Lucas and Fibonacci polynomials based approach for the study of one- and two-dimensional Burger and heat-type equations

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Abstract

In this work a numerical technique, combination of Lucas and Fibonacci polynomials, is proposed for the solution of one- and two-dimensional nonlinear heat type equations. In first round, for discretization, finite difference has been used for time and Crank Nicolson scheme for spatial part. In second round, the unknown functions have been approximated by Lucas polynomial while their derivatives by Fibonacci polynomials. With the help of these approximations, the nonlinear partial differential equation transforms to a system of algebraic equations which can be solved easily. Convergence of the method has been investigated numerically. Performance of the method has been studied by taking one- and two-dimensional heat and burger equations. Efficiency of the technique has been investigated in terms of root mean square (RMS), L_2 and L_∞ norms. The obtained results are then compared with those available in the literature.

Key words: Lucas polynomials; Fibonacci polynomials; Convergence analysis; Error norms.
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1 Introduction

Evolutionary differential equations having prodigious applications in various physical, chemical, biological, electrical and mechanical processes. These process are mostly modeled in the form of heat, Laplace, wave type and poison equations. Some of these partial differential equations (PDE's) have exact solution in regular domains and is very difficult to obtain analytical solution of these type equations in irregular domains. Therefore, researchers are trying to developed numerical techniques to get rid of domain problem. Here we deal with heat type time dependent PDEs of the form:

$$\partial_t Y(\xi,\eta,t)+\alpha Y(\xi,\eta,t)\nabla Y(\xi,\eta,t)+\beta\,\Delta Y(\xi,\eta,t)+\gamma\,Y(\xi,\eta,t)=g(\xi,\eta,t),\quad \xi,\ \eta\in\Gamma,\quad t\in[0,T],\tag{1}$$

with initial and boundary conditions

$$Y(\xi,\eta,0)=\phi_1(\xi,\eta),\quad \xi,\ \eta\in\Gamma,\quad \text{and}\quad Y(\xi,\eta,t)=\phi_2(\xi,\eta,t),\quad \xi,\ \eta\in\partial\Gamma,$$

where g is source term depends on space and time variable, Γ, $\partial\Gamma$ are spatial domain and boundary of the domain respectively, α and R are positive constants, Δ represents laplacian and ∇ stands for

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gradient operator. When $\alpha = 0$ and $\beta = -1$ Eq. (1) becomes the well known heat equation which arise in transient heat conduction problems. Mostly heat equation has been used to describe phenomena of heat distribution over a solid body. This equation also works in different forms like: in finance it is used in the form of Black-sholes model for estimation of price, in chemical process it is designed in the form of diffusion equation [1]. In the form of advection-diffusion equation it is used to describe heat transfer in a solid body and to describe dissipation of salt in ground water. In conjunction with fourier theory, heat equation is used to measure thermal diffusivity in polymer and to conclude heat distribution profile. In image processing it is used to handle image at different scales [2]. The solution of this equation plays a significant role to examine behavior of physical model. Different numerical methods have been used for solution of heat equation by many researchers. A comparative study between the classical finite difference and finite element method had been investigated by Benyam Mebrate [4]. B-spline and finite element method have been used by Darbel et al [3]. Hooshmandasl et al [1] considered the use of wavelets method to solve one dimensional heat equation, while Dehgan [5] used second order finite difference scheme for solution of two dimensional heat equation.
When $\alpha = 1$ and $\beta = -1/Re$ ('Re' is Reynold number), then Eq. (1) becomes well known Burger equation which had been experienced in the field of turbulence, shock wave theory, viscous fluid flow, gas dynamics, cosmology, traffic flow, quantum field and heat conduction [6]. The low kinematic viscosity shocks and the relation between cellular and large-scale structure of the universe have been described by one and three dimensional burger equation. When traffic is treated as one dimensional incompressible fluid then the density wave in traffic flow which changes from non-uniform to uniform distribution is described by burger equation [7]. The burger equation was first introduced by Bateman in viscous fluid flow, which was then extended by Burger in (1948) to examine turbulence phenomena that's why it is known as Burger equation [6]. Due to wide applications of burger equation many numerical methods have been implemented to study behavior of the model. One-dimensional burger equation has been solved using various techniques [8, 9]. Mittal and jiwari [6] implemented differential quadrature method for solution of burger type equation. Similarly El-sayed and Kaya [10] solved two-dimensional burger equation using decomposition method. Liao [12] used fourth order finite difference technique for the study of two-dimensional burger equation.
In this work, we study afore mentioned equations by using combination of Lucas and Fibonacci polynomials. These polynomials have a great importance in solution of different type of differential equations. Many researchers applied these polynomials for the solution of fractional differential equations (FDEs) such as Elhameed and Youssri [13] applied lucas polynomial in a caputo sense to fractional order differential equation. Moreover they computed solution of coupled FDEs using combined spectral tau method with Fibonacci polynomial sequence [14]. Cetin [15] used lucas polynomial approach to study a system of higher order differential equation where as Bayku [16] applied hybrid Taylor-Lucas collocation technique for delay differential equation. Mostefa [17] obtained solution of integro differential equation using lucas sequence. Similarly Farshid et al [18] had used Fibonacci polynomial for solution of voltera-fredholm integral equations. Omer oruc [19, 20] applied Lucas polynomial approach for numerical solution of evolutionary equation for the first time. Therefore, it is a kind of interest to examine efficiency of the proposed method for heat type equations. In this paper, we implement this polynomial approach to one and two-dimensional Burger and heat type equations and show accuracy of the method. Convergence of the method is also investigated here. Rest of the paper is organized as follows: In section 2 we discuss solution methodology of the proposed method. Convergence of the technique is described in section 3. Numerical experiments are presented in section 4 followed by the

conclusion of the work.

2 Solution Methodology

Consider the following evolutionary equation

$$\partial_t Y(\mathbf{x},t) + \pounds Y(\mathbf{x},t) = g(\mathbf{x},t) \quad \mathbf{x} \in \Gamma, \quad t \in [0,T] \tag{2}$$

where $\pounds$ is differential operator and $g(\mathbf{x},t)$ is a given smooth function. The initial and boundary conditions are given as

$$Y(\mathbf{x},0) = Y_0(\mathbf{x}), \quad \mathbf{x} \in \Gamma \quad \text{and} \quad \mathfrak{B}Y(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} \in \partial\Gamma \tag{3}$$

where $\mathfrak{B}$ is boundary operator, Γ and $\partial\Gamma$ stand for domain and boundary of the domain respectively. Applying finite difference scheme to temporal part and $\theta-$weighted scheme to spatial part of Eq. (2), one can write

$$\frac{1}{\delta t}[Y^{n+1}(\mathbf{x}) - Y^n(\mathbf{x})] + \theta \pounds Y^{n+1}(\mathbf{x}) + (1-\theta)\pounds Y^n(\mathbf{x}) = g^{n+1}(\mathbf{x}), \tag{4}$$

Approximating dependent variable $Y^n(\mathbf{x})$ in Eq. (2) by Lucas polynomial as follows:

$$Y^{n+1}(\mathbf{x}) = \sum_{k=1}^{N}\sum_{m=1}^{N} C_{km}^{n+1} L_k(\xi) L_m(\eta), \quad \mathbf{x} = (\xi,\eta) \tag{5}$$

where C_{km}^{n+1} are unknown coefficient and $L_\alpha(\Gamma)$ is Lucas polynomial defined by [19]

$$L_0(\gamma) = 2, \quad L_1(\gamma) = \gamma \quad \text{and} \quad L_\alpha(\gamma) = \gamma L_{\alpha-1}(\gamma) + L_{\alpha-2}(\gamma), \quad \text{for} \quad \alpha \geq 2.$$

Derivatives of Lucas polynomial $L_\alpha^{(n)}(\gamma)$ in term of Fibonacci $\mathcal{F}_\alpha(\gamma)$ polynomials are given by the following relation [19]

$$L_\alpha^{(n)}(\gamma) = \alpha \mathcal{F}_\alpha^{(n-1)}(\gamma), \text{with } \mathcal{F}^{(n)} = \mathcal{F}(\gamma) D^n \quad \text{for} \quad n \geq 1,$$

where Fibonacci polynomial are given by

$$\mathcal{F}_0(\gamma) = 0, \quad \mathcal{F}_1(\gamma) = 1 \quad \text{and} \quad \mathcal{F}_\alpha(\gamma) = \gamma \mathcal{F}_{\alpha-1}(\gamma) + \mathcal{F}_{\alpha-2}(\gamma) \quad \text{for} \quad \alpha \geqslant 2, \tag{6}$$

and D is differentiation matrix given by expression [19]

$$D = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & d & \\ 0 & & & \end{bmatrix},$$

where d is square matrix of order $N \times N$ defined as

$$d_{m,n} = \begin{cases} m \sin \frac{(n-m)\pi}{2}, & \text{if } n > m. \\ 0, & \text{otherwise.} \end{cases}$$

The unknown coefficients C_{km} can be computed by collocation method. Therefore at collocation points $\mathbf{x} = (\xi_i, \eta_j)$, Eq. (5) can be written as

$$Y^{n+1}(\xi_i, \eta_j) = \sum_{k=1}^{N} \sum_{m=1}^{N} C_{km}^{n+1} L_k(\xi_i) L_m(\eta_j), \quad i, j = 1, 2, \dots, N \tag{7}$$

Using Eq. (7) in Eq. (4), we have

$$\frac{1}{\delta t}\left[\sum_{k=1}^{N}\sum_{m=1}^{N} C_{km}^{n+1} L_k(\xi_i) L_m(\eta_j) - \sum_{k=1}^{N}\sum_{m=1}^{N} C_{km}^{(n)} L_k(\xi_i) L_m(\eta_j)\right] + \theta\left[\sum_{k=1}^{N}\sum_{m=1}^{N} C_{km}^{n+1} \pounds L_k(\xi_i) L_m(\eta_j)\right] + (1-\theta)\left[\sum_{k=1}^{N}\sum_{m=1}^{N} C_{km}^{n} \pounds L(\xi_i) L(\eta_j)\right] = g^{n+1}(\xi_i, \eta_j), \quad (\xi_i, \eta_j) \in \Gamma$$

The above equation can also be written as

$$\sum_{k=1}^{N}\sum_{m=1}^{N} \left[L_k(\xi_i) L_m(\eta_j) + \theta\ \delta t\ \pounds L_k(\xi_i) L_m(\eta_j)\right] C_{km}^{n+1} = \sum_{k=1}^{N}\sum_{m=1}^{N} \left[L_k(\xi_i) L_m(\eta_j) + (1-\theta)\ \delta t\ \pounds L(\xi_i) L(\eta_j)\right] C_{km}^{n} + g^{n+1}(\xi_i, \eta_j), \quad (\xi_i, \eta_j) \in \Gamma \tag{8}$$

The boundary conditions (3) transform to

$$\sum_{k=1}^{N}\sum_{m=1}^{N} C_{km}^{n+1} \mathfrak{B} L_k(\xi_i)) L_m(\eta_j)) = f^{n+1}(\xi_i, \eta_j), \quad (\xi_i, \eta_j) \in \partial\Gamma. \tag{9}$$

Matrix form of Eqs. (8) and (9) is given by

$$HC^{n+1} = GC^{n} + B^{n+1}, \tag{10}$$

In above equation, $C = [C_{11}, C_{21}, \dots, C_{N1}, \dots, C_{NN}]$ and for $k, m = 1, \dots, N$

$$H = \begin{cases} L_k(\xi_i) L_m(\eta_j) + \delta t \theta \pounds L_k(\xi_i) L_m(\eta_j), & (\xi_i, \eta_j) \in \Gamma, \\ \mathfrak{B} L_k(\xi_i) L_m(\eta_j), & (\xi_i, \eta_j) \in \partial\Gamma, \end{cases} \tag{11}$$

$$G = \begin{cases} L_k(\xi_i) L_m(\eta_j) + \delta t\ (1-\theta)\ \pounds L_k(\xi_i) L_m(\eta_j), & (\xi_i, \eta_j) \in \Gamma, \\ 0, & (\xi_i, \eta_j) \in \partial\Gamma, \end{cases} \tag{12}$$

$$B^{n+1} = \begin{cases} g^{n+1}(\xi_i, \eta_j), & (\xi_i, \eta_j) \in \Gamma, \\ f^{n+1}(\xi_i, \eta_j), & (\xi_i, \eta_j) \in \partial\Gamma, \end{cases} \tag{13}$$

where H, G and B are $N^2 \times N^2$ matrices. The unknown coefficient C can be obtained by solving Eq. (10). Once the values of unknown coefficient are computed the solution of the problem under consideration can be obtained from Eq. (5).

2.1 Convergence analysis

For convergence of the method the following lemmas are required.
Lemma 1
The infinitely differentiable function $Y(\xi)$ at the origin can be expressed in terms of locus polynomial as

$$Y(\xi) = \sum_{p=0}^{\infty}\sum_{p=0}^{\infty} \frac{(-1)^q \sigma_p Y^{(p+2q)}(0)}{q!(p+q)!} L_p(\xi) \tag{14}$$

Lemma 2.
For golden ratio $\sigma = \frac{1+\sqrt{5}}{2}$, the following relations hold

$$\mid L_p(\xi) \mid \leq 2\sigma^p \quad \text{and} \quad \mathcal{F}_p(\xi) \leq \sigma^{p-1} \tag{15}$$

Theorem 1
If $Y(\xi)$ is defined on a closed interval $[0,1]$ and there exist a positive constant β such that $\mid Y^j(\xi) \mid \leq \beta^i,\ i \geq 0$ and $Y(\xi)$ can be expanded as $Y(\xi) = \sum_{p=1}^{\infty} C_p \mathcal{F}_p(\xi)$ then

- $\mid C_k \mid \leq \frac{\beta^{p-1} cosh(\beta)}{2^{p-1}(p-1)!}$.
- The series converges absolutely.

when the function $Y(\xi)$ is expanded in term of locus polynomial then $\quad \mid C_k \mid \leq \frac{\beta^p cosh(2\beta)}{p!}$
For proof see [22, 23]
Theorem 2
When $Y(\xi)$ satisfies theorem 1, then for Fibonacci polynomial $\mathcal{F}(\xi)$ the global error estimation $e_N(x) = \sum_{p=N+1}^{\infty} C_p \mathcal{F}_p(\xi)$ is

$$\mid e_N(\xi) \mid \leq \frac{\alpha^{N+1} cosh(\beta) e^{\alpha}}{(N+1)!} \quad where \ \ \alpha = \frac{\beta\sigma}{2}$$

Similarly the global error estimation in terms of locus polynomial $e_N(\xi) = \sum_{p=N+1}^{\infty} C_p L_p(\xi)$ is

$$\mid e_N(\xi) \mid \leq \frac{2\alpha^{N+1} cosh(2\beta) e^{\alpha}}{(N+1)!} \quad where \ \ \alpha = \beta\sigma,$$

where N is the number of nodal points. From Theorem 2 it is clear that by increasing N global error decrease and solution converge. Proof of the theorem can be seen in [13]. In this paper we give convergence of the method in test examples.

3 Numerical examples

In order to show effectiveness of the technique problems of one and two dimensional burger and heat type equations have been solved. Performance of the method is examined by computing error norms, L_2, L_∞, and root mean square (RMS) for different collocation points M and time T. The obtained results are then compared with available results in literature.

Example 1.

When $\alpha = 0,\ \beta = -1,\ \gamma = 0$ and $g(\xi, t) = 0$ then Eq. (1) becomes heat equation given by

$$\partial_t Y(\xi, t) - \partial_{\xi\xi} Y(\xi, t) = 0, \tag{16}$$

with initial and boundary conditions

$$Y(\xi, 0) = \sin(\xi), \quad 0 \leq \xi \leq 1 \quad \text{and} \qquad Y(0, t) = 0, \quad Y(1, t) = \sin(1)e^{-t} \quad \text{for} \quad 0 \leq t \leq 1.$$

Exact solution is $Y(\xi, t) = \sin(\xi)e^{-t}$ [1].
Comparison of the above equations with Eqs. (2) and (3) give

$$\pounds = -\partial_{\xi\xi}, \quad g(\xi, t) = 0, \quad Y_0(\xi) = \sin(\xi), \quad \text{and} \quad f(\xi, t) = 0 \quad \text{for} \quad \xi = 0, 1.$$

Applying the technique discussed in Section 2, Eq. (8) takes the form

$$\sum_{k=1}^{N} C_k^{n+1} L_k(\xi) - \delta t\ \theta\ k\ \sum_{k=1}^{N} C_k^{n+1}\ \mathcal{F}_k'(\xi) = \sum_{k=1}^{N} C_k^{n} L_k(\xi) + \delta t\ (1-\theta)\ k\ \sum_{k=1}^{N} C_k^{n}\ \mathcal{F}_k'(\xi), \tag{17}$$

where $\mathcal{F}_k(\xi)$ represents Fibonacci polynomial given by Eq. (6). The matrices in Eq. (10) are $C = [C_1, C_2, \ldots, C_N]$ and for $k = 1, \ldots, N$,

$$H = \begin{cases} L_k(\xi_i) - \delta t\ \theta\ k\ \mathcal{F}_k'(\xi_i), & i = 2, ..., N-1, \\ L_k(\xi_i), & i = 1, N, \end{cases} \tag{18}$$

$$G = \begin{cases} L_k(\xi_i) + \delta t\ (1-\theta)\ k\ \mathcal{F}_k'(\xi_i), & i = 2, ..., N-1, \\ 0, & i = 1, N, \end{cases} \tag{19}$$

$$B^{n+1} = \begin{cases} 0, & i = 1, 2, \ldots, N-1, \\ \sin(1)e^{-t}, & N. \end{cases} \tag{20}$$

The problem has been solved for different value of M, and T. Error norms L_2, L_∞ and RMS are computed and comparison is made between the results obtained using the current method and the results of Chebyshev Wavelets Method [1] for $M = 16$, different values of T and are shown in Table 1. It is clear from the table that the results obtained using present method are better than that of [1] and this shows efficiency of the technique. In order to show space convergence of the proposed method the results for different values of M are given in Table 2. From the table it can be observed clearly that the solution converges as the nodal point M increase. i.e ($d\xi$ decrease). The graphs of solution profile and absolute errors are presented in Figure 1 which show effectiveness of the current technique.

Example 2.

Consider Eq. (1), with $\alpha = 0$, $\beta = -1$, $\gamma = -2$ and $g(\xi, t) = 0$, defined as homogeneous heat equation given as follows

$$\partial_t Y(\xi, t) = \partial_{\xi\xi} Y(\xi, t) + 2Y(\xi, t), \quad 0 \leq \xi \leq 1, \quad 0 \leq t \leq 1.$$

with initial and boundary conditions

$$Y(\xi,0)=\sinh(\xi), \quad \text{for} \quad 0\le\xi\le 1 \quad \text{and} \quad Y(0,t)=0, \quad Y(1,t)=\sinh(1)e^{-t} \quad \text{where} \quad 0\le t\le 1$$

The exact solution is given by $Y(\xi,t)=\sinh(\xi)e^{-t}$ [1]. The problem has been solved adopting the procedure discussed in section 2. The results are computed for different values of M and T. RMS and L_2, L_∞ error norms have been calculated and comparison is made with available results in literature [1] for different values of T, $M=16$ and $dt=0.001$ and are shown in Table 3. From the table it is straight forward that the results achieved using the proposed method are better than those available in literature which show proficiency of the method used. For convergence in space the results obtained are shown in Table 4 for different values of $d\xi$ showing that the solution converges as the value $d\xi$ decreases. The solution profile of approximate along with absolute errors for $T=1$ and M=15 are plotted in Figure 2.

Example 3.

Consider $\alpha=1,\ \beta=-1,\ \gamma=0,\ g(\mathbf{x},t)=0$ in Eq. (1), we get the following two-dimensional nonlinear burger equation

$$\partial_t Y(\xi,\eta,t)+Y(\xi,\eta,t)\big\{\partial_\xi Y(\xi,\eta,t)+\partial_\eta Y(\xi,\eta,t)\big\}=\beta\big\{\partial_{\xi\xi}Y(\xi,\eta,t)+\partial_{\eta\eta}Y(\xi,\eta,t)\big\} \tag{21}$$

There are two cases.
Case 1:
In this case the exact solution is given as [26]

$$Y(\xi,\eta,t)=\frac{1}{1+e^{\frac{\xi+\eta-t}{2\beta}}}, \quad 0\le\xi,\eta\le 1. \tag{22}$$

Initial and boundary conditions are extracted from the exact solutions. Applying the technique discussed in section 2 Eq. (8) takes the form

$$\begin{aligned}\sum_{k=1}^{N}\sum_{m=1}^{N}\Big[L_k(\xi_i)L_m(\eta_j)-\delta t\ \theta\ \big\{k\ \mathcal{F}_k^{'}(\xi_i)L_m(\eta_j)+m\ L_k(\xi_i)\mathcal{F}_m^{'}(\eta_j)\big\}\Big]C_{km}^{n+1}\\ =\sum_{k=1}^{N}\sum_{m=1}^{N}\Big[L_k(\xi_i)L_m(\eta_j)-\delta t\ (1-\theta)\ \big\{k\ \mathcal{F}_k^{'}(\xi_i)L_m(\eta_j)+m\ L_k(\xi_i)\mathcal{F}_m^{'}(\eta_j)\big\}\\ -\delta t\ \beta\ k\ Y^n\mathcal{F}_k(\xi_i)L_m(\eta_j)\Big]C_{km}^{n}, \quad 0<\xi,\eta<1\end{aligned}$$

The matrices $H,\ G$ and B in Eq. (10), for $k,m=1,\dots,N,$, are

$$H=\begin{cases}L_k(\xi_i)L_m(\eta_j)-\delta t\ \theta\ \big\{k\ \mathcal{F}_k^{'}(\xi_i)L_m(\eta_j)+m\ L_k(\xi_i)\mathcal{F}_m^{'}(\eta_j)\big\} & (\xi_i,\eta_j)\in\Gamma\\ L_k(\xi_i)L_m(\eta_j), & (\xi_i,\eta_j)\in\partial\Gamma\end{cases} \tag{23}$$

$$G=\begin{cases}L_k(\xi_i)L_m(\eta_j)-\delta t\ (1-\theta)\ \big\{k\ \mathcal{F}_k^{'}(\xi_i)L_m(\eta_j)+m\ L_k(\xi_i)\mathcal{F}_m^{'}(\eta_j)\big\} & \\ -\delta t\ \nu\ k\ Y^n\mathcal{F}_k(\xi_i)L_m(\eta_j), & (\xi_i,\eta_j)\in\Gamma,\\ 0, & (\xi_i,\eta_j)\in\partial\Gamma.\end{cases} \tag{24}$$

$$B^{n+1} = \begin{cases} 0, & (\xi_i, \eta_j) \in \Gamma, \\ \frac{1}{1+e^{\frac{\xi_i+\eta_j-t}{2\beta}}}, & (\xi_i, \eta_j) \in \partial\Gamma \end{cases} \tag{25}$$

The approximate solution of the problem has been computed for different values of β, T and N. Error norms L_2, L_∞ and RMS have been calculated and obtained results have been compared with the results of Haar wavlet [26] and differential quadrature [27]. The results are shown in Tables 5 and 6. From these tables it is obvious that proposed method works pretty well than those available in the literature. The solution profile and error plots for $T = 2$, $dt = 0.001$, $\beta = 1$ and nodal points $[20 \times 20]$ are shown in Figure 5 showing efficiency of the proposed technique.

Case 2:

In this case exact solution of Eq. (21) is given as [27]

$$Y(\xi, \eta, t) = 0.5 - \tanh\left(\frac{\xi + \eta - t}{2\beta}\right) \quad 0.5 \leq \xi, \eta \leq 0.5, \quad t \geq 0. \tag{26}$$

Here also initial and boundary conditions are extracted from the exact solutions. The method discussed in previous example is applied to solve this example for different value of viscosity β and M. Here also the error norms L_2, L_∞ and RMS have been computed and are compared with the results of meshless collocation method [27] available in literature for different values of M. The obtained results presented in Table 7 which shows that the present method gives better results than those available in literature. The graph of numerical and exact solution are shown in Figure 4 which shows efficiency of the current technique.

Example 4

Finally consider the case when $\alpha = 0$, $\beta = -1$, $\gamma = 0$ and $g(\xi, \eta, t) = 0$ then Eq. (1) takes the form

$$\partial_t Y(\xi, \eta, t) = \partial_{\xi\xi} Y(\xi, \eta, t) + \partial_{\eta\eta} Y(\xi, \eta, t) \tag{27}$$

The initial and boundary conditions are extracted from exact solution [25]

$$Y(\xi, \eta, t) = \sin(\pi\xi)\sin(\pi\eta)e^{-2\pi^2 t}.$$

The problem has been solved in domain $[0, 1] \times [0, 1]$ over the time interval $[0, 1]$ and the solutions have been obtained for different values of T and M. Error norms are computed and compared with the results of RBFs available in literature given in Tables 8 and 9. From these tables it is clear that the results got using the proposed technique are comparable with those of multiquadric RBFs and better than those of wedland RBFs [25]. The solution profile is plotted in Figure 5 when $T = 0.2$ showing efficiency of the method suggested.

4 Conclusion

In this paper, we studied a numerical technique based on Lucas and Fibonacci polynomials. For implementation of the method first we descritized temporal part of PDEs by finite difference and spatial part by $\theta-$ weighted scheme with $\theta = 1/2$ (Crank Nicolson method). After that the unknown functions are expanded by Lucas series while their derivatives are replaced by Fibonacci polynomials.

Performance and convergence of the method is investigated by test problems including one and two-dimensional linear and nonlinear examples. The results are compared with exact as well numerical results available in the literature. The comparison of the results justify efficiency and applicability of the proposed methodology.

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	Present Method			[1]	
T	L_2	L_∞	**RMS**	L_2	L_∞
0.1	2.14E-09	3.01E-09	2.08E-09	4.46E-06	1.07E-05
0.3	2.75E-09	3.82E-09	2.66E-09	2.15E-07	3.45E-06
0.5	2.39E-09	3.32E-09	2.31E-09	3.18E-06	5.13E-06
0.7	1.97E-09	2.74E-09	1.91E-09	4.71E-06	7.45E-06
0.9	1.62E-09	2.25E-09	1.57E-09	6.04E-06	9.47E-06
1	1.47E-09	2.04E-09	1.42E-09	6.55E-06	1.02E-05

Table 1: Error norms for $M = 16$, $dt = 0.001$ of example 1

$d\xi$	**Cpu time**	L_∞	L_2	**RMS**
1/3	0.0316	5.20E-04	3.67E-04	3.00E-04
1/4	0.0169	2.19E-04	2.31E-04	1.42E-04
1/5	0.0164	1.50E-05	1.42E-05	8.56E-06
1/6	0.0201	9.99E-07	1.62E-06	7.42E-07
1/7	0.0223	5.96E-08	6.90E-08	3.37E-08
1/8	0.0271	1.25E-09	1.60E-09	7.04E-10
1/9	0.0339	1.01E-09	1.71E-09	1.38E-09

Table 2: Space convergence when $T = 1$ and $dt = 0.001$ corresponding to example 1.

	Present Method			**[1]**	
T	L_∞	L_2	**RMS**	L_∞	L_2
0.1	3.20E-09	2.26E-09	2.20E-09	8.44E-03	6.01E-03
0.3	3.69E-09	2.66E-09	2.58E-09	8.10E-03	2.59E-03
0.5	3.13E-09	2.25E-09	2.18E-09	7.43E-03	4.78E-03
0.7	2.57E-09	1.85E-09	1.79E-09	9.18E-03	5.88E-03
0.9	2.11E-09	1.51E-09	1.47E-09	1.07E-03	6.89E-03
1	1.91E-09	1.37E-09	1.33E-09	1.15E-03	7.39E-03

Table 3: Error norms when $M = 16$, $dt = 0.001$ of example 2

$d\xi$	**Cpu time**	L_∞	L_2	**RMS**
1/3	0.0395	4.47E-04	3.16E-04	2.58E-04
1/4	0.0420	2.03E-04	1.48E-04	1.28E-04
1/5	0.0594	1.81E-05	1.17E-05	1.05E-05
1/6	0.0922	9.70E-07	7.37E-07	6.73E-07
1/7	0.1274	7.31E-08	4.47E-08	4.14E-08
1/8	0.1643	1.55E-09	7.11E-09	6.65E-09
1/9	0.2232	1.49E-09	1.35E-09	1.27E-09

Table 4: Space convergence for $T = 1$ and $dt = 0.001$ of example 2.

For T=0.5, dt=0.001 ν=1					
	Present Method			[26]	
Nodal points M	L_∞	L_2	RMS	L_∞	L_2
4×4	1.32E-05	1.25E-05	1.36E-04	5.81E-05	1.53E-04
8×8	1.98E-05	3.07E-05	6.75E-05	6.36E-05	2.77E-04
16×16	1.56E-05	3.33E-05	1.99E-05	6.30E-05	5.42E-04

For T=0.25, dt=0.0025						
	Present Method				[27]	
viscosity ν	Nodal points M	L_∞	L_2	RMS	L_∞	L_2
1	10×10	6.50E-05	1.03E-04	1.00E-04	7.88E-04	4.03E-03
	20×20	2.12E-05	1.85E-05	1.07E-05	8.55E-05	8.47E-04
0.1	10×10	1.79E-04	1.69E-04	1.29E-04	7.18E-04	3.68E-03
	20×20	5.88E-05	2.03E-05	1.12E-05	3.54E-04	1.96E-03

Table 5: Error norms of example 3 case 1

T	0.1	0.25	0.5	1
L_∞	7.17E-04	4.65E-05	3.66E-05	3.46E-05
L_2	0.001247	4.96E-05	1.87E-05	1.79E-05
RMS	1.37E-04	2.74E-05	1.68E-05	1.65E-05

Table 6: Error norms for different values of T when $M = 15$, and $dt = 0.01$, for example 3 case 1.

		Present Method			**[27]**	
ν	**M**	L_2	L_∞	**RMS**	L_2	L_∞
0.02	100	2.41E-03	1.38E-03	4.03E-04	2.49E-01	6.86E-01
	400	2.63E-04	7.55E-04	3.67E-05	1.42E-02	5.69E-02
	625	8.70E-05	2.32E-04	1.38E-05	1.44E-04	1.65E-03
0.013	100	2.58E-03	2.37E-03	3.99E-04	7.69E-01	2.07E+00
	400	2.64E-04	9.52E-04	3.68E-05	8.61E-02	3.48E-01
	625	5.96E-05	1.83E-04	1.14E-05	8.53E-03	5.68E-02
0.01	100	2.43E-03	2.35E-01	4.04E-04	1.83E+00	4.96E+00
	400	2.99E-04	8.36E-04	3.91E-05	2.10E-01	1.09E+01
	625	7.19E-05	3.11E-04	1.25E-05	4.31E-02	4.29E-01

Table 7: Error norms for $T = 0.1$, $dt = 0.001$ of example 3 case 2

Present Method				[25]			
				Multiquadric RBF		**Wedland RBF**	
M	L_∞	L_2	**RMS**	L_∞	**RMS**	L_∞	**RMS**
25	3.15E-04	3.83E-04	5.29E-04	2.59E-04	1.21E-04	1.96E-03	9.23E-04
64	1.40E-05	1.86E-05	5.25E-05	2.35E-05	1.22E-05	3.49E-04	1.82E-04
100	8.17E-07	8.60E-07	7.60E-06	6.43E-06	3.26E-06	1.18E-04	6.27E-05

Table 8: Error norms when $T = 0.2$, and $dt = 0.001$ for example 4

T	0.1	0.2	0.3	0.4	0.5
L_∞	5.01E-05	1.40E-05	2.98E-06	5.52E-07	9.54E-08
L_2	6.95E-05	1.86E-05	3.88E-06	7.11E-07	1.22E-07
RMS	1.02E-04	5.25E-05	2.40E-05	1.03E-05	4.26E-06

Table 9: Error norms when $M = 64$ and $dt = 0.001$ of example 4.

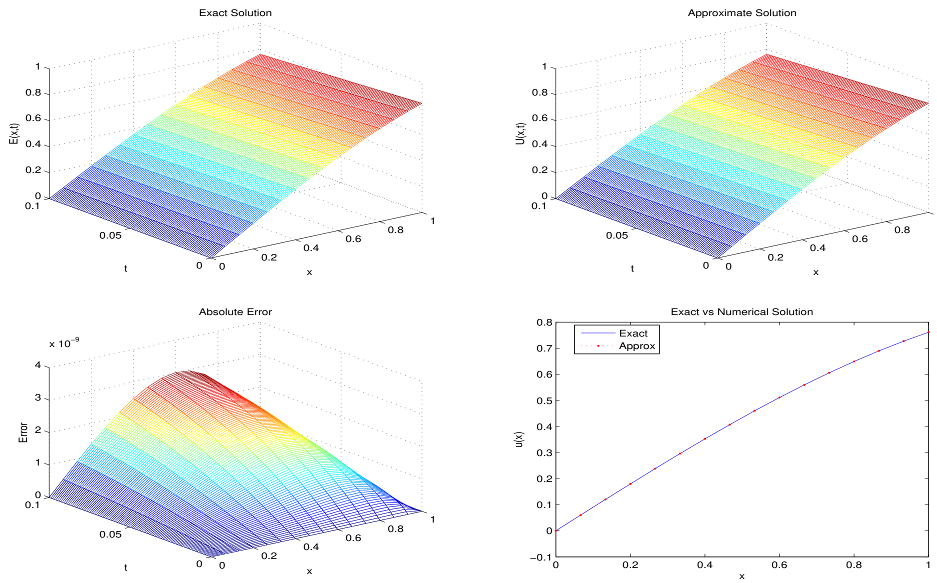

Figure 1: Solution Profile when $T = 0.1$, $dt = 0.001$, $M = 15$ of example 1

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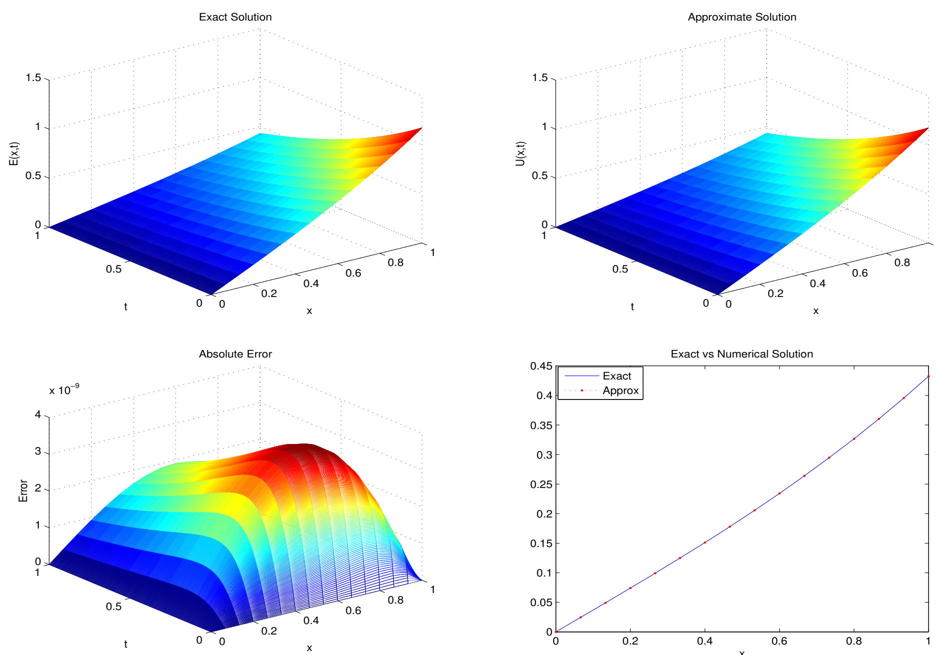

Figure 2: Solution Profile when $T = 1$, $dt = 0.001$, $M = 15$ of example 2

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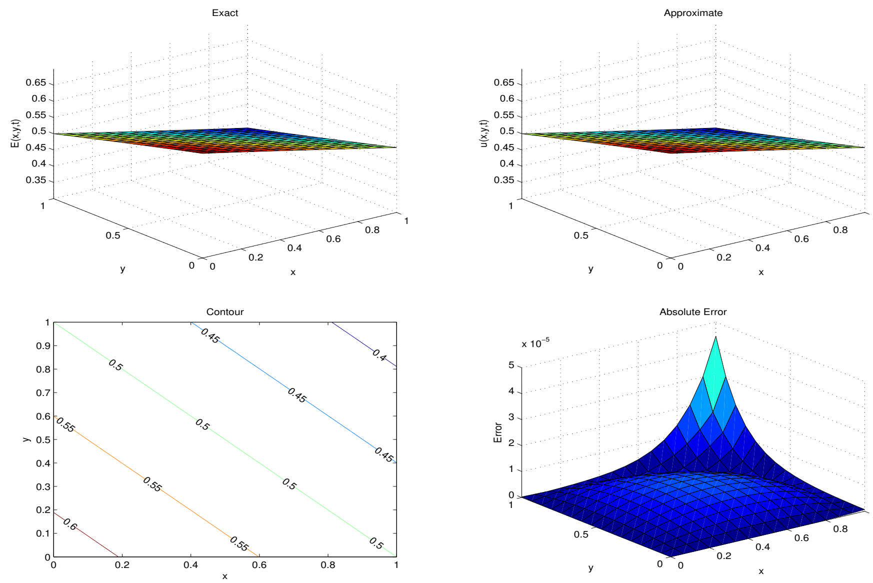

Figure 3: Solution Profile when $T = 1$, $dt = 0.01$, $M = [15 \times 15]$ of example 3 case 1

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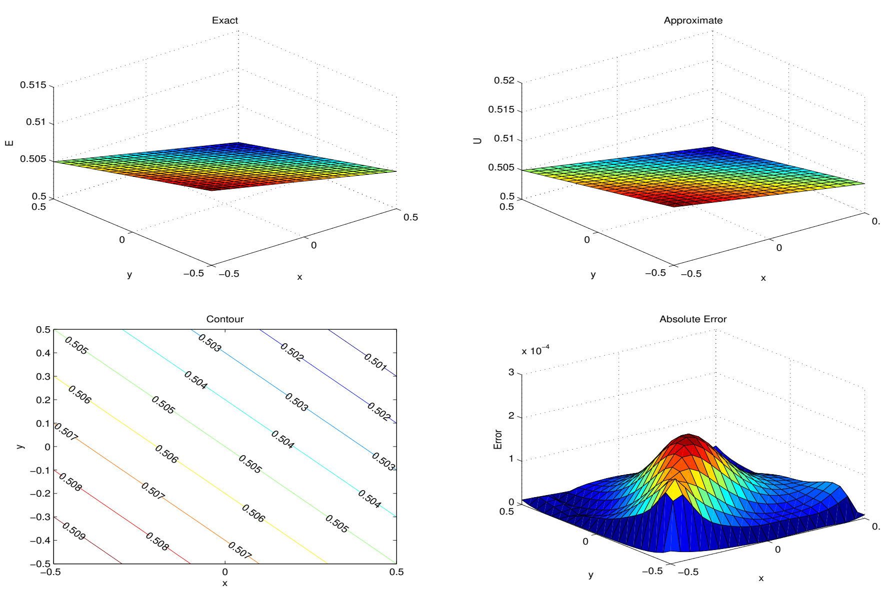

Figure 4: Solution Profile when $T = 1$, $dt = 0.01$, $\nu = 0.01$, $M = [20 \times 20]$ of example 3 case 2

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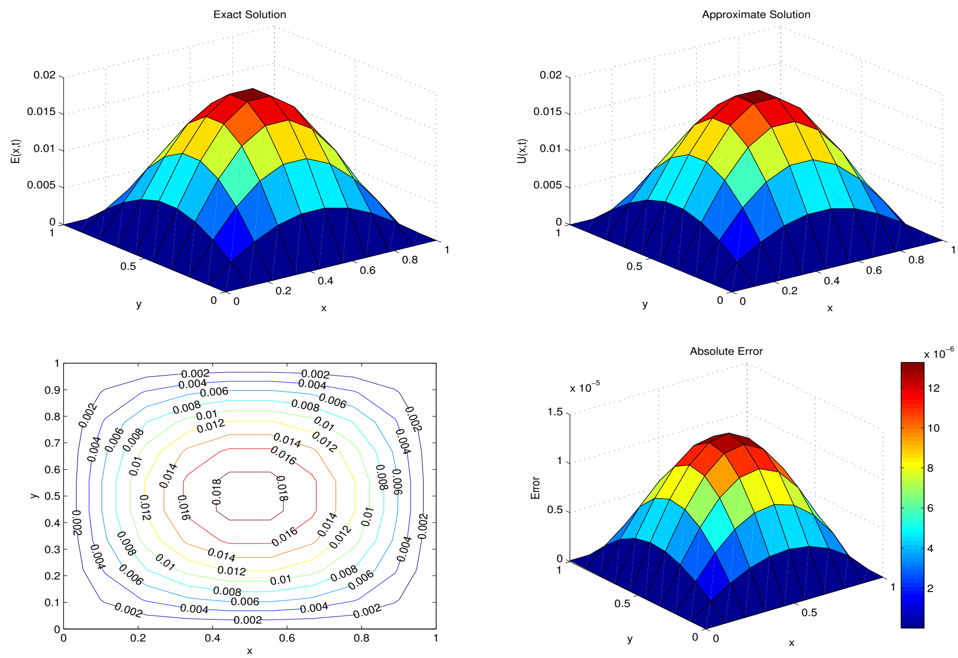

Figure 5: Solution Profile when $T = 0.2$, $dt = 0.001$, $M = 100$ of example 4
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