Exponential stability of implicit numerical solution for nonlinear neutral stochastic differential equations with time-varying delay and Poisson jumps

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Abstract

The aim of this work is to investigate the exponential mean-square stability for neutral stochastic differential equations with time-varying delay and Poisson jumps. We give some conditions that all the drift, diffusion and jumps coefficients can be nonlinear, to obtain the stability of the analytic solution. It is revealed that the implicit backward Euler-Maruyama numerical solution can reproduce the corresponding stability of the analytic solution under these nonlinear conditions. This is different from the explicit Euler-Maruyama numerical solution whose stability depends on the linear growth condition. With some requirements related to the delay function and the property of compensated Poisson process, we deal with time-varying delay and Poisson jumps. One highly nonlinear example is provided to confirm the

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effectiveness of our theory.
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1. Introduction

Since Poisson jumps can characterize the unpredictable abrupt disturbance in the real world, to describe the discontinuous random effects in the objective phenomena, they are usually introduced to stochastic systems. The systems driven by both Brownian motions and Poisson jumps are considered to be more accurate and extensive than the general stochastic systems only driven by Brownian motions. They have important applications in many fields such as economy, finance, medicine biology and so on [1–3].

As a class of specific models of stochastic systems, the neutral stochastic delay differential equations (NSDDEs) ([4–7]) provide basis and reference for the study of NSDDEs with Poisson jumps. For example, Milošević [6] studied the almost sure (a.s.) exponential stability and convergence of the backward Euler-Maruyama (BEM) numerical solution for NSDDEs. Liu et al. [7] revealed that the stability between the NSDDEs and the Euler-Maruyama (EM) numerical solution can be equivalent under certain conditions. When Poisson jumps are also considered in the equations, Tan et al. [8] proved the convergence of the EM method with strong order 1/2 to the analytic solution under the local Lipschitz condition. Mo et al. [9] investigated the exponential mean-square stability of the analytic solution and the split-step θ-numerical solution. These works have focused on the corresponding equations with constant time delay. Recently, in order to generalize the constant time delay to the case of time-varying delay, Milošević [10, 11] considered the convergence of EM method and BEM method, respectively. Later, the author obtained the a.s. exponential stability of EM method for the same equations (without jumps) by applying the semimartingale convergence theorem [12]. Since time-varying delay and Poisson jumps may be the source of instability, when they coexist, it is important to analyze the stability of such equations. However, little is known about the stability for neutral stochastic differential equations (NSDEs) with time-varying delay and Poisson jumps.

It should be mentioned that conditions are vital for deriving stability. With the linear growth condition on drift coefficient, the EM numerical solution was shown to reproduce the exponential stability of analytic solution for neutral stochastic functional (or delay) differential equation [13, 14]. Several works showed that the linear growth condition was necessary to ensure the stability of EM method [12, 13, 16, 17]. However, it was not required for the same type of stability of the BEM method. We saw that, with the one-sided Lipschitz condition, the BEM method was shown to preserve the corresponding stability of analytic solution [10, 18, 19], which indicated its superiority in condition compared to the EM method. In reality, the linear growth condition restricts the application of such equations, for the reasons that most realistic systems are nonlinear. Therefore, seeking nonlinear conditions for stability is in great demand. For example, Zhou et al. [20, 21] gave nonlinear conditions to investigate the stability and convergence of the BEM method. Later, the extended polynomial growth condition was proposed for the a.s. exponential stability of the BEM method for stochastic functional differential equation [22]. However, we want to know, under the nonlinear conditions, whether the BEM method can preserve the stability of analytic solution for the NSDEs with time-varying delay and Poisson jumps.

In this paper, we aim at establish the exponential mean-square stability of the analytic solution and BEM numerical solution for NSDEs with time-varying delay and Poisson jumps. The highlights of this work are twofold. One is that we will give nonlinear conditions to deal with the equation driven by both Brownian motion and Poisson jumps with time-varying delay, the other is that the implicit BEM numerical solution is investigated. Finally, we will show that the BEM method can reproduce the exponential mean-square stability of the analytic solution under these nonlinear conditions.

The structure of this paper has six parts. Section 2 presents some basic notation and assumptions. The exponential mean-square stability of the analytic solution is shown in Section 3. Section 4 introduces the BEM method. Section 5 proves the exponential stability of the BEM numerical solution. A nonlinear example is given in Section 6 for illustrating our theory.

2. Preliminaries

We introduce some basic notations. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. For a scalar Brownian motion $W(t)$ and a Poisson process $N(t)$, which are both defined on this probability space, we assume that they are independent with each other. $\lambda > 0$ denotes the intensity of Poisson process. For a given $\tau > 0$, $\mathcal{C}([-\tau, 0]; R^n)$ represents the family of all continuous R^n-valued functions ϕ on $[-\tau, 0]$, equipped with the norm $\|\phi\| = \sup_{-\tau\leqslant t\leqslant 0} |\phi(t)|$. Also, we use $\mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];R^n)$ to represent the family of all bounded, $\mathcal{F}_0$-measurable, $\mathcal{C}([-\tau,0];R^n)$ valued random variables. Let the Euclidean norm be denoted by $|\cdot|$. For x,y in R^n, their inner product is denoted by $\langle x,y\rangle$ or x^Ty. $a\vee b$ denotes $\max\{a,b\}$, and $a\vee b$ represents $\min\{a,b\}$.

We consider the following nonlinear NSDEs with time-varying delay and Poisson jumps

$$
\begin{aligned}
\mathrm{d}[x(t)-\upsilon(x(t-\delta(t)))] &= f(x(t),x(t-\delta(t)))\mathrm{d}t+g(x(t),x(t-\delta(t)))\mathrm{d}W(t) \\
&\quad +h(x(t),x(t-\delta(t)))\mathrm{d}N(t), \quad t\geqslant 0, \qquad (1)\\
x(t) &= \xi(t), \qquad -\tau\leqslant t\leqslant 0, \qquad (2)
\end{aligned}
$$

where the initial data $\xi(t)\in\mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];R^n)$. Functions $f,g,h:R^n\times R^n\to R^n, \upsilon:R^n\to R^n$ are all Borel measurable. $\upsilon(x(t-\delta(t)))$ means the neutral term. Delay function $\delta:[0,+\infty]\to[0,\tau]$ is also Borel measurable. In order to ensure Eq. (1) exists the trivial solution $x(t)=0$, we assume that $\upsilon(0)=0, f(0,0)=0$ and $g(0,0)=0$. Also, we introduce the compensated Poisson process

$$\widetilde{N}(t)=N(t)-\lambda t$$

to deal with jumps, which possesses the property of martingale.

To guarantee the existence and uniqueness of the solution processes, we should impose the local Lipschitz condition on functions f,g and h of Eq. (1). Moreover, we need some assumptions.

$\mathcal{A}_1$: There exists a constant $\rho\in(0,1)$ such that, for all $x,y\in R^n$,

$$|\upsilon(x)-\upsilon(y)| \leqslant \rho\,|x-y|\,. \qquad (3)$$

$\mathcal{A}_2$: There exist constants $a_1,a_2>0$ such that for $f\in C(R^n\times R^n;R^n)$ and all $q_1, q_2, y \in R^n$

$$\begin{aligned}\langle q_1 - q_2, f(q_1, y) - f(q_2, y)\rangle &\leqslant a_1|q_1 - q_2|^2,\\ \langle q_1 - q_2, f(y, q_1) - f(y, q_2)\rangle &\leqslant a_2|q_1 - q_2|^2.\end{aligned}$$

$\mathcal{A}_3$: The delay function δ is continuously differentiable with $\delta'(t) \leqslant \bar{\delta} < 1$.

$\mathcal{A}_4$: There exists a constant $\eta \in (0, 1)$ such that

$$|\delta(t) - \delta(s)| \leqslant \eta\,|t - s|\,, \quad t, s \geqslant 0.$$

$\mathcal{A}_5$: For all $x, y \in R^n$, there are positive constants $\alpha_i, \hat{\alpha}_i, k_i, \hat{k}_i, \alpha$ and constants $\sigma_i, \hat{\sigma}_i, i \in \{1, 2\}$ such that,

$$\begin{aligned}2\left(x - \upsilon(y)\right)^T f(x, y) &\leqslant -\alpha_1|x|^2 + \alpha_2|y|^2 - \hat{\alpha}_1|x|^{\alpha+2} + \hat{\alpha}_2|y|^{\alpha+2}, &(4)\\ 2(x - \upsilon(y))^T h(x, y) &\leqslant \sigma_1|x|^2 + \sigma_2|y|^2 + \hat{\sigma}_1|x|^{\alpha+2} + \hat{\sigma}_2|y|^{\alpha+2}, &(5)\\ |g(x, y)|^2 \vee |h(x, y)|^2 &\leqslant k_1|x|^2 + k_2|y|^2 + \hat{k}_1|x|^{\alpha+2} + \hat{k}_2|y|^{\alpha+2}. &(6)\end{aligned}$$

Definition 2.1.[23] The solution of Eq.(1) is said to be exponentially mean-square stable, if for any initial data $\xi(t) \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$, there exists a constant $\gamma > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t)|^2) \leqslant -\gamma.$$

3. Exponential stability of the analytic solution

In this section, we will give some conditions and show that the analytic solution of Eq. (1) is exponentially mean-square stable.

Theorem 3.1. *Let assumptions $\mathcal{A}_1, \mathcal{A}_3$ and $\mathcal{A}_5$ hold. If the parameters satisfy the following conditions*

$$\text{(I)} \begin{cases} \alpha_1 - k_1 - \lambda(k_1 + \sigma_1) - [k_2 + \alpha_2 + \lambda(k_2 + \sigma_2)]\frac{1}{1-\bar{\delta}} > 0 \\ \hat{\alpha}_1 - \hat{k}_1 - \lambda(\hat{k}_1 + \hat{\sigma}_1) - [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2 + \hat{\sigma}_2)]\frac{1}{1-\bar{\delta}} > 0, \end{cases}$$

$$
\text{(II)} \begin{cases} k_2 + \lambda(\sigma_2 + k_2) \geqslant 0 \\ \hat{k}_2 + \lambda(\hat{k}_2 + \hat{\sigma}_2) \geqslant 0, \end{cases}
$$

then the analytic solution of Eq. (1) has the property that

$$
\limsup_{t \to \infty} \frac{1}{t} \log(E|x(t)|^2) \leqslant -\gamma, \tag{7}
$$

where $\gamma \in (0, \gamma_1 \wedge \gamma_2 \wedge \gamma_3)$.

Proof. Let $z(t) = x(t) - \upsilon(x(t - \delta(t)))$ for simplicity. For certain $\gamma > 0$, applying the Itô formula to $e^{\gamma t} |z(t)|^2$ yields

$$
\begin{aligned}
& \mathrm{d}(e^{\gamma t} |z(t)|^2) \\
= \;& e^{\gamma t} \left[\gamma |z(t)|^2 + 2 \langle z(t), f(t, x(t), x(t - \delta(t))) \rangle \right] \mathrm{d}t \\
& + e^{\gamma t} |g(t, x(t), x(t - \delta(t)))|^2 \mathrm{d}t + 2 e^{\gamma t} \langle z(t), g(t, x(t), x(t - \delta(t))) \rangle \, \mathrm{d}W(t) \\
& + e^{\gamma t} \left(|z(t) + h(t, x(t), x(t - \delta(t)))|^2 - |z(t)|^2 \right) \mathrm{d}N(t).
\end{aligned} \tag{8}
$$

Based on $\widetilde{N}(t) = N(t) - \lambda t$, (8) implies

$$
\begin{aligned}
& e^{\gamma t} |z(t)|^2 \\
= \;& |z(0)|^2 + \int_0^t \gamma e^{\gamma s} |z(s)|^2 \mathrm{d}s \\
& + \int_0^t e^{\gamma s} \left[2 \langle z(s), f(s, x(s), x(s - \delta(s))) \rangle + |g(s, x(s), x(s - \delta(s)))|^2 \right] \mathrm{d}s \\
& + 2 \int_0^t e^{\gamma s} \langle z(s), g(s, x(s), x(s - \delta(s))) \rangle \, \mathrm{d}W(s) \\
& + \lambda \int_0^t e^{\gamma s} \left(|z(s) + h(s, x(s), x(s - \delta(s)))|^2 - |z(s)|^2 \right) \mathrm{d}s \\
& + \int_0^t e^{\gamma s} \left(|z(s) + h(s, x(s), x(s - \delta(s)))|^2 - |z(s)|^2 \right) \mathrm{d}\widetilde{N}(s).
\end{aligned}
$$

With conditions (4)-(6), it follows that

$$
e^{\gamma t} E|z(t)|^2
$$

$$\begin{aligned}
&\leqslant E|z(0)|^2+\gamma\int_0^t e^{\gamma s}E|z(s)|^2\mathrm{d}s+\int_0^t e^{\gamma s}E[(k_1-\alpha_1)|x(s)|^2+(k_2+\alpha_2)\\
&\quad\times|x(s-\delta(s))|^2+(\hat{k}_1-\hat{\alpha}_1)|x(s)|^{\alpha+2}+(\hat{k}_2+\hat{\alpha}_2)|x(s-\delta(s))|^{\alpha+2}]\mathrm{d}s\\
&\quad+\lambda\int_0^t e^{\gamma s}E[(k_1+\sigma_1)|x(s)|^2+(k_2+\sigma_2)|x(s-\delta(s))|^2\\
&\quad+(\hat{k}_1+\hat{\sigma}_1)|x(s)|^{\alpha+2}+(\hat{k}_2+\hat{\sigma}_2)|x(s-\delta(s))|^{\alpha+2}]\mathrm{d}s. \qquad (9)
\end{aligned}$$

Applying the Hölder inequality,

$$(a+b)^p\le(1+\varepsilon)^{p-1}(a^p+\varepsilon^{1-p}b^p),\quad a,b,\varepsilon>0,p>1,$$

for $p=2$, with assumption $\mathcal{A}_1$, we get

$$|z(s)|^2 = |x(s)-\upsilon(x(s-\delta(s)))|^2\leqslant(1+\varepsilon)|x(s)|^2+(1+\frac{1}{\varepsilon})\rho^2|x(s-\delta(s))|^2.$$

The estimate (9) gives

$$\begin{aligned}
&e^{\gamma t}E|z(t)|^2\\
&\leqslant E|z(0)|^2+\int_0^t\gamma e^{\gamma s}\left[(1+\varepsilon)E|x(s)|^2+(1+\frac{1}{\varepsilon})\rho^2E|x(s-\delta(s))|^2\right]\mathrm{d}s\\
&\quad+\int_0^t e^{\gamma s}\left[k_1-\alpha_1+\lambda(k_1+\sigma_1)\right]E|x(s)|^2\mathrm{d}s\\
&\quad+\int_0^t e^{\gamma s}\left[k_2+\alpha_2+\lambda(k_2+\sigma_2)\right]E|x(s-\delta(s))|^2\mathrm{d}s\\
&\quad+\int_0^t e^{\gamma s}\left[\hat{k}_1-\hat{\alpha}_1+\lambda(\hat{k}_1+\hat{\sigma}_1)\right]E|x(s)|^{\alpha+2}\mathrm{d}s\\
&\quad+\int_0^t e^{\gamma s}\left[\hat{k}_2+\hat{\alpha}_2+\lambda(\hat{k}_2+\hat{\sigma}_2)\right]E|x(s-\delta(s))|^{\alpha+2}\mathrm{d}s. \qquad (10)
\end{aligned}$$

Since $\int_0^t e^{\gamma s}E|x(s-\delta(s))|^2\mathrm{d}s \leqslant \frac{e^{\gamma\tau}}{1-\bar{\delta}}\left[\int_{-\tau}^0 e^{\gamma s}E|x(s)|^2\mathrm{d}s+\int_0^t e^{\gamma s}E|x(s)|^2\mathrm{d}s\right]$, and similarly,

$$\int_0^t e^{\gamma s}E|x(s-\delta(s))|^{\alpha+2}\mathrm{d}s\leqslant\frac{e^{\gamma\tau}}{1-\bar{\delta}}\left[\int_{-\tau}^0 e^{\gamma s}E|x(s)|^{\alpha+2}\mathrm{d}s+\int_0^t e^{\gamma s}E|x(s)|^{\alpha+2}\mathrm{d}s\right],$$

we obtain

$$
\begin{aligned}
& e^{\gamma t}E|z(t)|^2 \\
\leqslant\ & E|z(0)|^2 + [\gamma(1+\varepsilon) + (k_1-\alpha_1) + \lambda(k_1+\sigma_1)] \int_0^t e^{\gamma s}E|x(s)|^2 \mathrm{d}s \\
& + \left[\gamma(1+\frac{1}{\varepsilon})\rho^2 + (k_2+\alpha_2) + \lambda(k_2+\sigma_2)\right] \frac{e^{\gamma\tau}}{1-\bar{\delta}} [\int_{-\tau}^0 e^{\gamma s}E|x(s)|^2 \mathrm{d}s \\
& + \int_0^t e^{\gamma s}E|x(s)|^2] \mathrm{d}s + \int_0^t e^{\gamma s}\left[\hat{k}_1 - \hat{\alpha}_1 + \lambda(\hat{k}_1+\hat{\sigma}_1)\right] E|x(s)|^{\alpha+2}\mathrm{d}s \\
& + [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2+\hat{\sigma}_2)] \frac{e^{\gamma\tau}}{1-\bar{\delta}} \left[\int_{-\tau}^0 e^{\gamma s}E|x(s)|^{\alpha+2}\mathrm{d}s + \int_0^t e^{\gamma s}E|x(s)|^{\alpha+2}\mathrm{d}s\right],
\end{aligned}
$$

which can be reformed as

$$
\begin{aligned}
& e^{\gamma t}E|z(t)|^2 \\
\leqslant\ & E|z(0)|^2 + \left[\gamma(1+\frac{1}{\varepsilon})\rho^2 + (k_2+\alpha_2) + \lambda(k_2+\sigma_2)\right] \frac{e^{\gamma\tau}}{1-\bar{\delta}} \\
& \times \int_{-\tau}^0 e^{\gamma s}E|x(s)|^2 \mathrm{d}s + [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2+\hat{\sigma}_2)] \frac{e^{\gamma\tau}}{1-\bar{\delta}} \int_{-\tau}^0 e^{\gamma s}E|x(s)|^{\alpha+2}\mathrm{d}s \\
& + \left\{\gamma(1+\varepsilon) + (k_1-\alpha_1) + \lambda(k_1+\sigma_1) + \left[\gamma(1+\frac{1}{\varepsilon})\rho^2 + (k_2+\alpha_2)\right.\right. \\
& \left.\left. + \lambda(k_2+\sigma_2)\right] \frac{e^{\gamma\tau}}{1-\bar{\delta}}\right\} \int_0^t e^{\gamma s}E|x(s)|^2 \mathrm{d}s + \left\{\hat{k}_1 - \hat{\alpha}_1 + \lambda(\hat{k}_1+\hat{\sigma}_1)\right. \\
& \left. + [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2+\hat{\sigma}_2)] \frac{e^{\gamma\tau}}{1-\bar{\delta}}\right\} \int_0^t e^{\gamma s}E|x(s)|^{\alpha+2}\mathrm{d}s.
\end{aligned}
\tag{11}
$$

Denote

$$
\begin{aligned}
H(\gamma) &= \gamma(1+\varepsilon) + k_1 - \alpha_1 + \lambda(k_1+\sigma_1) + \left[\gamma(1+\frac{1}{\varepsilon})\rho^2 + k_2 + \alpha_2\right. \\
& \quad \left. + \lambda(k_2+\sigma_2)\right] \frac{e^{\gamma\tau}}{1-\bar{\delta}},
\end{aligned}
\tag{12}
$$

and

$$
Q(\gamma) = \hat{k}_1 - \hat{\alpha}_1 + \lambda(\hat{k}_1+\hat{\sigma}_1) + [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2+\hat{\sigma}_2)] \frac{e^{\gamma\tau}}{1-\bar{\delta}}, \tag{13}
$$

respectively. Let's analyze the properties of these two functions. We see that

$$H'(\gamma) = 1 + \varepsilon + (1 + \frac{1}{\varepsilon})\rho^2 \frac{e^{\gamma\tau}}{1-\bar{\delta}} + [\gamma(1 + \frac{1}{\varepsilon})\rho^2 + k_2 + \alpha_2 + \lambda(k_2 + \sigma_2)]\frac{\tau e^{\gamma\tau}}{1-\bar{\delta}}$$

and

$$H(0) = (k_1 - \alpha_1) + \lambda(k_1 + \sigma_1) + [k_2 + \alpha_2 + \lambda(k_2 + \sigma_2)]\frac{1}{1-\bar{\delta}}.$$

With the first condition of (II) and $\mathcal{A}_3$, we find that $H'(\gamma) > 0$. On the other hand, we have $H(0) < 0$ under the first condition of (I). Thus for function H, there exists a unique root γ_1 satisfying $H(\gamma_1) = 0$. In the same way, we compute

$$Q'(\gamma) = [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2 + \hat{\sigma}_2)]\frac{\tau e^{\gamma\tau}}{1-\bar{\delta}}$$

and

$$Q(0) = \hat{k}_1 - \hat{\alpha}_1 + \lambda(\hat{k}_1 + \hat{\sigma}_1) + [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2 + \hat{\sigma}_2)]\frac{1}{1-\bar{\delta}}.$$

Under the second conditions of (I) and (II), we get that $Q'(\gamma) > 0$ and $Q(0) < 0$, which means that for function Q, there exists a unique root γ_2 satisfying $Q(\gamma_2) = 0$. Thus, when choosing $\gamma \in (0, \gamma_1 \wedge \gamma_2)$, we have both $H(\gamma) < 0$ and $Q(\gamma) < 0$. So, (11) becomes

$$\begin{aligned}
& e^{\gamma t} E|z(t)|^2 \\
\leqslant\ & E|x(0) - \upsilon(x(-\delta(0)))|^2 + \left[\gamma(1 + \frac{1}{\varepsilon})\rho^2 + (k_2 + \alpha_2) + \lambda(k_2 + \sigma_2)\right] \\
& \times \frac{e^{\gamma\tau}}{1-\bar{\delta}} \int_{-\tau}^{0} e^{\gamma s} E|\xi|^2 \mathrm{d}s + [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2 + \hat{\sigma}_2)]\frac{e^{\gamma\tau}}{1-\bar{\delta}} \int_{-\tau}^{0} e^{\gamma s} E|\xi|^{\alpha+2} \mathrm{d}s.
\end{aligned}$$

Denoting the right side of this estimate as $\phi(\xi)$, we have $e^{\gamma t} E|z(t)|^2 \leqslant \phi(\xi)$. Bearing in mind that $x(s) = z(s) + \upsilon(x(s - \delta(s)))$, by virtue of the Hölder inequality with $p = 2$, one gets

$$|x(s)|^2 = |z(s) + \upsilon(x(s - \delta(s)))|^2 \leqslant (1 + \varepsilon)[|z(s)|^2 + \varepsilon^{-1}\rho^2 |x(s - \delta(s))|^2].$$

Let $\varepsilon = \frac{\rho}{1-\rho}$ in this inequality. For $t > s > 0$, taking the expectation yields

$$\begin{aligned} e^{\gamma s}E|x(s)|^2 &\leqslant (1-\rho)^{-1}e^{\gamma s}E|z(s)|^2 + \rho e^{\gamma s}E|x(s-\delta(s))|^2 \\ &\leqslant (1-\rho)^{-1}e^{\gamma s}E|z(s)|^2 + \rho e^{\gamma\tau}e^{\gamma(s-\delta(s))}E|x(s-\delta(s))|^2 \\ &\leqslant (1-\rho)^{-1}e^{\gamma s}E|z(s)|^2 + \rho e^{\gamma\tau}\sup_{s-\tau\leqslant\theta\leqslant s} e^{\gamma\theta}E|x(\theta)|^2 \\ &\leqslant (1-\rho)^{-1}\phi(\xi) + \rho e^{\gamma\tau}\sup_{-\tau\leqslant\theta\leqslant t} e^{\gamma\theta}E|x(\theta)|^2. \end{aligned}$$

Consequently, we have

$$\sup_{-\tau\leqslant s\leqslant t} e^{\gamma s}E|x(s)|^2 \leqslant (1-\rho)^{-1}\phi(\xi) + \rho e^{\gamma\tau}\sup_{-\tau\leqslant\theta\leqslant t} e^{\gamma\theta}E|x(\theta)|^2.$$

Let $\gamma_3 = -\frac{1}{\tau}\log\rho$, then for any $\gamma \in (0, \gamma_1 \wedge \gamma_2 \wedge \gamma_3)$,

$$e^{\gamma t}E|x(t)|^2 \leqslant (1-\rho e^{\gamma\tau})^{-1}(1-\rho)^{-1}\phi(\xi),$$

which gives the desired conclusion and the proof is completed. $\square$

Remark 3.2. The conditions in assumption $\mathcal{A}_5$ are inspired by paper [25], in which they are called superlinear growth conditions (or polynomial growth conditions in [22]). With the appearance of superlinear terms $|x|^{\alpha+2}$ and $|y|^{\alpha+2}$, more nonlinear functions may be involved and suitable to Eq.(1), which make our model more extensive than that with the linear growth conditions [17, 23]. Just for this reason, the given parameter conditions and the proving process are somewhat complicated.

4. Backward Euler-Maruyama method

Now, we give the BEM numerical approximations y_k for Eq. (1).

$$\begin{cases} y_k &= \xi(k\triangle), \quad k=-n^*,-n^*+1,\cdots,0 \\ y_{k+1} &= y_k + \upsilon(y_{k+1-[\delta((k+1)\Delta)/\Delta]}) - \upsilon(y_{k-[\delta(k\Delta)/\Delta]}) \\ & \quad +f(y_{k+1}, y_{k+1-[\delta((k+1)\Delta)/\Delta]})\triangle + g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\triangle W_k \\ & \quad +h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\triangle N_k,\ k=0,1,2,\cdots \end{cases} \tag{14}$$

$\Delta > 0$ is the stepsize with $\Delta = \tau/n^*$ for some integer $n^* > \tau$. $\Delta W_k := W(t_{k+1}) - W(t_k)$, which has zero mean and Δ variance; $\Delta N_k := N(t_{k+1}) - N(t_k)$, which has $\lambda\Delta$ mean and $\lambda\Delta$ variance.

In order to show that the BEM method is an implicit method clearly, we analyze the property of equation (14). For a fixed integer k, we introduce the indicative function I_A with $I_A = 1$ if $[\delta((k+1)\Delta)/\Delta] = 0$, and $I_A = 0$, otherwise. Then, equation (14) can be written as:

$$\begin{aligned} y_{k+1} &= y_k + \upsilon(y_{k+1})I_A + \upsilon(y_{k+1-[\delta((k+1)\Delta)/\Delta]})I_{A^c} - \upsilon(y_{k-[\delta(k\Delta)/\Delta]}) \\ & \quad +\Delta f(y_{k+1}, y_{k+1})I_A + \Delta f(y_{k+1}, y_{k+1-[\delta((k+1)\Delta)/\Delta]})I_{A^c} \\ & \quad +g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta W_k + h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta N_k. \end{aligned} \tag{15}$$

In fact, we can see for y_{k+1}, if equation (15) admits a solution, that is equivalent to the following equation exists a unique solution,

$$x = b + \upsilon(x)I_A + \Delta f(x,x)I_A + \Delta f(x,a)I_{A^c}, \tag{16}$$

for any $a, b \in R^n$. Then, we will give some conditions, which ensure equation (16) exists a unique solution. The proof of Lemma 4.1 can be obtained with the Brouwer's fixed point theorem, which is the same as that in [10]. So we only show the result.

Lemma 4.1. *Let assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold, if $(a_1 + a_2)\Delta + \rho < 1$, then there exists a unique solution to equation (16).*

With the help of Lemma 4.1, the BEM method (14) is solvable under the same conditions.

Definition 4.2. A discrete numerical solution y_k is said to be exponentially mean-square stable, if there exist constants $\gamma > 0$ and $\Delta^* > 0$ such that

$$\limsup_{k\to\infty} \frac{\log(E|y_k|^2)}{k\Delta} \leqslant -\gamma$$

for step size $\Delta \in (0, \Delta^*)$ and any bounded initial condition $\xi(k\Delta), k = -n^*, -n^* + 1, \cdots, 0$.

5. Exponential stability of the BEM numerical solution

In this section, the BEM numerical solution are shown to reproduce the exponential mean-square stability of the analytic solution of Eq.(1). To deal with the time-varying delay in the numerical analysis, by using the integral function, the subscript expression of the numerical solution, such as expressions $i - [\delta(i\Delta)/\Delta]$, may be equal for some $i \in \{0, 1, 2, ...\}$. So we need to estimate the maximum number of such indices i. The following lemma plays important roles in the proceeding stability analysis, which can be found in [12].

Lemma 5.1.[12] *Suppose that $\mathcal{A}_4$ holds. For an arbitrary but fixed $i \in \{0, 1, 2, ...\}$, let $i - [\delta(i\Delta)/\Delta] = a$, where $a \in \{-n^*, -n^* + 1, ...0, 1, ..., i\}$. Then,*

$$\#\{j \in \{0, 1, 2, ...\} : j - [\delta(j\Delta)/\Delta] = a\} \leqslant [(1 - \eta)^{-1}] + 1 \qquad (17)$$

where $\#S$ denotes the number of elements of the set S.

Theorem 5.2. *Suppose that assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4, \mathcal{A}_5$ hold, together with condition (II) and, instead of (I), assume the parameters satisfy*

$$\text{(III)}\begin{cases} \alpha_1 - \alpha_2 - k_1 - \lambda(k_1 + \sigma_1) - [k_2 + \alpha_2 + \lambda(k_2 + \sigma_2)]([(1-\eta)^{-1}] + 1) > 0 \\ \hat{\alpha}_1 - \hat{\alpha}_2 - \hat{k}_1 - \lambda(\hat{k}_1 + \hat{\sigma}_1) - [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2 + \hat{\sigma}_2)]([(1-\eta)^{-1}] + 1) > 0, \end{cases}$$

$$\text{(IV)}\begin{cases} k_1 + \lambda(\sigma_1 + k_1) \geqslant 0 \\ \hat{k}_1 + \lambda(\hat{\sigma}_1 + \hat{k}_1) \geqslant 0, \end{cases}$$

then there exists $\gamma \in (0, \log(C_1^* \wedge C_2^*))$ and stepsize bound $\Delta^* > 0$ such that for any stepsize $\Delta < \Delta^*$, the BEM numerical solution defined by (14) satisfies

$$\limsup_{k\to\infty} \frac{\log(E|y_k|^2)}{k\Delta} \leqslant -\gamma, \tag{18}$$

where C_1^* and C_2^* are the positive roots of equations $H(C,\Delta) = 0$ and $P(C,\Delta) = 0$, respectively; $\Delta^* = \frac{1-\rho}{a_1+a_2} \wedge \Delta_1 \wedge \Delta_2$.

Proof. Let $z_k = y_k - \upsilon(y_{k-[\delta(k\Delta)/\Delta]})$, based on (14), we have

$$\begin{aligned} & z_{k+1} - f(y_{k+1}, y_{k+1-[\delta((k+1)\Delta)/\Delta]})\Delta \\ = \;& z_k + g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta W_k + h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta N_k. \end{aligned} \tag{19}$$

Squaring both sides of the equality yields

$$\begin{aligned} |z_{k+1}|^2 \leqslant\; & |z_k|^2 + \left|g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta W_k\right|^2 + \left|h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta N_k\right|^2 \\ & +2\left\langle z_k, g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta W_k\right\rangle + 2\left\langle z_k, h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta N_k\right\rangle \\ & +2\left\langle g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta W_k, h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta N_k\right\rangle \\ & +2\left\langle z_{k+1}, f(y_{k+1}, y_{k+1-[\delta((k+1)\Delta)/\Delta]})\Delta\right\rangle. \end{aligned} \tag{20}$$

Based on the properties that $E(\Delta W_k) = 0, E(\Delta W_k)^2 = \Delta, E(\Delta N_k) = \lambda\Delta, E(\Delta N_k)^2 = \lambda\Delta(1+\lambda\Delta)$, and $y_k, y_{k-[\delta(k\Delta)/\Delta]}$ are all $\mathcal{F}_{t_k}$-measurable, we obtain

$$\begin{cases} E\left|g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta W_k\right|^2 = \Delta E\left|g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\right|^2, \\ E\left|h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta N_n\right|^2 = \lambda\Delta(1+\lambda\Delta)E\left|h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\right|^2, \\ E\left\langle z_k, g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta W_k\right\rangle = 0, \\ E\left\langle z_k, h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta N_k\right\rangle = \lambda\Delta E\left\langle z_k, h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\right\rangle, \\ E\left\langle g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta W_k, h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\Delta N_k\right\rangle = 0. \end{cases}$$

Taking expectation on both sides of (20), one gets

$$\begin{aligned} E|z_{k+1}|^2 \leqslant E|z_k|^2 &+ \Delta E\left|g(y_k, y_{k-[\delta(k\Delta)/\Delta]})\right|^2 \\ &+\lambda\Delta(1+\lambda\Delta)E\left|h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\right|^2 \\ &+2\lambda\Delta E\left\langle z_k, h(y_k, y_{k-[\delta(k\Delta)/\Delta]})\right\rangle \\ &+2E\left\langle z_{k+1}, f(y_{k+1}, y_{k+1-[\delta((k+1)\Delta)/\Delta]})\Delta\right\rangle, \end{aligned}$$

which with assumption $\mathcal{A}_5$, can be estimated as

$$\begin{aligned} & E|z_{k+1}|^2 \\ \leqslant\ & E|z_k|^2 + [1+\lambda(1+\lambda\Delta)]\Delta E[k_1|y_k|^2 + k_2\left|y_{k-[\delta(k\Delta)/\Delta]}\right|^2 + \hat{k}_1|y_k|^{\alpha+2} \\ & +\hat{k}_2\left|y_{k-[\delta(k\Delta)/\Delta]}\right|^{\alpha+2}] + \lambda\Delta E[\sigma_1|y_k|^2+\sigma_2\left|y_{k-[\delta(k\Delta)/\Delta]}\right|^2 + \hat{\sigma}_1|y_k|^{\alpha+2} \\ & +\hat{\sigma}_2\left|y_{k-[\delta(k\Delta)/\Delta]}\right|^{\alpha+2}] + \Delta E[-\alpha_1|y_{k+1}|^2 + \alpha_2\left|y_{k+1-[\delta((k+1)\Delta)/\Delta]}\right|^2 \\ & -\hat{\alpha}_1|y_{k+1}|^{\alpha+2} + \hat{\alpha}_2\left|y_{k+1-[\delta((k+1)\Delta)/\Delta]}\right|^{\alpha+2}]. \end{aligned} \tag{21}$$

Expression (21) can be rearranged to

$$\begin{aligned} & E|z_{k+1}|^2 \\ \leqslant\ & E|z_k|^2 + l_1E|y_k|^2 + l_2E\left|y_{k-[\delta(k\Delta)/\Delta]}\right|^2 + l_3E|y_k|^{\alpha+2} + l_4E\left|y_{k-[\delta(k\Delta)/\Delta]}\right|^{\alpha+2} \\ & -\alpha_1\Delta E|y_{k+1}|^2 + \alpha_2\Delta E\left|y_{k+1-[\delta((k+1)\Delta)/\Delta]}\right|^2 - \hat{\alpha}_1\Delta E|y_{k+1}|^{\alpha+2} \end{aligned}$$

$$+\hat{\alpha}_2\Delta E\big|y_{k+1-[\delta((k+1)\Delta)/\Delta]}\big|^{\alpha+2}, \tag{22}$$

where $l_1 = [1+\lambda(1+\lambda\Delta)]\Delta k_1 + \lambda\Delta\sigma_1, l_2 = [1+\lambda(1+\lambda\Delta)]\Delta k_2 + \lambda\Delta\sigma_2, l_3 = [1+\lambda(1+\lambda\Delta)]\Delta\hat{k}_1 + \lambda\Delta\hat{\sigma}_1, l_4 = [1+\lambda(1+\lambda\Delta)]\Delta\hat{k}_2 + \lambda\Delta\hat{\sigma}_2$. For any constant $C \geqslant 1$, we derive

$$\begin{aligned}
& C^{(k+1)\Delta}E|z_{k+1}|^2 - C^{k\Delta}E|z_k|^2 \\
\leqslant\ & (1-C^{-\Delta})C^{(k+1)\Delta}E|z_k|^2 + l_1C^{(k+1)\Delta}E|y_k|^2 + l_2C^{(k+1)\Delta}E\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^2 \\
& + l_3C^{(k+1)\Delta}E|y_k|^{\alpha+2} + l_4C^{(k+1)\Delta}E\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^{\alpha+2} - \alpha_1\Delta C^{(k+1)\Delta}E|y_{k+1}|^2 \\
& + \alpha_2\Delta C^{(k+1)\Delta}E\big|y_{k+1-[\delta((k+1)\Delta)/\Delta]}\big|^2 - \hat{\alpha}_1\Delta C^{(k+1)\Delta}E|y_{k+1}|^{\alpha+2} \\
& + \hat{\alpha}_2\Delta C^{(k+1)\Delta}E\big|y_{k+1-[\delta((k+1)\Delta)/\Delta]}\big|^{\alpha+2}.
\end{aligned} \tag{23}$$

Note that

$$|z_k|^2 = \big|y_k - \upsilon(y_{k-[\delta(k\Delta)/\Delta]})\big|^2 \leqslant 2|y_k|^2 + 2\rho^2\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^2.$$

Inequality (23) turns to be

$$\begin{aligned}
& C^{k\Delta}E|z_k|^2 \\
\leqslant\ & E|z_0|^2 + \sum_{j=0}^{k-1}(1-C^{-\Delta})C^{(j+1)\Delta}2(E|y_j|^2 + \rho^2E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^2) \\
& + l_1\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_j|^2 + l_2\sum_{j=0}^{k-1}C^{(j+1)\Delta}E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^2 \\
& + l_3\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_j|^{\alpha+2} + l_4\sum_{j=0}^{k-1}C^{(j+1)\Delta}E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^{\alpha+2} \\
& - \sum_{j=0}^{k-1}C^{(j+1)\Delta}\alpha_1\Delta E|y_{j+1}|^2 + \sum_{j=0}^{k-1}C^{(j+1)\Delta}\alpha_2\Delta E\big|y_{j+1-[\delta((j+1)\Delta)/\Delta]}\big|^2 \\
& - \sum_{j=0}^{k-1}C^{(j+1)\Delta}\hat{\alpha}_1\Delta E|y_{j+1}|^{\alpha+2} + \sum_{j=0}^{k-1}C^{(j+1)\Delta}\hat{\alpha}_2\Delta E\big|y_{j+1-[\delta((j+1)\Delta)/\Delta]}\big|^{\alpha+2}
\end{aligned} \tag{24}$$

We see that,

$$\sum_{j=0}^{k-1} C^{(j+1)\Delta} E\big|y_{j+1-[\delta((j+1)\Delta)/\Delta]}\big|^2 = \sum_{j=1}^{k} C^{j\Delta} E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^2$$

$$= \quad C^{-\Delta} \sum_{j=0}^{k-1} C^{(j+1)\Delta} E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^2 - E\big|y_{-[\delta(0)/\Delta]}\big|^2 + C^{k\Delta} E\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^2.$$

Similarly,

$$\sum_{j=0}^{k-1} C^{(j+1)\Delta} E\big|y_{j+1-[\delta((j+1)\Delta)/\Delta]}\big|^{\alpha+2}$$

$$= C^{-\Delta} \sum_{j=0}^{k-1} C^{(j+1)\Delta} E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^{\alpha+2} - E\big|y_{-[\delta(0)/\Delta]}\big|^{\alpha+2} + C^{k\Delta} E\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^{\alpha+2}.$$

Then the estimate (24) becomes

$$\begin{aligned}
& C^{k\Delta} E|z_k|^2 + \alpha_2 \Delta E\big|y_{-[\delta(0)/\Delta]}\big|^2 + \hat{\alpha}_2 \Delta E\big|y_{-[\delta(0)/\Delta]}\big|^{\alpha+2} \\
\leqslant \ & E|z_0|^2 + [l_1 + 2(1 - C^{-\Delta})] \sum_{j=0}^{k-1} C^{(j+1)\Delta} E|y_j|^2 + l_3 \sum_{j=0}^{k-1} C^{(j+1)\Delta} E|y_j|^{\alpha+2} \\
& + [l_2 + 2\rho^2(1 - C^{-\Delta})] \sum_{j=0}^{k-1} C^{(j+1)\Delta} E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^2 \\
& + l_4 \sum_{j=0}^{k-1} C^{(j+1)\Delta} E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^{\alpha+2} - \sum_{j=0}^{k-1} C^{(j+1)\Delta} \alpha_1 \Delta E|y_{j+1}|^2 \\
& + \alpha_2 \Delta C^{-\Delta} \sum_{j=0}^{k-1} C^{(j+1)\Delta} E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^2 + \alpha_2 \Delta C^{k\Delta} E\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^2 \\
& - \sum_{j=0}^{k-1} C^{(j+1)\Delta} \hat{\alpha}_1 \Delta E|y_{j+1}|^{\alpha+2} + \hat{\alpha}_2 \Delta C^{-\Delta} \sum_{j=0}^{k-1} C^{(j+1)\Delta} E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^{\alpha+2} \\
& + \hat{\alpha}_2 \Delta C^{k\Delta} E\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^{\alpha+2}. \qquad (25)
\end{aligned}$$

We find that

$$\alpha_2 \Delta C^{k\Delta} E\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^2$$

$$
\begin{aligned}
&\leqslant \alpha_2\Delta C^{k\Delta}E|y_k|^2+\alpha_2\Delta C^{k\Delta}E\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^2 I_{\{[\delta(k\Delta)/\Delta]\neq 0\}}\\
&\leqslant \alpha_2\Delta C^{k\Delta}E|y_k|^2+\alpha_2\Delta C^{([\delta(k\Delta)/\Delta]-1)\Delta}C^{(k-[\delta(k\Delta)/\Delta]+1)\Delta}E\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^2\\
&\leqslant \alpha_2\Delta C^{k\Delta}E|y_k|^2+\alpha_2\Delta C^{(n^*-1)\Delta}\sum_{j=-n^*}^{k-1}C^{(j+1)\Delta}E|y_j|^2. \qquad (26)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\hat{\alpha}_2\Delta C^{k\Delta}E\big|y_{k-[\delta(k\Delta)/\Delta]}\big|^{\alpha+2}\\
\leqslant\ &\hat{\alpha}_2\Delta C^{k\Delta}E|y_k|^{\alpha+2}+\hat{\alpha}_2\Delta C^{(n^*-1)\Delta}\sum_{j=-n^*}^{k-1}C^{(j+1)\Delta}E|y_j|^{\alpha+2}. \qquad (27)
\end{aligned}
$$

On the basis of (26) and (27), (25) can be read as

$$
\begin{aligned}
&C^{k\Delta}E|z_k|^2+\alpha_2\Delta E\big|y_{-[\delta(0)/\Delta]}\big|^2+\hat{\alpha}_2\Delta E\big|y_{-[\delta(0)/\Delta]}\big|^{\alpha+2}-\alpha_2\Delta C^{k\Delta}E|y_k|^2\\
&-\hat{\alpha}_2\Delta C^{k\Delta}E|y_k|^{\alpha+2}\\
\leqslant\ &E|z_0|^2+[l_1+2(1-C^{-\Delta})+\alpha_2\Delta C^{(n^*-1)\Delta}]\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_j|^2\\
&+(l_3+\hat{\alpha}_2\Delta C^{(n^*-1)\Delta})\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_j|^{\alpha+2}\\
&+[l_2+2\rho^2(1-C^{-\Delta})+\alpha_2\Delta C^{-\Delta}]\sum_{j=0}^{k-1}C^{(j+1)\Delta}E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^2\\
&+(l_4+\hat{\alpha}_2\Delta C^{-\Delta})\sum_{j=0}^{k-1}C^{(j+1)\Delta}E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^{\alpha+2}\\
&-\sum_{j=0}^{k-1}C^{(j+1)\Delta}\alpha_1\Delta E|y_{j+1}|^2-\sum_{j=0}^{k-1}C^{(j+1)\Delta}\hat{\alpha}_1\Delta E|y_{j+1}|^{\alpha+2}\\
&+\alpha_2\Delta C^{(n^*-1)\Delta}\sum_{j=-n^*}^{-1}C^{(j+1)\Delta}E|y_j|^2\\
&+\hat{\alpha}_2\Delta C^{(n^*-1)\Delta}\sum_{j=-n^*}^{-1}C^{(j+1)\Delta}E|y_j|^{\alpha+2}. \qquad (28)
\end{aligned}
$$

We also have that

$$\begin{aligned}
&\alpha_1\Delta\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_{j+1}|^2\\
=\ &\alpha_1\Delta C^{-\Delta}\sum_{j=1}^{k}C^{(j+1)\Delta}E|y_j|^2\\
=\ &\alpha_1\Delta C^{-\Delta}\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_j|^2-\alpha_1\Delta E|y_0|^2+\alpha_1\Delta C^{k\Delta}E|y_k|^2, \quad (29)
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\alpha}_1\Delta\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_{j+1}|^{\alpha+2}\\
=\ &\hat{\alpha}_1\Delta C^{-\Delta}\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_j|^{\alpha+2}-\hat{\alpha}_1\Delta E|y_0|^{\alpha+2}+\hat{\alpha}_1\Delta C^{k\Delta}E|y_k|^{\alpha+2}. \quad (30)
\end{aligned}$$

Substituting (29) and (30) into (28), we get

$$\begin{aligned}
&C^{k\Delta}E|z_k|^2+\alpha_2\Delta E\big|y_{-[\delta(0)/\Delta]}\big|^2+\hat{\alpha}_2\Delta E\big|y_{-[\delta(0)/\Delta]}\big|^{\alpha+2}\\
&+(\alpha_1-\alpha_2)\Delta C^{k\Delta}E|y_k|^2+(\hat{\alpha}_1-\hat{\alpha}_2)\Delta C^{k\Delta}E|y_k|^{\alpha+2}\\
\leqslant\ &E|z_0|^2+[l_1+2(1-C^{-\Delta})+\alpha_2\Delta C^{(n^*-1)\Delta}-\alpha_1\Delta C^{-\Delta}]\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_j|^2\\
&+(l_3+\hat{\alpha}_2\Delta C^{(n^*-1)\Delta}-\hat{\alpha}_1\Delta C^{-\Delta})\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_j|^{\alpha+2}\\
&+[l_2+2\rho^2(1-C^{-\Delta})+\alpha_2\Delta C^{-\Delta}]\sum_{j=0}^{k-1}C^{(j+1)\Delta}E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^2\\
&+(l_4+\hat{\alpha}_2\Delta C^{-\Delta})\sum_{j=0}^{k-1}C^{(j+1)\Delta}E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^{\alpha+2}\\
&+\alpha_1\Delta E|y_0|^2+\hat{\alpha}_1\Delta E|y_0|^{\alpha+2}+\alpha_2\Delta C^{(n^*-1)\Delta}\sum_{j=-n^*}^{-1}C^{(j+1)\Delta}E|y_j|^2
\end{aligned}$$

$$+\hat{\alpha}_2\Delta C^{(n^*-1)\Delta}\sum_{j=-n^*}^{-1}C^{(j+1)\Delta}E|y_j|^{\alpha+2}. \tag{31}$$

Using Lemma 5.1, we obtain

$$\begin{aligned}
&\sum_{j=0}^{k-1}C^{(j+1)\Delta}E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^2\\
\leqslant\ &\sum_{j=0}^{k-1}C^{n^*\Delta}C^{(j-[\delta(j\Delta)/\Delta]+1)\Delta}E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^2\\
\leqslant\ &(\big[(1-\eta)^{-1}\big]+1)C^{n^*\Delta}\sum_{j=-n^*}^{k-1}C^{(j+1)\Delta}E|y_j|^2,
\end{aligned} \tag{32}$$

and

$$\sum_{j=0}^{k-1}C^{(j+1)\Delta}E\big|y_{j-[\delta(j\Delta)/\Delta]}\big|^{\alpha+2}\ \leqslant\ (\big[(1-\eta)^{-1}\big]+1)C^{n^*\Delta}\sum_{j=-n^*}^{k-1}C^{(j+1)\Delta}E|y_j|^{\alpha+2}.$$

Thus, expression (31) can be reformed as

$$\begin{aligned}
&C^{k\Delta}E|z_k|^2+\alpha_2\Delta E\big|y_{-[\delta(0)/\Delta]}\big|^2+\hat{\alpha}_2\Delta E\big|y_{-[\delta(0)/\Delta]}\big|^{\alpha+2}\\
&+(\alpha_1-\alpha_2)\Delta C^{k\Delta}E|y_k|^2+(\hat{\alpha}_1-\hat{\alpha}_2)\Delta C^{k\Delta}E|y_k|^{\alpha+2}\\
\leqslant\ &E|z_0|^2+\alpha_1\Delta E|y_0|^2+\hat{\alpha}_1\Delta E|y_0|^{\alpha+2}+\alpha_2\Delta C^{(n^*-1)\Delta}\sum_{j=-n^*}^{-1}C^{(j+1)\Delta}E|y_j|^2\\
&+[l_2+2\rho^2(1-C^{-\Delta})+\alpha_2\Delta C^{-\Delta}]([(1-\eta)^{-1}]+1)C^{n^*\Delta}\sum_{j=-n^*}^{-1}C^{(j+1)\Delta}E|y_j|^2\\
&+[\hat{\alpha}_2\Delta C^{(n^*-1)\Delta}+(l_4+\hat{\alpha}_2\Delta C^{-\Delta})([(1-\eta)^{-1}]+1)C^{n^*\Delta}]\\
&\times\sum_{j=-n^*}^{-1}C^{(j+1)\Delta}E|y_j|^{\alpha+2}+H(C,\Delta)\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_j|^2\\
&+P(C,\Delta)\sum_{j=0}^{k-1}C^{(j+1)\Delta}E|y_j|^{\alpha+2},
\end{aligned} \tag{33}$$

where

$$
\begin{aligned}
H(C,\Delta) &= l_1 + 2(1 - C^{-\Delta}) + \alpha_2 \Delta C^{(n^*-1)\Delta} - \alpha_1 \Delta C^{-\Delta} \\
&\quad + [l_2 + 2\rho^2(1 - C^{-\Delta}) + \alpha_2 \Delta C^{-\Delta}] \times ([(1-\eta)^{-1}] + 1) C^{n^*\Delta}, \\
P(C,\Delta) &= l_3 + \hat{\alpha}_2 \Delta C^{(n^*-1)\Delta} - \hat{\alpha}_1 \Delta C^{-\Delta} + (l_4 + \hat{\alpha}_2 \Delta C^{-\Delta}) \\
&\quad \times ([(1-\eta)^{-1}] + 1) C^{n^*\Delta}.
\end{aligned}
$$

We analyze the properties of functions $H(C,\Delta)$ and $P(C,\Delta)$. Bearing in mind that $n^*\Delta = \tau$, one can easily obtain

$$
\begin{aligned}
&\frac{\partial}{\partial C} H(C,\Delta) \\
=\ & 2\Delta C^{-\Delta-1} + \alpha_2 \Delta(\tau - \Delta) C^{\tau-\Delta-1} + \alpha_1 \Delta^2 C^{-\Delta-1} \\
& + [l_2 \tau C^{\tau-1} + 2\rho^2(\tau C^{\tau-1} - (\tau - \Delta) C^{\tau-\Delta-1}) + \alpha_2 \Delta(\tau - \Delta) C^{\tau-\Delta-1}] \\
& \times ([(1-\eta)^{-1}] + 1),
\end{aligned}
$$

and

$$
\begin{aligned}
H(1,\Delta) &= l_1 + \alpha_2 \Delta - \alpha_1 \Delta + (l_2 + \alpha_2 \Delta)([(1-\eta)^{-1}] + 1) \\
&= \Delta \left\{ (1 + \lambda(1 + \lambda\Delta)) k_1 + \lambda \sigma_1 + \alpha_2 - \alpha_1 \right. \\
&\quad \left. + [(1 + \lambda(1 + \lambda\Delta)) k_2 + \lambda \sigma_2 + \alpha_2] \times ([(1-\eta)^{-1}] + 1) \right\}.
\end{aligned}
$$

With the first condition of (II), we have $l_2 = [1 + \lambda(1 + \lambda\Delta)]\Delta k_2 + \lambda\Delta\sigma_2 > 0$, which makes $\frac{\partial}{\partial C} H(C,\Delta) > 0$ for all $C \geqslant 1$. Let

$$
\Delta_1 = \frac{\alpha_1 - \alpha_2 - \lambda\sigma_1 - (1+\lambda)k_1 - [\alpha_2 + \lambda\sigma_2 + (1+\lambda)k_2]([(1-\eta)^{-1}] + 1)}{\lambda^2(k_1 + k_2([(1-\eta)^{-1}] + 1))},
$$

for any $\Delta \in (0, \Delta_1 \wedge \frac{1-\rho}{a_1 + a_2})$, we have $H(1,\Delta) < 0$ under the first condition of (III). So, there exists a unique $C_1^* > 1$ such that $H(C_1^*, \Delta) = 0$ for $\Delta \in$

$(0, \Delta_1 \wedge \frac{1-\rho}{a_1+a_2})$. Meanwhile, we observe that

$$\begin{aligned}\frac{\partial}{\partial C}P(C,\Delta) &= \hat{\alpha}_2\Delta(\tau-\Delta)C^{\tau-\Delta-1} + \hat{\alpha}_1\Delta^2 C^{-\Delta-1} \\ &\quad + [l_4\tau C^{\tau-1} + \hat{\alpha}_2\Delta(\tau-\Delta)C^{\tau-\Delta-1}]([(1-\eta)^{-1}]+1),\end{aligned}$$

and

$$\begin{aligned}P(1,\Delta) &= l_3 + \hat{\alpha}_2\Delta - \hat{\alpha}_1\Delta + (l_4+\hat{\alpha}_2\Delta)([(1-\eta)^{-1}]+1) \\ &= \Delta\Big\{\lambda\hat{\sigma}_1 + (1+\lambda(1+\lambda\Delta))\hat{k}_1 + \hat{\alpha}_2 - \hat{\alpha}_1 \\ &\quad + \Big[\lambda\hat{\sigma}_2 + (1+\lambda(1+\lambda\Delta))\hat{k}_2 + \hat{\alpha}_2\Big] \times ([(1-\eta)^{-1}]+1)\Big\}.\end{aligned}$$

When the second conditions of (II) and (III) are satisfied, we get $\frac{\partial}{\partial C}P(C,\Delta) > 0$ for all $C \geqslant 1$, and $P(1,\Delta) < 0$ for any $\Delta \in (0, \Delta_2 \wedge \frac{1-\rho}{a_1+a_2})$, where

$$\Delta_2 = \frac{\hat{\alpha}_1 - \hat{\alpha}_2 - \lambda\hat{\sigma}_1 - (1+\lambda)\hat{k}_1 - [\hat{\alpha}_2 + \lambda\hat{\sigma}_2 + (1+\lambda)\hat{k}_2]([(1-\eta)^{-1}]+1)}{\lambda^2(\hat{k}_1 + \hat{k}_2([(1-\eta)^{-1}]+1))}.$$

It is shown that there exists a unique $C_2^* > 1$ such that $P(C_2^*,\Delta) = 0$ for $\Delta \in (0, \Delta_2 \wedge \frac{1-\rho}{a_1+a_2})$. Thus, by choosing $C^* \in (1, C_1^* \wedge C_2^*)$, we have both $H(C^*,\Delta) < 0$ and $P(C^*,\Delta) < 0$ when $\Delta \in (0, \Delta_1 \wedge \Delta_2 \wedge \frac{1-\rho}{a_1+a_2})$. Therefore, inequality (33) becomes

$$\begin{aligned}&C^{k\Delta}E|z_k|^2 + \alpha_2\Delta E\big|y_{-[\delta(0)/\Delta]}\big|^2 + \hat{\alpha}_2\Delta E\big|y_{-[\delta(0)/\Delta]}\big|^{\alpha+2} \\ &+(\alpha_1-\alpha_2)\Delta C^{k\Delta}E|y_k|^2 + (\hat{\alpha}_1-\hat{\alpha}_2)\Delta C^{k\Delta}E|y_k|^{\alpha+2} \leqslant \varphi(\xi), \quad (34)\end{aligned}$$

where $\varphi(\xi) = (2 + 2\rho^2 + \alpha_1\Delta + \alpha_2\tau C^{\tau-\Delta})E\|\xi\|^2 + \hat{\alpha}_1\Delta E\|\xi\|^{\alpha+2} + [(1+\lambda(1+\lambda\Delta))k_2 + \lambda\sigma_2 + 2\rho^2(1-C^{-\Delta})/\Delta + \alpha_2 C^{-\Delta}]([(1-\eta)^{-1}]+1)C^\tau\tau E\|\xi\|^2 + \Big\{\hat{\alpha}_2 C^{\tau-\Delta} + [(1+\lambda(1+\lambda\Delta))\hat{k}_2 + \lambda\hat{\sigma}_2 + \hat{\alpha}_2 C^{-\Delta}]([(1-\eta)^{-1}]+1)C^\tau\Big\}\tau E\|\xi\|^{\alpha+2}$, which is a positive constant depending on the initial value ξ. Applying the condition (IV), which ensure that $\alpha_1 - \alpha_2 > 0$ and $\hat{\alpha}_1 - \hat{\alpha}_2 > 0$, we conclude from (34) that

$$(\alpha_1 - \alpha_2)\Delta C^{*k\Delta} E|y_k|^2 \leqslant \varphi(\xi).$$

Let $\gamma = \log C^*$,

$$E|y_k|^2 \leqslant \frac{1}{(\alpha_1 - \alpha_2)\Delta} \varphi(\xi) e^{-\gamma k\Delta},$$

which implies (18). $\square$

Remark 5.3. Let us analyze the conditions $\mathcal{A}_5$. Please note that the parameters $\sigma_i, \hat{\sigma}_i, (i \in \{1, 2\})$ related to jumps in (5) are constants, which are different from the parameters related to drift coefficient f in (4)(positive constants). In Theorem 3.1 and Theorems 5.2, to obtain the stability of analytic solution and numerical solution, we propose condition (II) and condition (IV), respectively, to strengthen constraints mainly for jumps parameters. It is not difficult to see, if we assume that $\sigma_i, \hat{\sigma}_i$ are all positive constants, then conditions (II) and (IV) will no longer be required in these two theorems, because they will hold obviously in the case that conditions (4) and (6) are unchanged. However, the proposed conditions (II) and (IV) are weaker than the assumptions of positive parameter, although they seem a little complicated. In other words, we give relatively weak conditions for jumps. Later, we will give an example to show that all these conditions can be satisfied.

Remark 5.4. Under some nonlinear conditions and parameter conditions, the BEM numerical solution is shown to reproduce the exponential mean-square stability of the analytic solution by Theorem 5.2. In particular, if Eq. (1) with no jumps is considered, that is $h(t, x(t), x(t - \delta(t))) = 0, \sigma_i = \hat{\sigma}_i = 0 (i \in \{1, 2\})$, and if the superlinear terms $|x|^{\alpha+2}$ and $|y|^{\alpha+2}$ are not involved, condition (III) will only become $\alpha_1 - \alpha_2 - k_1 - (\alpha_2 + k_2)([(1 - \eta)^{-1}] + 1) > 0$, which is the condition of formula (70) in [10] by letting $\alpha_1 = \beta_1, \alpha_2 = \beta_2, k_1 = \beta_3, k_2 = \beta_4$. It is worth noting that, the jumps and time-varying delay are dealt with under the nonlinear condition. The exponential mean-square stability we obtained can deduce the a.s. exponential stability with the linear growth condition by using the Chebyshev inequality as well as the Borel-Cantelli lemma [16]. Therefore, this work generalizes some results of [9, 10] to more general conditions and the case with Poisson jumps.

6. Illustrating example

Consider the nonlinear scalar NSDEs with time-varying delay and Poisson jumps:

$$\begin{cases} \mathrm{d}[x(t) - \frac{1}{9}\sin(x(t-\delta(t)))] &= [-3x(t) - 3x^3(t) - \frac{1}{2}x(t-\delta(t))]\mathrm{d}t \\ & \quad -\frac{1}{2}x^2(t-\delta(t))\mathrm{d}W(t) \\ & \quad -\frac{1}{2}(x(t) + x^2(t-\delta(t)))\mathrm{d}N(t), \ t \geqslant 0, \\ x(t) &= t+1, \qquad\qquad -1 \leqslant t \leqslant 0. \end{cases} \tag{35}$$

We show this example satisfies the presented conditions. Let y denotes $x(t-\delta(t))$, and $\delta(t) = 1 - \frac{1}{4}\sin t$, we deduce

$$\begin{aligned} & 2\left(x - \upsilon(y)\right)^T f(x,y) = 2(x - \frac{1}{9}\sin y)(-3x - 3x^3 - \frac{1}{2}y) \\ = \ & -6x^2 - 6x^4 - xy + \frac{2}{3}x\sin y + \frac{2}{3}x^3\sin y + \frac{1}{9}y\sin y \\ \leqslant \ & -6x^2 - 6x^4 + \frac{1}{2}(x^2+y^2) + \frac{1}{3}[x^2 + (\sin y)^2] + \frac{2}{3}[\frac{3}{4}x^4 + \frac{1}{4}(\sin y)^4] \\ & + \frac{1}{9}[\frac{1}{2}y^2 + \frac{1}{2}(\sin y)^2] \\ \leqslant \ & -6x^2 - 6x^4 + \frac{5}{6}x^2 + \frac{1}{2}y^2 + \frac{1}{3}y^2 + \frac{1}{2}x^4 + \frac{1}{6}y^4 + \frac{1}{9}y^2 \\ = \ & -\frac{31}{6}x^2 + \frac{17}{18}y^2 - \frac{11}{2}x^4 + \frac{1}{6}y^4, \end{aligned}$$

which implies that condition (4) holds with $\alpha_1 = \frac{31}{6}, \alpha_2 = \frac{17}{18}, \hat{\alpha}_1 = \frac{11}{2}, \hat{\alpha}_2 = \frac{1}{6}, \alpha = 2$. In the same way, we get

$$\begin{aligned} & 2\left(x - \upsilon(y)\right)^T h(x,y) = 2(x - \frac{1}{9}\sin y)\frac{1}{2}(-x - y^2) \\ = \; & -x^2 - xy^2 + \frac{1}{9}x\sin y + \frac{1}{9}y^2 \sin y \\ \leqslant \; & -x^2 + \frac{1}{2}(x^2 + y^4) + \frac{1}{18}[x^2 + (\sin y)^2] + \frac{1}{9}y^2 \\ \leqslant \; & -\frac{4}{9}x^2 + \frac{1}{6}y^2 + \frac{1}{2}y^4, \end{aligned}$$

which means that $\sigma_1 = -\frac{4}{9}, \sigma_2 = \frac{1}{6}, \hat{\sigma}_1 = 0, \hat{\sigma}_2 = \frac{1}{2}$ in condition (5). We also have that

$$|h(x,y)|^2 = \frac{1}{4}(x + y^2)^2 = \frac{1}{4}(x^2 + y^4 + 2xy^2) \leqslant \frac{1}{2}(x^2 + y^4),$$

which reveals that $k_1 = \frac{1}{2}, k_2 = 0, \hat{k}_1 = 0, \hat{k}_2 = \frac{1}{2}$ in condition (6).

Now, we set $\lambda = \frac{1}{4}$, and note that $\bar{\delta} = \frac{1}{4}, \eta = \frac{1}{4}$. It is easy to verify

$$\begin{aligned} & \alpha_1 - k_1 - \lambda(k_1 + \sigma_1) - [k_2 + \alpha_2 + \lambda(k_2 + \sigma_2)]\frac{1}{1 - \bar{\delta}} \\ = \; & \frac{31}{6} - \frac{1}{2} - \frac{1}{4}(\frac{1}{2} - \frac{4}{9}) - (\frac{17}{18} + \frac{1}{4}\cdot\frac{1}{6})\frac{1}{1 - 0.25} = \frac{31}{6} - \frac{49}{27} - \frac{1}{64} > 0, \end{aligned}$$

and

$$\begin{aligned} & \hat{\alpha}_1 - \hat{k}_1 - \lambda(\hat{k}_1 + \hat{\sigma}_1) - [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2 + \hat{\sigma}_2)]\frac{1}{1 - \bar{\delta}} \\ = \; & \frac{11}{2} - [\frac{1}{2} + \frac{1}{6} + \frac{1}{4}(\frac{1}{2} + \frac{1}{6})]\frac{1}{1 - 0.25} = \frac{77}{18} > 0. \end{aligned}$$

Besides, we find that $k_2 + \lambda(\sigma_2 + k_2) \geqslant 0$ and $\hat{k}_2 + \lambda(\hat{\sigma}_2 + \hat{k}_2) \geqslant 0$. So, the conditions of Theorem 3.1 hold, which means that the analytic solution of Eq.(1) is exponentially mean-square stable.

Then we check the conditions of stability about the numerical solution. Firstly, we observe that, for any $q_1, q_2, y \in R$,

$$\begin{aligned}
\langle q_1 - q_2, f(q_1, y) - f(q_2, y)\rangle &= -3(q_1 - q_2)^2(1 + q_1^2 + q_2^2 + q_1 q_2) \\
&\leqslant a_1|q_1 - q_2|^2, \\
\langle q_1 - q_2, f(y, q_1) - f(y, q_2)\rangle &= -\frac{1}{2}(q_1 - q_2)^2 \leqslant a_2|q_1 - q_2|^2,
\end{aligned}$$

which show that assumption $\mathcal{A}_2$ holds for any $a_1 > 0$ and $a_2 > 0$. So, the BEM method for Eq. (35) is well defined when step size $\Delta \in (0, 1)$. Secondly, we have that

$$\begin{aligned}
&\alpha_1 - \alpha_2 - k_1 - \lambda(k_1 + \sigma_1) - [k_2 + \alpha_2 + \lambda(k_2 + \sigma_2)]([(1 - \eta)^{-1}] + 1) \\
= &\frac{31}{6} - \frac{17}{18} - \frac{1}{2} - \frac{1}{4}(\frac{1}{2} - \frac{4}{9}) - [\frac{17}{18} + \frac{1}{4} \cdot \frac{1}{6}]([(1 - \frac{1}{4})^{-1}] + 1) = \frac{125}{72} > 0,
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\alpha}_1 - \hat{\alpha}_2 - \hat{k}_1 - \lambda(\hat{k}_1 + \hat{\sigma}_1) - [\hat{k}_2 + \hat{\alpha}_2 + \lambda(\hat{k}_2 + \hat{\sigma}_2)]([(1 - \eta)^{-1}] + 1) \\
= &\frac{11}{2} - \frac{1}{6} - [\frac{1}{2} + \frac{1}{6} + \frac{1}{4}(\frac{1}{2} + \frac{1}{2})]([(1 - \frac{1}{4})^{-1}] + 1) = \frac{21}{6} > 0.
\end{aligned}$$

At last, we have that $k_1 + \lambda(\sigma_1 + k_1) \geqslant 0$ and $\hat{k}_1 + \lambda(\hat{\sigma}_1 + \hat{k}_1) \geqslant 0$. Thus, the conditions of Theorem 5.2 are also fulfilled, which means that the BEM numerical solution can achieve their exponential mean-square stability.

For the purpose of testing the exponential stability of BEM numerical solution, we generate 500 sample paths (green color) with stepsize $\Delta = 0.018$, which is revealed in Fig.1. We can see that their mean-square curve (red color) tends to a common value, which means the numerical solutions are exponentially mean-square stable. For a clear show of one path, we have Fig.2, which reveals the trajectory tends to be stable as the time goes on.

Then we let the step size be $\Delta = 0.058$ and $\Delta = 0.07$, we obtain Fig. 3 and Fig. 4, respectively, which show that the numerical solutions achieve their exponential mean-square stability.

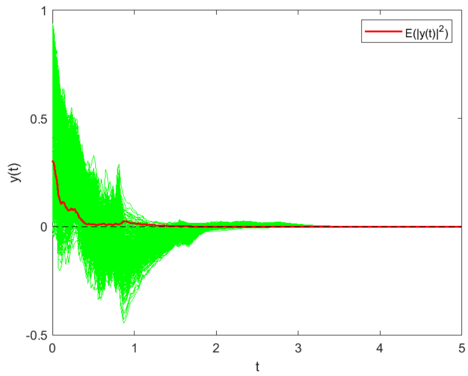

Fig. 1. Mean square stability of numerical solutions y_k with $\Delta = 0.018$.

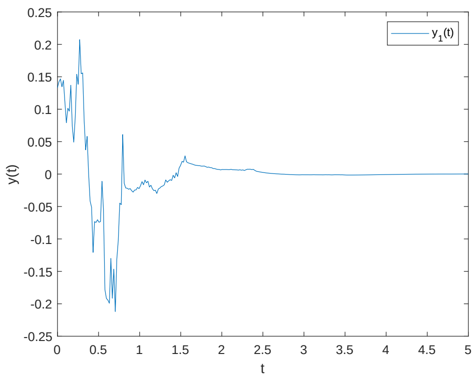

Fig. 2. A sample path of BEM numerical solution y_k with $\Delta = 0.018$.

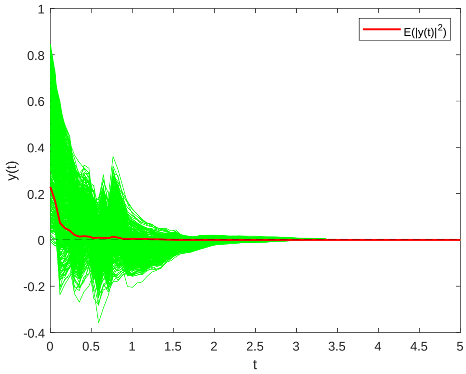

Fig. 3. Mean square stability of numerical solutions y_k with $\Delta = 0.058$.

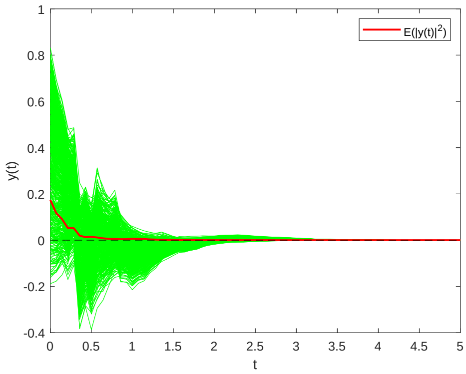

Fig. 4. Mean square stability of numerical solutions y_k with $\Delta = 0.07$.

7. Conclusion

In this paper, we proved that the analytic solution was exponential mean-square stable under some general nonlinear conditions, for NSDEs with time-varying delay and Poisson jumps. Our work reveals that the BEM method can reproduce the corresponding stability of the analytic solution without the linear growth condition, which is different from the EM method of some existing results. The exponential mean-square stability we have obtained is stronger than the a.s. exponential stability. The results generalize the stability of NSDEs with constant delay to the case of NSDEs with time-varying delay and Poisson jumps.

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