Almost periodicity in time-dependent and state-dependent delay differential equations

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Abstract

We study the response of various linear and nonlinear differential equations to different kinds of forced oscillations, specially the periodic and almost periodic oscillations. A special attention is given to differential equations with time-almost periodic type and state-dependent delays. To the best of our knowledge, there are no results in literature that address this problem.

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1 Introduction

The capability of ordinary and functional differential equations to mimic the dynamics in a periodically fluctuating environment has been the driving force for researchers to make valuable contributions to such problems. There has also been great interest in finding conditions under which periodic differential equations admit periodic solutions. Such problems have wide applications in many physical, chemical, biological, economic and ecological phenomena. This usually happens when the involved systems are strongly influenced by periodic environmental variations or external factors. Usually, the study of these phenomena require concepts that go beyond the concept of periodicity, which take into account the fact that these phenomena are not quite periodic. Therefore it is not unreasonable to consider the various parameters of systems to be changing "almost-periodically" rather than periodically with a common period.

In general, the occurrence of almost periodic oscillations in nature is actually much more common than the periodic ones. This shows the need for a mathematical theory which addresses those kind of oscillations.

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The theory of almost periodic functions was initiated between 1924 and 1926 by Danish mathematician Harald Bohr [10]. Bohr's work was preceded by the important investigations of P. Bohl and E. Esclangon [9, 18]. Subsequently, important contributions were made by A. Besicovitch, S. Bochner, J. von Neumann, V. Stepanov, and B. Levitan [6, 7, 8, 21, 28]. Recently, the existence of almost periodic solutions for differential equations has been extensively studied. In the literature, several books are devoted to almost periodic differential equations. For example, let us indicate the books of Amerio and Prouse [5], Corduneanu [14], Fink [19], Levitan and Zhikov [22] and Zaidman [31]. Recently, there has being a great interest in studying time periodic and almost periodic differential equations [3, 4, 11, 16, 24, 34].

Let us consider the following differential equation in $\mathbb{R}^n$:

$$x'(t) = G(t)x(t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

where the matrix $G(t)$ and the vector $f(t)$ are both continuous and ω-periodic for some $\omega > 0$. In [23], Massera proved that the existence of a bounded solution of Equation (1.1) on the positive real line is enough to get the existence of an ω-periodic solution. This result is known in the literature as the Massera theorem. Fixed point theory plays an important role in this kind of results.

For almost periodic equations, the situation is more complicated, since one cannot use fixed point arguments. Bohr and Neugebauer, see [19], extended Massera's theorem for Equation (1.1) to the almost periodic case when $G(t) = G$ a constant matrix. In addition, they proved what it is known in the literature as the Bohr-Neugebauer Theorem, namely, they showed that all bounded solutions of Equation (1.1) on $\mathbb{R}$ are almost periodic. We note that this result (the Bohr-Neugebauer Theorem) does not hold for the periodic case (see Remark 3.6).

Differential equations with time-dependent and state-dependent delays have its applications in ecology, biology and many other disciplines. Despite the role of such equations in describing real phenomena, we see very few works in literature in this direction. This is partly due to the problem of existence of solutions for such equations been not trivial, specially in the context of finding almost periodic solutions. This constituted a motivation for us to investigate sufficient conditions for the existence of almost periodic and pseudo almost periodic solutions for the following nonlinear differential equations:

$$x'(t) = -\alpha(t)x(t) + f(t, x(x(t)))$$

$$x'(t) = -\alpha(t)x(t) + f(t, x(t - \sigma(t)))$$

$$x'(t) = -\alpha(t)x(t) + f(t, x(t - \rho(x(t)))).$$

To the best of our knowledge, there are no results in literature that address this problem.

This work is organized as follows: In section 2, we give some preliminary results about exponential dichotomy and almost periodic type functions. In section 3, we study the nonautonomous linear differential equation $x' = A(t)x + b(t)$ both in the periodic and almost periodic case. Section 3 is devoted to the nonlinear case with nested states, time-dependent and state-dependent delay.

2 Preliminaries

Let X be a Banach space. Throughout the paper, we denote by $C(\mathbb{R}, X)$ space of continuous functions from $\mathbb{R}$ to X and by $C_b(\mathbb{R}, X)$ the space of functions $f \in C(\mathbb{R}, X)$ which are bounded on $\mathbb{R}$. We denote by $|f|_\infty$ the supremum norm of a function $f \in C_b(\mathbb{R}, X)$ defined by

$$|f|_\infty := \sup_{t\in\mathbb{R}} |f(t)|,$$

where $|.|$ is the norm on the space X. Let $T > 0$, we denote by $C_T(\mathbb{R}, X)$ the space of functions $f \in C(\mathbb{R}, X)$ which are T-periodic.

2.1 Almost periodic functions

Definition 2.1. [13] A continuous function $f : \mathbb{R} \to X$ is said to be almost periodic in Bohr's sense (or simply almost periodic) if for every $\epsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$|f(t+\tau) - f(t)| < \epsilon \quad \text{for } t \in \mathbb{R}.$$

We denote this class of functions by $AP(\mathbb{R}, X)$.

Consider the following spaces

$$PAP_0(\mathbb{R}, X) = \left\{ \varphi \in \mathcal{C}_b(\mathbb{R}, X),\ \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\varphi(s)|\, ds = 0 \right\},$$

$$\widetilde{PAP_0}(\mathbb{R}, X) = \left\{ \varphi \in \mathcal{C}(\mathbb{R}, X),\ \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\varphi(s)|\, ds = 0 \right\}.$$

Definition 2.2. [15] A function $f \in \mathcal{C}_b(\mathbb{R}, X)$ is said to be pseudo almost periodic if f is written in the form

$$f = f^{ap} + f^{e},$$

where $f^{ap} \in AP(\mathbb{R}, X)$ and $f^e \in PAP_0(\mathbb{R}, X)$. The functions f^{ap} and f^e are respectively called the almost periodic and ergodic components of f. We denote this class of functions by $PAP(\mathbb{R}, X)$.

It is easy to verify that $PAP(\mathbb{R}, X)$ is a translation invariant closed subspace of $C_b(\mathbb{R}, X)$ containing the constant functions. Furthermore,

$$PAP(\mathbb{R}, X) = AP(\mathbb{R}, X) \oplus PAP_0(\mathbb{R}, X).$$

2.2 Exponential dichotomy

Let $A(t)$ be a continuous $n \times n$ matrix and consider the following differential equation

$$x' = A(t)x, \tag{2.1}$$

and let $X(t)$ be its fundamental matrix satisfying $X(0) = I$, where I is the unit matrix.

Definition 2.3. [12] The differential equation (2.1) is said to possess an exponential dichotomy on an interval J, if there exists a projection matrix P (i.e. $P^2 = P$) and constants $k > 1$, $\alpha > 0$, such that

$$\begin{cases} \left|X(t)PX^{-1}(s)\right| \leq ke^{-\alpha(t-s)}, & \text{for } s \leq t, \text{ with } s, t \in J \\ \\ \left|X(t)(I-P)X^{-1}(s)\right| \leq ke^{-\alpha(s-t)}, & \text{for } t \leq s, \text{ with } s, t \in J. \end{cases} \tag{2.2}$$

We denote by (P, k, α) the parameters associated to this exponential dichotomy.

The differential equation (2.1) is said to have a bounded growth on an interval J if, for some fixed $h > 0$, there exists a constant $C \geq 1$ such that every solution $x(t)$ of (2.1) satisfies

$$|x(t)| \leq C\,|x(s)| \text{ for } s, t \in J \text{ and } s \leq t \leq s + h.$$

Remark 2.4. The equation (2.1) has bounded growth if and only if there exist real constants K, α such that its fundamental matrix $X(t)$ satisfies

$$\left|X(t)X^{-1}(s)\right| \leq Ke^{\alpha(t-s)} \text{ for } t \geq s.$$

If the coefficient matrix $A(t)$ is bounded, then equation (2.1) has a bounded growth.

Proposition 2.5. [12] *Suppose (2.1) has bounded growth. Then the homogeneous equation (2.1) has an exponential dichotomy on $\mathbb{R}$ if and only if the inhomogeneous equation*

$$y' = A(t)y + f(t)$$

has a unique bounded solution on $\mathbb{R}$ for every bounded continuous function f on $\mathbb{R}$.

Remark 2.6. When $A(t) = A$ is a constant matrix, the system (2.1) has an exponential dichotomy on an infinite interval if and only if the eigenvalues of A have a nonzero real part. When $A(t)$ is periodic, the system (2.1) has an exponential dichotomy on an infinite interval if and only if the Floquet multipliers lie off the unit circle. For more properties of exponential dichotomies, we refer to [1, 12, 25, 26].

We have the following perturbation result:

Lemma 2.7. *Assume that Equation (2.1) has a bounded growth and has an exponential dichotomy on $\mathbb{R}$ with parameters (P, k, α). If $E(t)$ is a bounded continuous $n \times n$ matrix with $|E|_\infty < \dfrac{\alpha}{2k}$, then the differential equation*

$$x' = (A(t) + E(t))\,x$$

has also an exponential dichotomy.

Proof. Let $X(t)$ be the fundamental matrix associated to Equation (2.1) and consider the function

$$G(t,s) = \begin{cases} X(t)PX^{-1}(s) & \text{if } t \geq s \\ -X(t)(I-P)X^{-1}(s) & \text{if } t < s. \end{cases}$$

Then $|G(t,s)| \leq ke^{-\alpha|t-s|}$. Let $f \in C_b(\mathbb{R}, \mathbb{R}^n)$ and consider the differential equation

$$x' = (A(t) + E(t))\, x + f(t). \tag{2.3}$$

Let $x \in C_b(\mathbb{R}, \mathbb{R}^n)$, since $f + Ex \in C_b(\mathbb{R}, \mathbb{R}^n)$ and (2.1) has an exponential dichotomy then the equation

$$y' = A(t)y + (f(t) + E(t)x(t))$$

has a unique bounded solution $y_x = Tx$ on $\mathbb{R}$ which is given by

$$(Tx)(t) = \int_{\mathbb{R}} G(t,s)(f(s) + E(s)x(s))ds.$$

Let $x_1, x_2 \in C_b(\mathbb{R}, \mathbb{R}^n)$, then

$$\begin{aligned} |(Tx_1 - Tx_2)(t)| &= \left| \int_{\mathbb{R}} G(t,s)E(s)(x_1(s) - x_2(s))ds \right| \\ &\leq k\, |E|_\infty \int_{\mathbb{R}} e^{-\alpha|t-s|} |\, |x_1(s) - x_2(s)|\, |ds. \end{aligned}$$

Thus

$$|Tx_1 - Tx_2|_\infty \leq \frac{2k\, |E|_\infty}{\alpha}\, |x_1 - x_2|_\infty .$$

Therefore by the contraction fixed point principle, Equation (2.3) has a unique bounded solution on $\mathbb{R}$. The exponential dichotomy follows then from Proposition 2.5. ■

Remark 2.8. In [12], in the general case, the author proved a variant of Lemma 2.7 using the following condition

$$|E|_\infty < \frac{\alpha}{4k^2}. \tag{2.4}$$

The condition in Lemma 2.7 improves (2.4). In fact, if we take $t = s = 0$ in (2.2) we get $|P| \leq k$ and $|I - P| \leq k$ which implies that $k \geq 1$.

Consider the following scalar differential equation

$$x'(t) = -\alpha(t)x(t). \tag{2.5}$$

Then, (2.5) has an exponential dichotomy if there exist k and $c > 0$ such that

$$e^{-\int_s^t \alpha(u)\ du} \leq ke^{-c(t-s)}, \quad \text{for all } t \geq s \tag{2.6}$$

or

$$e^{\int_s^t \alpha(u)\ du} \leq ke^{-c(t-s)}, \quad \text{for all } t \geq s. \tag{2.7}$$

Proposition 2.9. [2] *Let $\alpha \in C_b(\mathbb{R}, \mathbb{R})$. If there exists $r_0 > 0$ such that*

$$\inf_{t \in \mathbb{R}} \int_{t-r_0}^{t} \alpha(\xi)\ d\xi > 0,$$

then, Equation (2.5) has an exponential dichotomy.

3 The equation $x' = A(t)x + b(t)$

In this section, $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$ and $M_n(\mathbb{K})$ the space of $\mathbb{K}$-valued $n \times n$ matrices.

3.1 The periodic case

Let $T > 0$, $A \in C_T(\mathbb{R}, M_n(\mathbb{K}))$ and $b \in C_T(\mathbb{R}, \mathbb{K}^n)$. Consider the following differential equation

$$x' = A(t)x \tag{3.1}$$

and its associated inhomogeneous equation

$$x' = A(t)x + b(t). \tag{3.2}$$

The central question of this subsection is to investigate under what conditions does (3.2) have a T-periodic solution.

Lemma 3.1. *Let x be a solution of* (3.2)*. Then x is T-periodic if and only if $x(0) = x(T)$.*

Proof. The first implication is trivial. For the second implication, assume that x is a solution of (3.2) such that $x(0) = x(T)$. Then x and $\widetilde{x} : t \mapsto x(t+T)$ are two solutions of (3.2) having the same initial condition ($x(0) = \widetilde{x}(0)$). It follows by uniqueness of solutions that $x(t) = \widetilde{x}(t) = x(t+T)$ for all $t \in \mathbb{R}$. ■

Lemma 3.2. *The following properties are equivalent:*

(i) The zero solution is the unique T-periodic solution of (3.1)*.*

(ii) Equation (3.2) *has a unique T-periodic solution for each $b \in C_T(\mathbb{R}, \mathbb{K}^n)$.*

Proof. The implication (ii)⇒(i) is trivial. Let us prove the implication (i)⇒(ii). Assume that the zero solution is the unique T-periodic solution of (3.1). Let S_H be the n-dimensional space of solutions of (3.1) and $f : S_H \to \mathbb{K}^n$ be the linear operator defined for each $x \in S_H$ by

$$f(x) := x(T) - x(0).$$

It follows from Lemma 3.1 that $\ker f = \{0\}$. Since $\dim S_H = \dim \mathbb{K}^n$, we deduce that f is an isomorphism. Let x_P be a particular solution of (3.2). Thus there exists a unique solution x_H of (3.1) satisfying $f(x_H) = x_P(0) - x_P(T)$. Consider the function $x := x_P + x_H$. Then x is a solution of (3.2). Moreover x is T-periodic since

$$\begin{aligned} x(0) &= x_P(0) + x_H(0) \\ &= f(x_H) + x_P(T) + x_H(0) \\ &= x_H(T) + x_P(T) \\ &= x(T). \end{aligned}$$

Let $y := x_p + y_H$ be another T-periodic solution of (3.2). Then $y(0) = y(T)$ which implies that $f(y_H) = x_P(0) - x_P(T)$. Therefore by uniqueness of x_H we have $x_H = y_H$ and thus $x = y$. ■

Let A^* be the conjugate transpose of A. In what follows, we denote by $H(t)$ the matrix defined by

$$H(t) := A(t) + A^*(t)$$

and $\sigma(H(t))$ the set of its eigenvalues.

Lemma 3.3. *Assume that* $\bigcup_{t\in\mathbb{R}} \sigma(H(t)) \subset \mathbb{R}^+$ *(or* $\mathbb{R}^-$*). Let* x *be a* T*-periodic solution of* (3.1)*. Then* $x(t) \in \ker H(t)$ *for all* $t \in \mathbb{R}$.

Proof. Let x be a T-periodic solution of (3.1) and set $y(t) = |x(t)|^2$. Then

$$y' = \langle Ax, x\rangle + \langle x, Ax\rangle = \langle Hx, x\rangle.$$

It follows that y is monotone and T-periodic, so it is constant, that is, $y' = 0$ hence $\langle Hx, x\rangle = 0$. Since the eigenvalues of $H(t)$ have the same sign, we conclude that $H(t)x(t) = 0$. ■

Proposition 3.4. *Assume that* $\bigcup_{t\in\mathbb{R}} \sigma(H(t)) \subset \mathbb{R}^+$ *(or* $\mathbb{R}^-$*) and there exists* $t_0 \in \mathbb{R}$ *such that* $\ker H(t_0) = \{0\}$*. Then, for all* $b \in C_T(\mathbb{R}, \mathbb{K}^n)$ *Equation* (3.2) *has a unique* T*-periodic solution.*

Proof. By Lemma 3.2), it is sufficient to prove that the zero solution is the unique T-periodic solution of (3.1). Let x be a solution of (3.1). Using Lemma 3.3, we have $x(t) \in \ker H(t)$ for all $t \in \mathbb{R}$, in particular $x(t_0) \in \ker H(t_0) = \{0\}$. Thus $x(t_0) = 0$ and by the uniqueness of solution of the Cauchy problem

$$\begin{cases} x'(t) = A(t)x(t) \\ x(t_0) = 0 \end{cases}$$

we deduce that $x = 0$. ■

Proposition 3.5. *Assume that* $A(t)$ *is Hermitian positive (or Hermitian negative) for all* $t \in \mathbb{R}$*. Then, the following properties are equivalent:*

(i) Equation (3.2) *has a unique* T*-periodic solution for each* $b \in C_T(\mathbb{R}, \mathbb{K}^n)$.

(ii) $\bigcap_{t\in\mathbb{R}} \ker A(t) = \{0\}$.

Proof. (i)⇒(ii) Let $x_0 \in \bigcap_{t\in\mathbb{R}} \ker A(t)$, then the constant function $x(t) = x_0$ is a T-periodic solution of (3.1). But since the zero solution is the unique T-periodic solution of (3.1) (Lemma 3.2), then we have $x_0 = 0$. Therefore $\bigcap_{t\in\mathbb{R}} \ker A(t) = \{0\}$.
(ii)⇒(i) Using again Lemma 3.2), it is sufficient to prove that the zero solution is the unique T-periodic solution of (3.1). Let x be a T-periodic solution of (3.1). We have $A = A^*$, thus $H(t) = 2A(t)$ and $\bigcup_{t\in\mathbb{R}} \sigma(H(t)) \subset \mathbb{R}^+$ (or $\mathbb{R}^-$). It follows by Lemma 3.3 that

$$x(t) \in \ker H(t) = \ker 2A(t) = \ker A(t).$$

Hence $x'(t) = A(t)x(t) = 0$ and x is then constant, which implies that $x(0) = x(t) \in \ker A(t)$. Since $t \in \mathbb{R}$ is arbitrary, we deduce that $x(0) \in \bigcap_{t\in\mathbb{R}} \ker A(t) = \{0\}$ and thus $x = 0$. ■

Example. Let b be a 2π-periodic function and consider the following differential equation

$$x' = A(t)x + b(t), \tag{3.3}$$

where $A(t) = \begin{pmatrix} \cos^2 t & -\sin t \\ \sin t & 2\cos^2 t \end{pmatrix}$. Then (3.3) has a unique 2π-periodic solution..

Remark 3.6. In general, if Equation (3.3) has a periodic solution, the other bounded solutions on $\mathbb{R}$ are not necessarily periodic. This happens even in the autonomous case

$$x'(t) = Ax(t). \tag{3.4}$$

In fact, consider the following matrix

$$A = \begin{pmatrix} 0 & -3 & 1 & -3 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 1 & \frac{11}{9} & 0 \end{pmatrix}.$$

The trivial solution of (3.4) is of course periodic. However, the following solution

$$x(t) = e^{tA} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4\cos t + 5\cos\sqrt{2}t \\ -\sin t + \sqrt{2}\sin\sqrt{2}t \\ 3\cos t - 3\cos\sqrt{2}t \\ \cos t - \cos\sqrt{2}t - \frac{1}{3}\sin t + \frac{2\sqrt{2}}{3}\sin\sqrt{2}t \end{pmatrix}$$

is bounded on $\mathbb{R}$ but is not periodic. It belongs however to a larger class which is the class of almost periodic functions. In fact, all solutions of (3.4) are almost periodic. This is known in the literature as the Bohr-Neugebauer property [16, 17, 19, 27, 29, 30].

3.2 The almost periodic case

In this subsection, we assume that $A \in C(\mathbb{R}, M_n(\mathbb{R}))$ and $b \in C(\mathbb{R}, \mathbb{R}^n)$. Consider the following differential equation

$$x' = A(t)x + b(t). \tag{3.5}$$

Proposition 3.7. [1, 33] *Let $A \in AP(\mathbb{R}, M_n(\mathbb{R}))$ and $b \in AP(\mathbb{R}, \mathbb{R}^n)$ such that $x' = A(t)x$ has an exponential dichotomy. Then the differential equation (3.5) has a unique bounded solution which is in $AP(\mathbb{R}, \mathbb{R}^n)$.*

Remark 3.8. Without the exponential dichotomy, the existence of an almost periodic solution to the differential equation (3.5) can fail even if a bounded solution exists [20].

Proposition 3.9. [1, 33] *Let $A \in C(\mathbb{R}, M_n(\mathbb{R}))$ and $b \in PAP_0(\mathbb{R}, \mathbb{R}^n)$ such that $x' = A(t)x$ has an exponential dichotomy. Then the differential equation (3.5) has a unique bounded solution which is in $PAP_0(\mathbb{R}, \mathbb{R}^n)$.*

Proposition 3.10. *Let $A \in PAP(\mathbb{R}, M_n(\mathbb{R}))$ and $b \in PAP(\mathbb{R}, \mathbb{R}^n)$ such that both $x' = A^{ap}(t)x$ and $x' = A(t)x$ have an exponential dichotomy. Then the differential equation (3.5) has a unique bounded solution which is in $PAP(\mathbb{R}, \mathbb{R}^n)$.*

Proof. By Proposition 3.7 the equation

$$x' = A^{ap}(t)x + b^{ap}(t)$$

has a unique bounded solution $\varphi_1 \in AP(\mathbb{R}, \mathbb{R}^n)$.
On the other hand, since $A^e(t)\varphi_1 + b^e(t) \in PAP_0(\mathbb{R}, \mathbb{R}^n)$ then by Proposition 3.9 the equation

$$x' = A(t)x + (A^e(t)\varphi_1 + b^e(t))$$

has a unique bounded solution $\varphi_2 \in PAP_0(\mathbb{R}, \mathbb{R}^n)$. Let $\varphi := \varphi_1 + \varphi_2$, then

$$\begin{aligned}\varphi' &= A^{ap}(t)\varphi_1 + b^{ap}(t) + A(t)\varphi_2 + (A^e(t)\varphi_1 + b^e(t))\\ &= A(t)\varphi_1 + A(t)\varphi_2 + b(t)\\ &= A(t)\varphi + b(t).\end{aligned}$$

That is, φ is a solution of (3.5) which is in $PAP(\mathbb{R}, \mathbb{R}^n)$. The solution φ is the unique bounded solution of (3.5) on $\mathbb{R}$ because $x' = A(t)x$ has an exponential dichotomy (Proposition (2.5)). ■

Corollary 3.11. *Let $A \in PAP(\mathbb{R}, M_n(\mathbb{R}))$ and $b \in PAP(\mathbb{R}, \mathbb{R}^n)$ such that $x' = A^{ap}(t)x$ has an exponential dichotomy with parameters (P, k, α) and $\sup_{t \in \mathbb{R}} |A^e(t)| < \dfrac{\alpha}{2k}$. Then the differential equation (3.5) has a unique bounded solution which is in $PAP(\mathbb{R}, \mathbb{R}^n)$.*

Proof. The proof is a direct consequence of Proposition 3.10 and the perturbation result in Lemma 2.7. ■

4 The nonlinear case

4.1 Differential equations with nested states

Consider the following scalar differential equation

$$x'(t) = -\alpha(t)x(t) + f\left(t, x\left(x(t)\right)\right), \tag{4.1}$$

where $\alpha \in C(\mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Consider the following hypotheses:

(H1) $\alpha \in PAP(\mathbb{R}, \mathbb{R})$ with

$$c := \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du} ds < \infty. \tag{4.2}$$

(H2) $f(., x) \in PAP(\mathbb{R}, \mathbb{R})$ for all $x \in \mathbb{R}$.

(H3) There exists $L_f > 0$ such that for all $t, x, y \in \mathbb{R}$

$$|f(t, x) - f(t, y)| \leq L_f |x - y|.$$

Remark 4.1. If $\inf\limits_{t \in \mathbb{R}} \alpha(t) > 0$, then Condition (4.2) is satisfied.

Lemma 4.2. *Assume that* **(H1)** *holds. Then for all* $v \in PAP(\mathbb{R}, \mathbb{R})$*, the equation*

$$x' = -\alpha(t)x + v(t) \tag{4.3}$$

has a unique bounded solution on $\mathbb{R}$*. Furthermore this solution is in* $PAP(\mathbb{R}, \mathbb{R})$ *and is given by the formula*

$$x_v(t) = \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du} v(s) ds.$$

Proof. Let $v \in PAP(\mathbb{R}, \mathbb{R})$ and consider the function x_v defined by

$$x_v(t) = \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du} v(s) ds.$$

Since v is bounded, it is clear from **(H1)** that x_v is a bounded solution of (4.3) on $\mathbb{R}$. Let y_v be another bounded solution of (4.3) on $\mathbb{R}$. Then $x_v - y_v$ is a solution of the equation

$$x'(t) + \alpha(t)x(t) = 0.$$

That is

$$x_v(t) - y_v(t) = e^{-\int_0^t \alpha(u)du} \left(x_v(0) - y_v(0)\right).$$

By **(H1)**, the function $t \mapsto e^{\int_0^t \alpha(u)du}$ is integrable on $(-\infty, 0)$. This implies that $t \mapsto e^{-\int_0^t \alpha(u)du}$ cannot be bounded. In fact, if it is bounded then there exists $M > 0$ such that $e^{-\int_0^t \alpha(u)du} \leq M$, that is $e^{\int_0^t \alpha(u)du} \geq \frac{1}{M} > 0$ which contradicts the integrability of $t \mapsto e^{\int_0^t \alpha(u)du}$ on $(-\infty, 0)$. Therefore $t \mapsto e^{-\int_0^t \alpha(u)du}$ must not be bounded. Now, since $x_v(t) - y_v(t)$ is bounded, one must have

$x_v(0) - y_v(0) = 0$. Therefore x_v is the only bounded solution of (4.3) on $\mathbb{R}$. This implies that the equation $x'(t) = -\alpha(t)x(t)$ has an exponential dichotomy (Proposition 2.5). That is $\alpha(t)$ satisfies (2.6) (since it cannot satisfy both (2.7) and (4.2)). It follows from [2, Lemma 7.4] that the equation $x'(t) = -\alpha^{ap}(t)x(t)$ has also an exponential dichotomy. Hence we deduce from Proposition 3.10 that $x_v \in PAP(\mathbb{R}, \mathbb{R})$. ∎

Theorem 4.3. *Assume that* **(H1)-(H3)** *hold. Then Equation (4.1) has a unique pseudo almost periodic solution, provided that*

$$\sqrt{|f(\cdot, 0)|\, cL_f\, (c\,|\alpha|_\infty + 1)} < 1 - cL_f \tag{4.4}$$

where $|f(\cdot, 0)| := \sup_{t \in \mathbb{R}} |f(t, 0)|$.

Remark 4.4. Provided that $cL_f < 1$, one can see that (4.4) is equivalent to

$$|f(\cdot, 0)| < \frac{(1 - cL_f)^2}{cL_f\, (c\,|\alpha|_\infty + 1)}.$$

Thus (4.4) can be met if L_f and $|f(\cdot, 0)|$ are small enough.

In order to prove Theorem 4.3 one needs the following lemmas.

Lemma 4.5. *Let* $\varphi \in PAP(\mathbb{R}, \mathbb{R})$ *and* $\Phi : \mathbb{R} \to \mathbb{R}$ *be a continuous function. Then* $\Phi \circ \varphi \in PAP(\mathbb{R}, \mathbb{R})$.

Proof. The proof is similar to the one in [32], but here we do not need to assume Φ to be uniformly continuous. In fact, Φ is continuous on the range of φ which is a bounded subset of $\mathbb{R}$, thus by the Stone-Weierstrass theorem, Φ can be approximated uniformly on the range of φ by a sequence of polynomials Φ_n. Since the functions $\Phi_n \circ \varphi$ are in $PAP(\mathbb{R}, \mathbb{R})$, then their uniform limit $\Phi \circ \varphi$ is also in $PAP(\mathbb{R}, \mathbb{R})$. ∎

Lemma 4.6. [33] *Let* $\varphi \in PAP(\mathbb{R}, \mathbb{R})$ *and* $f : \mathbb{R}^2 \to \mathbb{R}$ *satisfy* **(H2)** *and* **(H3)**. Then $t \mapsto f(t, \varphi(t)) \in PAP(\mathbb{R}, \mathbb{R})$.

Proof of Theorem 4.3. Let Λ be the subset of $PAP(\mathbb{R}, \mathbb{R})$ defined by

$$\Lambda := \Big\{ x \in PAP(\mathbb{R}, \mathbb{R}) : x \text{ is } k\text{-Lipschitz and } |x|_\infty \leq m \Big\},$$

where $m = \dfrac{c\,|f(\cdot, 0)|}{1 - cL_f}$ and $k = \left(|\alpha|_\infty + \dfrac{1}{c} \right) m$. Remark that (4.4) implies that $cL_f < 1$ and thus $m \geq 0$. Hence Λ is not empty. Notice that a solution of Equation (4.1) which is bounded on $\mathbb{R}$ is necessarily k-Lipschitz and satisfies $|x|_\infty \leq m$. In fact, let x be such a solution. Then for all $t \geq \sigma$

$$x(t) = e^{-\int_\sigma^t \alpha(u)du} x(\sigma) + \int_\sigma^t e^{-\int_s^t \alpha(u)du} f(s, x(x(s)))ds. \tag{4.5}$$

From the proof of Lemma 4.2 one can see that $\alpha(t)$ satisfies (2.6), and since $x(t)$ is bounded on $\mathbb{R}$ then

$\lim_{\sigma \to -\infty} e^{-\int_\sigma^t \alpha(u)du} x(\sigma) = 0$. It follows by taking $\sigma \to -\infty$ in (4.5) that

$$x(t) = \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du} f(s, x(x(s)))ds.$$

Thus for all $t \in \mathbb{R}$ we have

$$|x(t)| \quad \leq \quad c \left(L_f \, |x|_\infty + |f(\cdot, 0)| \right).$$

That is

$$|x|_\infty \leq \frac{c\, |f(\cdot, 0)|}{1 - cL_f} = m.$$

Moreover since x satisfies Equation (4.1), we have

$$\begin{aligned} |x'(t)| &\leq |\alpha|_\infty \, |x|_\infty + L_f \, |x|_\infty + |f(t, 0)| \\ &\leq |\alpha|_\infty \, m + L_f m + |f(., 0)| \\ &= |\alpha|_\infty \, m + \frac{m}{c} = k. \end{aligned}$$

Let $PAP^u(\mathbb{R}, \mathbb{R})$ be the space of uniformly continuous pseudo almost periodic functions. Consider the operator $P : PAP^u(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ defined by

$$P(x)(t) = \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du} f(s, x(x(s)))ds \text{ for } t \in \mathbb{R}.$$

Using Lemma 4.2, Lemma 4.5 and Lemma 4.6, it is clear that P maps $PAP^u(\mathbb{R}, \mathbb{R})$ into itself. We claim that the operator P maps Λ into itself. In fact, for $x \in \Lambda$ and $t \in \mathbb{R}$ we have

$$|(Px)(t)| \quad \leq \quad c \left(L_f m + |f(\cdot, 0)| \right) = m.$$

We now verify that Px is k-Lipschitz. Since Px satisfies the differential equation

$$(Px)'(t) = -\alpha(t) Px(t) + f(t, x(x(t))),$$

then

$$\begin{aligned} \left| (Px)'(t) \right| &\leq |\alpha|_\infty \, m + L_f m + |f(t, 0)| \\ &= |\alpha|_\infty \, m + \frac{m}{c} \\ &= k. \end{aligned}$$

This means that $Px \in \Lambda$. Now it suffices to prove that P is a contraction on Λ. We have for $x, y \in \Lambda$

$$\begin{aligned} |P(x)(t) - P(y)(t)| &\leq \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du} |f(s, x(x(s))) - f(s, y(y(s)))| \, ds \\ &\leq L_f \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du} |x(x(s))) - y(y(s))| \, ds. \end{aligned}$$

But since

$$\begin{aligned} |x(x(s))) - y(y(s)))| &\leq |x(x(s)) - x(y(s))| + |x(y(s)) - y(y(s))| \\ &\leq (k+1) |x - y|_\infty , \end{aligned}$$

then

$$\begin{aligned} |P(x)(t) - P(y)(t)| &\leq cL_f (k+1) |x - y|_\infty \\ &= cL_f \left(\left(|\alpha|_\infty + \frac{1}{c} \right) \frac{c |f(\cdot, 0)|}{1 - cL_f} + 1 \right) |x - y|_\infty . \end{aligned}$$

Using the contraction principle on the closed subset Λ, we deduce that Equation (4.1) has a unique pseudo almost periodic solution. ■

Example. Consider the following differential equation

$$x'(t) = -\alpha(t)x(t) + \beta(t)\sin(x(x(t))) + \gamma(t) \quad \text{for } t \in \mathbb{R}, \tag{4.6}$$

where

$$\begin{aligned} \alpha(t) &= a + \sin(t) + \sin\left(\sqrt{2}t\right) + \frac{1}{1+t^2} \\ \beta(t) &= b + \sin(t) + \sin\left(\sqrt{2}t\right) + \frac{1}{1+t^2} \\ \gamma(t) &= \varepsilon \left(1 + \sin(t) + \sin\left(\sqrt{2}t\right) + \frac{1}{1+t^2} \right) \end{aligned}$$

with $b \geq 0$ and $a > b + 5$. We have $a - 2 \leq \alpha(t) \leq a + 3$, thus $\frac{1}{a+3} \leq c \leq \frac{1}{a-2}$. Set $f(t, x) = \beta(t)\sin(x) + \gamma(t)$, then $L_f \leq |\beta|_\infty \leq b + 3$. Therefore, the condition $cL_f < 1$ is met. Moreover since $|f(\cdot, 0)| = |\gamma|_\infty \leq 4\varepsilon$, then to guarantee (4.4) one can choose ε such that

$$\varepsilon < \frac{(a - b - 5)^2}{4(b+3)\,(2a+1)}.$$

Since $\alpha, \beta, \gamma \in PAP(\mathbb{R}, \mathbb{R})$, then Equation (4.6) has a unique pseudo almost periodic solution.

4.2 Differential equations with time-dependent delay

In what follows, we consider the following time-dependent delay differential equation

$$x'(t) = -\alpha(t)x(t) + f(t, x(t - \sigma(t))).$$

In order to prove an existence theorem for such an equation, we need to prove some composition results of pseudo almost periodic functions.

Lemma 4.7. *Let $y(.) \in AP(\mathbb{R},\mathbb{R})$ and $\sigma(.) \in AP(\mathbb{R},\mathbb{R})$. Then $t \mapsto y(t-\sigma(t)) \in AP(\mathbb{R},\mathbb{R})$.*

Proof. Let $(s_n)_n$ be a sequence of real numbers. Then there exist a subsequence $(s'_n)_n \subset (s_n)_n$, a function $\widetilde{y} : \mathbb{R} \to \mathbb{R}$ and a function $\widetilde{\sigma} : \mathbb{R} \to \mathbb{R}$ such that

$$y(t + s'_n) \to \widetilde{y}(t),$$

$$\widetilde{y}(t - s'_n) \to y(t),$$

$$\sigma(t + s'_n) \to \widetilde{\sigma}(t),$$

and

$$\widetilde{\sigma}(t - s'_n) \to \sigma(t),$$

as $n \to \infty$, where all the above convergences hold uniformly on $\mathbb{R}$. It follows that

$$\begin{aligned} |y((t+s'_n) - \sigma(t+s'_n)) - \widetilde{y}(t-\widetilde{\sigma}(t))| &\le |y(t - \sigma(t+s'_n) + s'_n) - \widetilde{y}(t - \sigma(t+s'_n))| \\ &\quad + |\widetilde{y}(t - \sigma(t+s'_n)) - \widetilde{y}(t - \widetilde{\sigma}(t))|, \\ &\le \sup_{s\in\mathbb{R}} |y(s+s'_n) - \widetilde{y}(s)| + |\widetilde{y}(t - \sigma(t+s'_n)) - \widetilde{y}(t-\widetilde{\sigma}(t))|. \end{aligned}$$

Now since $\widetilde{y}$ is uniformly continuous (it is in fact even almost periodic) we deduce that

$$\sup_{t\in\mathbb{R}} |y((t+s'_n) - \sigma(t+s'_n)) - \widetilde{y}(t-\widetilde{\sigma}(t))| \to 0$$

as $n \to \infty$. This implies the almost periodicity of the function $t \mapsto y(t - \sigma(t))$. ■

Lemma 4.8. *Let $\sigma \in \widetilde{PAP_0}(\mathbb{R})$ and $\varphi \in PAP_0(\mathbb{R})$ such that φ is uniformly continuous. Then $t \mapsto \varphi(t - \sigma(t))$ is also in $PAP_0(\mathbb{R})$.*

Proof. Let $\varepsilon > 0$, then there exists $\eta > 0$ such that if $|u| \le \eta$, then for all $x \in \mathbb{R}$

$$|\varphi(x+u) - \varphi(x)| \le \varepsilon.$$

Let

$$A_r = \{t \in [-r, r] : \ |\sigma(t)| > \eta\}.$$

Since $\sigma \in PAP_0(\mathbb{R})$, then $\lim_{r\to\infty} \frac{1}{2r}\mu(A_r) = 0$. Let $r_0 > 0$ be such that for all $r \ge r_0$, $\frac{1}{2r}\mu(A_r) \le \varepsilon$.

Then for $r \geq r_0$ one has

$$\begin{aligned}
\frac{1}{2r}\int_{-r}^{r}|\varphi(t-\sigma(t))-\varphi(t)|\,dt &\leq \frac{1}{2r}\int_{A_r}|\varphi(t-\sigma(t))-\varphi(t)|\,dt+\frac{1}{2r}\int_{[-r,r]\smallsetminus A_r}|\varphi(t-\sigma(t))-\varphi(t)|\,dt \\
&\leq 2\,|\varphi|_\infty\,\frac{1}{2r}\mu\left(A_r\right)+\frac{1}{2r}\int_{[-r,r]\smallsetminus A_r}\varepsilon dt \\
&\leq 2\,|\varphi|_\infty\,\frac{1}{2r}\mu\left(A_r\right)+\varepsilon.
\end{aligned}$$

Hence

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\varphi(t-\sigma(t))-\varphi(t)|\,dt=0.$$

But since $\varphi \in PAP_0\left(\mathbb{R}\right)$, then $t \mapsto \varphi(t-\sigma(t)) \in PAP_0\left(\mathbb{R}\right)$. $\blacksquare$

In Lemma 4.8, we assumed that $\sigma \in \widetilde{PAP_0}\left(\mathbb{R}\right)$, that is the delay σ is not necessarily bounded. If we assume that σ is bounded then we do not need σ to be in $PAP_0\left(\mathbb{R}\right)$ as the following lemma shows:

Lemma 4.9. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and $\varphi \in PAP_0\left(\mathbb{R}\right)$ such that φ is uniformly continuous. Then, $t \mapsto \varphi(t-\sigma(t))$ is also in $PAP_0\left(\mathbb{R}\right)$.*

Proof. For any fixed $\varepsilon > 0$, let $\delta > 0$ be such that

$$|\varphi\left(t\right)-\varphi\left(s\right)|<\varepsilon,$$

for all $t, s \in \mathbb{R}$ with $|t-s| \leq \delta$.
Let a, b be such that $a \leq \sigma(t) \leq b$ for all $t \in \mathbb{R}$. Then, there exist $\sigma_1, \ldots, \sigma_k \in [a,b]$ such that

$$[a,b]\subset\bigcup_{i=1}^{k}\left[\sigma_i-\delta,\sigma_i+\delta\right].$$

For each $t \in \mathbb{R}$, there exists $\sigma_{i(t)}$, $1 \leq i\left(t\right) \leq k$ such that $\left|\sigma(t)-\sigma_{i(t)}\right| \leq \delta$. Then, we get

$$\begin{aligned}
|\varphi(t-\sigma(t))| &\leq \left|\varphi(t-\sigma(t))-\varphi(t-\sigma_{i(t)})\right|+\left|\varphi(t-\sigma_{i(t)})\right| \\
&\leq \varepsilon+\sum_{i=1}^{k}|\varphi(t-\sigma_i)|\,.
\end{aligned}$$

Which gives

$$\frac{1}{2r}\int_{-r}^{r}|\varphi(t-\sigma(t))|\,dt \quad\leq\quad \varepsilon+\frac{1}{2r}\sum_{i=1}^{k}\int_{-r}^{r}|\varphi(t-\sigma_i)|\,dt.$$

Since $PAP_0(\mathbb{R})$ is translation invariant, we have $\varphi(t-\sigma_i) \in PAP_0(\mathbb{R})$, $i = 1 \ldots k$. Therefore

$$\limsup_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\varphi(t-\sigma(t))|\,dt\leq\varepsilon,$$

where ε is arbitrary. We deduce that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\varphi(t-\sigma(t))|\,dt=0.$$

That is $t\mapsto\varphi(t-\sigma(t))\in PAP_0(\mathbb{R})$. ■

Remark 4.10. Lemma 4.9 gives a property which is stronger than the well known translation invariance enjoyed by the space $PAP_0(\mathbb{R})$. It shows that ergodicity in not affected even by varying translations, as long as those translations are bounded and the function is uniformly continuous.

Lemma 4.11. *Let $y(.)\in PAP(\mathbb{R},\mathbb{R})$ be such that y is uniformly continuous and $\sigma(.)\in PAP(\mathbb{R},\mathbb{R})$. Then $t\mapsto y(t-\sigma(t))\in PAP(\mathbb{R},\mathbb{R})$.*

Proof. Let

$$\begin{aligned}y(t)&=\underbrace{y_1(t)}_{\in AP(\mathbb{R})}+\underbrace{y_2(t)}_{\in PAP_0(\mathbb{R})}\\ \sigma(t)&=\underbrace{\sigma_1(t)}_{\in AP(\mathbb{R})}+\underbrace{\sigma_2(t)}_{\in PAP_0(\mathbb{R})}\end{aligned}$$

Consider the following decomposition

$$\begin{aligned}y(t-\sigma(t))&=y_1(t-\sigma_1(t))+y(t-\sigma(t))-y_1(t-\sigma_1(t))\\&=\underbrace{y_1(t-\sigma_1(t))}_{\in AP(\mathbb{R})}+\underbrace{y(t-\sigma(t))-y(t-\sigma_1(t))}_{\in PAP_0(\mathbb{R})}+\underbrace{y_2(t-\sigma_1(t))}_{\in PAP_0(\mathbb{R})}\end{aligned}$$

By Lemma 4.7, we have $y_1(t-\sigma_1(t))\in AP(\mathbb{R},\mathbb{R})$. In the other hand, since y_1 and y are uniformly continuous, then y_2 is also uniformly continuous. It follows by Lemma 4.9 that $y_2(t-\sigma_1(t))\in PAP_0(\mathbb{R})$. For any fixed $\varepsilon>0$, let $\delta>0$ be such that

$$|y\,(t)-y\,(s)|<\varepsilon,$$

for all $t,s\in\mathbb{R}$ with $|t-s|\leq\delta$. Let

$$M_{r,\delta}\,(\sigma_2):=\{t\in[-r,r]:\ |\sigma_2(t)|\geq\delta\}$$

and

$$Y(t):=y(t-\sigma(t))-y(t-\sigma_1(t)).$$

Then we have

$$\begin{aligned}\frac{1}{2r}\int_{-r}^{r}|Y(t)|\,dt&=\frac{1}{2r}\int_{M_{r,\delta}(\sigma_2)}|Y(t)|\,dt+\frac{1}{2r}\int_{[-r,r]\setminus M_{r,\delta}(\sigma_2)}|Y(t)|\,dt\\&\leq\frac{|Y|_\infty}{2r}\mu\,(M_{r,\delta}\,(\sigma_2))+\varepsilon.\end{aligned}$$

Therefore

$$\limsup_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|Y(t)|\,dt\le\varepsilon,$$

where ε is arbitrary. We deduce that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|Y(t)|\,dt=0,$$

that is $Y(t)=y(t-\sigma(t))-y(t-\sigma_1(t))\in PAP_0(\mathbb{R})$. ■

We return to the aim of this subsection which is the investigation of the following equation:

$$x'(t)=-\alpha(t)x(t)+f(t,x(t-\sigma(t)))\ \text{ for } t\in\mathbb{R}. \tag{4.7}$$

Consider the following hypothesis:

(H4) $\sigma\in PAP(\mathbb{R},\mathbb{R})$.

Theorem 4.12. *Assume that* **(H1)-(H4)** *hold. Then Equation (4.7) has a unique pseudo almost periodic solution, provided that* $cL_f<1$.

Proof. Consider the operator $P: PAP^u(\mathbb{R},\mathbb{R})\to C(\mathbb{R},\mathbb{R})$ defined by

$$P(x)(t)=\int_{-\infty}^{t}e^{-\int_s^t\alpha(u)du}f(s,x(s-\sigma(s)))ds \text{ for } t\in\mathbb{R}.$$

From Lemma 4.2, Lemma 4.6 and Lemma 4.11, it is clear that P maps $PAP^u(\mathbb{R},\mathbb{R})$ into itself. For $x,y\in PAP^u(\mathbb{R},\mathbb{R})$, we have

$$\begin{aligned}|P(x)(t)-P(y)(t)| &\le \int_{-\infty}^{t}e^{-\int_s^t\alpha(u)du}\left|f(s,x(s-\sigma(s)))-f(s,y(s-\sigma(s)))\right|ds\\ &\le cL_f|x-y|.\end{aligned}$$

Using the contraction principle on the Banach space $PAP^u(\mathbb{R},\mathbb{R})$, we deduce that Equation (4.7) has a unique solution in $PAP^u(\mathbb{R},\mathbb{R})$. ■

4.3 Differential equations with state-dependent delay

Consider the following state-dependent delay differential equation

$$x'(t)=-\alpha(t)x(t)+f(t,x(t-\rho(x(t)))). \tag{4.8}$$

where $\rho:\mathbb{R}\to\mathbb{R}^+$ is L_ρ-Lipschitz.

Theorem 4.13. *Assume that* **(H1)-(H3)** *hold. Then Equation (4.8) has a unique pseudo almost periodic solution, provided that*

$$\sqrt{|f(\cdot,0)|\,cL_fL_\rho\left(c\,|\alpha|_\infty+1\right)}<1-cL_f. \tag{4.9}$$

Proof. Consider the operator $P : PAP^u(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ defined by

$$P(x)(t) = \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du} f(s, x(s-\rho(x(s))))ds \text{ for } t \in \mathbb{R}.$$

Using Lemma 4.2, Lemma 4.6 and Lemma 4.5, it is clear that P maps $PAP^u(\mathbb{R},\mathbb{R})$ into itself.
Let Λ be the subset of $PAP(\mathbb{R},\mathbb{R})$ defined by

$$\Lambda := \Big\{ x \in PAP(\mathbb{R},\mathbb{R}) : x \text{ is } k\text{-Lipschitz} \text{ and } |x| \leq m \Big\},$$

where $m = \dfrac{c\,|f(\cdot,0)|}{1-cL_f}$ and $k = \left(|\alpha|_\infty + \frac{1}{c}\right) m$. Using the same argument as in the proof of Theorem (4.3), one can prove that the operator P maps Λ into itself. It suffices then to prove that P is a contraction on Λ. We have For $x, y \in \Lambda$

$$\begin{aligned} |P(x)(t) - P(y)(t)| &\leq \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du} \left| f(s, x(s-\rho(x(s)))) - f(s, y(s-\rho(y(s)))) \right| ds \\ &\leq L_f \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du} \left| x(s-\rho(x(s))) - y(s-\rho(y(s))) \right| ds. \end{aligned}$$

But since

$$\begin{aligned} |x(s-\rho(x(s))) - y(s-\rho(y(s)))| &\leq |x(s-\rho(x(s))) - x(s-\rho(y(s)))| + |x(s-\rho(y(s))) - y(s-\rho(y(s)))| \\ &\leq k\,|\rho(x(s)) - \rho(y(s))| + |x-y|_\infty \\ &\leq (kL_\rho + 1)\,|x-y|_\infty\,, \end{aligned}$$

then

$$\begin{aligned} |P(x)(t) - P(y)(t)| &\leq cL_f\,(kL_\rho+1)\,|x-y|_\infty \\ &= cL_f \left(\left(|\alpha|_\infty + \frac{1}{c} \right) \frac{c\,|f(\cdot,0)|}{1-cL_f} L_\rho + 1 \right) |x-y|_\infty\,. \end{aligned}$$

Using the contraction principle on the closed subset Λ, we deduce that Equation (4.8) has a unique solution in $PAP^u(\mathbb{R},\mathbb{R})$. ■

Remark 4.14. Provided that $cL_f < 1$ Condition (4.9) is equivalent to

$$L_\rho < \frac{(1-cL_f)^2}{cL_f\,(c\,|\alpha|_\infty + 1)\,|f(\cdot,0)|},$$

which implies that Condition (4.9) can be met if L_ρ is small enough (the effect of the delay state-dependency is small).

Example. Consider the following hematopoiesis model

$$x'(t) = -\alpha(t)x(t) + \beta(t)\frac{x^2(t-\rho(x(t)))}{1+x^2(t-\rho(x(t)))} + \gamma(t) \quad \text{for } t \in \mathbb{R}, \tag{4.10}$$

where $x(t)$ denotes the density of mature cells in blood circulation at time t. The cells are lost from the circulation at a time-dependent rate $\alpha(t)$, the term $\beta(t)\dfrac{x^2(t)}{1+x^2(t)}$ represents the flux of the cells into the circulation from the stem cell compartment. One can see that this flux takes in consideration the density of mature cells in blood circulation with a time lag (delay) $\rho(x(t))$ that depends on the state itself. On the other hand, the term $\gamma(t)$ represents a flux of blood cells which does not take in consideration the density of blood cells in circulation (blood donation for example).

In the real-world phenomena, the periodic variations of the environment (e.g., seasonal effects of weather, resource availability, reproduction, food supplies, mating habits, etc.) plays an important role in many biological and ecological systems. In particular the effects of a periodically varying environment for such systems differ from those of a stable environment. Thus, the assumption of periodicity of the parameters are a way of incorporating the periodicity of the environment.

In what follows, we assume that $\alpha(.)$, $\beta(.)$ and $\gamma(t)$ are the pseudo almost periodic functions defined by

$$\alpha(t) = 4 + \sin(t) + \sin\left(\sqrt{2}t\right) + \frac{1}{1+t^2}$$

$$\beta(t) = \gamma(t) = \sin(t) + \sin\left(\sqrt{2}t\right) + \frac{1}{1+t^2}$$

The function ρ is defined by $\rho(x) = d\left|\cos(x)\right|$, with $d \geq 0$. We claim that

$$\inf_{t\in\mathbb{R}} \alpha(t) = 2.$$

In fact, on one hand we have $\alpha(t) \geq 2$ for all $t \in \mathbb{R}$. On the other hand, since $\sqrt{2} \notin \mathbb{Q}$, all numbers of the form $a\sqrt{2} - b$ where a and b are integers are dense in the real line. Thus there exist two integer sequences $(a_n)_n$ and $(b_n)_n$ such that

$$\lim_{n\to\infty} a_n\sqrt{2} - b_n = \frac{\sqrt{2}-1}{4}$$

with $\lim_{n\to\infty} |a_n| = \infty$. Let

$$t_n := -\frac{\pi}{2} + 2a_n\pi,$$

then we have $\lim_{n\to\infty} \alpha(t_n) = 2$. That is $(t_n)_n$ is a minimizing sequence and thus $\inf_{t\in\mathbb{R}} \alpha(t) = 2$. Therefore α satisfies **(H1)** with $\frac{1}{7} \leq c \leq \frac{1}{2}$. Set

$$f(t,x) = \beta(t)g(x) + \gamma(t),$$

where g is the function defined for each $x \in \mathbb{R}$ by $g(x) := \frac{x^2}{1+x^2}$. Since $\sup_{x\in\mathbb{R}} |g'(x)| = \frac{3\sqrt{3}}{8}$ and $|\beta| \leq 3$, then f satisfies **(H2)** with $L_f \leq \frac{9\sqrt{3}}{8} < 2$, thus $cL_f < 1$. Since $L_\rho \leq d$, then following the above-mentioned remark, we conclude that if we choose d small enough, Equation (4.10) has a unique pseudo almost periodic solution.

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