REGULARITY CRITERIA FOR THE 3D MAGNETIC BÉNARD EQUATIONS WITHOUT THERMAL DIFFUSION IN TERMS OF PRESSURE

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ABSTRACT. In this paper, the authors obtain some new blow up criteria for the smooth solution to the three dimensional magnetic Bénard equation without thermal diffusion in terms of pressure. We prove that if $\pi \in L^2(0,T;L^{\frac{3}{r}}(\mathbf{R}^3))$ with $0 < r \le 1$, then the strong solution (u,b,θ) to the magnetic Bénard equation can be extended beyond time $t = T$. Meanwhile we also show that provided that $\nabla\pi \in L^{\frac{9-2r}{2r}}(0,T;L^{\frac{3}{r}}(\mathbf{R}^3))$ with $0 < r \le 1$, the solution (u,b,θ) can also be extended smoothly beyond $t = T$. Finally, we also obtain the regularity criteria on Morrey space, Multiplier space, BMO space and Besov space by imposing some growth conditions only on the pressure field.

1. Introduction

This paper focuses on the regularity criteria for the following 3d magnetic Bénard equations

$$
\begin{cases}
\partial_t u + u\cdot\nabla u - \mu\Delta u + \nabla\pi = b\cdot\nabla b + \theta e_3, & x\in\mathbf{R}^3, t>0,\\
\partial_t b + u\cdot\nabla b - \nu\Delta b = b\cdot\nabla u, & x\in\mathbf{R}^3, t>0,\\
\partial_t\theta + u\cdot\nabla\theta - \kappa\Delta\theta = u\cdot e_3, & x\in\mathbf{R}^3, t>0,\\
\mathrm{div} u = 0, \mathrm{div} b = 0, & x\in\mathbf{R}^3, t>0,\\
(u,b,\theta)|_{t=0} = (u_0,b_0,\theta_0), & x\in\mathbf{R}^3,
\end{cases}
\tag{1.1}
$$

where u is the velocity field, b denotes the magnetic vector field and θ is temperature scalar field, π is pressure. μ,ν denote the viscosity and magnetic diffusivity respectively. Finally $e_3 = (0,0,1)$ represents the canoncial basis in $\mathbf{R}^3$.

It is well known that the magnetic Bénard equation (1.1) illuminates the heat convection phenomenon under the presence of the velocity, the magnetic field and the temperature(see [18], [19] and their reference therein). When the temperature scalar field $\theta = 0$, the magnetic Bénard equation (1.1) becomes the classical MHD equation. We also know that the magnetic Bénard equation (1.1) is really Boussinesq equation if the magnetic field $b = 0$. If both the magnetic field b and the temperature scalar field θ vanish, then the equation (1.1) is the incompressible Navier-Stokes equation. The first well-known regularity criterion is indebted to Serrin [21]: If u is the Leray-Hopf weak solution of the 3-D Navier-Stokes equations satisfying

$$u\in L^q(0,T;L^p(\mathbf{R}^3)),\quad \frac{2}{q}+\frac{3}{p}=1,\quad 3<p\le\infty,$$

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then the solution is regular on $(0,T]$. And a related well-known sufficient condition on velocity gradient

$$\nabla u \in L^q(0,T;L^p(\mathbf{R}^3)), \quad \frac{2}{q}+\frac{3}{p}=2, \quad \frac{3}{2}<p\le\infty,$$

was investigated by Beiráo da Veiga [3]. Similar regularity criteria for 3D MHD equation has been established(see [5], [6], [7], [12], [13] et al).

On the other hand, the regularity criterion of Navier-Stokes equations by imposing the growth conditions on pressure field seems more interesting. The similar Serrin's conditions in Lebesgue space on the pressure or on the gradient of pressure have been examined by, for example, Chae and Lee [4], Berselli and Galdi [1] and Zhou [23]-[25]. They proved that if the pressure π satisfies

$$\pi \in L^{\frac{2}{2-r}}(0,T;L^{\frac{3}{r}}(\mathbf{R}^3)), \quad for \quad 0<r\le 1, \tag{1.2}$$

or $\nabla\pi$ satisfies

$$\nabla\pi \in L^{\frac{2}{3-r}}(0,T;L^{\frac{3}{r}}(\mathbf{R}^3)), \quad for \quad 0<r\le 1, \tag{1.3}$$

then the solution to the three dimensional Navier-Stokes equation can be extended beyond $t=T$. Very recently, some important improvements to the Morrey space, multiplier space and Besov space are investigated by Fan [10]. Chen and Zhang [8] established the regularity criterion of weak solution to the Navier-Stokes on Besov space with negative index in term of the pressure. Duan[9] showed that if the pressure π satisfies (1.2) or (1.3), then the local strong solution (u,b) to the MHD equation can be extended smoothly beyond $t=T$.

In 2018, Ma [17] proved if $u \in L^{\frac{2}{1-r}}(0,T;X_r(\mathbf{R}^3))$ or if $\nabla u \in L^{\frac{2}{2-r}}(0,T;X_r(\mathbf{R}^3))$ with $0<r\le 1$, then the local strong solution (u,b,θ) to the 3D magnetic Bénard equation (1.1) can be extended beyond $t=T$. Here $X_r(\mathbf{R}^3)$ is the multiplier space. Motivated by [8], [9], [10]. [23], [24], [25] and [17], the aim of the paper is to give the complete description on the regularity criteria of the local strong solution to the following 3D magnetic Bénard equation without thermal diffusion

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u + \nabla\pi = b\cdot\nabla b + \theta e_3, & x\in\mathbf{R}^3, t>0,\\ \partial_t b + u\cdot\nabla b - \nu\Delta b = b\cdot\nabla u, & x\in\mathbf{R}^3, t>0,\\ \partial_t\theta + u\cdot\nabla\theta = 0, & x\in\mathbf{R}^3, t>0,\\ \mathrm{div} u = 0, \mathrm{div} b = 0, & x\in\mathbf{R}^3, t>0,\\ (u,b,\theta)|_{t=0} = (u_0,b_0,\theta_0), & x\in\mathbf{R}^3. \end{cases} \tag{1.4}$$

by imposing some critical growth conditions on the pressure field. Now, we state our main results.

Theorem 1.1. *Let $T>0$ and the initial data $(u_0,b_0)\in H^2(\mathbf{R}^3), \theta_0\in H^2(\mathbf{R}^3)\cap L^{\frac{6}{5-r}}(\mathbf{R}^3)$ satisfying $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Assume that (u,b,θ) is the strong solution to the magnetic Bénard equation (1.4) defined on $[0,T)$. If the pressure π satisfies*

$$\pi \in L^2(0,T;L^{\frac{3}{r}}(\mathbf{R}^3)), \quad 0<r\le 1. \tag{1.5}$$

Then (u,b,θ) can be extended smoothly beyond $t=T$.

Theorem 1.2. *Let $T > 0$ and the initial data $(u_0, b_0) \in H^2(\mathbf{R}^3), \theta_0 \in H^2(\mathbf{R}^3) \cap L^{\frac{9}{2r+3}}$ satisfy $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Assume that (u, b, θ) is the strong solution to the magnetic Bénard equation* (1.4) *defined on* $[0, T)$. *If $\nabla\pi$ satisfies*

$$\nabla\pi \in L^{\frac{9-2r}{2r}}(0, T; L^{\frac{3}{r}}(\mathbf{R}^3)), \quad 0 < r \le 1. \tag{1.6}$$

Then (u, b, θ) can be extended smoothly beyond $t = T$.

Remark 1.3. *Theorem 1.1 and Theorem 1.2 can be regarded as the generalization of* [4], [23],[24], [25] *and* [9]. *On the other hand, Theorem 1.2 answers the question raised by Liu in* [15].

Theorem 1.4. *Let $T > 0$ and the initial data $(u_0, b_0) \in H^2(\mathbf{R}^3), \theta_0 \in H^2(\mathbf{R}^3) \cap L^{\frac{6}{5}}(\mathbf{R}^3)$ satisfying $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Assume that (u, b, θ) is the strong solution to the magnetic Bénard equation* (1.4) *defined on* $[0, T)$. *If the pressure π satisfy*

$$\pi \in L^{\frac{4p}{4p-6}}(0, T; \dot{M}_{p,q}(\mathbf{R}^3)), \quad \frac{3}{2} < p \le \infty; \tag{1.7}$$

or

$$\pi \in L^2(0, T; \dot{X}^{-r}(\mathbf{R}^3)), \quad 0 < r \le 1. \tag{1.8}$$

Then (u, b, θ) can be extended smoothly beyond $t = T$.

Remark 1.5. *This theorem can be seen an improvement of the corresponding result in* [10].

Theorem 1.6. *Let $T > 0$ and the initial data $(u_0, b_0, \theta_0) \in H^2(\mathbf{R}^3)$ satisfying $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Assume that (u, b, θ) is the strong solution to the magnetic Bénard equation* (1.4) *defined on* $[0, T)$. *If the pressure π satisfies one of the following conditions:*

$$\nabla\pi \in L^2(0, T; BMO(\mathbf{R}^3)), \tag{1.9}$$

$$\pi \in L^2(0, T; \dot{B}^{-1}_{\infty,\infty}(\mathbf{R}^3)). \tag{1.10}$$

Then (u, b, θ) can be extended smoothly beyond $t = T$.

Remark 1.7. *Theorem 1.6 partially generalize the result in* [8].

The rest of this paper is organized as follows. Section 2 present some information on the function spaces and some crucial lemmas. The proof of Theorem 1.1 is introduced in Section 3. We will give the proof of Theorem 1.4 and Theorem 1.6 in Section 4 and Section 5, respectively.

2. Preliminaries

In this paper,we use the following usual function spaces.

Homogeneous Morrey space $\dot{M}_{p,q}(\mathbf{R}^3)$ is given as follows:

$$\|f\|_{\dot{M}_{p,q}} = \sup_{r>0} \sup_{y\in\mathbf{R}^3} r^{\frac{3}{p}-\frac{3}{q}} \left(\int_{|x-y|<r} |f(x)|^q dx \right)^{\frac{1}{q}} < \infty.$$

Homogeneous Bessel potential space is presented

$$\|f\|_{\dot{H}^r} = \|(-\triangle)^{\frac{r}{2}} f\|_{L^2}.$$

The multiplier space $\dot{X}^{-r}(\mathbf{R}^3)$, a homogeneous Banach space of bounded linear multipliers $f : \dot{H}^r(\mathbf{R}^3) \longmapsto L^2(\mathbf{R}^3)$ is denoted by

$$\|f\|_{\dot{X}^{-r}} = \sup_{\|g\|_{\dot{H}^r} \le 1} \left(\int_{\mathbf{R}^3} |fg|^2 dx \right)^{\frac{1}{2}}.$$

Homogeneous space of bounded mean oscillations BMO reads

$$\|f\|_{BMO} = \sup_{r>0, x\in\mathbf{R}^3} \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| f(y) - \frac{1}{|B_r(y)|} \int_{B_r(y)} f(z)dz \right| dy.$$

The definition of Besov space requires Littlewood-Paley decomposition which can be found in [16]. Define homogeneous space $\dot{B}^s_{p,q}(\mathbf{R}^3)$ as

$$\dot{B}^s_{p,q}(\mathbf{R}^3) = \{ f \in S'_h(\mathbf{R}^3) \mid \|f\|_{\dot{B}^s_{p,q}} < \infty \} \quad (s \in \mathbf{R}, 1 \le p, q \le \infty),$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = (2^{js} \|\dot{\triangle} f\|_{L^p})_{\ell^q}.$$

The above space definitions imply the continuous embedding:

$$\begin{aligned} &L^p(\mathbf{R}^3) \hookrightarrow \dot{M}_{p,q}(\mathbf{R}^3) \hookrightarrow \dot{X}^{-r}(\mathbf{R}^3) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbf{R}^3), p = \frac{3}{r} > q > 2, \\ &L^\infty(\mathbf{R}^3) \hookrightarrow BMO(\mathbf{R}^3) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbf{R}^3). \end{aligned}$$

The following lemma plays a crucial role in proving the regularity criterion for the magnetic Bénard equation (1.1).

Lemma 2.1. *Let $2 \le p \le \infty, s > d(\frac{1}{2} - \frac{1}{p})$. Then there exists a constant $C = C(d, p, s)$ such that for any d-dimensional functions $f \in H^s(\mathbf{R}^d)$,*

$$\|f\|_{L^p(\mathbf{R}^d)} \le C \|f\|^{1-\frac{d}{s}(\frac{1}{2}-\frac{1}{p})}_{L^2(\mathbf{R}^d)} \| \wedge^s f\|^{\frac{d}{s}(\frac{1}{2}-\frac{1}{p})}_{L^2(\mathbf{R}^d)}.$$

When $p \neq \infty$, the above inequality also holds for $s = d(\frac{1}{2} - \frac{1}{p})$.

Lemma 2.2. *([16]) Let $1 < p_1 \le p_2 \le \infty, p_3 \ge 2,$ and$\alpha = \frac{3}{p_2} + \frac{3}{p_3} - \frac{3}{2} \in (0, 1]$. Then the following inequality:*

$$\int_{\mathbf{R}^3} fghdx \le c\|f\|_{\dot{M}_{p_2,p_1}} \|g\|_{p_3} \|h\|_{\dot{H}^\alpha}$$

holds true provided that the right-hand side makes sense.

Lemma 2.3. *([11]) Let $1 < r < \infty, p_3 \ge 2,$ Then we have*

$$\|f \cdot g\|_{L^r} \le c(\|f\|_{L^r} \|g\|_{BMO} + \|g\|_{L^r} \|f\|_{BMO}),$$

for $f, g \in L^r \bigcap BMO$ with $c = c(r)$.

3. Proof of Theorem 1.1 and Theorem 1.2

In this section, we shall give the proof of Theorem 1.1 and Theorem 1.2.

Let $w^+ = u + b$ and $w^- = u - b$. We can convert the 3D magnetic Bénard equations (1.4) into the following form:

$$
\begin{cases}
\partial_t w^+ + w^- \cdot \nabla w^+ = \Delta w^+ - \nabla\pi + \theta e_3, \\
\partial_t w^- + w^+ \cdot \nabla w^- = \Delta w^- - \nabla\pi + \theta e_3, \\
\partial_t \theta + \frac{1}{2}(w^+ + w^-)\cdot\nabla\theta = 0, \\
\mathrm{div} w^+ = 0, \mathrm{div} w^- = 0, \\
(w^+, w^-, \theta)|_{t=0} = (w_0^+, w_0^-, \theta_0).
\end{cases}
\tag{3.1}
$$

Here we first present the proof of Theorem 1.1.

Proof. Takeing the inner products of $(3.1)_1$ with $w^+|w^+|^2$, of $(3.1)_2$ with $w^-|w^-|^2$ and of $(3.1)_3$ with $\theta|\theta|^2$, after integrating by parts and summing together to conclude

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int_{\mathbf{R}^3}(|w^+|^4 + |w^-|^4 + |\theta|^4)dx + \frac{1}{2}\int_{\mathbf{R}^3}((\nabla|w^+|^2)^2 + (\nabla|w^-|^2)^2)dx \\
&+\int_{\mathbf{R}^3}(|\nabla w^+|^2|w^+|^2 + |\nabla w^-|^2|w^-|^2)dx \\
=&-\int_{\mathbf{R}^3}\nabla\pi\cdot(w^+|w^+|^2 + w^-|w^-|^2)dx + \int_{\mathbf{R}^3}\theta e_3\cdot(w^+|w^+|^2 + w^-|w^-|^2)dx \\
=&I_1 + I_2.
\end{aligned}
\tag{3.2}
$$

We can infer from taking the inner product on both side of the third equation of (3.1) with $|\theta|^{p-2}\theta$,

$$
\|\theta\|_{L^p} \le \|\theta_0\|_{L^p}, \ for \quad 1 < p \le \infty.
\tag{3.3}
$$

Integrating by parts and using the Hölder inequality, note that $\mathrm{div} w^+ = \mathrm{div} w^- = 0$, one gets

$$
\begin{aligned}
I_1 &= \int_{\mathbf{R}^3}\pi\,\mathrm{div}(w^+|w^+|^2 + w^-|w^-|^2)dx \\
&\le \int_{\mathbf{R}^3}|\pi|^{\frac{1}{2}}|\pi|^{\frac{1}{2}}(|w^+ + w^-|)(|\nabla|w^+|^2| + |\nabla|w^-|^2|)dx \\
&\le C\|\pi^{\frac{1}{2}}\|_{L^{\frac{6}{r}}}\|\pi^{\frac{1}{2}}\|_{L^{\frac{12}{3-r}}}\|w^+ + w^-\|_{L^{\frac{12}{3-r}}}\|\nabla|w^+|^2 + \nabla|w^-|^2\|_{L^2} \\
&\le C\|\pi\|_{L^{\frac{3}{r}}}^{\frac{1}{2}}\|\pi\|_{L^{\frac{6}{3-r}}}^{\frac{1}{2}}\||w^+|^2 + |w^-|^2\|_{L^{\frac{6}{3-r}}}^{\frac{1}{2}}\|\nabla|w^+|^2 + \nabla|w^-|^2\|_{L^2}.
\end{aligned}
\tag{3.4}
$$

Thanks to the first equation in (3.1) and $\mathrm{div} w^+ = 0$, it follows that

$$
-\Delta\pi = \mathrm{div}(w^-\cdot\nabla w^+) + \mathrm{div}\theta e_3 = \sum_{i,j=1}^{3}\partial_i\partial_j(w_i^- w_j^+) + \partial_3\theta,
$$

which along with Hardy-Littlewood-Sobolev's inequality implies that

$$
\begin{aligned}
\|\pi\|_{L^{\frac{6}{3-r}}} &\le C\|w^- \otimes w^+\|_{L^{\frac{6}{3-r}}} + C\|\nabla^{-1}\theta\|_{L^{\frac{6}{3-r}}} \\
&\le C\||w^+|^2 + |w^-|^2\|_{L^{\frac{6}{3-r}}} + C\|\theta\|_{L^{\frac{6}{5-r}}} \\
&\le C\||w^+|^2 + |w^-|^2\|_{L^{\frac{6}{3-r}}} + C\|\theta_0\|_{L^{\frac{6}{5-r}}} \\
&\le C\||w^+|^2 + |w^-|^2\|_{L^{\frac{6}{3-r}}} + C_1 \\
&\le C(\||w^+|^2 + |w^-|^2\|_{L^{\frac{6}{3-r}}} + 1),
\end{aligned} \tag{3.5}
$$

where $\frac{6}{5-r} \in (\frac{6}{5}, \frac{3}{2}]$ for $0 < r \le 1$. Substituting (3.5) into (3.4) yields

$$
\begin{aligned}
I_1 &\le C\|\pi\|^{\frac{1}{2}}_{L^{\frac{3}{r}}}(\||w^+|^2 + |w^-|^2\|^{\frac{1}{2}}_{L^{\frac{6}{3-r}}} + 1)\||w^+|^2 + |w^-|^2\|^{\frac{1}{2}}_{L^{\frac{6}{3-r}}}\|\nabla|w^+|^2 + \nabla|w^-|^2\|_{L^2} \\
&\le C\|\pi\|^{\frac{1}{2}}_{L^{\frac{3}{r}}}\||w^+|^2 + |w^-|^2\|_{L^{\frac{6}{3-r}}}\|\nabla|w^+|^2 + \nabla|w^-|^2\|_{L^2} \\
&\quad + C\|\pi\|^{\frac{1}{2}}_{L^{\frac{3}{r}}}\||w^+|^2 + |w^-|^2\|^{\frac{1}{2}}_{L^{\frac{6}{3-r}}}\|\nabla|w^+|^2 + \nabla|w^-|^2\|_{L^2} \\
&:= I_{1,1} + I_{1,2}.
\end{aligned}
$$

Lemma 2.1 gives

$$
\|u\|_{L^{\frac{6}{3-r}}} \le \|u\|^{\frac{2-r}{2}}_{L^2}\|\nabla u\|^{\frac{r}{2}}_{L^2}, \tag{3.6}
$$

which together with the Young inequality yields

$$
\begin{aligned}
I_{1,1} &\le C\|\pi\|^{\frac{1}{2}}_{L^{\frac{3}{r}}}\||w^+|^2 + |w^-|^2\|^{\frac{2-r}{2}}_{L^2}\|\nabla|w^+|^2 + \nabla|w^-|^2\|^{\frac{2+r}{2}}_{L^2} \\
&\le C\|\pi\|^{\frac{2}{2-r}}_{L^{\frac{3}{r}}}\||w^+|^2 + |w^-|^2\|^2_{L^2} + \frac{1}{8}\|\nabla|w^+|^2 + \nabla|w^-|^2\|^2_{L^2} \\
&\le C\|\pi\|^{\frac{2}{2-r}}_{L^{\frac{3}{r}}}\||w^+|^2 + |w^-|^2\|^2_{L^2} + \frac{1}{4}(\|\nabla|w^+|^2\|^2_{L^2} + \|\nabla|w^-|^2\|^2_{L^2}) \\
&\le C\|\pi\|^{\frac{2}{2-r}}_{L^{\frac{3}{r}}}(\|w^+\|^4_{L^4} + \|w^-\|^4_{L^4}) + \frac{1}{4}(\|\nabla|w^+|^2\|^2_{L^2} + \|\nabla|w^-|^2\|^2_{L^2}).
\end{aligned} \tag{3.7}
$$

Similarly,

$$
\begin{aligned}
I_{1,2} &\le C\|\pi\|^{\frac{1}{2}}_{L^{\frac{3}{r}}}\||w^+|^2 + |w^-|^2\|^{\frac{2-r}{4}}_{L^2}\|\nabla|w^+|^2 + \nabla|w^-|^2\|^{\frac{4+r}{4}}_{L^2} \\
&\le C\|\pi\|^{\frac{4}{4-r}}_{L^{\frac{3}{r}}}\||w^+|^2 + |w^-|^2\|^{\frac{2(2-r)}{4-r}}_{L^2} + \frac{1}{8}\|\nabla|w^+|^2 + \nabla|w^-|^2\|^2_{L^2} \\
&\le C\|\pi\|^2_{L^{\frac{3}{r}}} + C(\|w^+\|^4_{L^4} + \|w^-\|^4_{L^4}) + \frac{1}{4}(\|\nabla|w^+|^2\|^2_{L^2} + \|\nabla|w^-|^2\|^2_{L^2}).
\end{aligned} \tag{3.8}
$$

Hölder's inequality and Young's inequality guarantee

$$
\begin{aligned}
I_2 &\le C\|\theta\|_{L^4}(\||w^+|^3\|_{L^{\frac{4}{3}}} + \||w^-|^3\|_{L^{\frac{4}{3}}}) \\
&\le C\|\theta\|_{L^4}(\|w^+\|^3_{L^4} + \|w^-\|^3_{L^4}) \\
&\le C(\|\theta\|^4_{L^4} + \|w^+\|^4_{L^4} + \|w^-\|^4_{L^4}).
\end{aligned} \tag{3.9}
$$

Substituting (3.7), (3.8) and (3.9) into (3.2),we can get

$$\begin{aligned}&\frac{d}{dt}(\|\theta\|_{L^4}^4+\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4)+2(\|\nabla|w^+|^2\|_{L^2}^2+\|\nabla|w^-|^2\|_{L^2}^2)\\&\le C(\|\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{2-r}}+1)(\|\theta\|_{L^4}^4+\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4)+C\|\pi\|_{L^{\frac{3}{r}}}^2,\end{aligned}\tag{3.10}$$

which together with the Gronwall inequality, we have

$$\begin{aligned}\sup_{0\le t\le T}(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+\|\theta\|_{L^4}^4)&\le C(\|w_0^+\|_{L^4}^4+\|w_0^-\|_{L^4}^4+\|\theta_0\|_{L^4}^4)\\&\times\exp\{C\int_0^T\|\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{2-r}}dt+T\}+\int_0^T\|\pi(\tau)\|_{L^{\frac{3}{r}}}^2d\tau.\end{aligned}$$

This helps us finish the proof of Theorem 1.1. □

Now we restore to prove Theorem 1.2.

Proof. Multiplying the first equation of (3.1) by $w^+|w^+|^{\frac{9-4r}{r}}$ and using integrating by parts again, we can verify

$$\begin{aligned}&\frac{r}{9-2r}\frac{d}{dt}\int_{\mathbf{R}^3}|w^+|^{\frac{9-2r}{r}}dx+\frac{4r(9-2r)}{(9-2r)^2}\int_{\mathbf{R}^3}|\nabla|w^+|^{\frac{9-2r}{2r}}|^2dx\\&+\int_{\mathbf{R}^3}|\nabla|w^+||^2|w^+|^{\frac{9-4r}{r}}dx\\&=-\int_{\mathbf{R}^3}\nabla\pi\cdot w^+|w^+|^{\frac{9-4r}{r}}dx+\int_{\mathbf{R}^3}\theta e_3\cdot w^+|w^+|^{\frac{9-4r}{r}}dx\\&=I_3+I_4.\end{aligned}\tag{3.11}$$

Integration by parts and using Hölder's inequality, we have

$$\begin{aligned}I_3&=-\int_{\mathbf{R}^3}\nabla\pi\cdot w^+|w^+|^{\frac{9-4r}{r}}dx\\&=\int_{\mathbf{R}^3}\pi w^+\cdot\nabla|w^+|^{\frac{9-4r}{r}}dx\\&=\frac{9-4r}{r}\int_{\mathbf{R}^3}\pi\cdot|\nabla|w^+||\cdot|w^+|^{\frac{9-4r}{r}}dx\\&=\frac{2(9-4r)}{9-2r}\int_{\mathbf{R}^3}\pi\cdot\nabla|w^+|^{\frac{9-2r}{2r}}\cdot|w^+|^{\frac{9-4r}{2r}}dx\\&\le\frac{2(9-4r)}{9-2r}(\int_{\mathbf{R}^3}|\pi|^2\cdot|w^+|^{\frac{9-4r}{r}}dx)^{\frac{1}{2}}(\int_{\mathbf{R}^3}|\nabla|w^+|^{\frac{9-2r}{2r}}|^2dx)^{\frac{1}{2}}.\end{aligned}\tag{3.12}$$

Thanks to the Young inequality, we can get

$$\begin{aligned}I_3=&I_3^{\frac{1}{2}}\times I_3^{\frac{1}{2}}\\\le&C(\int_{\mathbf{R}^3}|\nabla\pi|\cdot|w^+|^{\frac{9-3r}{r}}dx)^{\frac{1}{2}}(\int_{\mathbf{R}^3}|\pi|^2\cdot|w^+|^{\frac{9-4r}{r}}dx)^{\frac{1}{4}}(\int_{\mathbf{R}^3}|\nabla|w^+|^{\frac{9-2r}{2r}}|^2dx)^{\frac{1}{4}}\\\le&C(\int_{\mathbf{R}^3}|\nabla\pi|\cdot|w^+|^{\frac{9-3r}{r}}dx)^{\frac{2}{3}}(\int_{\mathbf{R}^3}|\pi|^2\cdot|w^+|^{\frac{9-4r}{r}}dx)^{\frac{1}{3}}\\&+\frac{r(9-4r)}{(9-2r)^2}\int_{\mathbf{R}^3}|\nabla|w^+|^{\frac{9-2r}{2r}}|^2dx.\end{aligned}\tag{3.13}$$

Similar to (3.5), we can get

$$\begin{aligned}\|\pi\|_{L^{\frac{9}{2r}}} &\le C\|w^-\otimes w^+\|_{L^{\frac{9}{2r}}} + C\|\nabla^{-1}\theta\|_{L^{\frac{9}{2r}}}\\ &\le C\||w^+|^2+|w^-|^2\|_{L^{\frac{9}{2r}}} + C\|\theta_0\|_{L^{\frac{9}{2r+3}}}\\ &\le C(\||w^+|^2+|w^-|^2\|_{L^{\frac{9}{2r}}}+1),\end{aligned}$$

where $\frac{9}{2r+3}\in[\frac{9}{5},3)$ for $0<r\le 1$, which together with the Hölder inequality gives rise to

$$\begin{aligned}\int_{\mathbf{R}^3}|\pi|^2\cdot|w^+|^{\frac{9-4r}{r}}dx \le&\||\pi|^2\|_{L^{\frac{9}{4r}}}\||w^+|^{\frac{9-4r}{r}}\|_{L^{\frac{9}{9-4r}}}\\ \le& C\|\pi\|^2_{L^{\frac{9}{2r}}}\|(|w^+|^2+|w^-|^2)^{\frac{9-4r}{2r}}\|_{L^{\frac{9}{9-4r}}}\\ \le& C\|\pi\|^2_{L^{\frac{9}{2r}}}\||w^+|^2+|w^-|^2\|^{\frac{9-4r}{2r}}_{L^{\frac{9}{2r}}}\\ \le& C\||w^+|^2+|w^-|^2\|^{\frac{9}{2r}}_{L^{\frac{9}{2r}}}+C\||w^+|^2+|w^-|^2\|^{\frac{9-4r}{2r}}_{L^{\frac{9}{2r}}}.\end{aligned}\tag{3.14}$$

With the help of the Hölder inequality, we get

$$\begin{aligned}\int_{\mathbf{R}^3}|\nabla\pi|\cdot|w^+|^{\frac{9-3r}{r}}dx \le&\|\nabla\pi\|_{L^{\frac{3}{r}}}\|(|w^+|^2+|w^-|^2)^{\frac{9-3r}{2r}}\|_{L^{\frac{3}{3-r}}}\\ \le& C\|\nabla\pi\|_{L^{\frac{3}{r}}}\||w^+|^2+|w^-|^2\|^{\frac{9-3r}{2r}}_{L^{\frac{9}{2r}}}.\end{aligned}\tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13) yields

$$\begin{aligned}I_3 \le&\frac{r(9-4r)}{(9-2r)^2}\int_{\mathbf{R}^3}|\nabla|w^+|^{\frac{9-2r}{2r}}|^2dx + C\|\nabla\pi\|^{\frac{2}{3}}_{L^{\frac{3}{r}}}\||w^+|^2+|w^-|^2\|^{\frac{9-2r}{2r}}_{L^{\frac{9}{2r}}}\\ &+C\|\nabla\pi\|^{\frac{2}{3}}_{L^{\frac{3}{r}}}\||w^+|^2+|w^-|^2\|^{\frac{27-10r}{6r}}_{L^{\frac{9}{2r}}}.\end{aligned}\tag{3.16}$$

By Hölder's inequality and Young's inequality, we have

$$\begin{aligned}I_4 \le&\|\theta\|_{L^{\frac{9-2r}{r}}}\||w^+|^{\frac{9-3r}{r}}\|_{L^{\frac{9-2r}{9-3r}}}\\ \le& C\|\theta\|_{L^{\frac{9-2r}{r}}}\|w^+\|^{\frac{9-3r}{r}}_{L^{\frac{9-2r}{r}}}\\ \le& C(\|\theta\|^{\frac{9-2r}{r}}_{L^{\frac{9-2r}{r}}}+\|w^+\|^{\frac{9-2r}{r}}_{L^{\frac{9-2r}{r}}})\\ \le& C(1+\|w^+\|^{\frac{9-2r}{r}}_{L^{\frac{9-2r}{r}}}).\end{aligned}\tag{3.17}$$

where $\frac{9-2r}{r}\in[7,\infty)$ for $r\in(0,1]$. Substituting (3.16) and (3.17) into (3.11),we can get

$$\begin{aligned}&\frac{d}{dt}\|w^+\|^{\frac{9-2r}{r}}_{L^{\frac{9-2r}{r}}}+\int_{\mathbf{R}^3}|\nabla|w^+|^{\frac{9-2r}{2r}}|^2dx\\ &\le C(\|\nabla\pi\|^{\frac{2}{3}}_{L^{\frac{3}{r}}}\||w^+|^2+|w^-|^2\|^{\frac{9-2r}{2r}}_{L^{\frac{9}{2r}}}+\|w^+\|^{\frac{9-2r}{r}}_{L^{\frac{9-2r}{r}}}\\ &+\|\nabla\pi\|^{\frac{2}{3}}_{L^{\frac{3}{r}}}\||w^+|^2+|w^-|^2\|^{\frac{27-10r}{6r}}_{L^{\frac{9}{2r}}}+1).\end{aligned}\tag{3.18}$$

Similarly,

$$
\begin{aligned}
&\frac{d}{dt}\|w^-\|_{L^{\frac{9-2r}{r}}}^{\frac{9-2r}{r}} + \int_{\mathbf{R}^3} |\nabla |w^-|^{\frac{9-2r}{2r}}|^2 dx \\
&\le C(\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3}} \||w^+|^2 + |w^-|^2\|_{L^{\frac{9}{2r}}}^{\frac{9-2r}{2r}} + \|w^-\|_{L^{\frac{9-2r}{r}}}^{\frac{9-2r}{r}} \\
&+ \|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3}} \||w^+|^2 + |w^-|^2\|_{L^{\frac{9}{2r}}}^{\frac{27-10r}{6r}} + 1).
\end{aligned}
\tag{3.19}
$$

Let$\mu = |w^+|^{\frac{9-2r}{2r}} + |w^-|^{\frac{9-2r}{2r}}$. Thanks to Lemma 2.1, we have

$$
\begin{aligned}
&\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3}} \||w^+|^2 + |w^-|^2\|_{L^{\frac{9}{2r}}}^{\frac{9-2r}{2r}} \le C\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3}} \|\mu\|_{L^{\frac{18}{9-2r}}}^2 \\
&\le C\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3}} \|\mu\|_{L^2}^{\frac{6-2r}{3}} \|\nabla\mu\|_{L^2}^{\frac{2r}{3}} \\
&\le \frac{1}{16}\|\nabla\mu\|_{L^2}^2 + C\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3-r}} \|\mu\|_{L^2}^2 \\
&\le \frac{1}{16}\int_{\mathbf{R}^3} (|\nabla |w^+|^{\frac{9-2r}{2r}}|^2 + |\nabla |w^-|^{\frac{9-2r}{2r}}|^2) dx + C\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3-r}} \|\mu\|_{L^2}^2.
\end{aligned}
\tag{3.20}
$$

Let $\delta = \frac{2(27-10r)}{3(9-2r)}$. Using Lemma 2.1 again, one has

$$
\begin{aligned}
&\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3}} \||w^+|^2 + |w^-|^2\|_{L^{\frac{9}{2r}}}^{\frac{27-10r}{6r}} \le C\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3}} \|\mu\|_{L^{\frac{18}{9-2r}}}^{\delta} \\
&\le C\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3}} \|\mu\|_{L^2}^{\frac{\delta(3-r)}{3}} \|\nabla\mu\|_{L^2}^{\frac{\delta r}{3}} \\
&\le \frac{1}{16}\|\nabla\mu\|_{L^2}^2 + C\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{4}{6-r\delta}} \|\mu\|_{L^2}^{\frac{2\delta(3-r)}{6-\delta r}} \\
&\le \frac{1}{16}\int_{\mathbf{R}^3} (|\nabla |w^+|^{\frac{9-2r}{2r}}|^2 + |\nabla |w^-|^{\frac{9-2r}{2r}}|^2) dx + C\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{4}{6-3\delta}} + C\|\mu\|_{L^2}^2,
\end{aligned}
\tag{3.21}
$$

which together with (3.18), (3.19) and (3.20) yields

$$
\begin{aligned}
&\frac{d}{dt}(\|w^+\|_{L^{\frac{9-2r}{r}}}^{\frac{9-2r}{r}} + \|w^-\|_{L^{\frac{9-2r}{r}}}^{\frac{9-2r}{r}} + 1) + \int_{\mathbf{R}^3} (|\nabla |w^+|^{\frac{9-2r}{2r}}|^2 + |\nabla |w^-|^{\frac{9-2r}{2r}}|^2) dx \\
&\le C(\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3-r}} + 1)(\|w^+\|_{L^{\frac{9-2r}{r}}}^{\frac{9-2r}{r}} + \|w^-\|_{L^{\frac{9-2r}{r}}}^{\frac{9-2r}{r}} + 1) + C\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{9-2r}{2r}}.
\end{aligned}
\tag{3.22}
$$

Gronwall's inequality guarantees

$$
\begin{aligned}
&\sup_{0\le t\le T} (\|w^+\|_{L^{\frac{9-2r}{r}}}^{\frac{9-2r}{r}} + \|w^-\|_{L^{\frac{9-2r}{r}}}^{\frac{9-2r}{r}} + 1) + \int_0^t \int_{\mathbf{R}^3} (|\nabla |w^+|^{\frac{9-2r}{2r}}|^2 + |\nabla |w^-|^{\frac{9-2r}{2r}}|^2) dx d\tau \\
&\le (\|w_0^+\|_{L^{\frac{9-2r}{r}}}^{\frac{9-2r}{r}} + \|w_0^-\|_{L^{\frac{9-2r}{r}}}^{\frac{9-2r}{r}} + 1) \exp\{C\int_0^T (\|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{2}{3-r}} + 1) dt\} + \int_0^T \|\nabla\pi\|_{L^{\frac{3}{r}}}^{\frac{9-2r}{2r}} dt,
\end{aligned}
$$

which completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.4

In this section, we shall show Theorem 1.4.

Proof. **Case 1: If** (1.7) **is true.** Integrating by parts, using the Hölder inequality and the Young inequality to conclude

$$\begin{aligned} I_1 =& -\int_{\mathbf{R}^3} \nabla\pi \cdot (w^+|w^+|^2 + w^-|w^-|^2)dx \\ =& \int_{\mathbf{R}^3} \pi w^+ \cdot (\nabla|w^+|^2)dx + \int_{\mathbf{R}^3} \pi w^- \cdot (\nabla|w^-|^2)dx \\ \le& \|\pi w^+\|_{L^2}\|\nabla|w^+|^2\|_{L^2} + \|\pi w^-\|_{L^2}\|\nabla|w^+|^2\|_{L^2} \\ \le& C(\|\pi w^+\|_{L^2}^2 + \|\pi w^-\|_{L^2}^2) + \frac{1}{8}(\|\nabla|w^+|^2\|_{L^2}^2 + \|\nabla|w^-|^2\|_{L^2}^2). \end{aligned} \tag{4.1}$$

We now estimate the terms $\|\pi w^+\|_{L^2}^2$ and $\|\pi w^-\|_{L^2}^2$. When $p \ge 3$, using the G-N interpolation inequality

$$\|v\|_{\dot{H}^\alpha} \le C\|v\|_{L^2}^{1-\alpha}\|\nabla v\|_{L^2}^{\alpha}, \alpha \in [0,1].$$

Similar to (3.5), we can get

$$\begin{aligned} \|\pi\|_{L^2} &\le C\||w^+|^2 + |w^-|^2\|_{L^2} + C\|\nabla^{-1}\theta\|_{L^2} \\ &\le C\||w^+|^2 + |w^-|^2\|_{L^2} + C\|\theta\|_{L^{\frac{6}{5}}} \\ &\le C\||w^+|^2 + |w^-|^2\|_{L^2} + C\|\theta_0\|_{L^{\frac{6}{5}}} \\ &\le C(\||w^+|^2 + |w^-|^2\|_{L^2} + 1). \end{aligned} \tag{4.2}$$

From Lemma 2.2 with $p_1 = q, p_2 = p, p_3 = 2, \alpha = \frac{3}{p} \in (0,1]$, and the Young inequality, one deduces

$$\begin{aligned} \|\pi w^+\|_{L^2}^2 \le& C\|\pi\|_{\dot{M}_{p,q}}\|\pi\|_{L^2}\||w^+|^2\|_{\dot{H}^{\frac{3}{p}}} \\ \le& C\|\pi\|_{\dot{M}_{p,q}}(\||w^+|^2 + |w^+|^2\|_{L^2} + 1)\||w^+|^2\|_{\dot{H}^{\frac{3}{p}}} \\ \le& C\|\pi\|_{\dot{M}_{p,q}}(\||w^+|^2 + |w^+|^2\|_{L^2} + 1)\||w^+|^2\|_{L^2}^{1-\frac{3}{p}}\|\nabla|w^+|^2\|_{L^2}^{\frac{3}{p}} \\ \le& C\|\pi\|_{\dot{M}_{p,q}}\||w^+|^2 + |w^+|^2\|_{L^2}^{2-\frac{3}{p}}\|\nabla|w^+|^2\|_{L^2}^{\frac{3}{p}} \\ &+ C\|\pi\|_{\dot{M}_{p,q}}\||w^+|^2\|_{L^2}^{1-\frac{3}{p}}\|\nabla|w^+|^2\|_{L^2}^{\frac{3}{p}} \\ \le& C\|\pi\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}(\|w^+\|_{L^4}^4 + \|w^-\|_{L^4}^4) + C\|\pi\|_{\dot{M}_{p,q}}^2 + C\|w^+\|_{L^4}^4 + \frac{1}{8}\|\nabla|w^+|^2\|_{L^2}^2 \\ \le& C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}} + \|\pi\|_{\dot{M}_{p,q}}^2 + 1)(\|w^+\|_{L^4}^4 + \|w^-\|_{L^4}^4 + 1) + \frac{1}{8}\|\nabla|w^+|^2\|_{L^2}^2. \end{aligned} \tag{4.3}$$

Similarly,

$$\|\pi w^+\|_{L^2}^2 \le C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}} + \|\pi\|_{\dot{M}_{p,q}}^2 + 1)(\|w^+\|_{L^4}^4 + \|w^-\|_{L^4}^4 + 1) + \frac{1}{8}\|\nabla|w^-|^2\|_{L^2}^2. \tag{4.4}$$

Substituting (4.3) and (4.4) into (4.1), we can get

$$\begin{aligned} I_1 \le& C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}} + \|\pi\|_{\dot{M}_{p,q}}^2 + 1)(\|w^+\|_{L^4}^4 + \|w^-\|_{L^4}^4 + 1) \\ &+ \frac{1}{4}\|\nabla|w^+|^2\|_{L^2}^2 + \frac{1}{4}\|\nabla|w^-|^2\|_{L^2}^2. \end{aligned} \tag{4.5}$$

which along with (3.9), (3.3) and (3.2) implies that

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)+\frac{1}{2}\int_{\mathbf{R}^3}((\nabla|w^+|^2)^2+(\nabla|w^-|^2)^2)dx\\&+\int_{\mathbf{R}^3}(|\nabla w^+|^2|w^+|^2+|\nabla w^-|^2|w^-|^2)dx\\&\le C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}+\|\pi\|_{\dot{M}_{p,q}}^2+1)(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1),\end{aligned}$$

which along with Gronwall's inequality guarantee

$$(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)\le(\|u_0\|_{L^4}^4+\|b_0\|_{L^4}^4+1)\exp\{\int_0^T(\|\pi\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}+\|\pi\|_{\dot{M}_{p,q}}^2+1)d\tau\},$$

which helps us to obtain the desired results when when $p>3$.

For the case $\frac{3}{2}<p<3$. Due to

$$\nabla\pi=(-\triangle)^{-1}\Big(div\nabla(w^-\cdot\nabla w^+)+div\nabla\theta e_3\Big),$$

which along with Hardy-Littlewood-Sobolev's inequality implies that

$$\begin{aligned}\|\nabla\pi\|_{L^2}\le&\|w^-\cdot\nabla w^+\|_{L^2}+\|\theta\|_{L^2}\\\le&C(\|w^-\cdot\nabla w^+\|_{L^2}+1).\end{aligned}\tag{4.6}$$

Using Lemma 2.2 with $q_1=q, q_3=6, \alpha=\frac{3}{p}-1\in(0,1)$ and the Young inequality, we can verify

$$\begin{aligned}\|\pi w^+\|_{L^2}^2\le&C\|\pi\|_{\dot{M}_{p,q}}\|\pi\|_{L^6}\||w^+|^2\|_{\dot{H}^{\frac{3}{p}-1}}\\\le&C\|\pi\|_{\dot{M}_{p,q}}\|\nabla|w^+|^2\|_{L^2}^{\frac{3}{p}-1}\||w^+|^2\|_{L^2}^{2-\frac{3}{p}}\|\nabla\pi\|_{L^2}\\\le&C\|\pi\|_{\dot{M}_{p,q}}\|\nabla|w^+|^2\|_{L^2}^{\frac{3}{p}-1}\||w^+|^2\|_{L^2}^{2-\frac{3}{p}}\|w^-\cdot\nabla w^+\|_{L^2}\\&+C\|\pi\|_{\dot{M}_{p,q}}\|\nabla|w^+|^2\|_{L^2}^{\frac{3}{p}-1}\||w^+|^2\|_{L^2}^{2-\frac{3}{p}}\\\le&C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{4p}{4p-6}}+1)\|w^+\|_{L^4}^4+C\|\pi\|_{\dot{M}_{p,q}}^2+\frac{1}{8}\|\nabla|w^+|^2\|_{L^2}^2+\frac{1}{8}\|w^-\cdot\nabla w^+\|_{L^2}^2\\\le&C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{4p}{4p-6}}+\|\pi\|_{\dot{M}_{p,q}}^2+1)(\|w^+\|_{L^4}^4+1)+\frac{1}{8}\|\nabla|w^+|^2\|_{L^2}^2+\frac{1}{8}\|w^-\cdot\nabla w^+\|_{L^2}^2.\end{aligned}\tag{4.7}$$

Similarly,

$$\|\pi w^-\|_{L^2}^2\le C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{4p}{4p-6}}+\|\pi\|_{\dot{M}_{p,q}}^2+1)(\|w^-\|_{L^4}^4+1)+\frac{1}{8}\|\nabla|w^-|^2\|_{L^2}^2+\frac{1}{8}\|w^-\cdot\nabla w^+\|_{L^2}^2.\tag{4.8}$$

Substituting (4.7) and (4.8) into (4.1), we can get

$$\begin{aligned}I_1\le&C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{4p}{4p-6}}+\|\pi\|_{\dot{M}_{p,q}}^2+1)(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)\\&+\frac{1}{4}\|\nabla|w^+|^2\|_{L^2}^2+\frac{1}{4}\|\nabla|w^-|^2\|_{L^2}^2+\frac{1}{4}\|w^-\cdot\nabla w^+\|_{L^2}^2.\end{aligned}\tag{4.9}$$

Combining (4.9), (3.9), (3.3) and (3.2) implies that

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)+\frac{1}{2}\int_{\mathbf{R}^3}((\nabla|w^+|^2)^2+(\nabla|w^-|^2)^2)dx\\&+\int_{\mathbf{R}^3}(|\nabla w^+|^2|w^+|^2+|\nabla w^-|^2|w^-|^2)dx\\&\le C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{4p}{4p-6}}+\|\pi\|_{\dot{M}_{p,q}}^2+1)(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)+\frac{1}{4}\|w^-\cdot\nabla w^+\|_{L^2}^2,\end{aligned}\tag{4.10}$$

which together with $w^+=u+b, w^-=u+b$, and the following triangle inequalities

$$\|w^+\|_{L^s}\le\frac{1}{2}(\|u\|_{L^s}+\|b\|_{L^s}),$$
$$\|w^-\|_{L^s}\le\frac{1}{2}(\|u\|_{L^s}+\|b\|_{L^s}),$$

give

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}(\|u\|_{L^4}^4+\|b\|_{L^4}^4+1)+\frac{1}{4}\int_{\mathbf{R}^3}((\nabla|u|^2)^2+(\nabla|b|^2)^2)dx\\&+\int_{\mathbf{R}^3}(|u|^2|\nabla u|^2+|u|^2|\nabla b|^2+|b|^2|\nabla u|^2+|b|^2|\nabla b|^2)dx\\&\le C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{4p}{4p-6}}+\|\pi\|_{\dot{M}_{p,q}}^2+1)(\|u\|_{L^4}^4+\|b\|_{L^4}^4+1)\\&+\frac{1}{2}(\|u\cdot\nabla u\|_{L^2}^2+\|b\cdot\nabla u\|_{L^2}^2+\|u\cdot\nabla b\|_{L^2}^2+\|b\cdot\nabla b\|_{L^2}^2).\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}(\|u\|_{L^4}^4+\|b\|_{L^4}^4+1)+\frac{1}{4}\int_{\mathbf{R}^3}((\nabla|u|^2)^2+(\nabla|b|^2)^2)dx\\&+\frac{1}{2}(\|u\cdot\nabla u\|_{L^2}^2+\|b\cdot\nabla u\|_{L^2}^2+\|u\cdot\nabla b\|_{L^2}^2+\|b\cdot\nabla b\|_{L^2}^2)\\&\le C(\|\pi\|_{\dot{M}_{p,q}}^{\frac{4p}{4p-6}}+\|\pi\|_{\dot{M}_{p,q}}^2+1)(\|u\|_{L^4}^4+\|b\|_{L^4}^4+1).\end{aligned}\tag{4.11}$$

Then the L^4-norms of (u,b,θ) follows from the standard Gronwall inequality.

Case 2: If (1.8) **is true.**

With the help of the definition of the multiplier space $\dot{X}^{-r}$, the G-N inequality, the Young inequality and (4.2), we see that

$$\begin{aligned}\|\pi w^+\|_{L^2}^2\le&C\|\pi\|_{\dot{X}^{-r}}\|\pi\|_{L^2}\||w^+|^2\|_{\dot{H}^r}\\\le&C\|\pi\|_{\dot{X}^{-r}}\|\nabla|w^+|^2\|_{L^2}^r\||w^+|^2\|_{L^2}^{1-r}(\||w^+|^2+|w^-|^2\|_{L^2}+1)\\\le&C(\|\pi\|_{\dot{X}^{-r}}^{\frac{r}{2-r}}+\|\pi\|_{\dot{X}^{-r}}^2+1)(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)+\frac{1}{8}\|\nabla|w^+|^2\|_{L^2}^2.\end{aligned}\tag{4.12}$$

Similarly,

$$\|\pi w^-\|_{L^2}^2\le C(\|\pi\|_{\dot{X}^{-r}}^{\frac{r}{2-r}}+\|\pi\|_{\dot{X}^{-r}}^2+1)(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)+\frac{1}{8}\|\nabla|w^-|^2\|_{L^2}^2.\tag{4.13}$$

Substituting (4.12) and (4.13) into (4.1), we can get

$$I_1\le C(\|\pi\|_{\dot{X}^{-r}}^{\frac{r}{2-r}}+\|\pi\|_{\dot{X}^{-r}}^2+1)(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)+\frac{1}{4}(\|\nabla|w^+|^2\|_{L^2}^2+\|\nabla|w^-|^2\|_{L^2}^2),$$

which together with (3.11), (3.3), and (3.2) ensures that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int_{\mathbf{R}^3}(|w^+|^4+|w^-|^4+1)dx+\frac{1}{2}\int_{\mathbf{R}^3}((\nabla|w^+|^2)^2+(\nabla|w^-|^2)^2)dx\\
&+\int_{\mathbf{R}^3}(|\nabla w^+|^2|w^+|^2+|\nabla w^-|^2|w^-|^2)dx\\
&\le C(\|\pi\|_{\dot{X}^{-r}}^{\frac{r}{2-r}}+\|\pi\|_{\dot{X}^{-r}}^2+1)(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1).
\end{aligned}$$

Hence, taking the Gronwall inequality into consideration, we obtain

$$\begin{aligned}
&\sup_{0\le t\le T}(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)\\
&\qquad\le C(\|w_0^+\|_{L^4}^4+\|w_0^-\|_{L^4}^4+1)exp\{C\int_0^T(\|\pi\|_{\dot{X}^{-r}}^{\frac{r}{2-r}}+\|\pi\|_{\dot{X}^{-r}}^2+1)dt\},
\end{aligned}$$

which completes the proof of Theorem 1.4. □

5. Proof of Theorem 1.6

This section will introduces the proof of Theorem 1.6.

Proof. **Case 1: If** (1.9) **is true.** The Hölder inequality, Young inequality, Lemma 2.3 and (4.6) guarantee

$$\begin{aligned}
I_1=&-\int_{\mathbf{R}^3}\nabla\pi\cdot(w^+|w^+|^2+w^-|w^-|^2)dx\\
\le&C\|\nabla\pi\|_{L^4}(\|w^+\|_{L^4}\||w^+|^2\|_{L^2}+\|w^-\|_{L^4}\||w^-|^2\|_{L^2})\\
\le&C\||\nabla\pi|^2\|_{L^2}^{\frac{1}{2}}(\|w^+\|_{L^4}^3+\|w^-\|_{L^4}^3)\\
\le&C\|\nabla\pi\|_{BMO}^{\frac{1}{2}}(\|w^-\cdot\nabla w^+\|_{L^2}^{\frac{1}{2}}+1)(\|w^+\|_{L^4}^3+\|w^-\|_{L^4}^3)\\
\le&C(\|\nabla\pi\|_{BMO}^{\frac{2}{3}}+\|\nabla\pi\|_{BMO}^2+1)(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)+\frac{1}{4}\|w^-\cdot\nabla w^+\|_{L^2}^2,
\end{aligned}$$

which combining (3.9), (3.3) and (3.2) implies that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)+\frac{1}{2}\int_{\mathbf{R}^3}((\nabla|w^+|^2)^2+(\nabla|w^-|^2)^2)dx\\
&+\int_{\mathbf{R}^3}(|\nabla w^+|^2|w^+|^2+|\nabla w^-|^2|w^-|^2)dx\\
&\le C(\|\nabla\pi\|_{BMO}^{\frac{2}{3}}+\|\nabla\pi\|_{BMO}^2+1)(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4+1)+\frac{1}{4}\|w^-\cdot\nabla w^+\|_{L^2}^2.
\end{aligned}\tag{5.1}$$

Then we get, by a similar derivation of (4.11), that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|u\|_{L^4}^4+\|b\|_{L^4}^4+1)+\frac{1}{4}\int_{\mathbf{R}^3}((\nabla|u|^2)^2+(\nabla|b|^2)^2)dx\\
&\quad+\frac{1}{2}(\|u\cdot\nabla u\|_{L^2}^2+\|b\cdot\nabla u\|_{L^2}^2+\|u\cdot\nabla b\|_{L^2}^2+\|b\cdot\nabla b\|_{L^2}^2)\\
&\quad\le C(\|\nabla\pi\|_{BMO}^{\frac{2}{3}}+\|\nabla\pi\|_{BMO}^2+1)(\|u\|_{L^4}^4+\|b\|_{L^4}^4+1).
\end{aligned}$$

The proof can be completed by applying the Gronwall inequality.

Case 2: If (1.10) **is true.**

Before going to the proof,we recall the following inequalities established in [20]:

$$\|f\|_{L^4}^2 \le C\|\nabla f\|_{L^2}\|f\|_{\dot{B}^{-1}_{\infty,\infty}}.$$

Integrating by parts and using (4.6), Hölder's inequality, Young's inequality guarantee

$$\begin{aligned}
I_1 &= \int_{\mathbf{R}^3} \pi(w^+\cdot\nabla|w^+|^2 + w^-\cdot\nabla|w^-|^2)dx \\
&\le \|\pi\|_{L^4}(\|w^+\|_{L^4}\|\nabla|w^+|^2\|_{L^2} + \|w^-\|_{L^4}\|\nabla|w^-|^2\|_{L^2}) \\
&\le \|\pi\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\pi\|^{\frac{1}{2}}_{L^2}(\|w^+\|_{L^4}\|\nabla|w^+|^2\|_{L^2} + \|w^-\|_{L^4}\|\nabla|w^-|^2\|_{L^2}) \\
&\le \|\pi\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}}(\|w^-\cdot\nabla w^+\|^{\frac{1}{2}}_{L^2}+1)(\|w^+\|_{L^4}\|\nabla|w^+|^2\|_{L^2} + \|w^-\|_{L^4}\|\nabla|w^-|^2\|_{L^2}) \\
&\le C(\|\pi\|^2_{\dot{B}^{-1}_{\infty,\infty}}+1)(\|w^+\|^4_{L^4}+\|w^-\|^4_{L^4}+1) \\
&\quad + \frac{1}{4}(\|\nabla|w^+|^2\|^2_{L^2}+\|\nabla|w^-|^2\|^2_{L^2}) + \frac{1}{4}\|w^-\cdot\nabla w^+\|^2_{L^2},
\end{aligned} \tag{5.2}$$

which combining (3.9), (3.3) and (3.2) implies that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|w^+\|^4_{L^4}+\|w^-\|^4_{L^4}+1) + \frac{1}{2}\int_{\mathbf{R}^3}((\nabla|w^+|^2)^2+(\nabla|w^-|^2)^2)dx \\
&+ \int_{\mathbf{R}^3}(|\nabla w^+|^2|w^+|^2 + |\nabla w^-|^2|w^-|^2)dx \\
&\le C(\|\pi\|^2_{\dot{B}^{-1}_{\infty,\infty}}+1)(\|w^+\|^4_{L^4}+\|w^-\|^4_{L^4}+1) + \frac{1}{4}\|w^-\cdot\nabla w^+\|^2_{L^2}.
\end{aligned} \tag{5.3}$$

By a similar derivation of (4.11), we see that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|u\|^4_{L^4}+\|b\|^4_{L^4}+1) + \frac{1}{4}\int_{\mathbf{R}^3}((\nabla|u|^2)^2+(\nabla|b|^2)^2)dx \\
&\quad + \frac{1}{2}(\|u\cdot\nabla u\|^2_{L^2}+\|b\cdot\nabla u\|^2_{L^2}+\|u\cdot\nabla b\|^2_{L^2}+\|b\cdot\nabla b\|^2_{L^2}) \\
&\quad \le C(\|\pi\|^2_{\dot{B}^{-1}_{\infty,\infty}}+1)(\|u\|^4_{L^4}+\|b\|^4_{L^4}+1),
\end{aligned}$$

which implies the desired estimate by Gronwall's inequality. Theorem 1.6 is completed. □

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