}\||\omega|^{\frac{1}{2}}\nabla\omega\|^2_{L^2} + \frac{1}{2}\||\omega|^{\frac{1}{2}}\nabla\cdot\omega\|^2_{L^2}\\ &\le |\int_{\mathbf{R}^3} \nabla \times u \cdot |\omega|\omega dx|,\end{aligned} \tag{3.2}$$

where we have used

$$\begin{aligned}\int_{\mathbf{R}^3} (\nabla \cdot \omega)\nabla \cdot (|\omega|\omega) dx &= \int_{\mathbf{R}^3} \nabla \cdot \omega(|\omega|\nabla\cdot\omega + \omega\nabla|\omega|)dx\\ &= \||\omega|^{\frac{1}{2}}\nabla\cdot\omega\|^2_{L^2} + \int_{\mathbf{R}^3} \nabla\cdot\omega \cdot \omega \cdot \nabla|\omega| dx\\ &\ge \frac{1}{2}\||\omega|^{\frac{1}{2}}\nabla\cdot\omega\|^2_{L^2} - \frac{1}{2}\||\omega|^{\frac{1}{2}}\nabla\omega\|^2_{L^2}.\end{aligned}$$

To estimate nonlinear term on the right hand side. Integrating by parts and using the Young inequality and (4.1) to obtain

$$
\begin{aligned}
\int_{\mathbf{R}^3} \nabla\times u\cdot|\omega|\omega dx \leq& \int_{\mathbf{R}^3}|u||\omega||\nabla|\omega||dx+\int_{\mathbf{R}^3}|u||\omega||\nabla\times\omega|dx\\
\leq& C\int_{\mathbf{R}^3}|u||\omega||\nabla\omega|dx\\
\leq& C\|u\|_{L^3}\||\omega|^{\frac12}\|_{L^6}\||\omega|^{\frac12}\nabla\omega\|_{L^2}\\
\leq& \frac14\||\omega|^{\frac12}\nabla\omega\|_{L^2}^2+C\|u\|_{L^3}^2\|\omega\|_{L^3}\\
\leq& \frac14\||\omega|^{\frac12}\nabla\omega\|_{L^2}^2+C\|u\|_{L^2}\|u\|_{L^6}\|\omega\|_{L^3}\\
\leq& \frac14\||\omega|^{\frac12}\nabla\omega\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\omega\|_{L^3}\\
\leq& \frac14\||\omega|^{\frac12}\nabla\omega\|_{L^2}^2+\frac12\|\omega\|_{L^3}^3+C\|\nabla u\|_{L^2}^{\frac32}\\
\leq& \frac14\||\omega|^{\frac12}\nabla\omega\|_{L^2}^2+\frac12\|\omega\|_{L^3}^3+C(\|\nabla u\|_{L^2}^2+1).
\end{aligned}
\tag{3.3}
$$

Substituting the above inequality into (3.2) gives

$$
\begin{aligned}
&\frac{d}{dt}\|\omega\|_{L^3}^3+\|\omega\|_{L^3}^3+\|\nabla|\omega|^{\frac32}\|_{L^2}^2+\frac34\||\omega|^{\frac12}\nabla\omega\|_{L^2}^2+\||\omega|^{\frac12}\nabla\cdot\omega\|_{L^2}^2\\
\leq& C(\|\nabla u\|_{L^2}^2+1).
\end{aligned}
$$

Integrating on $[0,t)$ and using (4.1) yield

$$
\begin{aligned}
\|\omega\|_{L^3}^3+\int_0^t\|\nabla|\omega|^{\frac32}\|_{L^2}^2d\tau \leq& \|\omega_0\|_{L^3}^3+C\int_0^t(\|\nabla u\|_{L^2}^2+1)d\tau\\
\leq& C(\|\omega_0\|_{L^3},\|\omega_0\|_{L^2},\|u_0\|_{L^2},\|b_0\|_{L^2},T).
\end{aligned}
$$

□

4. Proof of Theorem

In this section, we shall give the proof of Theorem1.1. We will assume that $\mu=\chi=\gamma=\kappa=1$ throughout this paper.

Proof. Let $[0,T^*)$ be the maximal time interval for the existence of the local smooth solution. If $T^*\geq T$, the conclusion is obviously valid, but for $T^*<T$,we would show that

$$
\limsup_{t\to T^*}(\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla\omega(\cdot,t)\|_{L^2}^2+\|\nabla b(\cdot,t)\|_{L^2}^2)\leq C
$$

under the assumption of (1.3) and (1.4). Hence, according to the definition of T^*, which leads to a contradiction.

Step 1: L^2-energy estimate A standard energy method says

$$
\begin{aligned}
&\|(u(t),\omega(t),b(t))\|_{L^2}^2+2\int_0^t\|\nabla u\|_{L^2}^2d\tau+2\int_0^t\|\nabla\omega\|_{L^2}^2d\tau\\
&\qquad+2\int_0^t\|\nabla\cdot\omega\|_{L^2}^2d\tau+2\int_0^t\|\omega\|_{L^2}^2d\tau\leq\|(u_0,\omega_0,b_0)\|_{L^2}^2.
\end{aligned}
\tag{4.1}
$$

Step2: H^1-Horizontal energy estimate

We first establish the horizontal gradient of the velocity u and magnetic field b. Taking ∇_h on both sides of equation$(1.1)_1$ and $(1.1)_3$, multiplying by $\nabla_h u$ and $\nabla_h b$ respectively and integrating over $\mathbf{R}^3$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2)+2\|\nabla_h\nabla u\|_{L^2}^2=-\int_{\mathbf{R}^3}\nabla_h(u\cdot\nabla u)\cdot\nabla_h u dx\\
&\quad+\int_{\mathbf{R}^3}\Big(\nabla_h(b\cdot\nabla b)\cdot\nabla_h u+\nabla_h(b\cdot\nabla u)\cdot\nabla_h b-\nabla_h(u\cdot\nabla b)\cdot\nabla_h b\Big)dx\\
&\quad+\int_{\mathbf{R}^3}\nabla_h(\nabla\times\omega)\cdot\nabla_h u dx\\
&\quad:=A_1+A_2+A_3.
\end{aligned}\tag{4.2}$$

We start to estimate A_2. From $(1.1)_4$ and Lemma 2.2 and the fact $\nabla\cdot u=\nabla\cdot b=0$, we know that $(p\geq 3)$

$$\begin{aligned}
A_2=&\int_{\mathbf{R}^3}\sum_{k=1}^{2}\partial_k(b\cdot\nabla b)\partial_k u dx+\int_{\mathbf{R}^3}\sum_{k=1}^{2}\partial_k(b\cdot\nabla u)\partial_k b dx-\int_{\mathbf{R}^3}\sum_{k=1}^{2}\partial_k(u\cdot\nabla b)\partial_k b dx\\
=&\int_{\mathbf{R}^3}\sum_{k=1}^{2}(\partial_k b\cdot\nabla b\partial_k u+\partial_k b\cdot\nabla u\partial_k b)dx-\int_{\mathbf{R}^3}\sum_{k=1}^{2}\partial_k u\cdot\nabla b\partial_k b dx\\
\leq& C\int_{\mathbf{R}^3}|\nabla b||\nabla u||\nabla b|dx\leq C\|\nabla b\|_{\dot{M}_{p,q}}\|\nabla b\|_{L^2}\|\nabla u\|_{\dot{H}^{\frac{3}{p}}}\\
\leq& C\|\nabla b\|_{\dot{M}_{p,q}}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}.
\end{aligned}\tag{4.3}$$

Thanks to the Hölder and Young inequality, one deduces

$$\begin{aligned}
A_3=&\int_{\mathbf{R}^3}\sum_{k=1}^{2}\partial_k(\nabla\times\omega)\cdot\partial_k u dx\\
=&-\int_{\mathbf{R}^3}\sum_{k=1}^{2}\nabla\times\omega\cdot\partial_k\partial_k u dx\\
\leq&\|\nabla\times\omega\|_{L^2}\|\nabla_h\nabla u\|_{L^2}\\
\leq&\frac{1}{2}\|\nabla_h\nabla u\|_{L^2}^2+C\|\nabla\omega\|_{L^2}^2.
\end{aligned}\tag{4.4}$$

Now it is time to deal with the first term A_1. Integrating by parts, we get

$$\begin{aligned}
A_1=&\int_{\mathbf{R}^3}u\cdot\nabla u\cdot\Delta_h u dx\\
=&\int_{\mathbf{R}^3}\sum_{k=1}^{2}\sum_{i,j=1}^{2}u_i\partial_i u_j\partial_k\partial_k u_j dx+\int_{\mathbf{R}^3}\sum_{k=1}^{2}\sum_{i=1}^{2}u_i\partial_i u_3\partial_k\partial_k u_3 dx\\
&+\int_{\mathbf{R}^3}\sum_{k=1}^{2}\sum_{j=1}^{3}u_3\partial_3 u_j\partial_k\partial_k u_j dx\\
:=&A_{11}+A_{12}+A_{13}.
\end{aligned}\tag{4.5}$$

Using integration by parts again and applying the fact that $\mathrm{div} u = 0$, it yields

$$
\begin{aligned}
A_{11} &= -\int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{i,j=1}^{2} \partial_k u_i \partial_i u_j \partial_k u_j + u_i \partial_i \partial_k u_j \partial_k u_j dx \\
&= \int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{i,j=1}^{2} \partial_k u_i \partial_i u_j \partial_k u_j dx + \frac{1}{2} \int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{i,j=1}^{2} \partial_i u_i \partial_k u_j \partial_k u_j dx \\
&= \int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{i,j=1}^{2} \partial_k u_i \partial_i u_j \partial_k u_j dx - \frac{1}{2} \int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{j=1}^{2} \partial_3 u_3 \partial_k u_j \partial_k u_j dx \\
&= -\int_{\mathbf{R}^3} \partial_3 u_3 \Big((\partial_1 u_2)^2 + \partial_2 u_1 \partial_1 u_2 + (\partial_2 u_1)^2 + (\partial_1 u_1)^2 - \partial_1 u_1 \partial_2 u_2 + (\partial_2 u_2)^2 \Big) dx \\
&\quad -\frac{1}{2} \int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{j=1}^{2} \partial_3 u_3 \partial_k u_j \partial_k u_j.
\end{aligned} \tag{4.6}
$$

and

$$
\begin{aligned}
A_{12} &= -\int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{i=1}^{2} (\partial_k u_i \partial_i u_3 \partial_k u_3 + u_i \partial_i \partial_k u_3 \partial_k u_3) dx \\
&= \int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{i=1}^{2} (\partial_k \partial_i u_i u_3 \partial_k u_3 + \partial_k u_i u_3 \partial_k \partial_i u_3) dx + \frac{1}{2} \int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{i=1}^{2} \partial_i u_i \partial_k u_3 \partial_k u_3 dx \\
&= \int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{i=1}^{2} (\partial_k \partial_i u_i u_3 \partial_k u_3 + \partial_k u_i u_3 \partial_k \partial_i u_3) dx - \frac{1}{2} \int_{\mathbf{R}^3} \sum_{k=1}^{2} \partial_3 u_3 \partial_k u_3 \partial_k u_3 dx \\
&= \int_{\mathbf{R}^3} \sum_{k=1}^{2} \sum_{i=1}^{2} (\partial_k \partial_i u_i u_3 \partial_k u_3 + \partial_k u_i u_3 \partial_k \partial_i u_3) dx + \int_{\mathbf{R}^3} \sum_{k=1}^{2} u_3 \partial_3 \partial_k u_3 \partial_k u_3 dx.
\end{aligned} \tag{4.7}
$$

Substituting (4.6) and (4.7) into (4.5) yields

$$
A_1 \leq C \int_{\mathbf{R}^3} |u_3| |\nabla u| |\nabla_h \nabla u| dx. \tag{4.8}
$$

Thanks to the Hölder inequality, (2.2) and the Young inequality, we obtain for $\alpha > 3$

$$
\begin{aligned}
A_1 &\leq C \|u_3\|_{L^\alpha} \|\nabla_h \nabla u\|_{L^2} \|\nabla u\|_{L^{\frac{2\alpha}{\alpha-2}}} \\
&\leq C \|u_3\|_{L^\alpha} \|\nabla_h \nabla u\|_{L^2} \|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}} \|\nabla u\|_{L^6}^{\frac{3}{\alpha}} \\
&\leq C \|u_3\|_{L^\alpha} \|\nabla_h \nabla u\|_{L^2} \|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}} \|\nabla_h \nabla u\|_{L^2}^{\frac{2}{\alpha}} \|\partial_3 \nabla u\|_{L^2}^{\frac{1}{\alpha}} \\
&\leq \frac{1}{2} \|\nabla_h \nabla u\|_{L^2}^2 + C \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}} \|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}}.
\end{aligned} \tag{4.9}
$$

which along with (4.3),(4.4) and (4.2) gives

$$
\begin{aligned}
&\frac{d}{dt} (\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) + 2 \|\nabla_h \nabla u\|_{L^2}^2 \\
\leq & C \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}} \|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}} + C \|\nabla \omega\|_{L^2}^2 \\
&+ C \|\nabla b\|_{\dot{M}_{p,q}} \|\nabla b\|_{L^2} \|\nabla u\|_{L^2}^{1-\frac{3}{p}} \|\Delta u\|_{L^2}^{\frac{3}{p}}.
\end{aligned} \tag{4.10}
$$

Integrating over $[0,t)$ and using (4.1), one can verify

$$\begin{aligned}&\sup_{0\le\tau\le t}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2)+2\int_0^t\|\nabla_h\nabla u\|_{L^2}^2d\tau\\ \le& C(\|\nabla_h u_0\|_{L^2}^2+\|\nabla_h b_0\|_{L^2}^2+1)+C\int_0^t\|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}}\|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}}d\tau\\ &+C\int_0^t\|\nabla b\|_{\dot M_{p,q}}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}d\tau.\end{aligned}\tag{4.11}$$

Step 3: H^1-full energy estimate

Multiplying $(1.1)_1,(1.1)_2$ and $(1.1)_3$ by $-\Delta u,-\Delta\omega$ and $-\Delta b$ respectively and integrating over $\mathbf{R}^3$, then adding them to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+2\|\Delta u\|_{L^2}^2+\|\Delta\omega\|_{L^2}^2+\|\nabla\nabla\cdot\omega\|_{L^2}^2\\ =&\int_{\mathbf{R}^3}u\cdot\nabla u\cdot\Delta udx-\int_{\mathbf{R}^3}b\cdot\nabla b\cdot\Delta udx+\int_{\mathbf{R}^3}u\cdot\nabla\omega\cdot\Delta\omega dx\\ &-2\int_{\mathbf{R}^3}\nabla\times u\cdot\Delta\omega dx+\int_{\mathbf{R}^3}u\cdot\nabla b\cdot\Delta bdx-\int_{\mathbf{R}^3}b\cdot\nabla u\cdot\Delta bdx\\ :=&B_1+B_2+B_3+B_4+B_5+B_6.\end{aligned}\tag{4.12}$$

where we have used the inequality

$$\begin{aligned}\int_{\mathbf{R}^3}(-\nabla\nabla\cdot\omega)(-\Delta\omega)dx=&\sum_{i,j,k=1}^3\int_{\mathbf{R}^3}\partial_i\partial_j\omega_j\partial_k\partial_k\omega_i dx\\ =&\sum_{i,j,k=1}^3\int_{\mathbf{R}^3}\partial_k\partial_j\omega_j\partial_k\partial_i\omega_i dx\\ =&\|\nabla\nabla\cdot\omega\|_{L^2}^2,\end{aligned}$$

and

$$\int_{\mathbf{R}^3}\nabla\times\omega\cdot\Delta udx=\int_{\mathbf{R}^3}\Delta\omega\cdot\nabla\times udx.$$

Now, we estimate B_2,B_5 and B_6. After applying integration by parts, $\mathrm{div}u=\mathrm{div}b=0$, the Hölder inequality , Lemma 2.2, the sobolev interpolation inequality and the Young inequality, we have

$$\begin{aligned}B_2+B_5+B_6=&\int_{\mathbf{R}^3}\nabla b\cdot\nabla b\cdot\nabla udx-\int_{\mathbf{R}^3}\nabla u\cdot\nabla b\cdot\nabla bdx+\int_{\mathbf{R}^3}\nabla b\cdot\nabla u\cdot\nabla bdx\\ \le& C\int_{\mathbf{R}^3}|\nabla b||\nabla u||\nabla b|dx\\ \le& C\|\nabla b\|_{\dot M_{p,q}}\|\nabla b\|_{L^2}\|\nabla u\|_{\dot H^{\frac{3}{p}}}\\ \le& C\|\nabla b\|_{\dot M_{p,q}}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}\\ \le&\frac{1}{4}\|\Delta u\|_{L^2}^2+C\|\nabla b\|_{\dot M_{p,q}}^{\frac{2p}{2p-3}}\|\nabla b\|_{L^2}^{\frac{2p}{2p-3}}\|\nabla u\|_{L^2}^{\frac{2(p-3)}{2p-3}}\\ \le&\frac{1}{4}\|\Delta u\|_{L^2}^2+C\|\nabla b\|_{\dot M_{p,q}}^{\frac{2p}{2p-3}}(\|\nabla b\|_{L^2}^2+\|\nabla u\|_{L^2}^2).\end{aligned}\tag{4.13}$$

We can infer from the Hölder and the Young inequality that

$$B_4 \leq 2\int_{\mathbf{R}^3} \nabla(\nabla\times u)\cdot\nabla\omega dx \leq \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\|\nabla\omega\|_{L^2}^2. \tag{4.14}$$

Applying integration by parts to obtain

$$\begin{aligned} B_3 =& -\int_{\mathbf{R}^3} \nabla u\cdot\nabla\omega\cdot\nabla\omega dx \\ =& \int_{\mathbf{R}^3} \nabla u\cdot\nabla(\nabla\omega)\cdot\omega dx + \int_{\mathbf{R}^3} \nabla(\nabla u)\cdot\nabla\omega\cdot\omega dx \\ :=& B_{31} + B_{32} \end{aligned} \tag{4.15}$$

Thanks to the Hölder inequality, interpolation inequality with $3 \leq \alpha \leq 9$ and Lemma 3.1, we arrives at

$$\begin{aligned} B_{31} \leq& \int_{\mathbf{R}^3} |\nabla u||\Delta\omega||\omega| dx \leq \|\Delta\omega\|_{L^2}\|\omega\|_{L^\alpha}\|\nabla u\|_{L^{\frac{2\alpha}{\alpha-2}}} \\ \leq& \|\Delta\omega\|_{L^2}\|\omega\|_{L^3}^{\frac{9-\alpha}{2\alpha}}\|\omega\|_{L^9}^{\frac{3\alpha-9}{2\alpha}}\|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}}\|\nabla u\|_{L^6}^{\frac{3}{\alpha}} \\ \leq& \|\Delta\omega\|_{L^2}\||\omega|^{\frac{3}{2}}\|_{L^6}^{1-\frac{3}{\alpha}}\|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}}\|\Delta u\|_{L^2}^{\frac{3}{\alpha}} \\ \leq& \|\Delta\omega\|_{L^2}\|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^{1-\frac{3}{\alpha}}\|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}}\|\Delta u\|_{L^2}^{\frac{3}{\alpha}} \\ \leq& \frac{1}{4}\|\Delta\omega\|_{L^2}^2 + C\|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^{2(1-\frac{3}{\alpha})}\|\nabla u\|_{L^2}^{2(1-\frac{3}{\alpha})}\|\Delta u\|_{L^2}^{\frac{6}{\alpha}} \\ \leq& \frac{1}{4}\|\Delta\omega\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^2\|\nabla u\|_{L^2}^2. \end{aligned} \tag{4.16}$$

Similarly, the term B_{32} can be bounded as follows:

$$\begin{aligned} B_{32} \leq& \int_{\mathbf{R}^3} |\Delta u||\nabla\omega||\omega| dx \leq \|\Delta u\|_{L^2}\|\omega\|_{L^\alpha}\|\nabla\omega\|_{L^{\frac{2\alpha}{\alpha-2}}} \\ \leq& \|\Delta u\|_{L^2}\|\omega\|_{L^3}^{\frac{9-\alpha}{2\alpha}}\|\omega\|_{L^9}^{\frac{3\alpha-9}{2\alpha}}\|\nabla\omega\|_{L^2}^{1-\frac{3}{\alpha}}\|\nabla\omega\|_{L^6}^{\frac{3}{\alpha}} \\ \leq& \|\Delta u\|_{L^2}\||\omega|^{\frac{3}{2}}\|_{L^6}^{1-\frac{3}{\alpha}}\|\nabla\omega\|_{L^2}^{1-\frac{3}{\alpha}}\|\Delta\omega\|_{L^2}^{\frac{3}{\alpha}} \\ \leq& \|\Delta u\|_{L^2}\|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^{1-\frac{3}{\alpha}}\|\nabla\omega\|_{L^2}^{1-\frac{3}{\alpha}}\|\Delta\omega\|_{L^2}^{\frac{3}{\alpha}} \\ \leq& \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^{2(1-\frac{3}{\alpha})}\|\nabla\omega\|_{L^2}^{2(1-\frac{3}{\alpha})}\|\Delta\omega\|_{L^2}^{\frac{6}{\alpha}} \\ \leq& \frac{1}{4}\|\Delta u\|_{L^2}^2 + \frac{1}{4}\|\Delta\omega\|_{L^2}^2 + C\|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^2\|\nabla\omega\|_{L^2}^2. \end{aligned} \tag{4.17}$$

For the term B_1, similar to A_1, we find that

(4.18)

$$
\begin{aligned}
B_1 =& \int_{\mathbf{R}^3} u\cdot\nabla u\cdot\Delta_h u dx + \int_{\mathbf{R}^3} u\cdot\nabla u\cdot\partial_3\partial_3 u dx\\
=& -\int_{\mathbf{R}^3}\sum_{k=1}^{2}\sum_{i,j=1}^{3}\partial_k u_i\partial_i u_j\partial_k u_j dx - \int_{\mathbf{R}^3}\sum_{k=1}^{2}\sum_{i,j=1}^{3} u_i\partial_i\partial_k u_j\partial_k u_j dx + \int_{\mathbf{R}^3}\sum_{i,j=1}^{3} u_i\partial_i u_j\partial_3\partial_3 u_j dx\\
=& -\int_{\mathbf{R}^3}\sum_{k=1}^{2}\sum_{i,j=1}^{3}\partial_k u_i\partial_i u_j\partial_k u_j dx - \int_{\mathbf{R}^3}\sum_{i,j=1}^{3}\partial_3 u_i\partial_i u_j\partial_3 u_j dx - \int_{\mathbf{R}^3}\sum_{i,j=1}^{3} u_i\partial_i\partial_3 u_j\partial_3 u_j dx\\
=& -\int_{\mathbf{R}^3}\sum_{k=1}^{2}\sum_{i,j=1}^{3}\partial_k u_i\partial_i u_j\partial_k u_j dx - \int_{\mathbf{R}^3}\sum_{j=1}^{3}\sum_{i=1}^{2}\partial_3 u_i\partial_i u_j\partial_3 u_j dx - \int_{\mathbf{R}^3}\sum_{j=1}^{3}\partial_3 u_3\partial_3 u_j\partial_3 u_j dx\\
\le& C\int_{\mathbf{R}^3}|\nabla_h u||\nabla u|^2 dx \le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^4}^2\\
\le& C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac12}\|\nabla u\|_{L^6}^{\frac32}\\
\le& C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12}\|\partial_2\nabla u\|_{L^2}^{\frac12}\|\partial_3\nabla u\|_{L^2}^{\frac12}\\
\le& C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac12}\|\nabla_h\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}.
\end{aligned}
$$

Combining (4.13), (4.14), (4.16), (4.17) and (4.18), then (4.12) becomes

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\Delta\omega\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2\\
\le& C(\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}+\|\nabla|\omega|^{\frac32}\|_{L^2}^2)(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\\
&+C\|\nabla\omega\|_{L^2}^2+C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac12}\|\nabla_h\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}.
\end{aligned}\tag{4.19}
$$

Integrating over $[0,t]$ yields

$$
\begin{aligned}
&(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\int_0^t\|\Delta u\|_{L^2}^2+\|\Delta\omega\|_{L^2}^2 d\tau\\
\le& C\int_0^t(\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}+\|\nabla|\omega|^{\frac32}\|_{L^2}^2)(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)d\tau\\
&+C(\|\nabla u_0\|_{L^2}^2+\|\nabla\omega_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2+1)+CG(t).
\end{aligned}\tag{4.20}
$$

where

$$
G(t)=\int_0^t\|\nabla_h u\|_{L^2}\|\nabla_h\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{\frac12}\|\Delta u\|_{L^2}^{\frac12}d\tau.
$$

We restore to estimate $G(t)$. From (4.1) and (4.11), we deduce that

$$
\begin{aligned}
G(t)\le& \sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}(\int_0^t\|\nabla_h\nabla u\|_{L^2}^2 d\tau)^{\frac12}(\int_0^t\|\nabla u\|_{L^2}^2)^{\frac14}(\int_0^t\|\Delta u\|_{L^2}^2 d\tau)^{\frac14}\\
\le& C(\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}^2+\int_0^t\|\nabla_h\nabla u\|_{L^2}^2 d\tau)(\int_0^t\|\Delta u\|_{L^2}^2 d\tau)^{\frac14}\\
\le& C\Bigg((\|\nabla_h u_0\|_{L^2}^2+\|\nabla_h b_0\|_{L^2}^2+1)+\int_0^t\|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}}\|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}}d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \|\nabla b\|_{\dot{M}_{p,q}}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}d\tau\Big)(\int_0^t \|\Delta u\|_{L^2}^2 d\tau)^{\frac14}\\
\le& C(\|\nabla_h u_0\|_{L^2}^{\frac83}+\|\nabla_h b_0\|_{L^2}^{\frac83}+1)+C(\int_0^t \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}}\|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}}d\tau)^{\frac43}\\
&+(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}d\tau)^{\frac43}+\frac14\int_0^t \|\Delta u\|_{L^2}^2 d\tau\\
\le& C(\|\nabla_h u_0\|_{L^2}^{\frac83}+\|\nabla_h b_0\|_{L^2}^{\frac83}+1)+\frac14\int_0^t \|\Delta u\|_{L^2}^2 d\tau\\
&+(\int_0^t \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}\|\nabla u\|_{L^2}^2 d\tau)^{\frac{4(\alpha-3)}{3(\alpha-2)}}(\int_0^t \|\Delta u\|_{L^2}^2 d\tau)^{\frac{4}{3(\alpha-2)}}\\
&+C(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}\|\nabla b\|_{L^2}^{\frac{2p}{2p-3}}\|\nabla u\|_{L^2}^{\frac{2(p-3)}{2p-3}}d\tau)^{\frac{2(2p-3)}{3p}}(\int_0^t \|\Delta u\|_{L^2}^2 d\tau)^{\frac{2}{p}}\\
\le& C(\|\nabla_h u_0\|_{L^2}^{\frac83}+\|\nabla_h b_0\|_{L^2}^{\frac83}+1)+\frac34\int_0^t \|\Delta u\|_{L^2}^2 d\tau\\
&+C(\int_0^t \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}\|\nabla u\|_{L^2}^2 d\tau)^{\frac{4(\alpha-3)}{3\alpha-10}}+C(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}\|\nabla b\|_{L^2}^{\frac{2p}{2p-3}}\|\nabla u\|_{L^2}^{\frac{2(p-3)}{2p-3}}d\tau)^{\frac{2(2p-3)}{3(p-2)}}\\
\le& C(\|\nabla_h u_0\|_{L^2}^{\frac83}+\|\nabla_h b_0\|_{L^2}^{\frac83}+1)+\frac34\int_0^t \|\Delta u\|_{L^2}^2 d\tau\\
&+C(\int_0^t \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}\|\nabla u\|_{L^2}^2 d\tau)^{\frac{4(\alpha-3)}{3\alpha-10}}+C(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}(\|\nabla b\|_{L^2}^2+\|\nabla u\|_{L^2}^2)d\tau)^{\frac{2(2p-3)}{3(p-2)}}\\
\le& C(\|\nabla_h u_0\|_{L^2}^{\frac83}+\|\nabla_h b_0\|_{L^2}^{\frac83}+1)+\frac34\int_0^t \|\Delta u\|_{L^2}^2 d\tau\\
&+C(\int_0^t \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}\|\nabla u\|_{L^2}^{\frac{3\alpha-10}{2(\alpha-3)}}\|\nabla u\|_{L^2}^{\frac{\alpha-2}{2(\alpha-3)}}d\tau)^{\frac{4(\alpha-3)}{3\alpha-10}}\\
&+C(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}(\|\nabla b\|_{L^2}^{\frac{3(p-2)}{2p-3}}+\|\nabla u\|_{L^2}^{\frac{3(p-2)}{2p-3}})(\|\nabla b\|_{L^2}^{\frac{p}{2p-3}}+\|\nabla u\|_{L^2}^{\frac{p}{2p-3}})d\tau)^{\frac{2(2p-3)}{3(p-2)}}\\
\le& C(\|\nabla_h u_0\|_{L^2}^{\frac83}+\|\nabla_h b_0\|_{L^2}^{\frac83}+1)+\frac34\int_0^t \|\Delta u\|_{L^2}^2 d\tau\\
&+C(\int_0^t \|u_3\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}\|\nabla u\|_{L^2}^2 d\tau)(\int_0^t \|\nabla u\|_{L^2}^2 d\tau)^{\frac{\alpha-2}{3\alpha-10}}\\
&+C(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}}^{\frac{4p}{3(p-2)}}(\|\nabla b\|_{L^2}^2+\|\nabla u\|_{L^2}^2)d\tau)(\int_0^t \|\nabla b\|_{L^2}^2+\|\nabla u\|_{L^2}^2 d\tau)^{\frac{p}{3(p-2)}}\\
\le& C(\|\nabla_h u_0\|_{L^2}^{\frac83}+\|\nabla_h b_0\|_{L^2}^{\frac83}+1)+\frac34\int_0^t \|\Delta u\|_{L^2}^2 d\tau\\
&+C\int_0^t (\|u_3\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}+\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{4p}{3(p-2)}})(\|\nabla u\|_{L^2}^2+\|\nabla \omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)d\tau.
\end{aligned}
$$

Inserting the above inequality into (4.20), we have

$$
\begin{aligned}
(\|\nabla u\|_{L^2}^2+&\|\nabla \omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\int_0^t (\|\Delta u\|_{L^2}^2+\|\Delta \omega\|_{L^2}^2)d\tau\\
&\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla \omega_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2+1)
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_0^t(\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}+\|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^2+\|u_3\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}+\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{4p}{3(p-2)}})\\
&\times(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)d\tau,
\end{aligned}
$$

Gronwall's inequality and Lemma 3.1 help to achieve

$$
\begin{aligned}
&(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\int_0^t(\|\Delta u\|_{L^2}^2+\|\Delta\omega\|_{L^2}^2)d\tau\\
&\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla\omega_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2+1)\\
&\times\exp\{\int_0^T(\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}+\|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^2+\|u_3\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}+\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{4p}{3(p-2)}})d\tau\}\\
&\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla\omega_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2+1)\exp\{C\int_0^T(1+\|u_3\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}+\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{4p}{3(p-2)}}d\tau\},
\end{aligned}
$$

which completes the proof of Theorem 1.1. □

Acknowledgments

The authors are partially supported by NSF of China under [Grant 11971209] and [Grant 11961032], the Natural Science Foundation of Jiangxi Provience[Grant (20191BAB201003)].

Conflict of interest The authors declare that there is no conflict of interest.

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