WEIGHTED HARDY-SOBOLEV INEQUALITY AND GLOBAL EXISTENCE RESULT OF THERMOELASTIC SYSTEM ON MANIFOLDS WITH CORNER-EDGE SINGULARITIES

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Abstract. This article concerns with the thermoelastic corner-edge type system with singular potential function on a wedge manifold with corner singularities. First, we introduce weighted $p-$Sobolev spaces on manifolds with corner-edge singularities. Then, we prove the corner-edge type Sobolev inequality , Poincaré inequality and Hardy inequality and obtain some results about the compactness of embedding maps on the weighted corner-edge Sobolev spaces. Finally, as an application of these results, we apply the potential well theory and the Faedo-Galerkin approximations to obtain the global weak solutions for the thermoelastic corner-edge type system 1.1.

1. Introduction

The present article deals with the global existence of solutions for thermoelastic corner-edge type system under suitable conditions. First, we introduce the corner-edge type weighted $p-$ Sobolev spaces and discuss the properties of the continuous embedding and compactness. Then, we obtain the corner-edge type Sobolev inequality , Poincaré inequality and Hardy inequality, which are important in the proof of main result about the global solution of thermoelastic type system on the manifolds with corner-edge singularities. In fact, by making use of our results in the preliminary sections we combine the Faedo-Galerkin method and the monotonicity-compactness method with some implications of the potential well theory and prove the existence of global solutions for a class of thermoelastic equations. More precisely, this article is concerned with following initial-boundary value problem for a thermoelastic system which contains corner-edge Laplacian and p-Laplacian type operators with potential

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function

$$\begin{cases} u_{tt} - \Delta_{p,\mathbb{K}} u - \epsilon V(\tilde{x})u + \psi = |u|^{\alpha-1}u, & (\tilde{x},t) \in int\mathbb{K} \times (0,T), \\ \psi_t - \Delta_{\mathbb{K}} u = u_t, & (\tilde{x},t) \in int\mathbb{K} \times (0,T), \\ u(\tilde{x},0) = u_0(\tilde{x}),\ u_t(\tilde{x},0) = u_1(\tilde{x}), & \\ \psi(\tilde{x},0) = \psi_0(\tilde{x}), & \tilde{x} \in int\mathbb{K}, \\ u(\tilde{x},t) = \psi(\tilde{x},t) = 0, & (\tilde{x},t) \in \partial\mathbb{K} \times (0,T), \end{cases} \tag{1.1}$$

where $u_0 \in \mathcal{H}_{p,0}^{1,(\frac{N-1}{p},\frac{N}{p})}(\mathbb{K})$, $u_1 \in L_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{K})$, $\psi_0 \in L_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{K})$, $T \in (0,\infty]$, $2 \leq p < \infty$, $1 < \alpha < 2^*$, $N = 1 + n + q_1 + 1 + q_2 \geq 3$ is the dimension of the stretched manifold $\mathbb{K}$ with respect to the manifold K with corner-edge singularity and $\tilde{x} = (r_1, x, y_1, r_2, y_2) \in \mathbb{K}$. Furthermore, the singular potential function $V(\tilde{x})$ is unbounded over $\mathbb{K}$ which is satisfied in weighted corner-edge Hardy inequality(see Proposition 3.4). The operator $\Delta_{p,\mathbb{K}} + \epsilon V(\tilde{x})$ with $p \neq 2$ arises from a diversity of physical phenomena. It is applied in reaction-diffusion problems, in nonlinear elasticity, in non-Newtonian fluids and petroleum extraction. The investigation of wave equations with the Laplacian and p-Laplacian operators emanates from the nonlinear model for the longitudinal vibrations of the viscoelastic materials. Furthermore, problems relating to wave propagation in generalized theories of thermoelasticity which admit finite speed of thermal signals (second sound effect) in elastic solids have been the subject of active research in recent years [14]. It is well-known that the question about the existence of a global solution for the wave equation of $p-$Laplacian type $u_{tt} - \Delta_p u = 0$, without any dissipation term is still an open problem. However, there is an extensive literature on the existence and non-existence global solutions to the wave equation of $p-$Laplacian type with potential function of the suitable domain $\Omega \subset \mathbb{R}^n$ under appropriate conditions. In [12], the author studied the local existence of the solution with respect to time variable and he showed by a counter example, that the global solution with respect to the time variable can not be obtained. In the case $p = 2$, Sattinger [27] proved the existence global weak solution of the following problem $u_{tt} - \Delta u = f(u)$ by using the potential well method. From then on,the potential well theory has become one of the most important methods for studying nonlinear evolution equations under the suitable assumptions on source term f and convenient initial-boundary conditions [20, 28, 31] . In [16], the authors investigated the global existence, nonexistence and asymptotic behaviour of solutions of the initial boundary value problem of semilinear hyperbolic equations with dissipative term $u_{tt} - \Delta u + \gamma u_t = f(u)$. The authors in [22] investigated a quasilinear wave equation with Kelvin-Voigt damping, $u_{tt} - \Delta_p u + \Delta u_t = f(u)$, where $2 < p < 3$. They proved existence of local weak solutions, which can be extended globally provided the damping term dominates the source in an appropriate sense. Moreover, a blow-up result is proved for solutions with negative initial total energy. In the case $p = 2$, Dafermos [10] studied the linear thermoelastic system defines a semigroup of contractions in an appropriated Hilbert space. In [23], the authors considered a thermal effect acting on the wave equation of $p-$Laplacian type and they obtained an global result for a nonlinear thermoelastic system on bounded open domain $\Omega \subset \mathbb{R}^n$. All of the aforementioned above works have been studied on suitable domain $\Omega \subset \mathbb{R}^n$, but in setting of the manifolds, Cavalcanti et al. [2], studied the asymptotic stability of the wave equation on a compact Riemannian manifold subject to the locally distributed viscoelastic effects. Boundary value problems in domains with conical singularity on the boundary were investigated by Kondratev [18]. Maž̌ya and Plamenevskii [21] studied elliptic boundary value problems of differential equations on manifolds with singularities of a sufficiently general nature. In [8], the authors studied multiple solutions of semilinear corner degenerate elliptic equations with singular potential term. Furthermore, Chen et al. studied about the cone Sobolev inequality and Dirichlet problems for nonlinear elliptic equations and multiple solutions for semilinear totally characteristic elliptic equations with subcritical or critical cone Sobolev exponents on the manifolds with conical and edge singularities, see [3] and the references therein. In [1], the authors studied the existence and totally characteristic properties of hypeelleptic equations with conical singularities. Moreover, multiple solutions for a class of nonhomogeneous semilinear equations with critical cone Sobolev exponent on manifolds with conical points is considered in [17]. Also, multiple sign changing solutions for semilinear corner degenerate elliptic equations were investigated by [9] and the references therein. In all of these papers, the authors were considered only the typical degenerate differential operators with cone, edge or corner degeneracy and obtained the existence results for stationary wave system. As far as we know, there have been no results up till now on the initial-boundary value problem for a hyperbolic form of wave equations and in a special case for a thermoelastic system which contains corner-edge Laplacian and p-Laplacian type operators with potential function on a manifold with corner-edge singularity. Hence, as an application of our results, we want to answer an open question about the global solution of the corner-edge thermoelastic equations in the last section. This paper is organized as follows. In the second section, we recall some basic definitions and concepts about the manifolds with singularities and then introduce the stretched corner-edge manifold $\mathbb{K}$. Moreover, in this section we introduce the corner-edge Sobolev spaces corresponding with the manifold $\mathbb{K}$. In the third section, we obtain two the most important inequalities so-called weighted corner-edge Sobolev and weighted corner-edge Hardy inequalities and prove a result about the compactness embedding theorem on the corner-edge Sobolev spaces. In the last section, we study about the existence of global solutions for a thermoelastic system which contains corner-edge Laplacian and p-Laplacian type operators with potential function on a manifold with corner-edge singularity.

2. Manifolds with corner-edge singularities

Consider X as a closed compact $C^\infty-$manifold of dimension n of the unit sphere in $\mathbb{R}^{n+1}$. We define an infinite cone in $\mathbb{R}^{n+1}$ as a quotient space $X^\Delta = \frac{\bar{\mathbb{R}}_+ \times X}{\{0\}\times X}$, with base X. The cylindrical coordinates $(r,\theta) \in X^\Delta - \{0\}$ in $\mathbb{R}^{n+1} - \{0\}$ are the standard coordinates. This gives us the description of $X^\Delta - \{0\}$ in the form $\mathbb{R}_+ \times X$. Then the stretched cone can be defined as $\bar{\mathbb{R}}_+ \times X = X^\wedge$. Now, consider $B = X^\Delta$ with a conical point, then by the similar way in [6, 25, 26], one can define the stretched manifold $\mathbb{B}$ with respect to B as a $C^\infty-$manifold with smooth boundary $\partial\mathbb{B} \cong X(0)$, where $X(0)$ is the cross section of singular point zero such that there is a diffeomorphism $B - \{0\} \cong \mathbb{B} - \partial\mathbb{B}$, the restriction of which to $U - \{0\} \cong V - \partial\mathbb{B}$ for an open neighbourhood $U \subset B$ near the conic point zero and a collar neighbourhood $V \subset \mathbb{B}$ with $V \cong [0,1) \times X(0)$. Therefore, we can take $\mathbb{B} = [0,1) \times X \subset \bar{\mathbb{R}}_+ \times X = X^\wedge$. In order to consider another type of a manifold with singularity of order one so-called wedge manifold, we consider a bounded domain Y_1 in $\mathbb{R}^{q_1}$. Set $W = X^\Delta \times Y_1 = B \times Y_1$. Then W is a corresponding wedge in $\mathbb{R}^{1+n+q_1}$. Therefore, the stretched wedge manifold $\mathbb{W}$ to W is $X^\wedge \times Y_1$ which is a manifold with smooth boundary $\{0\} \times X \times Y_1$. Set $(r_1, x) \in X^\wedge$. In order to define a finite wedge, it sufficient to consider the case $r_1 \in [0,1)$. Thus, we define a finite wedge as

$$E = \frac{[0,1) \times X}{\{0\} \times X} \times Y_1 \subset X^\Delta \times Y_1 = W.$$

The stretched wedge manifold with respect to E is

$$\mathbb{E} = [0,1) \times X \times Y_1 = \mathbb{B} \times Y_1 \subset X^\wedge \times Y_1 = W^\wedge,$$

with smooth boundary $\partial\mathbb{E} = \{0\} \times X \times Y_1$.

Again for a bounded domain $Y_2 \subset \mathbb{R}^{q_2}$, we take $M = (X^\Delta \times Y_1)^\Delta \times Y_2 = W^\Delta \times Y_2$ as a wedge, i.e., a Cartesian product between an infinite cone W^Δ and edge Y_2 which is as corner-edge in $\mathbb{R}^{1+n+q_1+1+q_2}$. Set $(r_1, x, y_1, r_2, y_2) \in W^\wedge \times Y_2$ where,

$$W^\wedge = \mathbb{W} \times \mathbb{R}_+ \times Y_2 = (X^\wedge \times Y_1) \times \mathbb{R}_+ \times Y_2 = (\mathbb{R}_+ \times X \times Y_1) \times \mathbb{R}_+ \times Y_2.$$

To define a finite corner-edge, we restrict ourselves to $r_2 \in [0,1)$ then we consider

$$K := \frac{E \times [0,1)}{E \times \{0\}} \times Y_2 \subset W^\Delta \times Y_2 = M$$

as a finite corner-edge. Then the stretched finite corner-edge to K is

$$\mathbb{K} = \mathbb{E} \times [0,1) \times Y_2 \subset E^\wedge \times \mathbb{R}_+ \times Y_2 \subset W^\wedge \times Y_2$$

with smooth boundary $\partial\mathbb{K} = \partial\mathbb{E} \times \{0\} \times Y_2$.

As introduced in [6, 7, 25, 26], the typical Fuchs operator A on the finite stretched wedge $\mathbb{E} \subset X^\wedge \times Y_1 \subset X^\wedge \times \mathbb{R}^{q_1}$ is as

$$A = r_1^{-\nu} \sum_{j+|\alpha|\leq\nu} a_{j\alpha}(r_1, y_1)(r_1\partial_{r_1})^j (r_1\partial_{y_1})^\alpha$$

with coefficients $a_{j\alpha} \in C^\infty\Big(\mathbb{R}_+ \times Y_1, Diff^{\nu-(j+|\alpha|)}(X)\Big)$. Let $Diff_e^\mu(\mathbb{E})$ denote the space of all edge-degenerate differential operators of order μ on finite stretched wedge $\mathbb{E} \subset X^\wedge \times Y_1$ equipped with the structure of a Fréchet space. Therefore, by looking at $(X^\wedge \times Y_1)^\wedge \times Y_2 = W^\wedge \times Y_2$ with a corner-edge metric

$$g_{ce} = dr_2^2 + r_2^2(dr_1^2 + r_1^2 g_X + dy_1^2) + dy_2^2$$

where g_X is a Riemannian metric on X, the typical corner-edge degenerate or corner-edge Fuchs operator on the open stretched corner-edge $\mathbb{K} = \mathbb{E} \times [0,1) \times Y_2$ is of the following form

$$B = r_2^{-\nu} \sum_{k+|\beta|\leq\nu} b_{k\beta}(r_2, y_2)(r_2\partial_{r_2})^k (r_2\partial_{y_2})^\beta$$

where $b_{k\beta} = r_1^{-(\nu-(k+|\beta|))} \sum_{j+|\alpha|\leq\nu-(k+|\beta|)} a_{j\alpha}(r_1, y_1)(r_1\partial_{r_1})^j (r_1\partial_{y_1})^\alpha$ which implies that

$$B = (r_1 r_2)^{-\nu} \sum_{j+|\alpha|+k+|\beta|\leq\nu} c_{j\alpha k\beta}(r_1, y_1, r_2, y_2)(-r_1\partial_{r_1})^j (r_1\partial_{y_1})^\alpha (-r_1 r_2 \partial_{r_2})^k (r_1 r_2 \partial_{y_2})^\beta \quad (2.1)$$

where

$$c_{j\alpha k\beta} \in C^\infty\Big(\bar{\mathbb{R}}_+ \times Y_1 \times \bar{\mathbb{R}}_+ \times Y_2, Diff^{\nu-(j+|\alpha|+k+|\beta|)}(X)\Big).$$

Suppose that $Diff_{ce}^\mu(\mathbb{K})$ denote the space of all corner-edge differential operators as 2.1 on $\mathbb{K}$, of order μ.

Let us consider $\mathbb{K} = [0,1) \times X \times Y_1 \times [0,1) \times Y_2 \subset \bar{\mathbb{R}}_+ \times X \times Y_1 \times \bar{\mathbb{R}}_+ \times Y_2$ and set $x = (x_1, ..., x_n) \in X$, $y_1 = (y_{1,1}, ..., y_{1,q_1}) \in Y_1$ and $y_2 = (y_{2,1}, ..., y_{2,q_2}) \in Y_2$. Thus with the coordinates $(r_1, x, y_1, r_2, y_2) \in \mathbb{K} \subset \mathbb{R}_+^N$ such that $N = 1 + n + q_1 + 1 + q_2$, the local model $\mathbb{K}$ can be regarded as a bounded subset in $\mathbb{R}_+^N$.

Consider $g_{ce}(r_1, y_1, r_2, y_2) = dr_2^2 + r_2^2(dr_1^2 + r_1^2 g_X + dy_1^2) + dy_2^2$ as a Riemannain corner-edge metric on manifold $\mathbb{K}$ which is infinity differentiable in $(r_1, r_2) \in [0,1) \times [0,1)$ and $g(0, y_1, 0, y_2)$

depend only on y_i and dy_i for $i = 1, 2$. Therefore, $(T\mathbb{K})_{ce}$ that is, the corner-edge tangent bundle which has a basis expressed in local coordinates as follows

$$\left\{ r_1\partial_{r_1}, \partial_x, r_1\partial_{y_1}, r_1r_2\partial_{r_2}, r_1r_2\partial_{y_2} \right\}$$

where $\partial_x = (\partial_{x_1}, ..., \partial_{x_n})$, $\partial_{y_1} = (\partial_{y_{1,1}}, ..., \partial_{y_{1,q_1}})$ and $\partial_{y_2} = (\partial_{y_{2,1}}, ..., \partial_{y_{2,q_2}})$. This basis induces a dual basis for the corner-edge cotangent bundle $(T^*\mathbb{K})_{ce}$ which is given by

$$\left\{ \frac{dr_1}{r_1}, dx, \frac{dy_1}{r_1}, \frac{dr_2}{r_1r_2}, \frac{dy_2}{r_1r_2} \right\}$$

where $dx = (dx_1, ..., dx_n)$, $dy_1 = (dy_{1,1}, ..., dy_{1,q_1})$ and $dy_2 = (dy_{2,1}, ..., dy_{2,q_2})$. According to the Riemannian corner-edge metric g_{ce} on the stretched manifold $\mathbb{K}$, the gradient and the divergence operators with respect to this metric are first order corner-edge operators that is

$$div_{\mathbb{K}} \in Diff^1_{ce}(\mathbb{K}, (T\mathbb{K})_{ce}, \mathbb{R}), \;\; \nabla_{\mathbb{K}} \in Diff^1_{ce}(\mathbb{K}, \mathbb{R}, (T\mathbb{K})_{ce}).$$

Therefore, we can consider the gradient and the divergence operators with corner-edge degeneracy as

$$\begin{aligned} \nabla_{\mathbb{K}} \quad &= \quad (r_1\partial_{r_1}, \partial_x, r_1\partial_{y_1}, r_1r_2\partial_{r_2}, r_1r_2\partial_{y_2}), \\ div_{\mathbb{K}}(.) \quad &= r_1\frac{\partial(.)}{\partial_{r_1}} + \sum_{j=1}^{n} \frac{\partial(.)}{\partial_{x_j}} + \sum_{l=1}^{q_1} r_1 \frac{\partial(.)}{\partial_{y_{1,l}}} + r_1r_2\frac{\partial(.)}{\partial_{r_2}} + \sum_{l=1}^{q_2} r_1r_2\frac{\partial(.)}{\partial_{y_{2,l}}}. \end{aligned}$$

Then the *corner-edge Laplacian* as elliptic differential operator of second order is defined by

$$\begin{aligned} \Delta_{\mathbb{K}} \quad &= \quad (r_1\partial_{r_1})^2 + \sum_{j=1}^{n} \partial^2_{x_j} + \sum_{l=1}^{q_1} (r_1\partial_{y_{1,l}})^2 + (r_1r_2\partial_{r_2})^2 + \sum_{l=1}^{q_2} (r_1r_2\partial_{y_{2,l}})^2 \\ &:= \quad (r_1\partial_{r_1})^2 + (\partial_x)^2 + (r_1\partial_{y_1})^2 + (r_1r_2\partial_{r_2})^2 + (r_1r_2\partial_{y_2})^2. \end{aligned} \tag{2.2}$$

In order to introduce the weighted corner-edge $p-$Sobolev spaces on $\mathbb{R}^N_+$ with $N = 1 + n + q_1 + 1 + q_2 \geq 3$, we first need the following L_p-spaces.

Definition 2.1. Suppose that $(r_1, x, y_1, r_2, y_2) \in \mathbb{R}^N_+ = \mathbb{R}_+ \times \mathbb{R}^n \times Y_1 \times \mathbb{R}_+ \times Y_2$ where, $Y_1 \subset \mathbb{R}^{q_1}$ and $Y_2 \subset \mathbb{R}^{q_2}$ are bounded domains and let $1 \leq p < +\infty$. We say that $u(r_1, x, y_1, r_2, y_2) \in \mathcal{D}'(\mathbb{R}^N_+)$ belongs to the space $L_p(\mathbb{R}^N_+, \frac{dr_1}{r_1} dx \frac{dy_1}{r_1} \frac{dr_2}{r_1r_2} \frac{dy_2}{r_1r_2} = d\mu)$ whenever,

$$\Big\| u \Big\|_{L_p} = \left(\int_{\mathbb{R}^N_+} |r_1^{\frac{N}{p}} r_2^{\frac{N}{p}} u(r_1, x, y_1, r_2, y_2)|^p d\mu \right)^{\frac{1}{p}} < \infty.$$

Now, we can define the weighted corner-edge $L_p^{\gamma_1,\gamma_2}$ on $\mathbb{R}^N_+$ as the following:

Definition 2.2. Suppose that $(r_1, x, y_1, r_2, y_2) \in \mathbb{R}_+^N = \mathbb{R}_+ \times \mathbb{R}^n \times Y_1 \times \mathbb{R}_+ \times Y_2$ and let $\gamma_1, \gamma_2 \in \mathbb{R}$ and $1 \leq p < \infty$. Then the weighted space $L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+^N, d\mu)$ denotes the space of all $u(r_1, x, y_1, r_2, y_2) \in \mathcal{D}'(\mathbb{R}_+^N)$ such that

$$\left\| u \right\|_{L_p^{\gamma_1,\gamma_2}} := \left\{ \int_{\mathbb{R}_+^N} |r_1^{\frac{N}{p}-\gamma_1} r_2^{\frac{N}{p}-\gamma_2} u(r_1, x, y_1, r_2, y_2)|^p d\mu \right\}^{\frac{1}{p}} < \infty.$$

Then, we can introduce the weighted corner-edge $p-$Sobolev on $\mathbb{R}_+^N$ by the above definitions.

Definition 2.3. Let $m \in \mathbb{N}$, γ_1, γ_2 and $1 \leq p < \infty$. The weighted corner-edge $p-$Sobolev space is defined as follows:

$$\begin{aligned}\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+^N) \quad = \quad & \Big\{ u(r_1, x, y_1, r_2, y_2) \in \mathcal{D}'(\mathbb{R}_+^N) \\ & \mid \quad (r_1\partial_{r_1})^l \partial_x^\alpha (r_1 \partial_{y_1})^\beta (r_1 r_2 \partial_{r_2})^k (r_1 r_2 \partial_{y_2})^\theta u(r_1, x, y_1, r_2, y_2) \in L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+^N, d\mu) \Big\}\end{aligned}$$

for $k, l \in \mathbb{N}$ and multi-indices $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^{q_1}$ and $\theta \in \mathbb{N}^{q_2}$ with $l + |\alpha| + |\beta| + k + |\theta| \leq m$.

Therefore, $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+^N)$ is a Banach space with the following norm,

$$\begin{aligned}\left\| u \right\|_{\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+^N)} \quad = \quad & \sum_{l+|\alpha|+|\beta|+k+|\theta| \leq m} \Big\{ \int_{\mathbb{R}_+^N} \\ & \mid \quad r_1^{\frac{N}{p}-\gamma_1} r_2^{\frac{N}{p}-\gamma_2} (r_1\partial_{r_1})^l \partial_x^\alpha (r_1 \partial_{y_1})^\beta (r_1 r_2 \partial_{r_2})^k (r_1 r_2 \partial_{y_2})^\theta u(r_1, x, y_1, r_2, y_2)|^p d\mu \Big\}^{\frac{1}{p}}.\end{aligned}$$

Furthermore, the closure of $C_0^\infty(\mathbb{R}_+^N)$ functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+^N)$ is indicated by $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+^N)$.

Proposition 2.4. *For any $u \in \mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+^N)$ we have an isomorphism*

$$S_p^{\gamma_1,\gamma_2} : \mathcal{H}_p^{\gamma_1,\gamma_2}(\mathbb{R}_+^N) \to W^{m,p}(\mathbb{R} \times \mathbb{R}^n \times Y_1 \times \mathbb{R} \times Y_2 = \mathbb{R}^N)$$

which is defined by

$$S_p^{\gamma_1,\gamma_2}(u(r_1, x, y_1, r_2, y_2)) := e^{-s_1(\frac{N}{p}-\gamma_1)} e^{-r_1 s_2(\frac{N}{p}-\gamma_2)} u(e^{-s_1}, xe^{-s_1}\xi_1, e^{-r_1 s_2}, e^{-s_1-s_2}\xi_2)$$

where $s_1 = -lnr_1$, $\xi_1 = e^{s_1} y_1$, $s_2 = -\frac{lnr_2}{r_1}$ and $\xi_2 = e^{s_1+s_2} y_2$.

Proof. Let us consider $v(s_1,x,\xi_1,s_2,\xi_2)=S_p^{\gamma_1,\gamma_2}(u(r_1,x,y_1,r_2,y_2))$. Then

$$\Big\|v(s_1,x,\xi_1,s_2,\xi_2)\Big\|^p_{W^{m,p}(\mathbb{R}^N))} = \sum_{l+|\alpha|+|\beta|+k+|\theta|\le m}\int_{\mathbb{R}^N}\Big|\partial^l_{s_1}\partial^\alpha_x\partial^\beta_{\xi_1}\partial^k_{s_2}\partial^\theta_{\xi_2}v(s_1,x,\xi_1,s_2,\xi_2)\Big|^p ds_1dxd\xi_1ds_2d\xi_2$$

$$= \sum_{l+|\alpha|+|\beta|+k+|\theta|\le m}\int_{\mathbb{R}^N}\Big|\partial^l_{s_1}\partial^\alpha_x\partial^\beta_{\xi_1}\partial^k_{s_2}\partial^\theta_{\xi_2}e^{-s_1(\frac{N}{p}-\gamma_1)}e^{-r_1s_2(\frac{N}{p}-\gamma_2)}$$

$$u(e^{-s_1},x,e^{-s_1}\xi_1,e^{-r_1s_2},e^{-s_1-s_2}\xi_2)\Big|^p ds_1dxd\xi_1ds_2d\xi_2$$

$$= \sum_{l+|\alpha|+|\beta|+k+|\theta|\le m}\int_{\mathbb{R}^N}\Big|e^{-s_1(\frac{N}{p}-\gamma_1)}e^{-r_1s_2(\frac{N}{p}-\gamma_2)}\partial^l_{s_1}\partial^\alpha_x\partial^\beta_{\xi_1}\partial^k_{s_2}\partial^\theta_{\xi_2}\times$$

$$u(e^{-s_1},x,e^{-s_1}\xi_1,e^{-r_1s_2},e^{-s_1-s_2}\xi_2)\Big|^p ds_1dxd\xi_1ds_2d\xi_2$$

$$\cong \sum_{l+|\alpha|+|\beta|+k+|\theta|\le m}\int_{\mathbb{R}^N}\Big|e^{-s_1(\frac{N}{p}-\gamma_1)}e^{-r_1s_2(\frac{N}{p}-\gamma_2)}\partial^l_{s_1}\partial^\alpha_x$$

$$(e^{s_1}\partial_{y_1})^\beta\partial^k_{\xi_2}(e^{s_1+s_2}\partial_{y_2})^\theta u(e^{-s_1},x,y_1,e^{-r_1s_2},y_2)\Big|^p ds_1dx\frac{dy_1}{r_1}ds_2\frac{dy_2}{r_1r_2}$$

$$\cong \sum_{l+|\alpha|+|\beta|+k+|\theta|\le m}\int_{\mathbb{R}^N_+}\Big|e^{-s_1(\frac{N}{p}-\gamma_1)}e^{-r_1s_2(\frac{N}{p}-\gamma_2)}(r_1\partial_1)^l\partial^\alpha_x$$

$$(r_1\partial_{y_1})^\beta(r_1r_2\partial^k_{r_2}(r_1r_2\partial_{y_2})^\theta u(r_1,x,y_1,r_2,y_2)\Big|^p\frac{dr_1}{r_1}dx\frac{dy_1}{r_1}\frac{dr_2}{r_1r_2}\frac{dy_2}{r_1r_2}$$

$$=\Big\|u\Big\|_{\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}^N_+)}.$$

□

Consider closed compact C^∞ manifold X and by the similar way, we can define the weighted corner-edge $p-$Sobolev on an open stretched corner-edge $W^\wedge\times Y_2=\mathbb{R}_+\times X\times Y_1\times\mathbb{R}_+\times Y_2$ as follows :

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(W^\wedge\times Y_2)=\Big\{u(r_1,x,y_1,r_2,y_2)\in\mathcal{D}'(W^\wedge\times Y_2)$$
$$\mid (r_1\partial_{r_1})^l\partial^\alpha_x(r_1\partial_{y_1})^\beta(r_1r_2\partial_{r_2})^k(r_1r_2\partial_{y_2})^\theta u(r_1,x,y_1,r_2,y_2)\in L_p^{\gamma_1,\gamma_2}(W^\wedge\times Y_2,d\mu)\Big\},$$

for $k,l\in\mathbb{N}$ and multi-indices $\alpha\in\mathbb{N}^n$, $\beta\in\mathbb{N}^{q_1}$ and $\theta\in\mathbb{N}^{q_2}$ with $l+|\alpha|+|\beta|+k+|\theta|\le m$, which is a Banach space with the following norm

$$\Big\|u\Big\|_{\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(W^\wedge\times Y_2)} = \sum_{l+|\alpha|+|\beta|+k+|\theta|\le m}\Big\{\int_{W^\wedge\times Y_2}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}$$
$$(r_1\partial_{r_1})^l\partial^\alpha_x(r_1\partial_{y_1})^\beta(r_1r_2\partial_{r_2})^k(r_1r_2\partial_{y_2})^\theta u(r_1,x,y_1,r_2,y_2)|^pd\mu\Big\}^{\frac{1}{p}}.$$

The closure of $C_0^\infty(W^\wedge\times Y_2)$ as a subspace of functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(W^\wedge\times Y_2)$ is denoted by $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(W^\wedge\times Y_2)$. Moreover, we indicate $W^{m,p}_{loc}(.)$ as the classical local Sobolev space on suitable space. Now, we define the weighted corner-edge $p-$Sobolev space on the stretched finite corner-edge manifold $\mathbb{K} = \mathbb{E} \times [0,1) \times Y_2 \subset W^{\wedge} \times Y_2$ with respect to K. Set $\mathbb{K}_0 = int\mathbb{K}$.

Definition 2.5. Let $m \in \mathbb{N}$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $1 \leq p < \infty$. The weighted corner-edge $p-$Sobolev space is defined as follows

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{K}) := \left\{ u(r_1, x, y_1, r_2, y_2) \in W_{loc}^{m,p}(\mathbb{K}_0) \mid (\omega_1\omega_2)u \in \mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(W^{\wedge} \times Y_2) \right\} \tag{2.3}$$

for any cut-off functions $\omega_1 = \omega(r_1)$ and $\omega_2 = \omega(r_2)$ supported by the collar neighborhoods of $(0,1) \times \partial\mathbb{K}$ and $\partial\mathbb{K} \times (0,1)$ respectively.

In fact, for cut-off functions ω_1 and ω_2, one can consider $\epsilon_1 \in (0,1)$ and $\epsilon_2 \in (0,1)$, depending only on ω_1 and ω_2 respectively, so that $\omega_1 = \omega(r_1) = 1$ for $r_1 \in supp(\omega_1) \cap (0, \epsilon_1]$ and $\omega_2 = \omega(r_2) = 1$ for $r_2 \in supp(\omega_2) \cap (0, \epsilon_2]$. Therefore,

$$\begin{aligned} \mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{K}) \quad &:= \quad [\omega_1][\omega_2]\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(W^{\wedge} \times Y_2) + [1-\omega_1][\omega_2]\mathcal{H}_{p,0}^{m,\gamma_2}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \\ &+ \quad [\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{m,\gamma_1}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \\ &+ \quad [1-\omega_1][1-\omega_2]W_0^{m,p}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \Lambda_2 \times Y_2) \end{aligned} \tag{2.4}$$

where, $\Lambda_{\epsilon_1} = (\epsilon_1, 1)$ and $\Lambda_{\epsilon_2} = (\epsilon_2, 1)$. Moreover, the weighted $p-$Sobolev spaces $\mathcal{H}_{p,0}^{m,\gamma_1}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)$ and $\mathcal{H}_{p,0}^{m,\gamma_2}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)$ are the closure of $C_0^{\infty}-$functions in the following weighted edge $p-$Sobolev spaces [3, 24]:

$$\begin{aligned} \mathcal{H}_p^{m,\gamma_1}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) := \Big\{ & u(r_1, x, y_1, r_2, y_2) \in W_{loc}^{m,p}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \\ & | r_1^{\frac{N}{p}-\gamma_1}(r_1\partial_{r_1})^l \partial_x^{\alpha} (r_1\partial_{y_1})^{\beta} (r_1\partial_{r_2})^k (r_1\partial_{y_2})^{\theta} u(r_1, x, y_1, r_2, y_2) \in \\ & L_p(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2, \ \frac{dr_1}{r_1} dx \frac{dy_1}{r_1} \frac{dr_2}{r_1} \frac{dy_2}{r_1}) \Big\} \end{aligned}$$

for $k, l \in \mathbb{N}$ and multi-indices $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^{q_1}$ and $\theta \in \mathbb{N}^{q_2}$ with $l + |\alpha| + |\beta| + k + |\theta| \leq m$, and also

$$\begin{aligned} \mathcal{H}_p^{m,\gamma_2}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) := \Big\{ & u(r_1, x, y_1, r_2, y_2) \in W_{loc}^{m,p}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \\ & | r_2^{\frac{N}{p}-\gamma_2}(r_2\partial_{r_1})^l \partial_x^{\alpha} (r_2\partial_{y_1})^{\beta} (r_2\partial_{r_2})^k (r_2\partial_{y_2})^{\theta} u(r_1, x, y_1, r_2, y_2) \in \\ & L_p(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2, \ \tfrac{dr_1}{r_2} dx \tfrac{dy_1}{r_2} \tfrac{dr_2}{r_2} \tfrac{dy_2}{r_2}) \Big\} \end{aligned}$$

for $k, l \in \mathbb{N}$ and multi-indices $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^{q_1}$ and $\theta \in \mathbb{N}^{q_2}$ with $l + |\alpha| + |\beta| + k + |\theta| \leq m$.

Remark 2.6. [8] *One can consider the Banach spaces* Ω_i *with norm* $\|.\|_i$ *for* $i = 1, 2, 3, 4$ *and a smooth partition unity* $\{\psi_i\}_{i=1}^4$. *Then* $\Omega = \psi_1\Omega_1 + \psi_2\Omega_2 + \psi_3\Omega_3 + \psi_4\Omega_4$ *will be a Banach space with norm* $\|u\|_\Omega^2 := \sum_{i=1}^4 \|\psi_i u\|_i^2$. *Since the cut-off functions* ω_1 *and* ω_2 *satisfy* $\omega_1\omega_2 + (1-\omega_1)\omega_2 + \omega_1(1-\omega_2) + (1-\omega_1)(1-\omega_2) = 1$, *then it implies that* $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{K})$ *is a Banach space for* $1 \le p < \infty$ *and a Hilbert space for* $p = 2$. *Moreover, one can obtain the property* $r_1^{\gamma_1'} r_2^{\gamma_2'} \mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{K}) = \mathcal{H}_p^{m,(\gamma_1+\gamma_1',\gamma_2+\gamma_2')}(\mathbb{K})$.

3. Corner-Edge Type Inequalities

Proposition 3.1. *(Corner-Edge Sobolev Inequality) Suppose that* $1 \le p < N$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$ *with* $N = 1 + n + q_1 + 1 + q_2$ *and* $\gamma_1, \gamma_2 \in \mathbb{R}$. *Then for any* $u(r_1, xy_1, r_2, y_2) \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{q_1} \times \mathbb{R}_+ \times \mathbb{R}^{q_2} = \mathcal{R}_+^N)$ *the following inequality holds :*

$$\begin{aligned}
\Big\|u\Big\|_{L_{p^*}^{\gamma_1^*,\gamma_2^*}(\mathcal{R}_+^N)} &\le \delta D_1\left[\sum_{i=1}^n \Big\|\partial_{x_i} u\Big\|_{L_p^{\gamma_1,\gamma_2}(\mathcal{R}_+^N)} + \sum_{l=1}^{q_1}\Big\|r_1\partial_{y_{1,l}} u\Big\|_{L_p^{\gamma_1,\gamma_2}(\mathcal{R}_+^N)} + \sum_{l=1}^{q_2}\Big\|r_1 r_2\partial_{y_{2,l}} u\Big\|_{L_p^{\gamma_1,\gamma_2}(\mathcal{R}_+^N)}\right] \\
&+ D_2\Big\|u\Big\|_{L_p^{\gamma_1-1,\gamma_2}(\mathcal{R}_+^N)} + D_3\Big\|u\Big\|_{L_p^{\gamma_1,\gamma_2}(\mathcal{R}_+^N)} \\
&+ \delta D_4\Big\|r_1\partial_{r_1} u\Big\|_{L_p^{\gamma_1,\gamma_2}(\mathcal{R}_+^N)} + \delta D_5\Big\|r_1 r_2\partial_{r_2} u\Big\|_{L_p^{\gamma_1,\gamma_2}(\mathcal{R}_+^N)} \qquad (3.1)
\end{aligned}$$

where $\gamma_1^* = \gamma_1 - 1$, $\gamma_2^* = \gamma_2 - 1$ *and* $\delta = \frac{(N-1)p}{N-p}$. *Moreover, the constants* D_1, D_2, D_3, D_4 *and* D_5 *are positive for which*

$$D_1 = \frac{1}{N} + \frac{1}{N}\left(\left|\frac{(N-1)^2(N-p\gamma_1)(N-p\gamma_2)}{(N-p)}\right|^{\frac{1}{N}} + \left|\frac{(N-1)(N-p\gamma_1)}{N-p}\right|^{\frac{1}{N}} + \left|\frac{(N-1)(N-p\gamma_2)}{N-p}\right|^{\frac{1}{N}}\right),$$

$$D_2 = \frac{1}{N}\left|\frac{(N-1)(N-p\gamma_2)}{N-p}\right|^{\frac{1}{N}}\left(1 + \left|\frac{(N-1)(N-p\gamma_1)}{N-p}\right|^{\frac{1}{N}}\right),$$

$$D_3 = \frac{1}{N}\left|\frac{(N-1)(N-p\gamma_1)}{N-p}\right|^{\frac{1}{N}}\left(1 + \left|\frac{(N-1)(N-p\gamma_2)}{N-p}\right|^{\frac{1}{N}}\right),$$

$$D_4 = \frac{1}{N} + \frac{1}{N}\left|\frac{(N-1)(N-p\gamma_2)}{N-p}\right|^{\frac{1}{N}},$$

$$D_5 = \frac{1}{N} + \frac{1}{N}\left|\frac{(N-1)(N-p\gamma_1)}{N-p}\right|^{\frac{1}{N}}.$$

Proof. To prove the inequality 3.1, first we consider $p = 1$, so $p^* = \frac{N}{N-1}$. Choose arbitrary $\gamma_1', \gamma_2' \in \mathbb{R}$ and $u(r_1, xy_1, r_2, y_2) \in C_0^\infty(\mathcal{R}_+^N)$ then

$$\begin{aligned}
\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} u(r_1, x, y_1, r_2, y_2)\Big| &= \Big|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}\int_0^{x_j} \partial_{x_j'} u\, dx_j'\Big| \\
&\le \Big|\int_{-\infty}^{x_j} \partial_{x_j'}(r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} u)\, dx_j'\Big| \le \int_{-\infty}^{\infty}\Big|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(\partial_{x_j} u)\Big| dx_j \\
&:= I_j, \qquad (3.2)
\end{aligned}$$

for $j = 1, ..., n$.

$$\begin{aligned}\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'} u(r_1, x, y_1, r_2, y_2)\Big| &= \Big|\int_0^{y_{1,l}} (r_1 \partial_{y'_{1,l}})(r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} u) \frac{dy'_{1,l}}{r_1}\Big| \\ \leq \quad & \int_{-\infty}^{\infty} \Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} (r_1 \partial_{y'_{1,l}}) u \Big| \frac{dy'_{1,l}}{r_1} := J_{1,l} \end{aligned} \tag{3.3}$$

for $l = 1, ..., q_1$, and

$$\begin{aligned}\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'} u(r_1, x, y_1, r_2, y_2)\Big| &= \Big|\int_0^{y_{2,l}} (r_1 r_2 \partial_{y'_{2,l}})(r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} u) \frac{dy'_{2,l}}{r_1 r_2}\Big| \\ \leq \quad & \int_{-\infty}^{\infty} \Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} (r_1 \partial_{y'_{2,l}}) u \Big| \frac{dy'_{2,l}}{r_1} := J_{2,l} \end{aligned} \tag{3.4}$$

for $l = 1, ..., q_2$. Now, we compute the similar estimates in r_1 and r_2-directions

$$\begin{aligned}\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'} u(r_1, x, y_1, r_2, y_2)\Big| &\leq \int_0^{\infty} \Big| (r_1 \partial_{r_1})(r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} u(r_1, x, y_1, r_2, y_2) \Big| \frac{dr_1}{r_1} \\ \leq \quad & \Big| N - 1 - \gamma_1' \Big| \int_0^{\infty} \Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} u \Big| \frac{dr_1}{r_1} + \int_0^{\infty} \Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} (r_1 \partial_{r_1} u) \Big| \frac{dr_1}{r_1} \\ := \quad & A_1 + A_2. \end{aligned} \tag{3.5}$$

And also,

$$\begin{aligned}\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'} u(r_1, x, y_1, r_2, y_2)\Big| &\leq \int_0^{\infty} \Big| (r_1 r_2 \partial_{r_2})(r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} u \Big| \frac{dr_2}{r_1 r_2} \\ \leq \quad & \Big| N - 1 - \gamma_2' \Big| \int_0^{\infty} \Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} u \Big| \frac{dr_2}{r_1 r_2} + \int_0^{\infty} \Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} (r_1 r_2 \partial_{r_2} u) \Big| \frac{dr_2}{r_1 r_2} \\ := \quad & A_3 + A_4. \end{aligned} \tag{3.6}$$

Hence, we have the number of $N = 1 + n + q_1 + 1 + q_2$ inequalities as the form 3.2 , 3.3 , 3.4 , 3.5 and 3.6. Multiplying these inequalities, one gets :

$$\begin{aligned}&\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'} u(r_1, x, y_1, r_2, y_2)\Big|^N \\ \leq \quad & \prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l} \ (A_1 + A_2)(A_3 + A_4) = \prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l} \ A_1 A_3 \\ + \quad & \prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l} \ A_1 A_4 + \prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l} \ A_2 A_3 \\ + \quad & \prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l} \ A_2 A_4. \end{aligned}$$

By assumption, we have $N = 1+n+q_1+1+q_2$, then $\frac{1}{N-1} < 1$ and thus we obtain the following estimates:

$$\begin{aligned}
&\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'} u(r_1,x,y_1,r_2,y_2)\Big|^{\frac{N}{N-1}} \qquad (3.7)\\
&\leq \quad \Big(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_1A_3\Big)^{\frac{1}{N-1}} + \Big(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_1A_4\Big)^{\frac{1}{N-1}}\\
&+ \quad \Big(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_2A_3\Big)^{\frac{1}{N-1}} + \Big(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_2A_4\Big)^{\frac{1}{N-1}}.
\end{aligned}$$

Now, let us integrate both side of 3.7 with respect to $\frac{dr_1}{r_1}$ then,

$$\begin{aligned}
&\int_{\mathbb{R}_+}\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'} u(r_1,x,y_1,r_2,y_2)\Big|^{\frac{N}{N-1}} \frac{dr_1}{r_1} \qquad (3.8)\\
&\leq \quad \int_{\mathbb{R}_+}\Big(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_1A_3\Big)^{\frac{1}{N-1}}\frac{dr_1}{r_1} + \int_{\mathbb{R}_+}\Big(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_1A_4\Big)^{\frac{1}{N-1}}\frac{dr_1}{r_1}\\
&+ \quad \int_{\mathbb{R}_+}\Big(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_2A_3\Big)^{\frac{1}{N-1}}\frac{dr_1}{r_1} + \int_{\mathbb{R}_+}\Big(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_2A_4\Big)^{\frac{1}{N-1}}\frac{dr_1}{r_1}.
\end{aligned}$$

For every term on the right hand side of the inequality 3.8 one obtains

$$\begin{aligned}
&\int_{\mathbb{R}_+}\Big(\quad \prod_{j=1}^{n} I_j \quad \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_1A_3\Big)^{\frac{1}{N-1}}\frac{dr_1}{r_1} \leq \prod_{j=1}^{n}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(\partial_{x_j}u)\Big|\frac{dr_1}{r_1}dx_j\Big)^{\frac{1}{N-1}}\\
&\times \quad \prod_{l=1}^{q_1}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}u)\Big|\frac{dr_1}{r_1}\frac{dy_{1,l}}{r_1}\Big)^{\frac{1}{N-1}}\\
&\times \quad \prod_{l=1}^{q_2}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}u)\Big|\frac{dr_1}{r_1}\frac{dy_{2,l}}{r_1r_2}\Big)^{\frac{1}{N-1}}\\
&\times \quad \Big|N-1-\gamma_1'\Big|^{\frac{1}{N-1}}\Big|N-1-\gamma_2'\Big|^{\frac{1}{N-1}}\Big(\int_{\mathbb{R}_+}\Big|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}u\Big|\frac{dr_1}{r_1}\Big)^{\frac{1}{N-1}}\\
&\times \quad \Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\Big|r_1^{N-\gamma_1'} r_2^{N-1-\gamma_2'}u\Big|\frac{dr_1}{r_1}\frac{dr_2}{r_1r_2}\Big)^{\frac{1}{N-1}}.
\end{aligned}$$

We use the similar way to obtain the same inequalities from the other three terms in 3.8 as follows:

$$\int_{\mathbb{R}_+}\left(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_1A_4\right)^{\frac{1}{N-1}}\frac{dr_1}{r_1} \leq \prod_{j=1}^{n}\left(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(\partial_{x_j}u)\right|\frac{dr_1}{r_1}dx_j\right)^{\frac{1}{N-1}}$$
$$\times\prod_{l=1}^{q_1}\left(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}u)\right|\frac{dr_1}{r_1}\frac{dy_{1,l}}{r_1}\right)^{\frac{1}{N-1}}$$
$$\times\prod_{l=1}^{q_2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}u)\right|\frac{dr_1}{r_1}\frac{dy_{2,l}}{r_1r_2}\right)^{\frac{1}{N-1}}$$
$$\times|N-1-\gamma_1'|^{\frac{1}{N-1}}\left(\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}u\right|\frac{dr_1}{r_1}\right)^{\frac{1}{N-1}}$$
$$\times\left(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\left|r_1^{N-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1r_2\partial_{r_2}u)\right|\frac{dr_1}{r_1}\frac{dr_2}{r_1r_2}\right)^{\frac{1}{N-1}}.$$

$$\int_{\mathbb{R}_+}\left(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_2A_3\right)^{\frac{1}{N-1}}\frac{dr_1}{r_1} \leq \prod_{j=1}^{n}\left(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(\partial_{x_j}u)\right|\frac{dr_1}{r_1}dx_j\right)^{\frac{1}{N-1}} \tag{3.9}$$
$$\times\prod_{l=1}^{q_1}\left(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}u)\right|\frac{dr_1}{r_1}\frac{dy_{1,l}}{r_1}\right)^{\frac{1}{N-1}}$$
$$\times\prod_{l=1}^{q_2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}u)\right|\frac{dr_1}{r_1}\frac{dy_{2,l}}{r_1r_2}\right)^{\frac{1}{N-1}}$$
$$\times|N-1-\gamma_2'|^{\frac{1}{N-1}}\left(\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1\partial_{r_1}u)\right|\frac{dr_1}{r_1}\right)^{\frac{1}{N-1}}$$
$$\times\left(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\left|r_1^{N-\gamma_1'} r_2^{N-1-\gamma_2'}u\right|\frac{dr_1}{r_1}\frac{dr_2}{r_1r_2}\right)^{\frac{1}{N-1}}.$$

$$\int_{\mathbb{R}_+}\left(\prod_{j=1}^{n} I_j \prod_{l=1}^{q_1} J_{1,l} \prod_{l=1}^{q_2} J_{2,l}\ A_2A_4\right)^{\frac{1}{N-1}}\frac{dr_1}{r_1} \leq \prod_{j=1}^{n}\left(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(\partial_{x_j}u)\right|\frac{dr_1}{r_1}dx_j\right)^{\frac{1}{N-1}} \tag{3.10}$$
$$\times\prod_{l=1}^{q_1}\left(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}u)\right|\frac{dr_1}{r_1}\frac{dy_{1,l}}{r_1}\right)^{\frac{1}{N-1}}$$
$$\times\prod_{l=1}^{q_2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}u)\right|\frac{dr_1}{r_1}\frac{dy_{2,l}}{r_1r_2}\right)^{\frac{1}{N-1}}$$
$$\times\left(\int_{\mathbb{R}_+}\left|r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1\partial_{r_1}u)\right|\frac{dr_1}{r_1}\right)^{\frac{1}{N-1}}$$
$$\times\left(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\left|r_1^{N-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1r_2\partial_{r_2}u)\right|\frac{dr_1}{r_1}\frac{dr_2}{r_1r_2}\right)^{\frac{1}{N-1}}.$$

Now, we can apply the similar integrations with respect to $dx_1, ..., dx_n$, $\frac{y_{1,1}}{r_1}, ..., \frac{dy_{1,q_1}}{r_1}$, $\frac{r_2}{r_1r_2}$ and

$\frac{dy_{2,1}}{r_1 r_2}, ..., \frac{dy_{2,q_2}}{r_1 r_2}$ and the obtain the following estimates

$$\begin{aligned}
\int_{\mathbb{R}^N_+}\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'} u\Big|^{\frac{N}{N-1}} d\mu \leq & \prod_{j=1}^{n}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(\partial_{x_j} u)\Big| d\mu\Big)^{\frac{1}{N-1}} \\
\times \quad & \prod_{l=1}^{q_1}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}} u)\Big| d\mu\Big)^{\frac{1}{N-1}} \\
\times \quad & \prod_{l=1}^{q_2}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1 r_2\partial_{y_{2,l}} u)\Big| d\mu\Big)^{\frac{1}{N-1}} \\
\times \quad & |N-1-\gamma_1'\Big|^{\frac{1}{N-1}}\Big| N-1-\gamma_2'\Big|^{\frac{1}{N-1}}\Big(\int_{\mathbb{R}_+}\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} u\Big| d\mu\Big)^{\frac{1}{N-1}} \\
\times \quad & \Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\Big| r_1^{N-\gamma_1'} r_2^{N-1-\gamma_2'} u\Big| d\mu\Big)^{\frac{1}{N-1}} +
\end{aligned}$$

$$\begin{aligned}
\prod_{j=1}^{n}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'}(\partial_{x_j} u)\Big| \frac{dr_1}{r_1} dx_j\Big)^{\frac{1}{N-1}} & \\
\times \quad & \prod_{l=1}^{q_1}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}} u)\Big| \frac{dr_1}{r_1}\frac{dy_{1,l}}{r_1}\Big)^{\frac{1}{N-1}} \\
\times \quad & \prod_{l=1}^{q_2}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1 r_2\partial_{y_{2,l}} u)\Big| \frac{dr_1}{r_1}\frac{dy_{2,l}}{r_1 r_2}\Big)^{\frac{1}{N-1}} \\
\times \quad & |N-1-\gamma_1'\Big|^{\frac{1}{N-1}}\Big(\int_{\mathbb{R}_+}\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'} u\Big| \frac{dr_1}{r_1}\Big)^{\frac{1}{N-1}} \\
\times \quad & \Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\Big| r_1^{N-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1 r_2\partial_{r_2} u)\Big| \frac{dr_1}{r_1}\frac{dr_2}{r_1 r_2}\Big)^{\frac{1}{N-1}} +
\end{aligned}$$

$$\begin{aligned}
\prod_{j=1}^{n}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'}(\partial_{x_j} u)\Big| \frac{dr_1}{r_1} dx_j\Big)^{\frac{1}{N-1}} & \\
\times \quad & \prod_{l=1}^{q_1}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}} u)\Big| \frac{dr_1}{r_1}\frac{dy_{1,l}}{r_1}\Big)^{\frac{1}{N-1}} \\
\times \quad & \prod_{l=1}^{q_2}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1 r_2\partial_{y_{2,l}} u)\Big| \frac{dr_1}{r_1}\frac{dy_{2,l}}{r_1 r_2}\Big)^{\frac{1}{N-1}} \\
\times \quad & |N-1-\gamma_2'\Big|^{\frac{1}{N-1}}\Big(\int_{\mathbb{R}_+}\Big| r_1^{N-1-\gamma_1'} r_2^{N-1-\gamma_2'}(r_1\partial_{r_1} u)\Big| \frac{dr_1}{r_1}\Big)^{\frac{1}{N-1}} \\
\times \quad & \Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\Big| r_1^{N-\gamma_1'} r_2^{N-1-\gamma_2'} u\Big| \frac{dr_1}{r_1}\frac{dr_2}{r_1 r_2}\Big)^{\frac{1}{N-1}} +
\end{aligned}$$

$$
\prod_{j=1}^{n}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big| \quad r_1^{N-1-\gamma_1'} \quad r_2^{N-1-\gamma_2'}(\partial_{x_j}u)\Big|\frac{dr_1}{r_1}dx_j\Big)^{\frac{1}{N-1}}
$$

$$
\times \qquad \prod_{l=1}^{q_1}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}u)\Big|\frac{dr_1}{r_1}\frac{dy_{1,l}}{r_1}\Big)^{\frac{1}{N-1}}
$$

$$
\times \qquad \prod_{l=1}^{q_2}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}u)\Big|\frac{dr_1}{r_1}\frac{dy_{2,l}}{r_1r_2}\Big)^{\frac{1}{N-1}}
$$

$$
\times \qquad \Big(\int_{\mathbb{R}_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{r_1}u)\Big|\frac{dr_1}{r_1}\Big)^{\frac{1}{N-1}}
$$

$$
\times \qquad \Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\Big|r_1^{N-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{r_2}u)\Big|\frac{dr_1}{r_1}\frac{dr_2}{r_1r_2}\Big)^{\frac{1}{N-1}} = I+II+III+IV.
$$

For any $a_i \geq 0$, $i=1,...,N$, we have the inequality $(\prod_{i=1}^{N})^{\frac{1}{N}} \leq \frac{1}{N}\sum_{i=1}^{N} a_i$ and since $\frac{N-1}{N}<1$ we can get the following estimates:

$$
\Big(\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}u\Big|^{\frac{N}{N-1}}d\mu\Big)^{\frac{N-1}{N}} \leq \frac{1}{N}\Big|N-1-\gamma_1'\Big|^{\frac{1}{N}}\Big|N-1-\gamma_2'\Big|^{\frac{1}{N}}
$$

$$
\Big[\sum_{j=1}^{n}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(\partial_{x_j}u)\Big|d\mu+\sum_{l=1}^{q_1}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}u)\Big|d\mu
$$

$$
+\sum_{l=1}^{q_2}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}u)\Big|d\mu+\int_{\mathbb{R}^N_+}\Big|r_1^{N-\gamma_1'}r_2^{N-1-\gamma_2'}u\Big|d\mu+\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}u\Big|d\mu\Big]+
$$

$$
\frac{1}{N}|N-1-\gamma_1'|^{\frac{1}{N}}\Big[\sum_{j=1}^{n}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(\partial_{x_j}u)\Big|d\mu+\sum_{l=1}^{q_1}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}u)\Big|d\mu+
$$

$$
\sum_{l=1}^{q_2}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}u)\Big|d\mu+\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}u\Big|d\mu+
$$

$$
\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{r_2}u)\Big|d\mu\Big]+\frac{1}{N}|N-1-\gamma_2'|^{\frac{1}{N}}\Big[\sum_{j=1}^{n}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(\partial_{x_j}u)\Big|d\mu
$$

$$
+\sum_{l=1}^{q_1}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}u)\Big|d\mu+\sum_{l=1}^{q_2}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}u)\Big|d\mu
$$

$$
+\int_{\mathbb{R}^N_+}\Big|r_1^{N-\gamma_1'}r_2^{N-1-\gamma_2'}u\Big|d\mu+\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{r_1}u)\Big|d\mu\Big]+
$$

$$\begin{aligned}
&\frac{1}{N}\Bigg[\sum_{j=1}^{n}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(\partial_{x_j}u)\Big|d\mu+\sum_{l=1}^{q_1}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}u)\Big|d\mu\\
&\quad+\sum_{l=1}^{q_2}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}u)\Big|d\mu\\
&\quad+\int_{\mathbb{R}^N_+}\Big|r_1^{N-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{r_1}u)\Big|d\mu+\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{r_2}u)\Big|d\mu\Bigg]\\
&\quad=C_1\big[C_2C_3+C_2+C_3\big]\Bigg[\sum_{j=1}^{n}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(\partial_{x_j}u)\Big|d\mu+\sum_{l=1}^{q_1}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}u)\Big|d\mu\\
&\quad+\sum_{l=1}^{q_2}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}u)\Big|d\mu\Bigg]+C_1C_3(C_2+1)\int_{\mathbb{R}^N_+}\Big|r_1^{N-\gamma_1'}r_2^{N-1-\gamma_2'}u\Big|d\mu\\
&\quad+C_1C_2(1+C_3)\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}u\Big|d\mu+C_1(C_3+1)\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{r_1}u)\Big|d\mu\\
&\quad+C_1(C_2+1)\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{r_2}u)\Big|d\mu
\end{aligned}$$

where $C_1=\frac{1}{N}$, $C_2=|N-1-\gamma_1'|^{\frac{1}{N}}$ and $C_3=|N-1-\gamma_2'|^{\frac{1}{N}}$. This follows that the assertion holds for $p=1$. Now, we show that the inequality hold for the case $1<p<\infty$. Let $v=|u|^\alpha$ where $\alpha>1$ and will be determined in the proof process. Then $v\in C_0^\infty(\mathbb{R}_+)$ and we have the following calculations

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^\alpha\Big|^{\frac{N}{N-1}}d\mu\right)^{\frac{N-1}{N}}\le C_1\big[C_2C_3+C_2+C_3\big]\Bigg[\sum_{j=1}^{n}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(\partial_{x_j}|u|^\alpha)\Big|d\mu\\
&\quad+\sum_{l=1}^{q_1}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{y_{1,l}}|u|^\alpha)\Big|d\mu+\sum_{l=1}^{q_2}\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{y_{2,l}}|u|^\alpha)\Big|d\mu\Bigg]\\
&\quad+C_1C_3(C_2+1)\int_{\mathbb{R}^N_+}\Big|r_1^{N-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^\alpha\Big|d\mu+C_1C_2(1+C_3)\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^\alpha\Big|d\mu\\
&\quad+C_1(C_3+1)\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1\partial_{r_1}|u|^\alpha)\Big|d\mu+C_1(C_2+1)\int_{\mathbb{R}^N_+}\Big|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(r_1r_2\partial_{r_2}|u|^\alpha)\Big|d\mu.
\end{aligned}$$

On the other hand, one can consider $|u|^\alpha=(u.u)^{\frac{\alpha}{2}}$ and the obtain the following computations:

$$\begin{aligned}
&\Big|\partial_{x_j}|u|^\alpha\Big|=\Big|\partial_{x_j}(u.u)^{\frac{\alpha}{2}}\Big|\le\alpha|u|^{\alpha-1}|\partial_{x_j}u| &&\Big|(r_1\partial_{y_{1,l}}|u|^\alpha\Big|\le\alpha|u|^{\alpha-1}|(r_1\partial_{y_{1,l}}u|\\
&\Big|(r_1r_2\partial_{y_{2,l}}|u|^\alpha)\Big|\le\alpha|u|^{\alpha-1}|(r_1r_2\partial_{y_{2,l}}u)| &&\Big|(r_1\partial_{r_1}|u|^\alpha)\Big|\le\alpha|u|^{\alpha-1}|(r_1\partial_{r_1}u)|\\
&\Big|(r_1r_2\partial_{r_2}|u|^\alpha)\Big|\le\alpha|u|^{\alpha-1}|(r_1r_2\partial_{r_2}u)\Big|.
\end{aligned}$$

Therefore,

$$\left(\int_{\mathbb{R}^N_+}\left|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^{\alpha}\right|^{\frac{N}{N-1}}d\mu\right)^{\frac{N-1}{N}} \leq \alpha C_1\Big[1+C_2C_3+C_2+C_3\Big]$$
$$\left(\sum_{j=1}^{n}\int_{\mathbb{R}^N_+}\left|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^{\alpha-1}|\partial_{x_j}u|\right|d\mu+\sum_{l=1}^{q_1}\int_{\mathbb{R}^N_+}\left|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^{\alpha-1}|r_1\partial_{y_{1,l}}u|\right|d\mu\right.$$
$$\left.+\sum_{l=1}^{q_2}\int_{\mathbb{R}^N_+}\left|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^{\alpha-1}|r_1r_2\partial_{y_{2,l}}u|\right|d\mu\right)+C_1C_3(C_2+1)\int_{\mathbb{R}^N_+}\left|r_1^{N-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^{\alpha-1}|u|d\mu\right.$$
$$+C_1C_2(C_3+1)\int_{\mathbb{R}^N_+}\left|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^{\alpha-1}|u|d\mu+\alpha C_1(1+C_3)\int_{\mathbb{R}^N_+}\right|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^{\alpha-1}|r_1\partial_{r_1}u|d\mu$$
$$+\alpha C_1(1+C_2)\int_{\mathbb{R}^N_+}\left|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}|u|^{\alpha-1}|r_1r_2\partial_{r_2}u|d\mu.\right.$$

We consider $\alpha=\frac{(N-1)p}{N-p}>1$ and choose γ_1^* and γ_2^* such that $\gamma_1^*=\frac{(N-p)\gamma_1'}{(N-1)p}$ and $\gamma_2^*=\frac{(N-p)\gamma_2'}{(N-1)p}$. Furthermore, we set $\varphi_ip=N-(\gamma_i^*+1)p$ and $\phi_i\frac{p}{p-1}=\frac{N(N-1-\gamma_i')}{N-1}$ for $i=1,2$. Since $\frac{1}{p^*}=\frac{1}{p}-\frac{1}{N}=\frac{N-p}{Np}=\frac{N-1}{N}-\frac{p-1}{p}$, then one can obtain $\varphi_i=\frac{(N-p)(N-1-\gamma_i')}{(N-1)p}$ and $\phi_i=\frac{N(p-1)(N-1-\gamma_i')}{(N-1)p}$. Therefore, $\varphi_i+\phi_i=N-1-\gamma_i'=n+q_1+q_2-\gamma_i'$ for $i=1,2$. Hence, by the above considerations and by the Hölder inequality we obtain the following calculations:

$$\left(\int_{\mathbb{R}^N_+}\left|r_1^{N-1-\gamma_1'}r_2^{N-1-\gamma_2'}(|u|^{\alpha})^{\frac{N}{N-1}}d\mu\right)^{\frac{N-1}{N}} \leq \alpha C_1\Big[1+C_2C_3+C_2+C_3\Big]\right.$$
$$\left[\sum_{j=1}^{n}\left(\int_{\mathbb{R}^N_+}\big(r_1^{\varphi_1}r_2^{\varphi_2}|\partial_{x_j}u|\big)^pd\mu\right)^{\frac{1}{p}}\times\left(\int_{\mathbb{R}^N_+}\big(r_1^{\phi_1}r_2^{\phi_2}|u|^{\alpha-1}\big)^{\frac{p}{p-1}}d\mu\right)^{\frac{p-1}{p}}\right.$$
$$+\sum_{l=1}^{q_1}\left(\int_{\mathbb{R}^N_+}\big(r_1^{\varphi_1}r_2^{\varphi_2}|(r_1\partial_{y_{1,l}}u)|\big)^pd\mu\right)^{\frac{1}{p}}\times\left(\int_{\mathbb{R}^N_+}\big(r_1^{\phi_1}r_2^{\phi_2}|u|^{\alpha-1}\big)^{\frac{p}{p-1}}d\mu\right)^{\frac{p-1}{p}}+$$
$$\left.\sum_{l=1}^{q_2}\left(\int_{\mathbb{R}^N_+}\big(r_1^{\varphi_1}r_2^{\varphi_2}|(r_1r_2\partial_{y_{2,l}}u)|\big)^pd\mu\right)^{\frac{1}{p}}\times\left(\int_{\mathbb{R}^N_+}\big(r_1^{\phi_1}r_2^{\phi_2}|u|^{\alpha-1}\big)^{\frac{p}{p-1}}d\mu\right)^{\frac{p-1}{p}}\right]+$$
$$C_1C_2(1+C_3)\left(\int_{\mathbb{R}^N_+}\big(r_1^{\varphi_1}r_2^{\varphi_2}|u|\big)^pd\mu\right)^{\frac{1}{p}}\times\left(\int_{\mathbb{R}^N_+}\big(r_1^{\phi_1}r_2^{\phi_2}|u|^{\alpha-1}\big)^{\frac{p}{p-1}}d\mu\right)^{\frac{p-1}{p}}+$$
$$C_1C_3(1+C_2)\left(\int_{\mathbb{R}^N_+}\big(r_1^{\varphi_1}r_2^{\varphi_2}|u|\big)^pd\mu\right)^{\frac{1}{p}}\times\left(\int_{\mathbb{R}^N_+}\big(r_1^{\phi_1}r_2^{\phi_2}|u|^{\alpha-1}\big)^{\frac{p}{p-1}}d\mu\right)^{\frac{p-1}{p}}+$$

$$\alpha C_1(1+C_3)\left(\int_{\mathbb{R}^N_+}\big(r_1^{\varphi_1}r_2^{\varphi_2}|r_1\partial_{r_1}u|\big)^pd\mu\right)^{\frac{1}{p}}\times\left(\int_{\mathbb{R}^N_+}\big(r_1^{\phi_1}r_2^{\phi_2}|u|^{\alpha-1}\big)^{\frac{p}{p-1}}d\mu\right)^{\frac{p-1}{p}}+$$
$$\alpha C_1(1+C_2)\left(\int_{\mathbb{R}^N_+}\big(r_1^{\varphi_1}r_2^{\varphi_2}|r_1r_2\partial_{r_2}u|\big)^pd\mu\right)^{\frac{1}{p}}\times\left(\int_{\mathbb{R}^N_+}\big(r_1^{\phi_1}r_2^{\phi_2}|u|^{\alpha-1}\big)^{\frac{p}{p-1}}d\mu\right)^{\frac{p-1}{p}}.$$

Finally, we set $\gamma_1 = \gamma_1^* + 1$ and $\gamma_2 = \gamma_2^* + 1$. Therefore,

$$\left\|u\right\|_{L^{\gamma_1^*,\gamma_2^*}_{p^*}(\mathbb{R}^N_+)} = \left(\int_{\mathbb{R}^N_+}\left|r_1^{\frac{N}{p^*}-\gamma_1^*}r_2^{\frac{N}{p^*}-\gamma_2^*}u\right|^{p^*}d\mu\right)^{\frac{1}{p^*}} \leq$$

$$\alpha C_1\Big[1 + C_2C_3 + C_2C_3\Big]\Bigg[\sum_{j=1}^{n}\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(\partial_{x_j}u)|^p d\mu\right)^{\frac{1}{p}}$$

$$+\sum_{l=1}^{q_1}\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(r_1\partial_{y_{1,l}}u)|^p d\mu\right)^{\frac{1}{p}} + \sum_{l=1}^{q_2}\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(r_1r_2\partial_{y_{2,l}}u)|^p d\mu\right)^{\frac{1}{p}}\Bigg]$$

$$+C_1C_3(1+C_2)\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}u|^p d\mu\right)^{\frac{1}{p}} + C_1C_2(1+C_3)\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}u|^p d\mu\right)^{\frac{1}{p}}$$

$$+\alpha C_1(1+C_3)\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(r_1\partial_{r_1}u)|^p d\mu\right)^{\frac{1}{p}}$$

$$+\alpha C_1(1+C_2)\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(r_1r_2\partial_{r_2}u)|^p d\mu\right)^{\frac{1}{p}}.$$

Hence, we can regular the coefficients of the previous calculations and then write it as follows:

$$\alpha D_1\Bigg[\sum_{j=1}^{n}\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(\partial_{x_j}u)|^p d\mu\right)^{\frac{1}{p}}$$

$$+\sum_{l=1}^{q_1}\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(r_1\partial_{y_{1,l}}u)|^p d\mu\right)^{\frac{1}{p}} + \sum_{l=1}^{q_2}\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(r_1r_2\partial_{y_{2,l}}u)|^p d\mu\right)^{\frac{1}{p}}\Bigg]$$

$$+D_2\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}u|^p d\mu\right)^{\frac{1}{p}} + D_3\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}u|^p d\mu\right)^{\frac{1}{p}}$$

$$+\alpha D_4\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(r_1\partial_{r_1}u)|^p d\mu\right)^{\frac{1}{p}} + \alpha D_5\left(\int_{\mathbb{R}^N_+}|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(r_1r_2\partial_{r_2}u)|^p d\mu\right)^{\frac{1}{p}}$$

where,

$$D_1 = \frac{1}{N} + \frac{1}{N}\left(|\frac{(N-1)^2(N-p\gamma_1)(N-p\gamma_2)}{(N-p)}|^{\frac{1}{N}} + |\frac{(N-1)(N-p\gamma_1)}{N-p}|^{\frac{1}{N}} + |\frac{(N-1)(N-p\gamma_2)}{N-p}|^{\frac{1}{N}}\right),$$

$$D_2 = \frac{1}{N}|\frac{(N-1)(N-p\gamma_2)}{N-p}|^{\frac{1}{N}}\left(1 + |\frac{(N-1)(N-p\gamma_1)}{N-p}|^{\frac{1}{N}}\right),$$

$$D_3 = \frac{1}{N}|\frac{(N-1)(N-p\gamma_1)}{N-p}|^{\frac{1}{N}}\left(1 + |\frac{(N-1)(N-p\gamma_2)}{N-p}|^{\frac{1}{N}}\right),$$

$$D_4 = \frac{1}{N} + \frac{1}{N}|\frac{(N-1)(N-p\gamma_2)}{N-p}|^{\frac{1}{N}},$$

$$D_5 = \frac{1}{N} + \frac{1}{N}|\frac{(N-1)(N-p\gamma_1)}{N-p}|^{\frac{1}{N}}.$$

□

Remark 3.2. *In the case of $\gamma_1 = \gamma_2 = \frac{N}{p}$, one can obtain that the constants $D_2 = D_3 = 0$ and the constants $D_1 = D_4 = D_5 = \frac{1}{N}$, then the Hölder inequality implies that for every $u(r_1, x, y_1, r_2, y_2) \in \mathcal{H}_p^{1,(\frac{N}{p},\frac{N}{p})}(\mathbb{R}_+^N)$*

$$\Big\|u\Big\|_{L_{p^*}^{\gamma_1^*,\gamma_2^*}} \leq C\Big\|\nabla_{\mathbb{K}}u\Big\|_{L_p^{\gamma_1,\gamma_2}}$$

where $\nabla_{\mathbb{K}} = (r_1\partial_{r_1}, \partial_x, r_1\partial_{y_1}, r_1r_2\partial_{r_2}, r_1r_2\partial_{y_2})$ is the corner-edge type gradient operator on the stretched corner-edge manifold $\mathbb{K}$ and the constant $C = \alpha C_1 = \frac{(N-1)p}{N(N-p)}$ is the best constant to the corner-edge type Sobolev inequality.

Proposition 3.3. *(Poincaré Inequality) Let $\mathbb{K} = \mathbb{E} \times [0, 1) \times Y_2$ be a bounded subset in $\mathbb{R}_+^N = \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{q_1} \times \mathbb{R}_+ \times \mathbb{R}^{q_2}$. For $u(r_1, x, y_1, r_2, y_2) \in \mathcal{H}_{p,0}^{1,(\gamma_1,\gamma_2)}(\mathbb{K})$, $1 \leq p < \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$, the following inequality holds*

$$\Big\|u\Big\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{K})} \leq d\Big\|\nabla_{\mathbb{K}}u\Big\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{K})}$$

where d is the diameter of $\mathbb{K}$.

Proof. First we consider

$$\Gamma = \Big\{(r_1, x, y_1, r_2, y_2) \in \mathbb{R}^N \mid 0 < r_1 < d,\ a_i < x_i < a_i + d,\ i = 1, ..., n,$$
$$0 < r_2 < d,\ y_1 \in Y_1 \subset \mathbb{R}^{q_1},\ y_2 \in Y_2 \subset \mathbb{R}^{q_2}\Big\}$$

where $d \in \mathbb{R}_+$ is large enough such that $\mathbb{K} \subset \Gamma$. Suppose that $u(r_1, x, y_1, r_2, y_2) \in C_0^\infty(\mathbb{K})$. Then for every $(r_1, x, y_1, r_2, y_2) \in \mathbb{K} \subset \Gamma$ one gets:

$$u(r_1, x, y_1, r_2, y_2) = \int_{a_1}^{x_1} \partial_{x_1}u(r_1, s, x_2, ..., x_n, y_1, r_2, y_2)ds.$$

Now we can use the Höder inequality and obtain the following inequalities:

$$\begin{aligned} \Big|u(r_1, x, y_1, r_2, y_2)\Big|^p &\leq \Big(\int_{a_1}^{x_1} 1^{\frac{p}{p-1}}ds\Big)^{\frac{p-1}{p}} \Big(\int_{a_1}^{x_1} \big|\partial_{x_1}u(r_1, s, x_2, ..., x_n, y_1, r_2, y_2)\big|^p ds\Big)^{\frac{1}{p}} \\ &\leq d^p\Big(\int_{a_1}^{x_1} \big|\partial_{x_1}u(r_1, s, x_2, ..., x_n, y_1, r_2, y_2)\big|^p ds\Big)^{\frac{1}{p}}. \end{aligned} \tag{3.11}$$

Therefore, one can apply the mean value theorem for the inequality 3.12 and obtain that

$$\Big|u(r_1, x, y_1, r_2, y_2)\Big|^p \leq d^p\big|\partial_{x_1}u(r_1, x_1', x_2, ..., x_n, y_1, r_2, y_2)\big|^p,$$

for $a_1 < x_1' < a_1 + d$. Multiplying the both sides with term $r_1^{N-\gamma_1 p}r_2^{N-\gamma_2 p}$ and then integrating with respect to $\frac{dr_1}{r_1}dx\frac{dy_1}{r_1}\frac{dr_2}{r_1r_2}\frac{dy_2}{r_1r_2} = d\mu$ on Γ. Hence,

$$\int_\Gamma \big|r_1^{N-\gamma_1 p}r_2^{N-\gamma_2 p}u(r_1, x, y_1, r_2, y_2)\big|^p d\mu \leq d^p \int_\Gamma \big|r_1^{N-\gamma_1 p}r_2^{N-\gamma_2 p}\partial_{x_1}u(r_1, x_1', x_2, ..., x_n, y_1, r_2, y_2)\big|^p d\mu.$$

According to the definition of Γ and the assumption $u \in C_0^\infty(\mathbb{K})$ one obtains

$$\int_{\mathbb{K}} \left| r_1^{N-\gamma_1 p} r_2^{N-\gamma_2 p} u(r_1, x, y_1, r_2, y_2) \right|^p d\mu \leq d^p \int_{\mathbb{K}} \left| r_1^{N-\gamma_1 p} r_2^{N-\gamma_2 p} \partial_{x_1} u(r_1, x_1', x_2, ..., x_n, y_1, r_2, y_2) \right|^p d\mu.$$

Therefore, the density of $C_0^\infty(\mathbb{K})$ in $\mathcal{H}_{p,0}^{1,(\gamma_1,\gamma_2)}(\mathbb{K})$, for every $u \in \mathcal{H}_{p,0}^{1,(\gamma_1,\gamma_2)}(\mathbb{K})$ we have the following inequality :

$$\left\| u(r_1, x, y_1, r_2, y_2) \right\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{K})} \leq d \left\| \nabla_{\mathbb{K}} u \right\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{K})}.$$

□

Proposition 3.4. *(Weighted Corner-Edge Hardy Inequality) Let* $(r_1, x, y_1, r_2, y_2) \in \mathbb{K}$ *and* $n \geq 2$, $N = 1 + n + q_1 + 1 + q_2$ *and let* $V = \frac{1}{\psi}$ *where*

$$\psi(r_1, x, y_1, r_2, y_2) = r_1^2 + \sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{q_1} y_{1,j}^2 + r_2^2 + \sum_{j=1}^{q_2} y_{2,j}^2.$$

Then for every $u(r_1, x, y_1, r_2, y_2) \in \mathcal{H}_{2,0}^{1,(\frac{\gamma_1}{2},\frac{\gamma_2}{2})}(\mathbb{K})$, *where* $\gamma_1 \geq q_2$ *and* $\gamma_2 \geq -1$, *we have the following Hardy estimate*

$$\int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} V |u|^2 d\mu \leq \left(\frac{2}{N - 2 - q_1 - q_2} \right)^2 \int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} |\nabla_{\mathbb{K}} u|^2 d\mu.$$

Proof. Since $C_0^\infty(int\mathbb{K})$ is dense in $\mathcal{H}_{2,0}^{1,(\frac{\gamma_1}{2},\frac{\gamma_2}{2})}(\mathbb{K})$, it is sufficient to prove the weighted corner-edge Hardy inequality for $u(r_1, x, y_1, r_2, y_2) \in C_0^\infty(\int \mathbb{K})$. To do this, we introduce an operator as follows:

$$R := r_1^2 \partial_{r_1} + \sum_{i=1}^{n} x_i \partial x_i + \sum_{j=1}^{q_1} r_1 \ y_{1,j} \partial_{y_{1,j}} + r_1 r_2^2 \partial_{r_2} + \sum_{j=1}^{q_2} r_1 r_2 \ y_{2,j} \partial_{y_{2,j}}.$$

Now, we use the operator R and obtain the following estimate

$$RV = R(\frac{1}{\psi}) = \frac{2}{\psi^2} A - 2V,$$

where $A = r_1^2(1 - r_1) + r_2^2(1 - r_1 r_2) + \sum_{j=1}^{q_1} y_{1,j}^2 (1 - r_1) + \sum_{j=1}^{q_2} y_{2,j}^2 (1 - r_1 r_2)$. Therefore,

$$\int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} V |u|^2 d\mu = \int_{\mathbb{K}} \frac{r_1^{\gamma_1} r_2^{\gamma_2} A}{\psi^2} |u|^2 d\mu - \frac{1}{2} \int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} RV |u|^2 d\mu.$$

By using of $u = 0$ near the boundary of $\mathbb{K}$, one can obtain

$$\begin{aligned} \int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} RV |u|^2 d\mu \quad &= \quad \int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} \Big[r_1^2 \partial_{r_1}(V) + \sum_{i=1}^{n} x_i \partial_{x_i}(V) + \sum_{j=1}^{q_1} r_1 \ y_{1,j} \partial_{y_{1,j}}(V) + r_1 r_2^2 \partial_{r_2}(V) \\ &+ \quad \sum_{j=1}^{q_2} r_1 r_2 \ y_{2,j} \partial_{y_{2,j}}(V) \Big] |u|^2 d\mu = I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

where

$$
\begin{aligned}
I_1 &= \int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} r_1^2 \partial_{r_1}(V)|u|^2 d\mu \\
&= -(\gamma_1 - q_1 - q_2)\int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} r_1 V|u|^2 d\mu - 2\int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2}(r_1 V u)(r_1\partial_{r_1} u) d\mu,
\end{aligned}
$$

$$
I_2 = \int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} \sum_{i=1}^{n} x_i \partial_{x_i}(V)|u|^2 d\mu = -n\int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} V|u|^2 d\mu - 2\int_{\mathbb{K}} \sum_{i=1}^{n} r_1^{\gamma_1} r_2^{\gamma_2}(x_i V u)(\partial_{x_i} u) d\mu,
$$

$$
\begin{aligned}
I_3 &= \int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} \sum_{j=1}^{q_1} r_1\ y_{1,j}\partial_{y_{1,j}}(V)|u|^2 d\mu = -q_1\int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} r_1 V|u|^2 d\mu \\
&- 2\int_{\mathbb{K}} \sum_{j=1}^{q_1} r_1^{\gamma_1} r_2^{\gamma_2}(y_{1,j} V u)(r_1 \partial_{y_{1,j}} u) d\mu,
\end{aligned}
$$

$$
I_4 = \int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} r_1 r_2^2 \partial_{r_2}(V)|u|^2 d\mu = -(\gamma_2 + 1 - q_2)\int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2}\ r_1 r_2 - 2\int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2}(r_2 V u)(r_1 r_2 \partial_{r_2} u) d\mu,
$$

$$
\begin{aligned}
I_5 &= \int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} \sum_{j=1}^{q_2} r_1 r_2\ y_{2,j}\partial_{y_{2,j}}(V)|u|^2 d\mu = -q_2\int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} r_1 r_2 V|u|^2 d\mu \\
&- 2\int_{\mathbb{K}} \sum_{j=1}^{q_2} r_1^{\gamma_1} r_2^{\gamma_2}(y_{2,j} V u)(r_1 r_2 \partial_{y_{2,j}} u) d\mu.
\end{aligned}
$$

Since $\frac{A}{\psi} \leq 1$, it follows from the assumptions and the above calculations that

$$
\begin{aligned}
\int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2} & \Big[\frac{(q_2-\gamma_1)r_1}{2} - \frac{n}{2} - \frac{(\gamma_2+1)r_1 r_2}{2}\Big] V|u|^2 d\mu \leq \int_{\mathbb{K}} r_1^{\gamma_1} r_2^{\gamma_2}\Big[(r_1 V u)(r_1\partial_{r_1} u) \\
&+ \sum_{i=1}^{n}(x_i V u)(\partial_{x_i} u) + \sum_{j=1}^{q_1}(y_{1,j} V u)(r_1\partial_{y_{1,j}} u) + (r_2 V u)(r_1 r_2 \partial_{r_2} u) \\
&+ \sum_{j=1}^{q_2}(y_{2,j} V u)(r_1 r_2 \partial_{y_{2,j}} u)\Big] d\mu.
\end{aligned}
$$

By making use of the Cauchy-Schwartz inequality on the right hand side of the above inequality and for all $r_1, r_2 \in [0,1)$, we obtain that

$$\begin{aligned}
\frac{n}{2}\int_{\mathbb{K}} \quad r_1^{\gamma_1} \quad r_2^{\gamma_2}V|u|^2d\mu &\le \int_{\mathbb{K}} r_1^{\gamma_1}r_2^{\gamma_2}\left[\frac{n}{2}+\frac{(\gamma_1-q_2)r_1}{2}+\frac{(\gamma_2+1)r_1r_2}{2}\right]V|u|^2d\mu \\
&\le \quad \Bigg(\int_{\mathbb{K}} r_1^{\gamma_1}r_2^{\gamma_2}\Bigg[(r_1\partial_{r_1}u)^2+\sum_{i=1}^{n}(\partial_{x_i}u)^2+\sum_{j=1}^{q_1}(r_1\partial_{y_{1,j}}u)^2 \\
&+ \quad (r_1r_2\partial_{r_2}u)^2+\sum_{j=1}^{q_2}(r_1r_2\partial_{y_{2,j}}u)^2\Bigg]d\mu\Bigg)^{\frac{1}{2}} \times \\
&\quad \left(\int_{\mathbb{K}} r_1^{\gamma_1}r_2^{\gamma_2}\left[(r_1Vu)^2\sum_{i=1}^{n}(x_iVu)^2+\sum_{j=1}^{q_1}(y_{1,j}Vu)^2+(r_2Vu)^2\sum_{j=1}^{q_2}(y_{2,j}Vu)^2\right]d\mu\right)^{\frac{1}{2}} \\
&= \quad \left(\int_{\mathbb{K}} r_1^{\gamma_1}r_2^{\gamma_2}|\nabla_{\mathbb{K}}u|^2d\mu\right)^{\frac{1}{2}} \times \left(\int_{\mathbb{K}} r_1^{\gamma_1}r_2^{\gamma_2}BV|u|^2d\mu\right)^{\frac{1}{2}},
\end{aligned}$$

where $B=\left[r_1^2+\sum_{i=1}^{n}x_i^2+r_1^2\sum_{j=1}^{q_1}y_{1,j}^2+r_2^2+r_1^2r_2^2\sum_{j=1}^{q_2}y_{2,j}^2\right]V\le 1$. Hence,

$$\frac{n}{2}\int_{\mathbb{K}} r_1^{\gamma_1}r_2^{\gamma_2}V|u|^2d\mu \le \left(r_1^{\gamma_1}r_2^{\gamma_2}|\nabla_{\mathbb{K}}u|^2d\mu\right)^{\frac{1}{2}} \times \left(\int_{\mathbb{K}} r_1^{\gamma_1}r_2^{\gamma_2}V|u|^2d\mu\right)^{\frac{1}{2}}.$$

Therefore,

$$\int_{\mathbb{K}} r_1^{\gamma_1}r_2^{\gamma_2}V|u|^2d\mu \le \left(\frac{2}{N-2-q_1-q_2}\right)^2\int_{\mathbb{K}} r_1^{\gamma_1}r_2^{\gamma_2}|\nabla_{\mathbb{K}}u|^2d\mu.$$

□

Theorem 3.5. *Suppose that m, m', $\gamma_1, \gamma_1^{'}$ and $\gamma_2, \gamma_2^{'}$ are real numbers such that $m^{'} \ge m$, $\gamma_1^{'} \ge \gamma_1$ and $\gamma_2^{'} \ge \gamma_2$, then the embedding map*

$$\mathcal{H}_{p,0}^{m^{'},(\gamma_1^{'},\gamma_2^{'})}(\mathbb{K}) \hookrightarrow \mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{K})$$

is continuous and this map is compact if $m^{'} > m$, $\gamma_1^{'} \ge \gamma_1$ and $\gamma_2^{'} \ge \gamma_2$.

Proof. The weighted corner-edge Sobolev spaces $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{K})$ are in the form of non-direct sum as the Definition 2.4. For the classical Sobolev spaces $W_0^{m,p}(\Lambda_{\epsilon_1}\times X\times Y_1\times\Lambda_{\epsilon_2}\times Y_2)$ is continuous for $m^{'}\ge m$ and is compact for $m^{'}>m$. Now, we prove the similar properties for the embedding map

$$[\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{m^{'},\gamma_1^{'}}(\mathbb{R}_+\times X\times Y_1\times\Lambda_{\epsilon_2}\times Y_2) \hookrightarrow [\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{m,\gamma_1}(\mathbb{R}_+\times X\times Y_1\times\Lambda_{\epsilon_2}\times Y_2).$$

To do this, we define the following map for any $u\in\mathcal{H}_{p,0}^{m,\gamma_1}(\mathbb{R}_+\times X\times Y_1\times\Lambda_{\epsilon_2}\times Y_2)$,

$$S_p^{\gamma_1}(u) := e^{-s_1(\frac{N}{p}-\gamma_1)}u(e^{-s_1},x,e^{-s_1}\xi_1,r_2,y_2) = V(s_1,x,\xi_1,r_2,y_2)$$

with $\xi_1 = e^{s_1} y_1$ and $r_1 = e^{-s_1}$. Therefore, we have an isomorphism as

$$S_p^{\gamma_1} : [\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{m',\gamma_1'}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \hookrightarrow [\tilde{\omega}_1][1-\omega_2]W_0^{m,p}(\mathbb{R}_+ \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2),$$

where $\tilde{\omega}_1 = \tilde{\omega}(s_1) = \omega(e^{-s_1}) = \omega(r_1)$ and $\xi_1 \in \tilde{Y}_1$ if and only if $y_1 \in e^{-s_1}\xi_1 \in Y_1$ for $r_1 \in supp\tilde{\omega}(s_1)$. Hence, $S_p^{\gamma_1}$ induces an isomorphism as follows :

$$S_p^{\gamma_1} : [\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{m',\gamma_1'}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \hookrightarrow [\tilde{\omega}_1][1-\omega_2]e^{-s_1(\gamma_1'-\gamma_1)}W_0^{m',p}(\mathbb{R}_+ \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2).$$

In fact, for every $u \in \mathcal{H}_{p,0}^{m',\gamma_1'}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)$ we have

$$\begin{aligned} S_p^{\gamma_1} u &= e^{-s_1(\frac{N}{p}-\gamma_1)}u(e^{-s_1}, x, e^{-s_1}\xi_1, r_2, y_2) = e^{-s_1(\gamma_1'-\gamma_1)}e^{-s_1(\frac{N}{p}-\gamma_1')}u(e^{-s_1}, x, e^{-s_1}\xi_1, r_2, y_2) \\ &= e^{-s_1(\gamma_1'-\gamma_1)}S_p^{\gamma_1} = v \in W_0^{m',p}(\mathbb{R}_+ \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2). \end{aligned}$$

This means that

$$[\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{m',\gamma_1'}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \hookrightarrow [\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{m,\gamma_1}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)$$

is a continuous embedding for $m' \geq m$, $\gamma_1' \geq \gamma_1$. Now for compactness of the embedding map, for $\gamma_1' > \gamma_1$ we set $e^{-s_1(\gamma_1'-\gamma_1)}s_1^{\delta} = \chi_\delta(s_1)$. Hence, all its derivatives in s_1 are uniformly bounded over $supp(\tilde{\omega})$ for $\delta > 0$. This implies that

$$\begin{aligned} &[\tilde{\omega}_1][1-\omega_2]e^{-s_1(\gamma_1'-\gamma_1)}W_0^{m',p}(\mathbb{R}_+ \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2) \\ &\quad= [\tilde{\omega}_1][1-\omega_2]e^{-s_1(\gamma_1'-\gamma_1)}[\chi_\delta]s_1^{-\delta}W_0^{m',p}(\mathbb{R}_+ \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2) \\ &\quad\hookrightarrow [\tilde{\omega}_1][1-\omega_2]e^{-s_1(\gamma_1'-\gamma_1)}s_1^{-\delta}W_0^{m',p}(\mathbb{R}_+ \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2) \\ &\quad\hookrightarrow [\tilde{\omega}_1][1-\omega_2]e^{-s_1(\gamma_1'-\gamma_1)}W_0^{m,p}(\mathbb{R}_+ \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2) \end{aligned}$$

where, the last embedding is compact for $m' > m$, $\gamma_1' > \gamma_1$. Moreover, one can obtain the similar result for the embedding map

$$[1-\omega_1][\omega_2]\mathcal{H}_{p,0}^{m',\gamma_2'}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \hookrightarrow [1-\omega_1][\omega_2]\mathcal{H}_{p,0}^{m,\gamma_2}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2),$$

if the similar isomorphism for any $u \in \mathcal{H}_{p,0}^{m,\gamma_2}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)$ is defined by

$$S_p^{\gamma_2} u := e^{-s_2(\frac{N}{p}-\gamma_2)}u(r_1, x, y_1, e^{-r_1 s_2}, e^{-s_1-s_2}\xi_2) = w(r_1, x, y_1, s_2, \xi_2),$$

with $\xi_2 = e^{-s_1-s_2}y_2$ and $r_2 = e^{-r_1 s_2}$. Now, it is enough to show that the embedding

$$[\omega_1][\omega_2]\mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \hookrightarrow [\omega_1][\omega_2]\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)$$

is continuous for $m' \geq m$, $\gamma_1' \geq \gamma_1$, $\gamma_2' \geq \gamma_2$ and is compact for $m' > m$, $\gamma_1' \geq \gamma_1$, $\gamma_2' \geq \gamma_2$ and any cut-off functions ω_1 and ω_2 with supports in the collar neighborhoods of $(0,1) \times \partial\mathbb{K}$

and $\partial\mathbb{K}\times(0,1)$ respectively. According to the isomorphisms $S_p^{\gamma_1}$ and $S_p^{\gamma_2}$, we consider the transform $T:\Lambda\to\tilde{\Lambda}$ which is defined by

$$T(r_1,x,y_1,r_2,y_2)=(s_1,x,\xi_1,s_2,\xi_2)=(-\ln r_1,x,\frac{y_1}{r_1},\frac{-\ln r_2}{r_1},\frac{y_2}{r_1r_2}),$$

where

$$\Lambda=\Big\{(r_1,x,y_1,r_2,y_2)\in\mathbb{R}_+\times X\times Y_1\times\mathbb{R}_+\times Y_2\mid(r_1,x,y_1)\in Supp(\omega_1)\ and\ (x,r_2,y_2)\in Supp(\omega_2)\Big\}$$
$$\tilde{\Lambda}=\Big\{(s_1,x,\xi_1,s_2,\xi_2)\in\mathbb{R}_+\times X\times\tilde{Y}_1\times\mathbb{R}_+\times\tilde{Y}_2\mid(s_1,x,\xi_1)\in Supp(\tilde{\omega}_1)\ and\ (x,s_2,\xi_2)\in Supp(\tilde{\omega}_2)\Big\}.$$

Therefore, we can define the following map

$$S_p^{\gamma_1,\gamma_2}:=S_p^{\gamma_1}oS_p^{\gamma_2}:u(r_1,x,y_1,r_2,y_2)\mapsto e^{-s_1(\frac{N}{p}-\gamma_1)}e^{-r_1s_2(\frac{N}{p}-\gamma_2)}u(e^{-s_1},x,e^{-s_1}\xi,e^{-r_1s_2},e^{-s_1-s_2}\xi_2)$$

which gives an isomorphism as follows

$$S_p^{\gamma_1,\gamma_2}:[\omega_1][\omega_2]\mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{R}_+\times X\times Y_1\times\mathbb{R}_+\times Y_2)\hookrightarrow[\tilde{\omega}_1][\tilde{\omega}_2]W_0^{m,p)}(\mathbb{R}_+\times X\times Y_1\times\mathbb{R}_+\times Y_2). \tag{3.12}$$

According to the definition of the transform T and the using the chain rule we obtain that

$$\begin{aligned}\partial_{s_1}&=-r_1\partial_{r_1}-y_1\partial_{y_1}-\frac{\ln r_2}{r_1}\partial_{r_2}-y_2\partial_{y_2}\ and\\ \partial_{s_2}&=-r_1r_2\partial_{r_2}-y_2\partial_{y_2}.\end{aligned}$$

Hence, if we consider the Jacobian matrix of the transformation T that is

$$J_T=\begin{pmatrix}\frac{-1}{r_1}&0&0&0&0\\0&1&0&0&0\\\frac{-y_1}{r_1^2}&0&\frac{1}{r_1}&0&0\\\frac{\ln r_2}{r_1^2}&0&0&\frac{-1}{r_1r_2}&0\\\frac{-y_2}{r_1r_2}&0&0&\frac{-y_2}{r_1r_2}&\frac{1}{r_1r_2}\end{pmatrix}.$$

Then the determinate of J_T, is $det(J_T) = \frac{1}{r_1^4 r_2^2}$. Therefore,

$$\begin{aligned}
\left\| w(s_1,x,\xi_1,s_2,\xi_2) \right\|^p_{W_0^{m,p}(\tilde{\Lambda})} &= \sum_{l+|\alpha|+k+|\beta|+|\theta|\le m} \int_{\tilde{\Lambda}} \left| \partial_\xi^l \partial_x^\alpha \partial_{\xi_1}^\beta \partial_{s_2}^k \partial_{\xi_2}^\theta w(s_1,x,\xi_1,s_2,\xi_2) \right|^p ds_1 dx d\xi_1 ds_2 d\xi_2 \\
&= \sum_{l+|\alpha|+k+|\beta|+|\theta|\le m} \int_{\tilde{\Lambda}} \Big| \partial_\xi^l \partial_x^\alpha \partial_{\xi_1}^\beta \partial_{s_2}^k \partial_{\xi_2}^\theta e^{-s_1(\frac{N}{p}-\gamma_1)} e^{-r_1 s_2(\frac{N}{p}-\gamma_2)} \\
&\quad \times u(e^{-s_1},x,e^{-s_1}\xi_1,e^{-r_1 s_2},e^{-s_1-s_2}\xi_2) \Big|^p ds_1 dx d\xi_1 ds_2 d\xi_2 \\
&\cong \sum_{l+|\alpha|+k+|\beta|+|\theta|\le m} \int_{\tilde{\Lambda}} \Big| e^{-s_1(\frac{N}{p}-\gamma_1)} e^{-r_1 s_2(\frac{N}{p}-\gamma_2)} \partial_\xi^l \partial_x^\alpha \partial_{\xi_1}^\beta \partial_{s_2}^k \partial_{\xi_2}^\theta \\
&\quad \times u(e^{-s_1},x,e^{-s_1}\xi_1,e^{-r_1 s_2},e^{-s_1-s_2}\xi_2) \Big|^p ds_1 dx d\xi_1 ds_2 d\xi_2 \\
&\cong \sum_{l+|\alpha|+k+|\beta|+|\theta|\le m} \int_{\tilde{\Lambda}} \Big| e^{-s_1(\frac{N}{p}-\gamma_1)} e^{-r_1 s_2(\frac{N}{p}-\gamma_2)} \partial_\xi^l \partial_x^\alpha (e^{-s_1 y_1})^\beta \partial_{s_2}^k (e^{-r_1 s_2}\partial_{y_2})^\theta \\
&\quad \times u(e^{-s_1},x,y_1,e^{-r_1 s_2},y_2) \Big|^p ds_1 dx \frac{dy_1}{r_1} ds_2 \frac{dy_2}{r_1 r_2} \\
&\cong \sum_{l+|\alpha|+k+|\beta|+|\theta|\le m} \int_{\Lambda} \Big| r_1^{\frac{N}{p}-\gamma_1} r_2^{\frac{N}{p}-\gamma_2} (r_1\partial_{r_1})^l \partial_x^\alpha (r_1\partial_{y_1})^\beta (r_1 r_2 \partial_{r_2})^k (r_1 r_2 \partial_{y_2})^\theta \\
&\quad \times u(r_1,x,y_1,r_2,y_2) \Big|^p d\mu = \| u(r_1,x,y_1,r_2,y_2) \|^p_{\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\Lambda)}.
\end{aligned}$$

On the other hand, one can use the similar way to prove the same result for $\mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)$. In fact, suppose that $u(r_1,x,y_1,r_2,y_2) \in \mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)$ then we set $u(r_1,xy_1,r_2,y_2) = \omega_1\omega_2 u(_1,x,y_1,r_2,y_2)$. Hence,

$$\begin{aligned}
S_p^{\gamma_1,\gamma_2} u &= e^{-s_1(\frac{N}{p}-\gamma_1)} e^{-r_1 s_2(\frac{N}{p}-\gamma_2)} u(e^{-s_1},x,e^{-s_1}\xi_1,e^{-r_1 s_2},e^{-s_1-s_2}\xi_2) \\
&= e^{-s_1(\gamma_1'-\gamma_1)} e^{-r_1 s_2(\gamma_2'-\gamma_2)} h(e^{-s_1},x,e^{-s_1}\xi_1,e^{-r_1 s_2},e^{-s_1-s_2}\xi_2)
\end{aligned}$$

where, $h(e^{-s_1},x,e^{-s_1}\xi_1,e^{-r_1 s_2},e^{-s_1-s_2}\xi_2) = e^{-s_1(\frac{N}{p}-\gamma_1)} e^{-r_1 s_2(\frac{N}{p}-\gamma_2)} u(e^{-s_1},x,e^{-s_1}\xi_1,e^{-r_1 s_2},e^{-s_1-s_2}\xi_2)$ such that $h \in W_0^{m',p}(\mathbb{R} \times X \times Y_1 \times \mathbb{R} \times Y_2)$. Indeed, $S_p^{\gamma_1,\gamma_2}$ induces an isomorphism from $\mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)$ to $W_0^{m',p}(\mathbb{R} \times X \times Y_1 \times \mathbb{R} \times Y_2)$. Thus, for $s_1 \in Supp(\tilde{\omega}_1)$ and $s_2 \in Supp(\tilde{\omega}_2)$ we have the following embedding

$$[\tilde{\omega}_1][\tilde{\omega}_2] e^{-s_1(\gamma_1'-\gamma_1)} e^{-r_1 s_2(\gamma_2'-\gamma_2)} W_0^{m',p}(\mathbb{R} \times X \times Y_1 \times \mathbb{R} \times Y_2) \hookrightarrow \tilde{\omega}_2] W_0^{m,p}(\mathbb{R} \times X \times Y_1 \times \mathbb{R} \times Y_2)$$

which is continuous for $m' \ge m$, $\gamma_1' \ge \gamma_1$, $\gamma_2' \ge \gamma_2$ and is compact for $m' > m$, $\gamma_1' \ge \gamma_1$, $\gamma_2' \ge \gamma_2$. Therefore, the required properties hold for the following embedding :

$$[\omega_1][\omega_2] \mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \hookrightarrow [\omega_1][\omega_2] \mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2).$$

□

Proposition 3.6. *For $1 < p < q < 2^*$, the following embedding*

$$\mathcal{H}_{p,0}^{1,(\frac{N-1}{p},\frac{N}{p})}(\mathbb{K}) \hookrightarrow \mathcal{H}_{q,0}^{0,(\frac{N-1}{q},\frac{N}{q})}(\mathbb{K})$$

is compact.

Proof. For $\mathbb{K} = \mathbb{E} \times [0,1) \times Y_2 \subset W^\wedge \times Y_2$ and according to the definition of the space $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{K})$, we can write $\mathcal{H}_{p,0}^{1,(\frac{N-1}{p},\frac{N}{p})}(\mathbb{K})$ and $\mathcal{H}_{q,0}^{0,(\frac{N-1}{q},\frac{N}{q})}(\mathbb{K})$ as follows

$$\begin{aligned}\mathcal{H}_{p,0}^{1,(\frac{N-1}{p},\frac{N}{p})}(\mathbb{K}) := {}&[\omega_1][\omega_2]\mathcal{H}_{p,0}^{1,(\frac{N-1}{p},\frac{N}{p})}(W^\wedge \times Y_2) + [1-\omega_1][\omega_2]\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)\\ &+ [\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{1,\frac{N-1}{p}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)\\ &+ [1-\omega_1][1-\omega_2]W_0^{1,p}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)\end{aligned}$$

and also

$$\begin{aligned}\mathcal{H}_{q,0}^{0,(\frac{N-1}{q},\frac{N}{q})}(\mathbb{K}) := {}&[\omega_1][\omega_2]\mathcal{H}_{q,0}^{0,(\frac{N-1}{q},\frac{N}{q})}(W^\wedge \times Y_2) + [1-\omega_1][\omega_2]\mathcal{H}_{q,0}^{1,\frac{N}{q}}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)\\ &+ [\omega_1][1-\omega_2]\mathcal{H}_{q,0}^{1,\frac{N-1}{q}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)\\ &+ [1-\omega_1][1-\omega_2]L^q(\Lambda_{\epsilon_1} \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2).\end{aligned}$$

From Rellich-Kondrachov theorem, for the classical Sobolev space $W_0^{1,p}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)$ we have the compactness of the following embedding

$$[1-\omega_1][1-\omega_2]W_0^{1,p}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \hookrightarrow [1-\omega_1][1-\omega_2]L^q(\Lambda_{\epsilon_1} \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2).$$

Now, we have to show that the embedding

$$[1-\omega_1][\omega_2]\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \hookrightarrow [1-\omega_1][\omega_2]\mathcal{H}_{q,0}^{0,\frac{N}{q}}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \tag{3.13}$$

$$[\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{1,\frac{N-1}{p}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \hookrightarrow [\omega_1][1-\omega_2]\mathcal{H}_{q,0}^{0,\frac{N-1}{q}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \tag{3.14}$$

are compact. To do this, we have the following isomorphism

$$S_q^{\frac{N-1}{q}} : [\omega_1][1-\omega_2]\mathcal{H}_{q,0}^{0,\frac{N-1}{q}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \hookrightarrow [\tilde{\omega}_1][1-\omega_2]L^q(\mathbb{R}_+ \times X \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2), \tag{3.15}$$

where $\tilde{\omega}_1 = \tilde{\omega}(s_1) = \omega(e^{-s_1})$ and it holds that $\xi_1 \in \tilde{Y}_1$ if and only if $y_1 = e^{-s_1}\xi_1 \in Y_1$ for $s_1 \in Supp(\tilde{\omega}_1)$. Using Proposition 2.4, for $u(r_1, x, y_1, r_2, y_2) \in \mathcal{H}_{q,0}^{0,\frac{N-1}{q}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)$ we have

$$\left(S_q^{\frac{N-1}{q}} \omega_1(1-\omega_2)u(r_1, x, y_1, r_2, y_2)\right)(s_1, x, \xi_1, r_2, y_2) = \omega(e^{-s_1})e^{-s_1(\frac{N-1}{q}, \frac{N}{q})}u(e^{-s_1}, x, e^{-s_1}\xi_1, r_2, y_2).$$

By the similar way, we obtain that

$$\left(S_p^{\frac{N-1}{p}} \omega_1(1-\omega_2)u(r_1, x, y_1, r_2, y_2)\right)(s_1, x, \xi_1, r_2, y_2) = \omega(e^{-s_1})e^{-s_1(\frac{N-1}{p}, \frac{N}{p})}u(e^{-s_1}, x, e^{-s_1}\xi_1, r_2, y_2)$$

such that implies an isomorphism as follows

$$S_p^{\frac{N-1}{p}} : [\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{1,\frac{N-1}{p}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \to [\tilde{\omega}_1][1-\omega]H_0^1(\mathbb{R} \times X \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2).$$

On the other hand, for $u(r_1, x, y_1, r_2, y_2) \in \mathcal{H}_{p,0}^{1,\frac{N-1}{p}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)$ one can imply that

$$\begin{aligned}(S_q^{\frac{N-1}{q}} u)(s_1, x, \xi_1, r_2, y_2) &= \omega(e^{-s_1})e^{-s_1(\frac{N-1}{q}, \frac{N-1}{q})}u(e^{-s_1}, x, e^{-s_1}\xi_1, r_2, y_2)\\ &= \omega(e^{-s_1})e^{-s_1(\frac{N-1}{p}, \frac{N-1}{p})}u(e^{-s_1}, x, e^{-s_1}\xi_1, r_2, y_2) = (S_p^{\frac{N-1}{p}} u)(s_1, x, \xi_1, r_2, y_2).\end{aligned}$$

Therefore,

$$S_q^{\frac{N-1}{q}} : [\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{1,\frac{N-1}{p}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)[\tilde{\omega}_1][1-\omega_2]H_0^1(\mathbb{R} \times X \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2)$$

is a an isomorphism. Because of the compactness of the embedding

$$[\tilde{\omega}_1][1-\omega_2]W_0^{1,p}(\mathbb{R} \times X \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2) \to [\tilde{\omega}_1][1-\omega_2]L^q(\mathbb{R} \times X \times \tilde{Y}_1 \times \Lambda_{\epsilon_2} \times Y_2) \quad (3.16)$$

is compact for $1 < p < q < 2^*$, we can imply that the embedding map

$$[\omega_1][1-\omega_2]\mathcal{H}_{p,0}^{1,\frac{N-1}{p}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2) \hookrightarrow \mathcal{H}_{q,0}^{0,\frac{N-1}{q}}(\mathbb{R}_+ \times X \times Y_1 \times \Lambda_{\epsilon_2} \times Y_2)$$

is also compact by the isomorphisms 3.15 and 3.16. By the similar method one can consider the following isomorphisms

$$S_q^{\frac{N}{q}} : [1-\omega_1][\omega_2]\mathcal{H}_{q,0}^{0,\frac{N}{q}}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \to [1-\omega_1][\tilde{\omega}_2]L^q(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times \tilde{Y}_2)$$

where $\tilde{\omega}_2 = \tilde{\omega}(s_2) = \omega(e^{-r_1 s_2})$ and $\xi_2 \in \tilde{Y}_2$ if and only if $y_2 = e^{-s_1-s_2}\xi_2 \in Y_2$ for $s_2 \in Supp(\tilde{\omega}_2)$. Also

$$S_p^{\frac{N}{p}} : [1-\omega_1][\omega_2]\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \to [1-\omega_1][\tilde{\omega}_2]W_0^{1,p}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times \tilde{Y}_2).$$

Then one obtains that the embedding

$$[1-\omega_1][\omega_2]\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \hookrightarrow [1-\omega_1][\omega_2]\mathcal{H}_{q,0}^{0,\frac{N}{q}}(\Lambda_{\epsilon_1} \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)$$

is compact for $1 < q < 2^*$. Finally, it is enough to show the compactness of the embedding

$$[\omega_1][\omega_2]\mathcal{H}_{p,0}^{1,(\frac{N-1}{p},\frac{N}{p})}(W^\wedge \times Y_2) \hookrightarrow [\omega_1][\omega_2]\mathcal{H}_{q,0}^{0,(\frac{N-1}{q},\frac{N}{q})}(W^\wedge \times Y_2).$$

According to the transform T, for $u(r_1,x,y_1,r_2,y_2) \in [\omega_1][\omega_2]\mathcal{H}^{0,(\frac{N-1}{q},\frac{N}{q})}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)$, the mapping

$$\begin{aligned} S_q^{(\frac{N-1}{q},\frac{N}{q})} u(r_1,x,y_1,r_2,y_2) &:= e^{-s_1(\frac{N}{q}-\frac{N-1}{q})} e^{-r_1 s_2(\frac{N}{q}-\frac{N}{q})} u(e^{-s_1},x,e^{-s_1}\xi_1,e^{-r_1 s_2},e^{-s_1-s_2}\xi_2) \\ &:= w(s_1,x,\xi_1,s_2,\xi_2) \end{aligned}$$

induces an isomorphism

$$S_q^{(\frac{N-1}{q},\frac{N}{q})} : [\omega_1][\omega_2]\mathcal{H}_{q,0}^{0,(\frac{N-1}{q},\frac{N}{q})}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \to [\tilde{\omega}_1][\tilde{\omega}_2]L^q(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \tag{3.17}$$

where $r_1 = e^{-s_1}$ for $r_1 \in Supp(\omega_1)$, $\tilde{\omega}(s_1) = \omega(e^{-s_1})$ and $\tilde{\omega}(s_2)$ is the cut-off function in $s_2 = \frac{-\ln r_2}{r_1}$ with $r_1 \in Supp(\omega(r_2))$ and $r_2 \in Supp(\omega(r_1))$. On the other hand, for $u(r_1,x,y_1,r_2,y_2) \in [\omega_1][\omega_2]\mathcal{H}_{p,0}^{1,(\frac{N-1}{p},\frac{N}{p})}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)$ we obtain

$$\begin{aligned} S_q^{(\frac{N-1}{q},\frac{N}{q})} u(r_1,x,y_1,r_2,y_2) &= e^{-s_1(\frac{N}{q}-\frac{N-1}{q})} e^{-r_1 s_2(\frac{N}{q}-\frac{N}{q})} u(e^{-s_1},x,e^{-s_1}\xi_1,e^{r_1 s_2},e^{-s_1-s_2}\xi_2) \\ &= e^{-s_1(\frac{1}{q}-\frac{1}{2})} S_p^{\frac{N-1}{p},\frac{N}{p}} u(r_1,x,y_1,r_2,y_2). \end{aligned}$$

Upon the isomorphism 3.12, $S_q^{\frac{N-1}{q},\frac{N}{q}}$ gives the following isomorphism

$$S_q^{\frac{N-1}{q},\frac{N}{q}} : [\omega_1][\omega_2]\mathcal{H}_{p,0}^{1,(\frac{N-1}{p},\frac{N}{p})}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \to [\tilde{\omega}_1][\tilde{\omega}_2]W_0^{1,p}(\mathbb{R} \times X \times Y_1 \times \mathbb{R} \times Y_2). \tag{3.18}$$

Moreover, the embedding

$$[\tilde{\omega}_1][\tilde{\omega}_2]W_0^{1,p}(\mathbb{R} \times X \times Y_1 \times \mathbb{R} \times Y_2) \hookrightarrow L^q(\mathbb{R} \times X \times Y_1 \times \mathbb{R} \times Y_2)$$

is compact for $1 < p < q < 2^*$ and $s_1 \in Supp(\tilde{\omega}_1)$ and $s_2 \in Supp(\tilde{\omega}_2)$. Therefore, the

$$[\omega_1][\omega_2]\mathcal{H}_{p,0}^{1,(\frac{N-1}{p},\frac{N}{p})}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2) \hookrightarrow [\omega_1][\omega_2]\mathcal{H}_{q,0}^{0,(\frac{N-1}{q},\frac{N}{q})}(\mathbb{R}_+ \times X \times Y_1 \times \mathbb{R}_+ \times Y_2)$$

is compact from the isomorphisms 3.17 and 3.18. $\square$

4. Global existence solution

In this section, we concerned with the following initial-boundary value problem for a thermoelastic system contains corner-edge Laplacian and $p-$Laplacian type operators with potential term as

$$\begin{cases} u_{tt} - \Delta_{p,\mathbb{K}}u - \epsilon V(\tilde{x})u + \psi = |u|^{\alpha-1}u, & (\tilde{x},t) \in int\mathbb{K} \times (0,T), \\ \psi_t - \Delta_{\mathbb{K}}u = u_t, & (\tilde{x},t) \in int\mathbb{K} \times (0,T), \\ u(\tilde{x},0) = u_0(\tilde{x}),\ u_t(\tilde{x},0) = u_1(\tilde{x}), & \\ \psi(\tilde{x},0) = \psi_0(\tilde{x}), & \tilde{x} \in int\mathbb{K}, \\ u(\tilde{x},t) = \psi(\tilde{x},t) = 0, & (\tilde{x},t) \in \partial\mathbb{K} \times (0,T), \end{cases} \tag{4.1}$$

where $u_0 \in \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})$, $u_1 \in L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})$, $\psi_0 \in L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})$, $T \in (0,\infty]$, $2 \leq p < \infty$, $\gamma_1 = N-(q_1+q_2), \gamma_2 = N - q_2$, $1 < \alpha < 2^*$, $N = 1+n+q_1+1+q_2 \geq 3$ is the dimension of $\mathbb{K}$, and $\tilde{x} = (r_1, x, y_1, r_2, y_2) \in \mathbb{K}$. We assume that $V(\tilde{x})$ is a singular potential function on the manifold $\mathbb{K}$ with corner-edge singularity and by making use of Proposition 3.1, Remark 3.2 and Proposition 3.4, we consider

$$C^* = \sup\left\{ \frac{\left\|\sqrt{V(\tilde{x})}u\right\|_{L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})}}{\left\|\nabla_{\mathbb{K}}u\right\|_{L^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})}}\ ;\ u \in \mathcal{H}_{2,,0}^{1,(\frac{\gamma_1}{2},\frac{\gamma_2}{2})}(\mathbb{K}),\ \left\|\nabla_{\mathbb{K}}u\right\|_{L^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})} \neq 0 \right\}, \tag{4.2}$$

and then we suppose that $1 < \epsilon < \frac{1}{C^{*2}}$. Upon the corner-edge type Laplacian operator in 2.2, the gradient and the divergence operators in the section 1, we introduce an operator

$$\Delta_{p,\mathbb{K}}(.) := div_{\mathbb{K}}\Big(|\nabla_{\mathbb{K}}(.)|^{p-2}\nabla_{\mathbb{K}}(.)\Big)$$

for all $2 \leq p < \infty$ as a *corner-edge* $p-$*Laplacian* operator on the stretched manifold $\mathbb{K}$, which can be extended to a monotone, bounded, hemicontinuous and coercive operator between the spaces $\mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})$ and its dual by

$$\begin{aligned} -\Delta_{p,\mathbb{K}} \quad &: \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K}) \to \mathcal{H}_q^{-1,(\frac{\gamma_1}{q},\frac{\gamma_2}{q})}(\mathbb{K}), \\ \langle -\Delta_{p,\mathbb{K}}u \quad &, v\rangle_{p,\mathbb{K}} := \int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |\nabla_{\mathbb{K}}u|^{p-2}\nabla_{\mathbb{K}}u.\nabla_{\mathbb{K}}v d\mu, \end{aligned} \tag{4.3}$$

where $d\mu = \frac{dr_1}{r_1}dx\frac{dy_1}{r_1}\frac{dr_2}{r_1r_2}\frac{dy_2}{r_1r_2}$ and in particular case of the corner-edge $p-$Sobolev space, we consider the space $\mathcal{H}_{p,0}^{0,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K}) = L_p^{\frac{\gamma_1}{p},\frac{\gamma_2}{p}}(\mathbb{K})$ by the following norm

$$\left\|u\right\|_{L_p^{\frac{\gamma_1}{p},\frac{\gamma_2}{p}}(\mathbb{K})} := \left(\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |u|^p d\mu\right)^{\frac{1}{p}}.$$

Similar to the classical case [27], we introduce the following functionals on the corner-edge Sobolev space $\mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})$:

$$\begin{aligned} J(u) &= \frac{1}{p}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2}|\nabla_{\mathbb{K}} u|^p d\mu - \frac{\epsilon}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} V(\tilde{x})|u|^2 d\mu - \frac{1}{\alpha+1}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2}|u|^{\alpha+1} d\mu, \\ K(u) &= \int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2}|\nabla_{\mathbb{K}} u|^p d\mu - \epsilon\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} V(\tilde{x})|u|^2 d\mu - \int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2}|u|^{\alpha+1} d\mu. \end{aligned} \tag{4.4}$$

Then the functionals $J(u)$ and $K(u)$ are well-defined and belong to $C^1(\mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K});\mathbb{R})$. According to the above functionals, we consider $J(\lambda u)$ for every $\lambda > 0$ and define the corner-edge *Nehari Manifold* set as follows

$$\begin{aligned} \mathcal{N}_{ce} &= \left\{ u \in \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})\ ;\ \left[\frac{dJ(\lambda u)}{d\lambda}\right]_{\lambda=1} = 0,\ u \neq 0 \right\} \\ &= \left\{ u \in \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})\ ;\ K(u) = 0,\ u \neq 0 \right\}. \end{aligned} \tag{4.5}$$

According to the Mountain Pass theorem [29] and the conception of the depth of the potential well in [15, 27], we take

$$d = \inf\left\{ \sup_{\lambda \geq 0} J(\lambda u)\ ;\ u \in \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K}),\ u \neq 0 \right\}.$$

This is the well-known result that for $1 < \alpha < 2^* = \frac{2N}{N-2}$ the depth of the potential well d is positive constant [29] and $d = \inf_{u\in\mathcal{N}} J(u)$.

By making use of the functionals above, we introduce the following corner-edge potential well

$$\mathcal{W}_{ce} = \left\{ u \in \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})\ ;\ J(u) < d \right\} \cup \{0\}$$

and partition it into two subsets

$$\mathcal{G}_{ce} = \left\{ u \in \mathcal{W}_{ce}\ ;\ J(u) > 0 \right\} \cup \{0\}, \quad \mathcal{B}_{ce} = \left\{ u \in \mathcal{W}_{ce}\ ;\ J(u) < 0 \right\}$$

such that we refer to $\mathcal{G}_{ce}$ and $\mathcal{B}_{ce}$ as the "Good" and "Bad" parts of the corner-edge potential well $\mathcal{W}_{ce}$ respectively. Furthermore, we can define the set of stability for the problem 4.1 by the good corner-edge potential well $\mathcal{G}_{ce}$ and the total energy of the problem 4.1 is given by

$$I(t) = \frac{1}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2}|u_t(t)|^2 d\mu + \frac{1}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2}|\psi(t)|^2 d\mu + J(u(t)).$$

In order to prove our main result in this section we need the concept Krasnoselskii genus from the index theory. Let $\mathcal{V}$ be a Banach space and define

$$\mathcal{A} := \left\{ A \subset \mathcal{V}\ ;\ A\ closed,\ A = -A \right\}$$

to be the class of closed symmetric subsets of $\mathcal{V}$.

Definition 4.1. For $A \in \mathcal{A}$, $A \neq \emptyset$, following Coffman [5], let

$$\gamma(A) = \begin{cases} \inf\Big\{ m \in \mathbb{N}\ ;\ \exists h \in C^0(A;\mathbb{R}^m - \{0\}),\ h(-u) = -h(u) \Big\} \\ \infty\ ;\ if\ \{..\} = \emptyset,\ in\ particular,\ if\ 0 \in A, \end{cases} \tag{4.6}$$

and define $\gamma(\emptyset) = 0$.

As stated in [29], one can extended any odd map $h \in C^0(A;\mathbb{R}^m)$ to a map $\tilde{h} \in C^0(\mathcal{V};\mathbb{R}^m)$ by the Tietze extension theorem for every $A \in \mathcal{A}$. $\gamma(A)$ is called the Krasnoselskii genus of A.

Now, we can state our main result about the global existence of the solution of the problem 4.1.

Theorem 4.2. *Suppose that the initial data* $u_0 \in \mathcal{G}_{ce}$, $u_1 \in L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})$, $\psi_0 \in L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})$ *and* $1 < \alpha < 2^*$ *are given, then there exist functions* $u, \psi : \mathbb{K} \times (0,T) \to \mathbb{R}$ *such that*

$$\begin{aligned} u \quad &\in L^\infty\Big((0,T)\ ;\ \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})\Big),\ \ u_t \in L^\infty\Big((0,T)\ ;\ L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})\Big) \\ \psi \quad &\in L^\infty\Big((0,T)\ ;\ L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})\Big), \end{aligned} \tag{4.7}$$

$$\begin{aligned} u(\tilde{x},0) \quad &= u_0(\tilde{x}),\ u_t(\tilde{x},0) = u_1(\tilde{x}),\ \psi(\tilde{x},0) = \psi_0(\tilde{x}),\ a.e.\ in\ \mathbb{K} \\ \tfrac{d}{dt}\langle u_t, \phi\rangle_{\mathbb{K}} \quad &+ \langle -\Delta_{p,\mathbb{K}} u, \phi\rangle_{p,\mathbb{K}} + \langle -\epsilon V(\tilde{x}) u, \phi\rangle_{\mathbb{K}} = \langle |u|^{\alpha-1} u, \phi\rangle_{\mathbb{K}}\ \forall \phi \in \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K}), \\ \tfrac{d}{dt}\langle \psi, \varphi\rangle_{\mathbb{K}} \quad &+ \langle -\Delta_{\mathbb{K}} \psi, \varphi\rangle_{\mathbb{K}} = \langle u_t, \varphi\rangle_{\mathbb{K}}\ \ \forall \varphi \in L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K}). \end{aligned} \tag{4.8}$$

Proof. By making use of the Definition 4.1, for $j \in \mathbb{N}$ we consider

$$\Sigma_j = \Big\{ A \subset G\ ;\ A\ is\ compact,\ symetric\ and\ \gamma(A) \geq j \Big\},$$

where $G = \big\{ u \in L_p^{\frac{\gamma_1}{p},\frac{\gamma_2}{p}}(\mathbb{K})\ ;\ \|u\|_{L_p^{\frac{\gamma_1}{p},\frac{\gamma_2}{p}}(\mathbb{K})} = 1 \big\}$. It follows from [29] (see proof of Theorem 2.6 page 181) that

$$\lambda_j = \inf_{A\in\sigma_j} \sup_{u\in A} \|\nabla_{\mathbb{K}} u\|^p_{L_p^{\frac{\gamma_1}{p},\frac{\gamma_2}{p}}(\mathbb{K})}, \quad \mu_j = \inf_{A\in\sigma_j} \sup_{u\in A} \|\nabla_{\mathbb{K}} u\|^2_{L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})}$$

are two sequences of eigenvalues of the elliptic corner-edge $p-$Laplacian $-\Delta_{p,\mathbb{K}} : \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K}) \to \mathcal{H}_q^{-1,(\frac{\gamma_1}{q},\frac{\gamma_2}{q})}(\mathbb{K})$ and elliptic corner-edge Laplacian $-\Delta_{\mathbb{K}} : \mathcal{H}_{2,0}^{1,(\frac{\gamma_1}{2},\frac{\gamma_2}{2})}(\mathbb{K}) \to \mathcal{H}_2^{-1,(\frac{\gamma_1}{2},\frac{\gamma_2}{2})}(\mathbb{K})$ respectively. These operators are monotone, coercive and hemicontinuous. Hence, by making use of the Minty-Browder theorem, there exists a basis $\{\chi_j\}_{j=1}^{\infty}$ for $\mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})$ given by the solutions of the stationary problem

$$-\Delta_{p,\mathbb{K}}\chi_j(\tilde{x}) - \epsilon V(\tilde{x})\chi_j(\tilde{x}) = \lambda_j\chi_j(\tilde{x}), \quad \chi_j(\tilde{x}) = 0, \ \ on\ \partial\mathbb{K}.$$

As similar result in [13], there exists a basis $\{\xi_j\}_{j=1}^{\infty}$ to corner-edge Laplacian operator given by

$$-\Delta_{\mathbb{K}}\xi_j(\tilde{x}) = \mu_j\xi_j(\tilde{x}) \quad \xi_j(\tilde{x}) = 0, \ on\ \partial\mathbb{K}.$$

Thus by orthogonallization process, the both bases are the Galerkin bases for $-\Delta_{p,\mathbb{K}}$ and $-\Delta_{\mathbb{K}}$ in $\mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})$ and $\mathcal{H}_{2,0}^{1,(\frac{\gamma_1}{2},\frac{\gamma_2}{2})}(\mathbb{K})$ respectively. Now, for every $m \in \mathbb{N}$, we set $\Upsilon_m := Span\{\chi_1, ..., \chi_m\}$ and $\Theta_m := Span\{\xi_1, ..., \xi_m\}$. We want to find the functions

$$u_m(t) = \sum_{j=1}^{m} a_{jm}(t)\chi_j \quad \psi_m(t) = \sum_{j=1}^{m} b_{jm}(t)\xi_j$$

such that for any $\chi \in \Upsilon_m$ and $\xi \in \Theta_m$, $u_m(t)$ and $\psi_m(t)$ satisfies the following approximation equations

$$\begin{aligned} \langle u_m^{''}(t), \chi\rangle_{\mathbb{K}} \quad &+ \quad \langle -\Delta_{p,\mathbb{K}}u_m(t), \chi\rangle_{p,\mathbb{K}} + \langle -\epsilon V(\tilde{x})u_m(t), \chi\rangle_{\mathbb{K}} + \langle \psi_m(t), \chi\rangle_{\mathbb{K}} \\ &= \quad \langle |u_m(t)|^{\alpha-1}u_m(t), \chi\rangle_{\mathbb{K}}, \end{aligned} \tag{4.9}$$

$$\langle \psi_m^{'}(t), \xi\rangle_{\mathbb{K}} + \langle -\Delta_{\mathbb{K}}\psi_m(t), \xi\rangle_{\mathbb{K}} = \langle u_m^{'}(t), \xi\rangle_{\mathbb{K}} \tag{4.10}$$

with the initial conditions $u_m(0) = u_{0m}$, $u_m^{'}(0) = u_{1m}$ and $\psi_m(0) = \psi_{0m}$ where u_{0m}, u_{1m} and ψ_{0m} are chosen in Υ_m and Θ_m such that $u_{0m} \to u_0$ in $\mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})$, $u_{1m} \to u_1$ in $L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})$ and $\psi_{0m} \to \psi_0$ in $L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})$. Now, we take $\chi = \chi_i$, and $\xi = \xi_i$ for $i = 1, ..., m$ and by making use of the following

$$\begin{aligned} u_m^{''}(t) \quad &= \quad \sum_{j=1}^{m} a_{jm}(t)\chi_j(\tilde{x}), \quad \Delta_{p,\mathbb{K}}u_m(t) = \sum_{j=1}^{m} a_{jm}(t)\Delta_{p,\mathbb{K}}\chi_j(\tilde{x}) \\ \psi_m^{'}(t) \quad &= \quad \sum_{j=1}^{m} b_{jm}(t)\xi_j(\tilde{x}), \quad \Delta_{\mathbb{K}}\psi_m(t) = \sum_{j=1}^{m} b_{jm}(t)\Delta_{\mathbb{K}}\xi_j(\tilde{x}), \end{aligned}$$

we obtain that the equations 4.9 and 4.10 approach to a ODE's system in the variable t such that it accepts a local solution $u_m(t)$, $\psi_m(t)$ in an interval $(0, T_m)$ via Carathéodery's theorem. In order to extend this local solution to the whole interval $[0, T]$ for $T > 0$, we calculate some estimates for them. Hence, putting $\chi = u_m^{'}(t)$, $\xi = \psi_m(t)$ in the approximation equations 4.9

and 4.10, then

$$\begin{aligned}\langle u''_m(t), u'_m(t)\rangle_{\mathbb{K}} &- \langle \Delta_{p,\mathbb{K}} u_m(t), u'_m(t)\rangle_{p,\mathbb{K}} - \langle \epsilon V(\tilde{x}) u_m(t), u'_m(t)\rangle_{\mathbb{K}} \\ &= \langle |u_m(t)|^{\alpha-1} u_m(t), u'_m(t)\rangle_{\mathbb{K}}, \\ & \langle \psi'_m(t), \psi_m(t)\rangle_{\mathbb{K}} - \langle \Delta_{\mathbb{K}} \psi_m(t), \psi_m(t)\rangle_{\mathbb{K}} = \langle u'_m(t), \psi_m(t)\rangle_{\mathbb{K}}. \quad (4.11)\end{aligned}$$

By simple calculations, one can obtain the following relations:

$$\begin{aligned}\frac{d}{dt} & \frac{1}{2} \int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |u'_m(t)|^2 d\mu + \frac{d}{dt}\frac{1}{p}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |\nabla_{\mathbb{K}} u_m(t)|^p d\mu \\ - & \frac{d}{dt}\frac{\epsilon}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} V(\tilde{x})|u_m(t)|^2 d\mu = \frac{d}{dt}\frac{1}{\alpha+1}\int_{\mathbb{K}} |u_m(t)|^{\alpha+1} d\mu, \quad (4.12)\end{aligned}$$

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |\psi_m(t)|^2 d\mu + \int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |\nabla_{\mathbb{K}}\psi_m(t)|^2 d\mu = \int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} u'_m(t)\psi_m(t) d\mu. \quad (4.13)$$

Therefore, one can substitute the estimates 4.12 and 4.13 into the approximation equations 4.11 and then obtains that $\frac{d}{dt} I_m(t) = -\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |\nabla_{\mathbb{K}}\psi_m(t)|^2 d\mu$ where the approximation total energy as follows:

$$I_m(t) = \frac{1}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |u'_m(t)|^2 d\mu + \frac{1}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |\psi_m(t)|^2 d\mu + J(u_m(t))$$

which satisfies the following energy inequality

$$I_m(t) \le I_m(0) = \frac{1}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |u'_m(0)|^2 d\mu + \frac{1}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |\psi_m(0)|^2 d\mu + J(u_m(0)). \quad (4.14)$$

From convergence of the sequences in the initial data and this fact that $J(u_m(0)) < d$ in the good potential well $\mathcal{G}_{ce}$, there exists a positive constant C independent of t and m such that

$$\frac{1}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |u'_m(0)|^2 d\mu + \frac{1}{2}\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |\psi_m(0)|^2 d\mu \le C.$$

Therefore, it follows from the inequality 4.14 that $I_m(t) \le I_m(0) \le C$ and we can extend the approximation solutions $u_m(t)$ and $\psi_m(t)$ to the interval $[0,T]$, for $T > 0$. On the other hand, from the inequality

$$\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |\nabla_{\mathbb{K}}\psi_m(t)|^2 d\mu \le \int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |\nabla_{\mathbb{K}}\psi_m(t)|^2 d\mu + I_m(t) \le I_m(0) \le C, \quad (4.15)$$

we obtain the bounded sequences

$$\begin{aligned}\{u_m(t)\} &\in L^\infty\Big((0,T); L_{\alpha+1}^{\frac{\gamma_1}{\alpha+1},\frac{\gamma_2}{\alpha+1}}(\mathbb{K})\Big) \cap L^\infty\Big((0,T); \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})\Big) \quad (4.16) \\ \{u'_m(t)\} &\in L^\infty\Big((0,T); L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})\Big), \ \{-\Delta_{p,\mathbb{K}} u_m(t)\} \in L^\infty\Big((0,T); \mathcal{H}_q^{-1,(\frac{\gamma_1}{q},\frac{\gamma_2}{q})}(\mathbb{K})\Big).\end{aligned}$$

Moreover, by the constant C^* in 4.2 and the inequality 4.15 we get the following bounded sequences

$$
\begin{aligned}
\{-\epsilon V(\tilde{x})|u_m(t)|^2\} &\in L^\infty\Big((0,T); L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})\Big),\\
\{|u_m(t)|^{\alpha-1}u_m(t)\} &\in L^\infty\Big((0,T); L_{\frac{\alpha+1}{\alpha}}^{\frac{\alpha(\gamma_1)}{\alpha+1},\frac{\alpha\gamma_1}{\alpha+1}}(\mathbb{K})\Big).
\end{aligned}
\tag{4.17}
$$

Again from the inequality 4.15 we obtain that the sequences

$$
\begin{aligned}
\{\psi_m(t)\} &\in L^\infty\Big((0,T); L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})\Big) \cap L^\infty\Big((0,T); \mathcal{H}_{p,0}^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})\Big)\\
\{-\Delta_{\mathbb{K}}\psi_m(t)\} &\in L^\infty\Big((0,T); \mathcal{H}_2^{-1,(\frac{\gamma_1}{2},\frac{\gamma_2}{2})}(\mathbb{K})\Big)
\end{aligned}
\tag{4.18}
$$

are bounded in their corresponding spaces respectively. Then, we want to obtain an estimate for $u_m^{''}(t)$. From the standard projection argument in the Hilbert space as described in [19], we use our Galerkin basis which was taken in Hilbert space $L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})$, the sequences in 4.16, 4.17 with approximation equations 4.9, 4.10 and obtain that $\{u_m^{''}(t)\}$ is bounded $L^\infty\Big((0,T); \mathcal{H}_q^{-1,(\frac{\gamma_1}{q},\frac{\gamma_2}{q})}(\mathbb{K})\Big)$. For the sequences in 4.16, 4.17 and 4.18, there exist the subsequences corresponding to them which we still denote in the same way and there exist $u(t)$, $\psi(t)$, $\eta_1(t)$, $\eta_2(t)$ and $\eta_3(t)$ such that they converge as a weakly star in the suitable spaces as follows:

$$
\begin{aligned}
u_m(t) &\rightharpoonup u(t) \quad in\ L^\infty\Big((0,T); L_{\alpha+1}^{\frac{\gamma_1}{\alpha+1},\frac{\gamma_2}{\alpha+1}}(\mathbb{K})\Big) \cap L^\infty\Big((0,T); \mathcal{H}_p^{1,(\frac{\gamma_1}{p},\frac{\gamma_2}{p})}(\mathbb{K})\Big)\\
u_m^{'}(t) &\rightharpoonup u^{'}(t) \quad in\ L^\infty\Big((0,T); L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})\Big)\\
-\Delta_{p,\mathbb{K}}u_m(t) &\rightharpoonup \eta_1(t) \quad in\ L^\infty\Big((0,T); \mathcal{H}_q^{-1,(\frac{\gamma_1}{q},\frac{\gamma_2}{q})}(\mathbb{K})\Big)\\
\epsilon V(\tilde{x})|u_m|^2 &\rightharpoonup \eta_2(t) \quad in\ L^\infty\Big((0,T); L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})\Big)\\
|u_m|^{\alpha-1}u_m(t) &\rightharpoonup \eta_3(t) \quad in\ L^\infty\Big((0,T); L_{\frac{\alpha+1}{\alpha}}^{\frac{\alpha(\gamma_1)}{\alpha+1},\frac{\alpha\gamma_2}{\alpha+1}}(\mathbb{K})\Big)\\
\psi_m(t) &\rightharpoonup \psi(t) \quad in\ L^\infty\Big((0,T); L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})\Big)\\
-\Delta_{\mathbb{K}}\psi_m(t) &\rightharpoonup -\Delta_{\mathbb{K}}\psi(t) \quad in\ L^\infty\Big((0,T); \mathcal{H}_2^{-1,(\frac{\gamma_1}{2},\frac{\gamma_2}{2})}(\mathbb{K})\Big).
\end{aligned}
\tag{4.19}
$$

By making use of Theorem 3.5, Proposition 3.6 and the Lions-Aubin compactness lemma [19], one can get from the boundedness of the sequence $\{u_m^{''}(t)\}$ in $L^\infty\Big((0,T); \mathcal{H}_q^{-1,(\frac{\gamma_1}{q},\frac{\gamma_2}{q})}(\mathbb{K})\Big)$ and the first two convergence in 4.19

$$\begin{aligned} u_m(t) &\to u(t) \quad \textit{strongly in } L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}\Big((0,T);L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})\Big) \\ u_m^{'}(t) &\to u^{'}(t) \quad \textit{strongly in } L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}\Big((0,T);L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})\Big). \end{aligned} \tag{4.20}$$

First, we show that $\eta_1(t)=-\Delta_{p,\mathbb{K}}u(t)$. In order to do this, we apply the following inequalities

$$\Big||x|^{\frac{p-2}{2}}x-|y|^{\frac{p-2}{2}}y\Big|\leq C\Big(|x|^{\frac{p-2}{2}}+|y|^{\frac{p-2}{2}}\Big)|x-y| \quad \forall x,y\in\mathbb{R}, p\geq 2, \tag{4.21}$$

$$\Big||x|^{p-2}x-|y|^{p-2}y\Big|\leq C\Big(|x|^{\frac{p-2}{2}}+|y|^{\frac{p-2}{2}}\Big)\Big||x|^{\frac{p-2}{2}}x-|y|^{\frac{p-2}{2}}y\Big| \quad \forall x,y\in\mathbb{R}, p\geq 2. \tag{4.22}$$

By making use of the Hölder generalized inequality with $\frac{1}{2}+\frac{1}{p}+\frac{p-2}{4p}+\frac{p-2}{4p}=1$ and applying the inequalities 4.21 and 4.22

$$\begin{aligned} \Big|\langle &- \Delta_{p,\mathbb{K}}u_m(t),v\rangle_{p,\mathbb{K}}-\langle-\Delta_{p,\mathbb{K}}u(t),v\rangle_{p,\mathbb{K}}\Big| \\ &= \Big|\int_{\mathbb{K}} r_1^{q_1+q_2}r_2^{q_2}\Big(|\nabla_{\mathbb{K}}u_m(t)|^{p-2}\nabla_{\mathbb{K}}u_m(t)-|\nabla_{\mathbb{K}}|^{p-2}\nabla_{\mathbb{K}}u(t)\Big)\nabla_{\mathbb{K}}v d\mu\Big| \\ &\leq C\int_{\mathbb{K}} r_1^{q_1+q_2}r_2^{q_2}\Big(|\nabla_{\mathbb{K}}u_m(t)|^{\frac{p-2}{2}}+|\nabla_{\mathbb{K}}u(t)|^{\frac{p-2}{2}}\Big) \\ &\times \Big||\nabla_{\mathbb{K}}u_m(t)|^{\frac{p-2}{2}}\nabla_{\mathbb{K}}u_m(t)-|\nabla_{\mathbb{K}}u(t)|^{\frac{p-2}{2}}\nabla_{\mathbb{K}}u(t)\Big||\nabla_{\mathbb{K}}v(t)|d\mu \\ &\leq C\int_{\mathbb{K}} r_1^{q_1+q_2}r_2^{q_2}\Big(|\nabla_{\mathbb{K}}u_m(t)|^{\frac{p-2}{2}}+|\nabla_{\mathbb{K}}u(t)|^{\frac{p-2}{2}}\Big)\Big|\nabla_{\mathbb{K}}u_m(t)-\nabla_{\mathbb{K}}u(t)\Big||\nabla_{\mathbb{K}}v|d\mu \\ &\leq C\Big(\Big\|\nabla_{\mathbb{K}}u_m(t)\Big\|^{\frac{p-2}{8}}_{L_{\frac{p}{2}}^{\frac{2\gamma_1}{p},\frac{2\gamma_2}{p}}(\mathbb{K})}+\Big\|\nabla_{\mathbb{K}}u(t)\Big\|^{\frac{p-2}{8}}_{L_{\frac{p}{2}}^{\frac{2\gamma_1}{p},\frac{2\gamma_2}{p}}(\mathbb{K})}\Big)^2 \\ &\times \Big\|\nabla_{\mathbb{K}}u_m(t)-\nabla_{\mathbb{K}}u(t)\Big\|_{L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})}\Big\|\nabla_{\mathbb{K}}v\Big\|_{L_p^{\frac{\gamma_1}{p},\frac{\gamma_2}{p}}(\mathbb{K})}\leq C\Big\|\nabla_{\mathbb{K}}u_m(t)-\nabla_{\mathbb{K}}u(t)\Big\|_{L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})}. \end{aligned}$$

Therefore, from the first convergence in 4.20, we obtain that $u_m(t)\to u(t)$ almost everywhere in $\mathbb{K}\times(0,T)$ such that it follows $\eta_1(t)=-\Delta_{p,\mathbb{K}}u(t)$. Now, we prove that $\eta_2(t)=-\epsilon V(\tilde{x})|u|^2$. Let us consider an arbitrary $\varphi\in L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})$. Then,

$$\begin{aligned} \Big|\langle-\epsilon V(\tilde{x})|u_m|^2,\varphi\rangle_{\mathbb{K}} &- \langle-\epsilon V(\tilde{x})|u|^2\rangle_{\mathbb{K}}\Big| = \Big|\int_{\mathbb{K}} r_1^{q_1+q_2}r_2^{q_2}V(\tilde{x})|u_m|^2\varphi d\mu-\int_{\mathbb{K}} r_1^{q_1+q_2}r_2^{q_2}V(\tilde{x})|u|^2\varphi d\mu\Big| \\ &= \Big|\int_{\mathbb{K}} r_1^{q_1+q_2}r_2^{q_2}V(\tilde{x})\varphi(|u_m|^2-|u|^2)d\mu\Big| \\ &\leq C^{*2}\|\nabla_{\mathbb{K}}\varphi\|^2_{L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})}\|u_m-u\|^2_{L_2^{\frac{\gamma_1}{2},\frac{\gamma_2}{2}}(\mathbb{K})}. \end{aligned}$$

Therefore, by making use the strongly convergence in 4.20, it implies that $\eta_2(t) = -\epsilon V(\tilde{x})|u|^2$. Furthermore, to obtain $\eta_3(t) = |u(t)|^{\alpha-1}u(t)$, we have

$$\int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} \left| |u_m(t)|^{\alpha-1}u_m(t) \right|^{\frac{\alpha+1}{\alpha}} d\mu = \int_{\mathbb{K}} r_1^{q_1+q_2} r_2^{q_2} |u_m|^{\alpha+1} d\mu \leq C.$$

Hence, $|u_m(t)|^{\alpha-1}u_m(t) \to |u(t)|^{\alpha-1}u(t)$ almost everywhere in $\mathbb{K} \times [0,T)$. Thus, from Lemma 1.3 in [19], we obtain

$$|u_m(t)|^{\alpha-1}u_m(t) \rightharpoonup |u(t)|^{\alpha-1}u(t) \quad weakly\ in\ L^{\frac{\alpha\gamma_1}{\alpha+1}, \frac{\alpha\gamma_2}{\alpha+1}}_{\frac{\alpha+1}{\alpha}}\left((0,T); L^{\frac{\alpha\gamma_1}{\alpha+1}, \frac{\alpha\gamma_2}{\alpha+1}}_{\frac{\alpha+1}{\alpha}}(\mathbb{K})\right). \tag{4.23}$$

It follows from the fifth weakly convergence in 4.19 and the weakly convergence in 4.23 that $\eta_3(t) = |u(t)|^{\alpha-1}u(t)$. Therefore, by making use the weak convergence in 4.19, one can pass to the limit in the approximation equations in 4.9,4.10 and obtains the assertions 4.7 and 4.8. The verification of the initial conditions is straightforward. □

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