ORIGINAL ARTICLE
Journal Section

Fractional differential equation modeling a viscoelastic fluid in mass-spring-magnetorheological damper mechanical system

Escalante-Martínez J. E.[1] | Morales-Mendoza L. J.[2] | Cruz-Orduña M. I.[1] | Rodriguez-Achach M.[3] | Behera-D.[4] | Laguna-Camacho J. R.[1] | López-Calderón H. D.[5] | López-Cruz V. M.[1]

[1]Faculty of Mechanical and Electrical Engineering, Universidad Veracruzana, 93390, Poza Rica, México

[2]Faculty of Electronics and Communications Engineering, Universidad Veracruzana, 93390, Poza Rica, México

[3]Marist University of Merida, 97300. Mérida, Yucatán, México

[4]Department of Mathematics, The University of the West Indies, Mona, Kingston 7, Jamaica

[5]Institute of Biotechnology, Universidad Autónoma de Nuevo León, México

Correspondence
Escalante-Martínez J.E., Faculty of Mechanical and Electrical Engineering, Universidad Veracruzana, 93390, Poza Rica, México
Email: jeescalante@uv.mx

Funding information
No financial support was received

The mass-spring-damper system is the minimum complexity scenario that characterizes almost all the mechanical vibration phenomena, it is well known that a second-order differential equation model its dynamics. However, if the damper has a magnetorheological fluid in the presence of a magnetic field then the fluid shows viscoelastic properties. Hence the mathematical model that best reflects the dynamics of this system is a fractional order differential equation. Naturally, the Mittag-Leffler function appears as analytical solution. Accordingly we present here the mathematical modeling of the mass-spring-magnetorheological damper system. The main result of our investigation is to show how the fractional order γ changes when the viscosity damping coefficient β changes, this was found when varying current intensity in the range of 0.2 to 2 Amperes. A

Abbreviations: ABC, a black cat; DEF, doesn't ever fret; GHI, goes home immediately.

* Equally contributing authors.

Helmholtz coil is used to produce the magnetic field. We consider that this document has a high pedagogical value in connecting the fractional calculation to mechanical vibrations and can be used as a starting point for a more advanced treatment of *fractional mechanical oscillations*.

KEYWORDS
Fractional Calculus, Mass-Spring-Damper, Magnetorheological Damper, Viscoelastic Fluids, Mittag-Leffler Function, Tracker video analysis, Caputo Derivative.

1 | INTRODUCTION

The mass spring damper system is a physical system used to study mechanical vibrations. When the viscous properties of the fluid in the buffer remain constant, an ordinary differential equation of the second order serves as a mathematical model; that is to say, the evolution of said mechanical oscillations can be predicted when studying the analytical solution of said equation. Three typical behaviors are observed from the values in the elastic spring constants, the value of the mass and the viscous damping coefficient, called: overdamped, critically damped and underdamped movement.

The viscous damping coefficient is an artificial parameter, which can not be accessed by a physical measurement such as the mass or spring elasticity constant. It is an ad hoc parameter, which serves to justify all the energy losses present in the movement of the mass, for example, the friction with the medium and the increase in the temperature of the spring. If the height measured from the point of repose of the mass is $x(t)$, by Newton's second law, Hooke's law and assuming that the viscosity of the fluid in the damper directly affects the velocity of motion, then it is well known that:

$$mx''(t) + \beta x'(t) + kx(t) = 0 \tag{1}$$

is an equation that models the movement of the mechanical mass-spring-damper classic system. Here m is the value of the mass, k is the elasticity constant of the spring and β is the viscous damping coefficient and analytical solution is

$$x(t) = x_0 \cdot \exp\left(-\frac{\beta}{2m}t\right) \cdot \cos\left(\sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}\; t\right) \tag{2}$$

One of the most obvious applications of this model is the design of shock absorbers that support earthquakes [1], [2]. However, classical dampers have the limitation that their viscous damping coefficient is constant, unlike magnetorheological damper capable of varying its viscosity. The equation 1 is analyzed for construct microfabricated electromagnetic energy harvesters [3]. The research presented here could help medical researchers refine their approaches, for example [4] where they consider the viscous drag of blood flow.

In this paper we will report what was observed experimentally, about the change of the fractional order of derivation versus the coefficient of viscous damping.

The way we are going to proceed is:

1. Record the mass-spring-magnetorheological damper experiment
2. Obtain the experimental data using the open source physics software Tracker Video Analysis and Modeling Tool [5].
3. Adjust a Mittag-Leffler function of a parameter using the routine by Igor Podlubny described in the article [6],[7],[8], this will allow us to determine the fractional order of derivation.
4. Discuss the change in the fractional order of derivation with respect to the viscous damping coefficient.

We will begin by analyzing the change that the value of the viscous damping coefficient undergoes, when we tune the value of the electric current in a range of 0.2 amps up to 2 amps, which induces different magnetic field intensities in the Helmholtz coil, since fluid inside the damper is magnetorheological.

2 | FRACTIONAL CALCULUS. BASIC TOOLS

Fractional calculus is a branch of mathematical analysis dedicated to study integrals and derivatives of arbitrary order even complex order (the term “fractional” is kept only for historical reasons), must be considered a branch of mathematical physics which deals with integro-differential equations, where integrals are of convolution type and exhibit singular or non singular kernels of power law type [9], [10]. Interesting applications of the non-integer derivative can see [11]-[12].

Most historians agree that the fractional calculus was developed for the first time on September 30, 1695 when the Marquis de L’Hôpital writes a letter to Leibniz asking about the notation for the n-th derivative of a function: What would happen if n were $1/2$? [13], [14]. Since then many important mathematicians, Euler (1738), Fourier (1820), Abel (1826), Liouville (1832), Riemann (1847), Laurent (1884), just for mention some of the twentieth century: Weyl (1917), Hardy and Littlewood (1928), Riesz (1936), Erdélyi (1940, 1963, 1972), Caputo (1969), Podlubny (1996) have continued the study of the derivative of arbitrary order and its applications.

Smart mechanical devices involve mathematical models more accurate to fitting experimental data [15]. It is the case of magnetorheological fluids, wich are viscoelastic fluids capable of changing their mechanical properties in the presence of a magnetic field. Its viscosity depends on the density of field lines hence non-locally, this is why the fractional derivative is better for modeling this phenomenon. However commercial applications are still not so common because in general it is difficult to build them and keep stable [16]. There are suspensions of different magnetic materials, for example iron particles [17] and some constructed from inkjet fluids [18]. In our experiment we have used toner powder given its magnetic properties. This dust suspended in seawater due to its electrical conduction properties, resulting a fluid that increases its viscous damping coefficient to a threshold value when the magnetic field intensity is increased.

Fractional model may cover a wide range of viscoelastic behavior. There are many rigorous work done by physicists, applieds mathematicians and engineers etc. can be found [19]-[20], [21].

2.1 | Magnetorheological fluids have viscoelastic properties

Magnetorheological fluids are suspensions of magnetic particles in a carrier fluid. In the presence of magnetic fields it change their mechanical properties, in particular their viscosity presents both viscous and elastic behavior [22]. Viscoelasticity is a property possessed by bodies. It exhibit both viscous and elastic behavior through simultaneous dissipation and storage of mechanical energy [23]-[24], [25]. The magnetorheological fluids are commonly used in intelligent devices that adjust their viscosity in view of the requirement to which they are subjected. For example dampers in the bases of a building can react to the intensity of an earthquake, see [26] for interesting seismic control of structures. Moreover next example is a vehicle with magnetorheological dampers can respond to different driving situations such as a tire puncture or a hole in the road [27]. By their nature, magnetorheological fluids are modeled with greater precision by derivatives of non-integer order [28]-[29].

3 | FRACTIONAL DIFFERENTIAL EQUATION MODEL

It is necessary to make some assumptions in order to keep the model as simple as possible but without sacrificing generality. We will assume that spring's mass is negligible, that the movement of the mass is along a fixed vertical line and it will be assumed that friction with air is included in the absorption of force by the damper. According to Hooke's Law, the spring exerts a restoring force that opposes the movement of the mass, F_s, it follows that:

$$F_s(t) = -kx(t). \tag{3}$$

In addition, the mass is connected to a magnetorheological damper that tends to slow the movement, this force is proportional and of opposite direction to the speed of the mass, F_d, that is to say:

$$F_d(t) = -\beta \frac{d^\gamma}{dt^\gamma} x(t), \quad 0 < \gamma \leq 1. \tag{4}$$

Where γ represents the order of the fractional temporal operator. It is at this point of the mathematical model that the fractional derivative becomes important, since the derivative of the integer order does not reflect the viscoelastic nature of the shock absorber that we are considering, because its viscosity depends on non-local effects in the magnetic field. There are experimental results that support this fact, can be found in [26], [28], [30] and some more in their references.

From Newton's second law $F = ma$, we have that the mass experiences a force $F = F_s + F_d$. To consider the non-local effects of the viscoelasticity of the magnetorheological damper, the acceleration a, will be replaced by its fractional generalization, $a = \dfrac{d^{2\gamma}}{dt^{2\gamma}}$, so the equation of the mass-spring-damper with magnetorheological fluid is:

$$m \frac{d^{2\gamma}}{dt^{2\gamma}} x(t) + \beta \frac{d^\gamma}{dt^\gamma} x(t) + kx(t) = 0, \quad 0 < \gamma \leq 1. \tag{5}$$

The mass spring damper system has been modeled by fractional calculation by many teams of researchers. For example, Chakraverty and Behera [31] took the fractional derivative and the homotopy method to analyze it. Morales-Delgado and Gómez-Aguilar [32] use Atangana-Koca fractional derivatives with variable- and constant-order to obtain the analytical solutions of the mass-spring-damper system. More recently, in the article [33] heat transfer of a ferrofluid with magnetic nanoparticles is studied by partial differential equations.

This mathematical model was reported in the papers [34], [35], despite this, here we use a fractional damper whose phenomenological translation to the equations is possible using derivatives of non- integer order, indeed we take Caputo's derivate to more accurately model the mechanical system. To keep the dimensionality physically plausible a new parameter σ was introduced in the following way

$$\frac{d}{dt} \to \frac{1}{\sigma^{1-\gamma}} \cdot \frac{d^{\gamma}}{dt^{\gamma}}, \quad m-1 < \gamma \leq m, m \in \mathbb{Z}^{+} \tag{6}$$

and

$$\frac{d^2}{dt^2} \to \frac{1}{\sigma^{2(1-\gamma)}} \cdot \frac{d^{2\gamma}}{dt^{2\gamma}}, \; m-1 < \gamma \leq m, m \in \mathbb{Z}^{+} \tag{7}$$

where σ is the dimension of seconds, this auxiliary parameter is associated with the temporal components in the system (these components change the time constant of the system). The authors of [36] used the Planck time, $t_p = 5.39106 \times 10^{-44}$ seconds, as a way to preserve the dimensional compatibility. Recently, the discussion about the use of physical units in mathematical models that use fractional derivatives has become more formal and rigorous, even in magneto-electrical phenomena [37]. Following [36] the σ parameter corresponds to the t_p in our calculations. For the case $\gamma = 1$ the expressions (6) and (7) become ordinary temporal operators. Following this idea, the fractional equation of the mass-spring-damper with magnetorheological fluid mechanical system represented in Figures 1, 2 is given by

$$\frac{m}{{t_p}^{2(1-\gamma)}} {}_0^C D_t^{2\gamma} x(t) + \frac{\beta}{{t_p}^{1-\gamma}} {}_0^C D_t^{\gamma} x(t) + kx(t) = 0, \qquad 0 < \gamma \leq 1, \tag{8}$$

where the mass is m, the viscous damping coefficient is β and the spring constant is k. The equation (8) may be written as follows

$${}_0^C D_t^{2\gamma} x(t) + \lambda_\gamma \, {}_0^C D_t^{\gamma} x(t) + \omega_\gamma^2 \, x(t) = 0, \quad 0 < \gamma \leq 1, \tag{9}$$

where

$$\lambda_\gamma = \frac{\beta}{2m} {t_p}^{1-\gamma}, \qquad \omega_\gamma^2 = \frac{k}{m} {t_p}^{2(1-\gamma)}. \tag{10}$$

The solution of the equation (9) is

$$x(t) = x_0 \cdot E_\gamma \left\{ -\frac{\beta}{2m} {t_p}^{1-\gamma} t^{\gamma} \right\} \cdot E_{2\gamma} \left\{ -\left[\frac{k}{m} - \frac{\beta^2}{4m^2} \right] {t_p}^{2(1-\gamma)} t^{2\gamma} \right\} \tag{11}$$

where $E_\gamma := E_{\gamma,1}$ is a two parametric function of the Mittag-Leffler type is defined by the series expansion [21], given by (12)

$$E_{\gamma,\varphi}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \varphi)}, \ (\gamma > 0, \ \varphi > 0) \tag{12}$$

For the classical case $(\gamma = \varphi = 1)$, the expression (11) becomes (2). To solve the fractional differential equation under the definition of the Caputo derivative we start from the equation (8). Making a change of variable,

$$\xi_\lambda = \frac{\beta}{m} t_p^{1-\gamma}, \qquad \zeta_\lambda = \frac{k}{m} t_p^{2(1-\gamma)}$$

and,

$$x_1(t) = x(t), \quad x_2(t) = {}_0^C D_t^\gamma x(t),$$

the equation (8) can be rewritten as the following system

$$\begin{aligned} {}_0^C D_t^\gamma x_1(t) &= x_2(t) \\ {}_0^C D_t^\gamma x_2(t) &= -\zeta_\gamma \, x_1(t) - \xi_\gamma \, x_2(t) \end{aligned} \tag{13}$$

This system is solved by Garrappa [38], the solution curve is shown in the Figures 7 and 6 together with the Mittag-Leffler function graphs and experimental data.

4 | MATERIALS AND METHODS

The experiments were carried out with the mechanical system spring-mass-damper assembled in a Maxwell type arrangement (See Figure 2). The mass was weighted in kilograms, $m = 0.2197\ kg$, spring's elastic constant $k = 2.35\ N/m$ was calculated by Hooke's Law; that is, the elongation produced by a mass of known weight was measured. The value of elastic constant k, is the quotient of weight divided by the elongation produced.

The Damper used is one graduated cylinder with $500\ ml$ capacity, $35\ cm$ in height and $5\ cm$ in internal diameter, the piston is a thin wooden tube $50\ cm$ long, $0.6\ cm$ in diameter, for one side the mass is held and on the other side it has embedded a circular disc of plastic of thickness $1\ mm$ and $2\ cm$ radius, its center is located in the wooden tube. Magnetorheological fluid is composed of seawater and toner, a mixture of $400\ ml$ seawater and $20.54\ gr$ of toner was used. To obtain a homogeneous mixture, it is liquefied for one minute. The DC power supplies is a BK Precision 1672 model. Gaussmeter F.W. Bell 5180 model, see Figure 3.

The coil is Helmholtz type which guarantees a uniform magnetic field zone along the space where piston moves [39], [40]. The coil consists of 200 turns of 22 AWG magnet wire on each side, each measuring $2\ cm$ wide and there is $3\ cm$ of separation between them, they are mounted on a thin cardboard paper and its diameter is $6\ cm$, see Figure 4.

FIGURE 1 Real mechanical system analyzed in the laboratory.

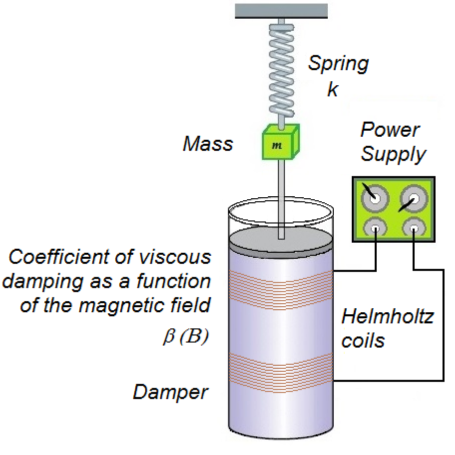

FIGURE 2 Scheme of mechanical oscillator considered, m is the mass, the viscous damping coefficient is β depended of magnetic field intensity B and spring's constant k.

FIGURE 3 Power supply, Gaussmeter and Helmholtz's coil. To measure the intensity of the magnetic field, we put the probe connected to the Gaussmeter inside coil, while adjusting the values of current intensity at the power supply.

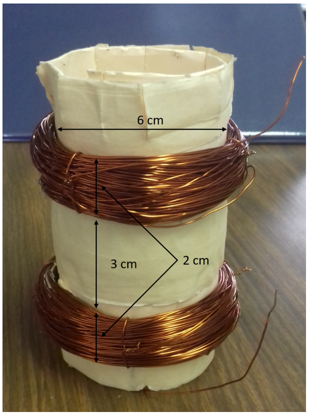

FIGURE 4 Helmholtz's Coil. 22 AWG magneto wire. A Helmholtz coil was used to have a uniform magnetic field section, which measures approximately 7 cm considering that the maximum displacement of the mass is 5 cm.

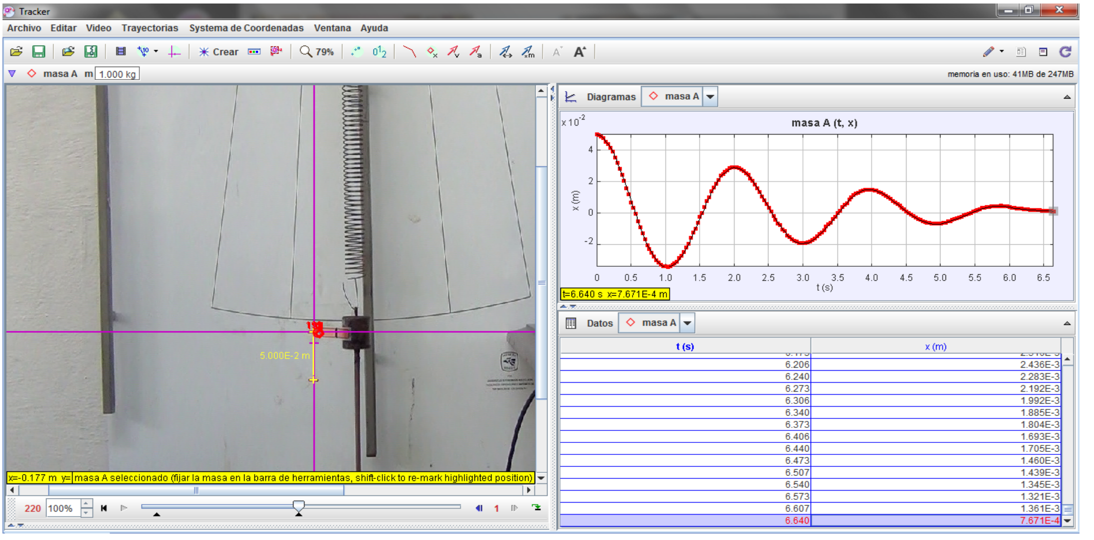

FIGURE 5 Screenshot of Tracker.

To collect the experimental data, we videotaped the damping spring mass system, which was released from rest at a distance of 0.05 meters below its equilibrium point; that is, in terms of initial conditions for the differential equation we have:

$$x(0) = 0.05, \quad x'(0) = 0.$$

Experiments were performed under identical conditions for each current intensity value from $0.2, 0.4, 0.6, \ldots, 2$ amps. The videos were analyzed by Tracker, where we obtained the data, see Fig 5.

5 | CURRENT INTENSITY-MAGNETIC FIELD VS DAMPING COEFFICIENT

The viscous damping coefficient β, changes by changing the magnetic field B, which is measured in Gauss. At the same time, the magnetic field varies with the increase in current I, measured in Amperes. Table 1 shows the values of magnetic field and the viscous damping coefficient for different values of current intensity, in the range of 0 to 2 amperes, varying from $0.2\ A$ in each experiment. The way to calculate β is described in [35].

Figure 6 shows the values of viscous damping coefficient β on the vertical axis and the amperage I supplied to the coil on the horizontal axis. Points A, B,..., K correspond to a ordered pairs (I_s, β_s) for $s = 1, 2, \ldots, 11$ respectively. For example, $C = (I_3, \beta_3) = (0.4, 0.1502748)$ in Table 1.

An adjustment was made by polynomials for the points of graph (Figure 6), which has a R-square value of 0.9757 for the grade 3 polynomial f function in the Figure 6. In addition, it was adjusted by a five grade polynomial function g in the Figure 6, this interpolation had an R-square value of 0.9922. Both adjustments were made through curve fitting tool of Matlab. The functions are:

	I (Amperes)	B (mT)	β Damping Coefficient
A	0.0	0.0	0.1308683
B	0.2	0.42194	0.1432004
C	0.4	0.84391	0.1502748
D	0.6	1.2658	0.1582718
E	0.8	1.6877	0.1743978
F	1.0	2.1098	0.1843283
G	1.2	2.5316	0.1855586
H	1.4	2.9535	0.1861737
I	1.6	3.3755	0.1899086
J	1.8	3.7975	0.1940390
K	2.0	4.2195	0.2056392

TABLE 1 Current intensity I measured in Amperes, Magnetic field B measured in miliTeslas and Viscous Damping Coefficient β.

$$\begin{aligned} f(x) &= 0.006x^3 - 0.031x^2 + 0.073x + 0.129 \\ g(x) &= 0.017x^5 - 0.06x^4 + 0.049x^3 - 0.001x^2 + 0.044x + 0.131 \end{aligned} \tag{14}$$

6 | FITTING OF EXPERIMENTAL DATA

The adjustment of the experimental data is made by Podlubny [6]. Curve fitting $x_f(t)$ is

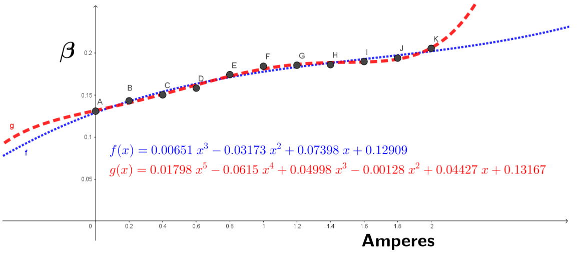

FIGURE 6 The viscous damping coefficient β seen as a function of the current intensity I measure in amperes in X-axis.

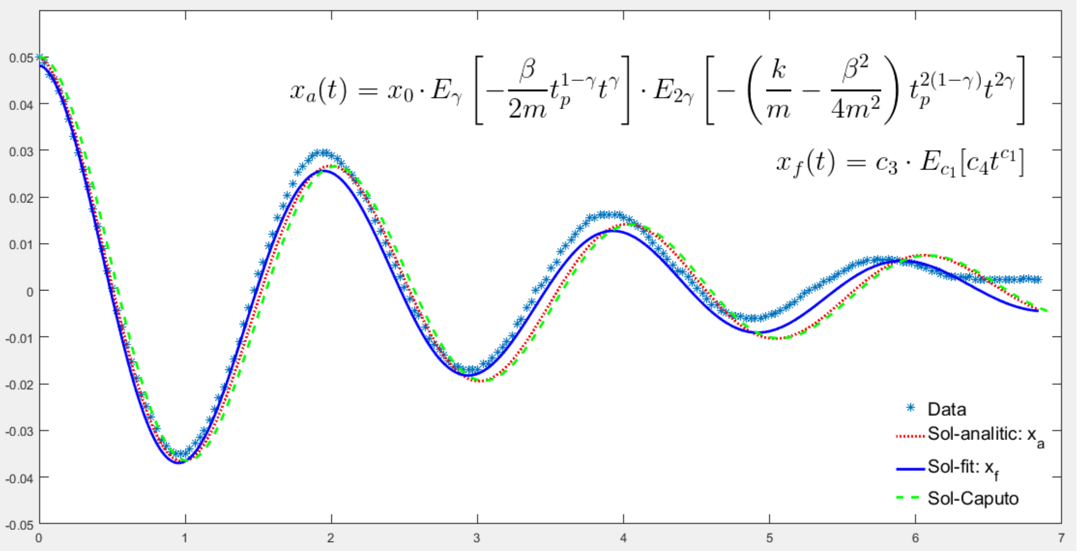

FIGURE 7 Adjustment of the analytical solution to the experimental data. Non-integer order of derivation is $\gamma = 0.9995$

$$x_f(t) = c_3 \cdot E_{c_1}\left[c_4 t^{c_1}\right]. \tag{15}$$

where, $C = [c_1, c_2, c_3, c_4]$ is a vector with the parameters that best fit the data (see Table 2), in every case $c_2 = 1$. In the Figures 7, 6 is shown Mittag-Leffler curve that best fit the data, analytic solution (11), together with the solution of system (13) using Caputo derivative, this was solved by the Garrappa routine outlined in the article.[38].

This procedure was performed for all the experiments where magnetic field intervened, the table 2 contains the value of viscous damping coefficient in the first column, fractional derivation orders (FDO) calculated as in [35] in the second column. From the third to fifth columns are Mittag-Leffler parameter that best fitting the experimental data as in [6], $c_2 = 1$ in every case for have a one parameter Mittag-Leffler function (see (12)). Table 2 does not show the row A of the table 1 because in that experiment no current was applied to the coil, therefore there was no presence of magnetic field and consequently the fluid not was magnetorheologic.

The similarity of the curves can be explained by the following theorem

Theorem. The Mittag-Leffler function, denoted by $E_\gamma(z)$, defined by $E_\gamma := E_{\gamma,\varphi}$ expressed in the equation (12), satisfy

$$E_{2\gamma}(z^2) = \frac{1}{2}\left[E_\gamma(z) + E_\gamma(-z)\right].$$

	β VDC	$(0<\gamma<1)$ FDO	$c_1=\gamma$ FDO-FIT	c_3	c_4
B	0.1432	0.9995	1.8681	0.0481	−8.744
C	0.1502	0.9995	1.8660	0.0490	−8.369
D	0.1582	0.9995	1.8577	0.476	−8.289
E	0.1743	0.9995	1.8455	0.475	−8.274
F	0.1843	0.9998	1.8411	0.444	−8.559
G	0.1855	0.9995	1.8240	0.460	−8.008
H	0.1861	0.9996	1.8228	0.477	−7.54
I	0.1899	0.9995	1.8155	0.491	−7.433
J	0.1940	0.9995	1.8109	0.485	−7.169
K	0.2056	0.9995	1.8221	0.463	−7.315

TABLE 2 Viscous Damping Coefficient β, Fractional Order of Derivation γ of analitic solution and Fractional Order of Derivation by fit Mittag-Leffler function.

Proof. First, notice that

$$\sum_{r=0}^{m-1} \exp\left(\frac{i2\pi kr}{m}\right) = \begin{cases} m, & \text{if } k = 0 \ (mod\ m) \\ 0, & \text{if } k \neq 0 \ (mod\ m) \end{cases}$$

then,

$$\sum_{r=0}^{m-1} E_\gamma(ze^{i2\pi r/m}) = mE_{\gamma m}(z^m), \quad m \in \mathbb{N}$$

wich can be written as

$$E_\gamma(z) = \frac{1}{m}\sum_{r=0}^{m-1} E_{\gamma/m}(z^{1/m}e^{i2\pi r/m}), \quad m \in \mathbb{N}$$

taking $\gamma = m/n$, it follows that

$$E_{m/n}(z) = \frac{1}{m}\sum_{r=0}^{m-1} E_{1/n}\left(z^{1/m}e^{i2\pi r/m}\right)$$

now, if $\gamma = m/n$

$$E_\gamma(z) = \frac{1}{m}\sum_{r=0}^{m-1} E_{\gamma/m}\left(z^{1/m}e^{i2\pi r/m}\right)$$

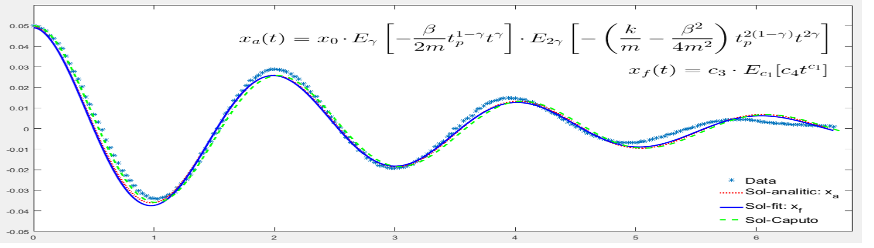

FIGURE 8 $\gamma = 0.9995$

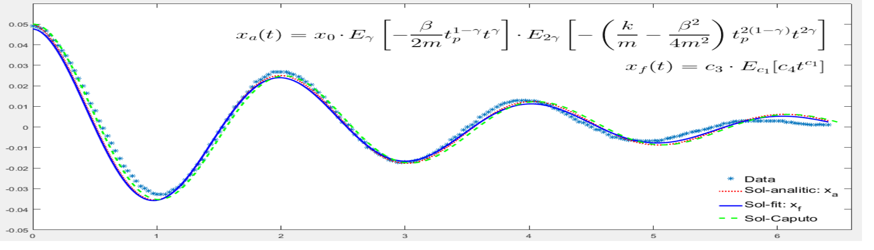

FIGURE 9 $\gamma = 0.9995$

in particular case, $m = 2$ and for Euler's identity

$e^{i\pi} + 1 = 0$

$$\begin{aligned} E_\gamma(z) &= \frac{1}{2}\sum_{r=0}^{1} E_{\gamma/2}\left(z^{1/2}e^{i2\pi r/2}\right) \\ &= \frac{1}{2}\left[E_{\gamma/2}(z^{1/2}) + E_{\gamma/2}(z^{1/2}(-1))\right] \end{aligned}$$

equivalent to what we wanted to demonstrate

$$E_{2\gamma}(z^2) = \frac{1}{2}\left[E_\gamma(z) + E_\gamma(-z)\right]$$

This is why the analytical solution (11) and the adjustment curve are very similar.

7 | CONCLUSION

The classical mathematical model (1) does not reflect the viscoelastic properties of the magnetoreological fluid, whereas the fractional model (9) captures the complex nature of the damper by non-integer derivatives ($0 < \gamma < 1$). Magnetorhelogical dampers are important for the design of structures capable of support earthquakes given their ability to adjust their viscosity. Other commercial applications are the design of shock absorbers for motorcycles [41].

The way in which the viscous damping coefficient varied was as a increasing function. There is a threshold value

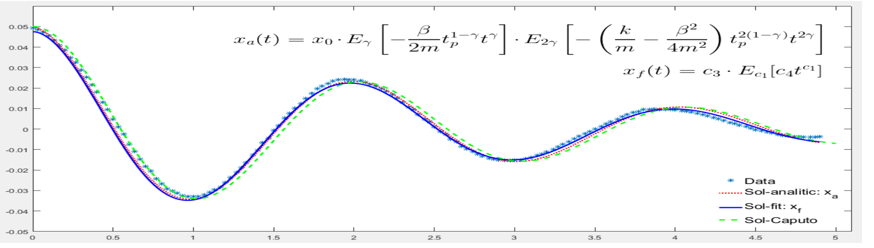

FIGURE 10 $\gamma = 0.9995$

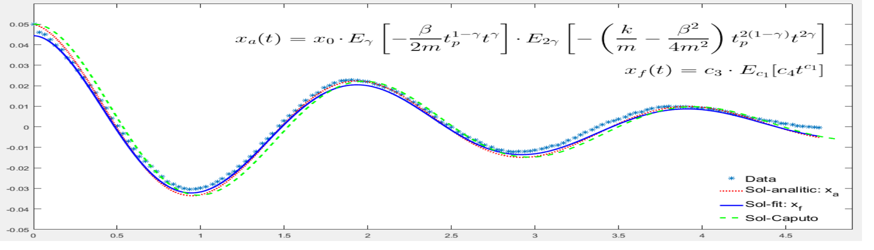

FIGURE 11 $\gamma = 0.9998$

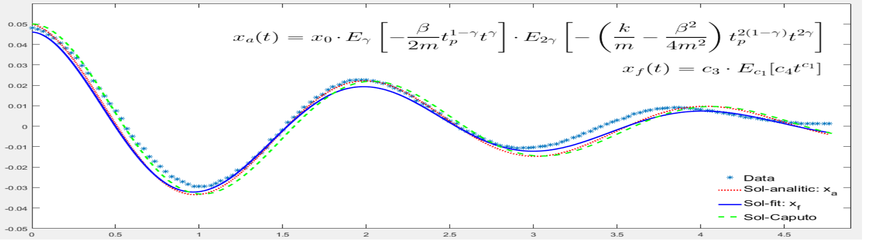

FIGURE 12 $\gamma = 0.9995$

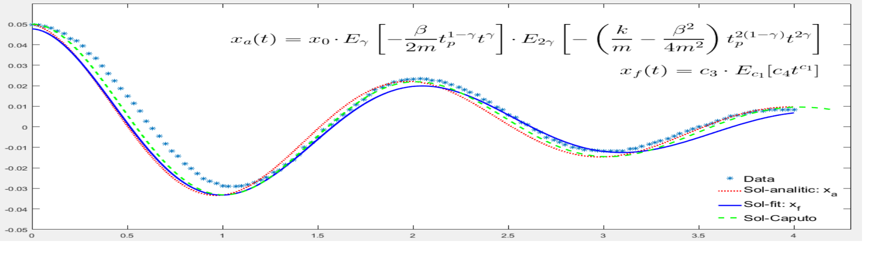

FIGURE 13 $\gamma = 0.9996$

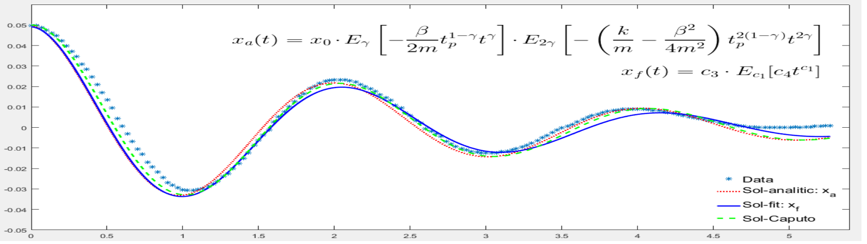

FIGURE 14 $\gamma = 0.9995$

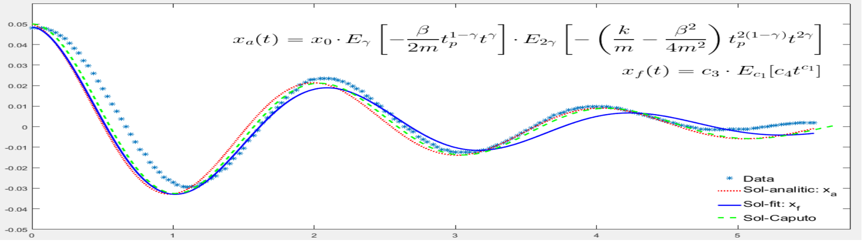

FIGURE 15 $\gamma = 0.9995$

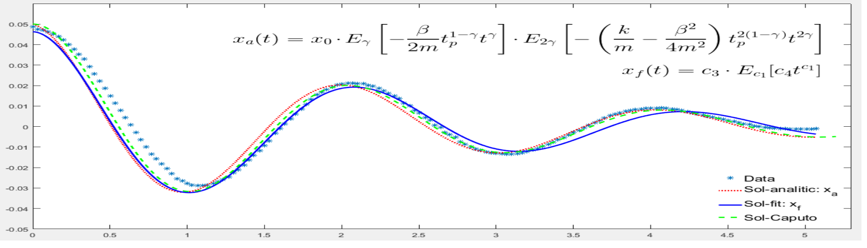

FIGURE 16 $\gamma = 0.9995$

in the current I which triggers a threshold magnetic field B for which the viscous damping coefficient β its value grows as a power of the current intensity; that is, the viscous damping coefficient it has better fit as a fifth degree polynomial of the current intensity as one can see in Table 1. On the other hand, there are other definitions of derivative with variable order that can to extend the analysis, when using the Caputo Derivative, the initial conditions are expressed in terms of derivatives of an integer order, so we have used only that definition.

The analysis shown here is the simplest possible, corresponds to an effort to compare the mathematical fractional differential equation model with a mechanical mass-spring-magnetorheological damper system, our hope this analysis will help to deepen the study of new materials and devices.

acknowledgements

We would like to thank to Marisol Martínez-Bello for the interesting discussions.

conflict of interest

The authors declare no potential conflict of interests.

references

[1] AlMusbahi S, Güngör A. A composite building isolation system for earthquake protection. Engineering Science and Technology, an International Journal 2019;22(2):399–404.

[2] Raheem SEA. Mitigation measures for earthquake induced pounding effects on seismic performance of adjacent buildings. Bulletin of earthquake engineering 2014;12(4):1705–1724.

[3] Zhang Q, Kim ES. Microfabricated electromagnetic energy harvesters with magnet and coil arrays suspended by silicon springs. IEEE Sensors Journal 2015;16(3):634–641.

[4] Gusenbauer M, Kovacs A, Reichel F, Exl L, Bance S, Özelt H, et al. Self-organizing magnetic beads for biomedical applications. Journal of Magnetism and Magnetic Materials 2012;324(6):977–982.

[5] Tracker Video Analysis and Modeling Tool;. `https://physlets.org/tracker`.

[6] Fitting data using the Mittag-Leffler function;. `https://www.mathworks.com/matlabcentral/fileexchange/32170-fitting-data-using-the-mittag-\\leffler-function`.

[7] Sierociuk D, Podlubny I, Petras I. Experimental evidence of variable-order behavior of ladders and nested ladders. IEEE Transactions on Control Systems Technology 2012;21(2):459–466.

[8] Petras I, Sierociuk D, Podlubny I. Identification of parameters of a half-order system. IEEE Transactions on Signal Processing 2012;60(10):5561–5566.

[9] Gómez-Aguilar J, Escobar-Jiménez R, López-López M, Alvarado-Martínez V. Analysis of projectile motion: A comparative study using fractional operators with power law, exponential decay and Mittag-Leffler kernel. The European Physical Journal Plus 2018;133(3):103.

[10] Morales-Delgado V, Gómez-Aguilar J, Taneco-Hernandez M. Analytical solutions for the motion of a charged particle in electric and magnetic fields via non-singular fractional derivatives. The European Physical Journal Plus 2017;132(12):527.

[11] Tiwana M, Maqbool K, Mann A. Homotopy perturbation Laplace transform solution of fractional non-linear reaction diffusion system of Lotka-Volterra type differential equation. Engineering Science and Technology, an International Journal 2017;20(2):672–678.

[12] Kumar M, Rawat TK. Design of fractional order differentiator using type-III and type-IV discrete cosine transform. Engineering Science and Technology, an International Journal 2017;20(1):51–58.

[13] Muñoz JMS. Génesis y desarrollo del Cálculo Fraccional. Pensamiento Matemático 2011;(1):4.

[14] Machado JT, Kiryakova V, Mainardi F. Recent history of fractional calculus. Communications in Nonlinear Science and Numerical Simulation 2011;16(3):1140–1153.

[15] Özsoy K, Usal MR. A mathematical model for the magnetorheological materials and magneto reheological devices. Engineering science and technology, an international journal 2018;21(6):1143–1151.

[16] Ashtiani M, Hashemabadi S, Ghaffari A. A review on the magnetorheological fluid preparation and stabilization. Journal of magnetism and Magnetic Materials 2015;374:716–730.

[17] Danas K, Kankanala S, Triantafyllidis N. Experiments and modeling of iron-particle-filled magnetorheological elastomers. Journal of the Mechanics and Physics of Solids 2012;60(1):120–138.

[18] Vadillo D, Tuladhar T, Mulji A, Mackley M. The rheological characterization of linear viscoelasticity for ink jet fluids using piezo axial vibrator and torsion resonator rheometers. Journal of Rheology 2010;54(4):781–795.

[19] Bagley RL, Torvik P. A theoretical basis for the application of fractional calculus to viscoelasticity. Journal of Rheology 1983;27(3):201–210.

[20] Gómez-Aguilar J, Atangana A. New insight in fractional differentiation: power, exponential decay and Mittag-Leffler laws and applications. The European Physical Journal Plus 2017;132(1):13.

[21] Podlubny I, Fractional differential equations, vol. 198 of Mathematics in Science and Engineering. Academic Press, San Diego, Calif, USA; 1999.

[22] Chen B, Li C, Wilson B, Huang Y. Fractional modeling and analysis of coupled MR damping system. IEEE/CAA Journal of Automatica Sinica 2016;3(3):288–294.

[23] Nakano M, Yamamoto H, Jolly MR. Dynamic viscoelasticity of a magnetorheological fluid in oscillatory slit flow. International Journal of Modern Physics B 1999;13(14n16):2068–2076.

[24] Li W, Zhou Y, Tian T. Viscoelastic properties of MR elastomers under harmonic loading. Rheologica acta 2010;49(7):733–740.

[25] Wharmby AW, Bagley RL. Generalization of a theoretical basis for the application of fractional calculus to viscoelasticity. Journal of Rheology 2013;57(5):1429–1440.

[26] Bhatti AQ. Performance of viscoelastic dampers (VED) under various temperatures and application of magnetorheological dampers (MRD) for seismic control of structures. Mechanics of Time-Dependent Materials 2013;17(3):275–284.

[27] Du H, Lam J, Cheung K, Li W, Zhang N. Direct voltage control of magnetorheological damper for vehicle suspensions 2013;22(10).

[28] Jolly MR, Carlson JD, Munoz BC. A model of the behaviour of magnetorheological materials. Smart Materials and Structures 1996;5(5):607.

[29] Nadzharyan T, Sorokin V, Stepanov G, Bogolyubov A, Kramarenko EY. A fractional calculus approach to modeling rheological behavior of soft magnetic elastomers. Polymer 2016;92:179–188.

[30] Agirre-Olabide I, Kuzhir P, Elejabarrieta M. Linear magneto-viscoelastic model based on magnetic permeability components for anisotropic magnetorheological elastomers. Journal of Magnetism and Magnetic Materials 2018;446:155–161.

[31] Chakraverty S, Behera D. Dynamic responses of fractionally damped mechanical system using homotopy perturbation method. Alexandria Engineering Journal 2013;52(3):557–562.

[32] Morales-Delgado V, Gomez-Aguilar J, Taneco-Hernandez M, Escobar-Jimenez R. A novel fractional derivative with variable-and constant-order applied to a mass-spring-damper system. The European Physical Journal Plus 2018;133(2):78.

[33] Abro KA, Khan I, Gómez-Aguilar J. Heat transfer in magnetohydrodynamic free convection flow of generalized ferrofluid with magnetite nanoparticles. Journal of Thermal Analysis and Calorimetry 2020;p. 1–10.

[34] Gómez-Aguilar JF, Yépez-Martínez H, Calderón-Ramón C, Cruz-Orduña I, Escobar-Jiménez RF, Olivares-Peregrino VH. Modeling of a mass-spring-damper system by fractional derivatives with and without a singular kernel. Entropy 2015;17(9):6289–6303.

[35] Escalante-Martínez J, Gómez-Aguilar J, Calderón-Ramón C, Morales-Mendoza L, Cruz-Orduna I, Laguna-Camacho J. Experimental evaluation of viscous damping coefficient in the fractional underdamped oscillator. Advances in Mechanical Engineering 2016;8(4):1687814016643068.

[36] Ertik H, Calik A, Şirin H, Şen M, Öder B. Investigation of electrical RC circuit within the framework of fractional calculus. Revista mexicana de física 2015;61(1):58–63.

[37] Sikora R. Czy pochodne ułamkowe unieważniły prawa Maxwella? Maszyny Elektryczne: zeszyty problemowe 2016;(4 (112)):209–212.

[38] Garrappa R. Numerical solution of fractional differential equations: A survey and a software tutorial. Mathematics 2018;6(2):16.

[39] Fano WG, Alonso R, Quintana G. El Campo Magnético Generado por las Bobinas de Helmholtz y sus Aplicaciones a Calibración de Sondas. Elektron 2017;1(2):91–96.

[40] JaeáKim H, HoáSong K, et al. Alternating magnetic field mediated micro reaction system for palladium-catalyzed coupling reactions. RSC advances 2017;7(59):37181–37184.

[41] Chen C, Chan YS, Zou L, Liao WH. Self-powered magnetorheological dampers for motorcycle suspensions. Proceedings of the Institution of Mechanical Engineers, Part D: Journal of Automobile Engineering 2018;232(7):921–935.