Well posedness for the Kawahara equation on the half-line

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We study the low-regularity properties of the Kawahara equation on the half line. We obtain the local existence, uniqueness, and continuity of the solution. Moreover, We obtain that the nonlinear terms of the solution are smoother than the initial data.

Keywords: Kawahara equation, Initial-Boundary value problem, Contraction map, Smoothing, Restricted norm method

MSC: 35A01, 35B45, 35Q53

1 Introduction

1.1 The model on the half-line.

We consider the following initial-boundary value problem on the half line, called Kawahara equation:

$$\begin{cases} u_t - u_{5x} - (u^3)_x = 0, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ u(x,0) = \phi(x), & x \in \mathbb{R}^+, \\ u(0,t) = h_1(t), u_x(0,t) = h_2(t), & t \in (0,T) \end{cases} \tag{1.1}$$

with the compatibility $h_1(0) = \phi(0)$, when $\frac{1}{2} < s < \frac{3}{2}$, or $h_1(0) = \phi(0), h_2(0) = \phi'(0)$ when $\frac{3}{2} < s < \frac{5}{2}$. The data (ϕ, h_1, h_2) will be in the space $H_x^s(\mathbb{R}^+), H_t^{\frac{s+2}{5}}(\mathbb{R}^+), H_t^{\frac{s+1}{5}}(\mathbb{R}^+)$. The equation was first proposed by Kawahara [1], it arises in the theory of shallow water waves and

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it is regarded as a singular perturbation of KdV equation, we refer to [2]- [5] and references therein.

There are many results about the Cauchy problem and IBVP for Kawahara equation. Here we only give some previous works. The LWP for Cauchy problem of Kawahara equation was first studied by Cui and Tao [6], they obtained the LWP in $H^s(\mathbb{R})$, $s > \frac{1}{4}$, and GWP in $H^2(\mathbb{R})$. Later, Cui [7] improved the previous result to $H^s(\mathbb{R})$, $s > -1$. In Chen [8] and Huo [9], they proved the LWP in $H^s(\mathbb{R})$, $s > -\frac{7}{4}$ independently. Kato [10] proved the LWP in $H^s(\mathbb{R})$, $s \geq -2$. We also refer to Kenig [11] for more details. In the previous works on the IBVP of the Kawahara equation and its related equations posed on the right half-line, e.g. [12]- [14] and references therein, authors proved local and global well-posedness in the high regularity function spaces with exponential decay property, or weighted Sobolev space for at least nonnegative regularity.

Kwak [15] proved the LWP of (1.1) in $H^s(\mathbb{R})$ on the half line, $s \geq -\frac{1}{4}$ by using Duhamel boundary forcing operator, for more details, we can refer to [16], [17].

In this paper, we consider Kawahara equation on the half line by using Laplace transportation instead of Duhamel boundary operator, and we have to note that the nonlinear term of this paper is smoother than initial value and we improved the regularity that obtained in [15]. We will extend the data into the whole line and then use Laplace transform to get an equivalent equation on the whole line. Then we use the restricted norm and tools that are available on the whole line. This method is standard and readers can refer to [18]- [20] and references therein for more details.

1.2 Structure of the paper

This paper is organized as follows. In section 1, we will introduce the function space we will work in and some notations in this paper, and give the main result at last. In the next section, we will recall some linear and nonlinear estimates needed in the proofs of our theorem. In section 3, we will give a prior estimates on linear and nonlinear terms that needed for the establish of the contraction map. Then in section 4, we will use the estimates above to get the main result, and in the last section, we will show that the uniqueness of the solution is independent in the extension.

1.3 Function spaces and notation.

Denoting D_0 representing evaluation at $x=0$, i.e., $D_0[u(x,t)] = u(0,t)$. The solution to the linear problem $u_t - u_{5x} = 0$ on $\mathbb{R}$ with initial data $u(x,0) = \phi(x)$ will be denoted by

$$W_R^t \phi(x) = \mathcal{F}_x^{-1}\big(e^{i\xi^5 t}\widehat{\phi}(\xi)\big)(x),$$

$\widehat{\phi}(\xi)$ denotes the Fourier transform

$$\widehat{\phi}(\xi) = (\mathcal{F}_x\phi)(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\phi(x)dx.$$

It is clear that the solution of (1.1) is

$$u(x,t) = W_R^t\phi^*(x) + W_0^t(0,h_1,h_2) + \int_0^t W_R^{t-t'}G(u)dt', \tag{1.2}$$

where

$$G(u) = (u^3)_x,$$

and ϕ^* is the extension of ϕ in the full line $\mathbb{R}$,

$$p_1(t) = D_0\left[W_R^t\phi^*(x)\right], \quad q_1(t) = D_0\left[\int_0^t W_R^{t-t'}G(u)dt'\right],$$
$$p_2(t) = D_0\left[W_R^t\phi^*(x)\right]_x, \quad q_2(t) = D_0\left[\int_0^t W_R^{t-t'}G(u)dt'\right]_x,$$

Note that

$$W_0^t(\phi,h_1,h_2) + W_R^t\phi^*$$

is the solution of

$$\begin{cases} u_t - u_{5x} = 0, & (x,t)\in\mathbb{R}^+\times(0,T), \\ u(x,0) = \phi(x), & x\in\mathbb{R}^+, \\ u(0,t) = h_1(t), u_x(0,t) = h_2(t), & t\in(0,T). \end{cases} \tag{1.3}$$

It is clear that the solution u of (1.1) satisfy $\Phi(u) = u$, for $t\le T$, where the operator Φ is defined by

$$\Phi(u(x,t)) = \eta(\frac{t}{T})W_R^t\phi^* + \eta(\frac{t}{T})W_0^t(0,h_1-p_1-q_1,h_2-p_2-q_2) + \eta(\frac{t}{T})\int_0^t W_R^{t-t'}G(u)dt', \tag{1.4}$$

where $\eta\in C^\infty$, such that $\eta = 1$ in $[-1,1]$, $\eta = 0$ in $(-\infty,-2)\cup(2,+\infty)$.

Next, we will use a fixed point argument to obtain a unique solution to $\Phi(u(x,t)) = u(x,t)$ in a suitable function space. We will work on the space $X^{s,b}$ where

$$\|u\|_{X^{s,b}} = \left\| \langle \xi \rangle^s \langle \tau - \xi^5 \rangle^b \widehat{u}(\xi, \tau) \right\|_{L^2_{\xi,\tau}},$$

and $\widehat{u}(\xi,\tau)$ denotes the Fourier transform over time-space

$$\widehat{u}(\xi,\tau) = \mathcal{F}_x\mathcal{F}_t[u(x,t)] = \iint_{\mathbb{R}^2} e^{-ix\xi}e^{-it\tau}u(x,t)dxdt,$$

we denote $\langle \xi \rangle = \sqrt{1+|\xi|^2}$, and the characteristic function on $(0,\infty)$ is denoted by χ. We define the Sobolev space H^s norm by

$$\|u\|^2_{H^s(\mathbb{R})} = \int \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 d\xi$$

and the Sobolev space $H^s(\mathbb{R}^+)$ on the half line is defined as

$$H^s(\mathbb{R}^+) = \{f \in \mathcal{D}(\mathbb{R}^+) : \text{there exists } \tilde{f} \in H^s(\mathbb{R}) \text{ with } \tilde{f}\chi = f\},$$

$$\|g\|_{H^s(\mathbb{R}^+)} = \inf\{\|\tilde{g}\|_{H^s(\mathbb{R})} : \tilde{g}\chi = g\}.$$

Finally, $a \lesssim b$ denotes that $a \le Cb$ for some constant C, and if $C \ll 1$ we use the symbol $a \ll b$ instead of $\lesssim$. $a \sim b$ indicates that $a \lesssim b$ and $b \lesssim a$, $a+$ indicates $a + \varepsilon$, where ε can be arbitrarily small, $a-$ denotes similarly.

Definition 1.1. *We define that the equation solution (1.1) is locally well-posed in $H^s(\mathbb{R}^+)$ if for any initial-boundary data $(\phi, h_1, h_2) \in H^s_x(\mathbb{R}^+) \times H_t^{\frac{s+2}{5}}(\mathbb{R}^+) \times H_t^{\frac{1+s}{5}}(\mathbb{R}^+)$, with the compatibility condition that $h_1(0) = \phi(0)$, when $\frac{1}{2} < s < \frac{3}{2}$, or $h_1(0) = \phi(0), h_2(0) = \phi'(0)$ when $\frac{3}{2} < s < \frac{5}{2}$. The equation $\Phi(u) = u$, where Φ has been defined in (1.4), has a unique solution $u \in X^{s,b} \cap C^0_t H^s_x \cap C^0_x H_t^{\frac{s+2}{5}}$ for $\frac{1}{2} - b > 0$ and T both sufficiently small. Furthermore, the solution u depends only on the initial-boundary data.*

Remark 1.2. *The main goal in the paper is to show the local well-posedness of (1.1) in the low regularity Sobolev space, so we only consider the regularity for $s < \frac{5}{2}$. The compatibility conditions for high regularities are negligible. See [5] for the comparison.*

Theorem 1.3. *For any $s \in (-\frac{2}{3}, \frac{5}{2}) \backslash \{\frac{1}{2}, \frac{3}{2}\}$, the equation solution (1.1) is locally well-posed in $H^s(\mathbb{R})$. For $a < \min\{2s + \frac{1}{2}, -1 - s\}$,*

$$u - W_0^t(\phi, h_1, h_2) \in C^0_t H^{s+a}_x.$$

2 Lemmas

We will render the problem (1.1) in the following:

$$\begin{cases} u_t - u_{5x} = 0, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ u(x,0) = \phi(x), & x \in \mathbb{R}^+, \\ u(0,t) = 0, u_x(0,t) = 0, & t \in (0,T), \end{cases} \tag{2.1}$$

$$\begin{cases} u_t - u_{5x} = f(x,t), & (x,t) \in \mathbb{R}^+ \times (0,T), \\ u(x,0) = 0, & x \in \mathbb{R}^+, \\ u(0,t) = 0, u_x(0,t) = 0, & t \in (0,T) \end{cases} \tag{2.2}$$

where $f(x,t) = (u^3)_x$, and

$$\begin{cases} u_t - u_{5x} = 0, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ u(x,0) = 0, & x \in \mathbb{R}^+, \\ u(0,t) = h_1(t), u_x(0,t) = h_2(t), & t \in (0,T). \end{cases} \tag{2.3}$$

Next, we will show the estimates in $X^{s,b}$.

Lemma 2.1. *([5]) Let $s, b \in \mathbb{R}$. If $\phi \in H^s(\mathbb{R})$, then*

$$\|\eta(t) W_R^t \phi\|_{X^{s,b}} \lesssim \|\phi\|_{H^s(\mathbb{R})}.$$

Lemma 2.2. *([21]) For any $-\frac{1}{2} < b' \leq 0 \leq b \leq b'+1$ and $T \in (0,1)$, the estimate holds*

$$\left\| \eta(t/T) \int_0^t W_R^{t-t'} G(u) dt' \right\|_{X^{s,b}} \lesssim T^{1-(b-b')} \|G(u)\|_{X^{s,b'}}.$$

Also, for any $-\frac{1}{2} < b_1 < b_2 < \frac{1}{2}$ and $F \in X^{s,b_2}$

$$\|\eta(t/T) F\|_{X^{s,b_1}} \lesssim T^{b_2-b_1} \|F\|_{X^{s,b_2}}. \tag{2.4}$$

To obtain the solution of IBVP (1.1), we also need the lemma about extensions of $H^s(\mathbb{R}^+)$.

Lemma 2.3. *([22]) Let $h \in H^s(\mathbb{R}^+)$,*

(1) If $-\frac{1}{2} < s < \frac{1}{2}$, then $\|\chi h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$,

(2) If $\frac{1}{2} < s < \frac{3}{2}$, $h(0) = 0$, then $\|\chi h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.

Lemma 2.4. *If $\beta > 1$, $\gamma \geq 0$, $\beta \geq \gamma$, then we have*

$$\int \frac{1}{\langle x-a \rangle^\beta \langle x-b \rangle^\gamma} dx \lesssim \langle a-b \rangle^{-\gamma}.$$

3 A prior estimates

To obtain the local theory, we need the linear estimates and nonlinear estimates. We will give the linear estimates firstly.

Lemma 3.1. *There holds that*

$$\|\eta(t)\partial_x^j W_R^t \phi^*(x)\|_{L_x^\infty H_t^{\frac{s+2-j}{5}}} \lesssim \|\phi^*(x)\|_{H_x^s}$$

for any s and $j = 0, 1, 2$.

Proof. We can refer [5] for details.

□

Lemma 3.2. *For any $s \in \mathbb{R}$ and $b < \frac{1}{2}$,*

$$\|\eta(t)W_0^t(0, h_1, h_2)\|_{X^{s,b}} \lesssim \|\chi\vec{h}\|_X$$

where $\vec{h} = (h_1, h_2)$,

$$\|\vec{h}\|_X = \|h_1\|_{H_t^{\frac{s+2}{5}}(\mathbb{R})} + \|h_2\|_{H_t^{\frac{s+1}{5}}(\mathbb{R})}.$$

Proof. Indeed, we need to calculate $W_0^t(0, h_1, h_2)$ firstly. Note that the Laplace transform of the function u over $[0, \infty)$ is

$$\widetilde{u}(s) = \int_0^\infty e^{-st}u(t)dt.$$

Taking the Laplace transform with respect to t of (2.3) yields

$$\begin{cases} s\widetilde{u}(x,s) - \widetilde{u}_{5x}(x,s) = 0, & x \in \mathbb{R}^+, s \in \mathbb{R}^+ \\ \widetilde{u}(x,0) = 0, & x \in \mathbb{R}^+, \\ \widetilde{u}(0,s) = \tilde{h}_1(s), \widetilde{u}_x(0,s) = \tilde{h}_2(s), & s \in \mathbb{R}^+. \end{cases} \tag{3.1}$$

The characteristic equation is $s - \lambda^5 = 0$ and $\lambda_j(s)(j = 1, 2, \cdots, 5)$ are the solutions.

As both $\widetilde{u}(x,s), \widetilde{u}_x(x,s) \to 0$, when $x \to \infty$, it is concluded that for any s with Re$s > 0$

$$\widetilde{u}(x,s) = \frac{1}{\lambda_1 - \lambda_2}\left[(\lambda_1\tilde{h}_1(s) - \tilde{h}_2(s))e^{\lambda_2 x} - (\lambda_2\tilde{h}_1(s) - \tilde{h}_2(s))e^{\lambda_1 x}\right].$$

Thus, for any fixed $\gamma > 0$, we have

$$u(x,t) = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} e^{st}\widetilde{u}(x,s)ds.$$

By the continuity of γ at $\gamma = 0$ we have

$$u(x,t) = \frac{1}{2\pi i}\int_0^{i\infty} e^{st}\widetilde{u}(x,s)ds + \frac{1}{2\pi i}\int_{-i\infty}^{0} e^{st}\widetilde{u}(x,s)ds.$$

On the positive imaginary axis, take $s = i\mu^5$, $1 \le \mu < \infty$, then

$$\lambda_1(s) = \mu e^{i\frac{17\pi}{10}}, \quad \lambda_2(s) = \mu e^{i\frac{\pi}{10}},$$

so

$$\begin{aligned}
u(x,t) &= [U(t)h](x) + \overline{[U(t)h](x)} \\
&= 2\mathrm{Re}\frac{1}{2\pi i}\int_0^{i\infty} e^{i\mu^5 t}\frac{1}{\lambda_1-\lambda_2}\left[(\lambda_1\tilde{h}_1(s) - \tilde{h}_2(s))e^{\lambda_2 x} - (\lambda_2\tilde{h}_1(s) - \tilde{h}_2(s))e^{\lambda_1 x}\right] ds,
\end{aligned}$$

where

$$[U(t)h](x) = \frac{1}{2\pi i}\int_0^{i\infty} e^{i\mu^5 t}\frac{1}{\lambda_1-\lambda_2}\left[(\lambda_1\tilde{h}_1(s) - \tilde{h}_2(s))e^{\lambda_2 x} - (\lambda_2\tilde{h}_1(s) - \tilde{h}_2(s))e^{\lambda_1 x}\right] ds.$$

Since, $s = i\mu^5$, $ds = 5i\mu^4 d\mu$, then

$$\begin{aligned}
u(x,t) = &\frac{5}{2\pi}\int_0^{\infty} \frac{e^{i\mu^5 t}\mu^4}{\mu(e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}})}\left[\left(\mu e^{i\frac{17\pi}{10}}\int_0^{\infty} e^{-st}h_1(t)dt - \int_0^{\infty} e^{-st}h_2(t)dt\right)e^{\mu x e^{i\frac{\pi}{10}}}\right] d\mu \\
&- \frac{5}{2\pi}\int_0^{\infty} \frac{e^{i\mu^5 t}\mu^4}{\mu(e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}})}\left[\left(\mu e^{i\frac{\pi}{10}}\int_0^{\infty} e^{-st}h_1(t)dt - \int_0^{\infty} e^{-st}h_2(t)dt\right)e^{\mu x e^{i\frac{17\pi}{10}}}\right] d\mu.
\end{aligned}$$

Denote

$$\mathrm{I} = \int_0^{\infty} \frac{e^{i\mu^5 t}\mu^4 e^{i\frac{17\pi}{10}} e^{\mu e^{i\frac{\pi}{10}}x}}{e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}}}\int_0^{\infty} e^{-st}h_1(t)dtd\mu = \int_0^{\infty}\frac{\mu^4 e^{i\mu^5 t + \mu x e^{i\frac{\pi}{10}}}}{e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}}}e^{i\frac{17\pi}{10}}\widehat{h}_1(\mu^5)d\mu,$$

$$\mathrm{II} = \int_0^{\infty}\frac{\mu^3 e^{i\mu^5 t + \mu x e^{i\frac{\pi}{10}}}}{e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}}}\widehat{h}_2(\mu^5)d\mu, \qquad \mathrm{III} = \int_0^{\infty}\frac{\mu^4 e^{i\mu^5 t + \mu x e^{i\frac{17\pi}{10}}} e^{i\frac{\pi}{10}}}{e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}}}\widehat{h}_1(\mu^5)d\mu,$$

$$\mathrm{IV} = \int_0^{\infty}\frac{\mu^3 e^{i\mu^5 t + \mu x e^{i\frac{17\pi}{10}}}}{e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}}}\widehat{h}_2(\mu^5)d\mu.$$

Then,

$$u(x,t) = \frac{5}{2\pi}\mathrm{Re}\ (\mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV}).$$

Since $\frac{2\pi}{5}W_0^t(0,h_1,h_2) = \mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV}$. We now consider I. Assume first $s = 0, b = \frac{1}{2}-$, $f(y) = e^{-y}\rho(y)$, where $\rho \in C^\infty$ be a cut-off function with $\rho = 1$ in $[0,\infty)$ and 0 in $(-\infty,-1)$. Therefore,

$$\mathrm{I}(x,t) = \frac{e^{i\frac{17\pi}{10}}}{e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}}}\int \mu^4 e^{it\mu^5} f(-\mu x e^{i\frac{\pi}{10}})\widehat{h}_1(\mu^5)d\mu,$$

and

$$\widehat{\eta \mathrm{I}}(\xi,\tau) = \frac{e^{i\frac{17\pi}{10}}}{e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}}} \int \widehat{\eta}(\tau-\mu^5)\mu^4 \widehat{h}_1(\mu^5)\mathcal{F}_x(f(-\mu x e^{i\frac{\pi}{10}})(\xi))d\mu$$
$$= \frac{e^{i\frac{17\pi}{10}}}{e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}}} \int \widehat{\eta}(\tau-\mu^5)\mu^4 \widehat{h}_1(\mu^5)\widehat{f}(\frac{\xi}{-\mu e^{i\frac{\pi}{10}}})d\mu.$$

Since f is a Schwarz function, we have

$$\left|\widehat{f}(\frac{\xi}{-\mu e^{i\frac{\pi}{10}}})\right| \lesssim \frac{1}{1+\xi^5\mu^5 e^{i\frac{\pi}{5}}} \lesssim \frac{\mu^5 e^{i\frac{\pi}{5}}}{\xi^5+\mu^5 e^{i\frac{\pi}{5}}},$$

and η is also a Schwarz function,

$$|\widehat{\eta}(\tau-\mu^5)| \lesssim \frac{1}{\langle\tau-\mu^5\rangle^2}\frac{1}{\langle\tau-\xi^5\rangle^{\frac{2}{5}+}}\langle\xi^5-\mu^5\rangle^{\frac{2}{5}-}.$$

Then, using variable changes $z=\mu^5$ we have

$$\|\eta\mathrm{I}\|_{X^{0,\frac{4}{5}-}} \lesssim \left\| \frac{\mu^5 e^{i\frac{\pi}{5}}}{\xi^5+\mu^5 e^{i\frac{\pi}{5}}}\langle\tau-\xi^5\rangle^{\frac{4}{5}-}\int |\widehat{\eta}(\tau-\mu^5)\mu^4\widehat{h}_1(\mu^5)|d\mu\right\|_{L^2_{\xi,\tau}}$$
$$\lesssim \left\|\int_{\mathbb{R}} \frac{1}{\langle\tau-\mu^5\rangle^2}\mu^4\mu^2|\widehat{h}_1(\mu^5)|d\mu\right\|_{L^2_\tau}$$
$$\lesssim \left\|\int_{\mathbb{R}} \frac{1}{\langle\tau-\mu^5\rangle^2}z^{\frac{2}{5}}|\widehat{h}_1(z)|dz\right\|_{L^2_\tau}$$
$$\lesssim \left(\int_{-\infty}^{\infty}\langle z\rangle^{\frac{4}{5}}|\widehat{h}_1(z)|^2dz\right)^{\frac{1}{2}} \lesssim \|h_1\|_{H_t^{\frac{2}{5}}(\mathbb{R})}.$$

Similarly, we also get

$$|\widehat{\eta\mathrm{II}}(\xi,\tau)| \lesssim \|h_2\|_{H_t^{\frac{1}{5}}(\mathbb{R})}, \quad |\widehat{\eta\mathrm{III}}(\xi,\tau)| \lesssim \|h_1\|_{H_t^{\frac{2}{5}}(\mathbb{R})}, \quad |\widehat{\eta\mathrm{IV}}(\xi,\tau)| \lesssim \|h_2\|_{H_t^{\frac{1}{5}}(\mathbb{R})}.$$

We still need to obtain bounds on I,III and II,IV in $X^{s,\frac{2}{5}-}$ and $X^{s,\frac{1}{5}-}$ respctively for general s. For any $s\in\mathbb{N}$,

$$\partial_x^s(\eta\mathrm{I})(x,t) = \frac{e^{i\frac{17\pi}{10}}}{e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}}}\int_{\mathbb{R}}\eta(t)\mu^4e^{it\mu^5}f^{(s)}(-\mu x e^{i\frac{\pi}{10}})(-\mu e^{i\frac{\pi}{10}})^s\widehat{h}_1(\mu^5)d\mu$$
$$\lesssim \int_{\mathbb{R}}\eta(t)\mu^4e^{it\mu^5}f^{(s)}(-\mu x e^{i\frac{\pi}{10}})\mu^s\widehat{h}_1(\mu^5)d\mu,$$

then

$$\begin{aligned}
\|\partial_x^s(\eta \mathrm{I})(x,t)\|_{X^{0,\frac{4}{5}-}} &\lesssim \left\| \mu^s \frac{\mu^5 e^{i\frac{\pi}{5}}}{\xi^5+\mu^5 e^{i\frac{\pi}{5}}} \langle \tau-\xi^5\rangle^{\frac{4}{5}-} \int_{\mathbb{R}} \left|\widehat{\eta}(\tau-\xi^5)\mu^4 \widehat{h}_1(\mu^5)\right| d\mu \right\|_{L^2_{\xi,\tau}} \\
&\lesssim \left\| \int_{\mathbb{R}} \frac{\mu^s}{\langle \tau-\xi^5\rangle^2} \mu^4\mu^2 \left|\widehat{h}_1(\mu^5)\right| d\mu \right\|_{L^2_\tau} \\
&\lesssim \left\| \int_{\mathbb{R}} \frac{1}{\langle \tau-\xi^5\rangle^2} z^{\frac{s+2}{5}} \left|\widehat{h}_1(z)\right| dz \right\|_{L^2_\tau} \lesssim \|h_1\|_{H_t^{\frac{2+s}{5}}(\mathbb{R})}.
\end{aligned}$$

Analogously, we have $\partial_x^s(\eta \mathrm{II})(x,t)$, $\partial_x^s(\eta \mathrm{III})(x,t)$ and $\partial_x^s(\eta \mathrm{IV})(x,t)$ bounds. We omit the proof.

□

Lemma 3.3. *For any $s\in\mathbb{R}$, the initial data such that $(\chi h_1, \chi h_2)\in H_t^{\frac{2}{5}}(\mathbb{R})\times H_t^{\frac{1}{5}}(\mathbb{R})$ we have*

$$W_0^t(0,h_1,h_2)\in \mathcal{C}_t^0 H_x^s(\mathbb{R}\times\mathbb{R}), \quad \eta(t)W_0^t(0,h_1,h_2)\in C_x^0 H_t^{\frac{s+2}{5}}(\mathbb{R}\times\mathbb{R}).$$

Proof. Since the proofs of II, III, IV are similar to I, so we only give the proof of I. Note that

$$\mathrm{I}(x,t) = \frac{e^{i\frac{17\pi}{10}}}{e^{i\frac{17\pi}{10}} - e^{i\frac{\pi}{10}}} \int \mu^4 e^{it\mu^5} f(-\mu x e^{i\frac{\pi}{10}}) \widehat{h}_1(\mu^5) d\mu,$$

let $\phi_I = \mu^4 \widehat{h}_1(\mu^5)$, then combing variable changes $z=\mu^5$ we have

$$\begin{aligned}
\|\phi_I\|^2_{H_x^s} = \int_{-\infty}^{\infty} \langle\mu\rangle^{2s+8} |\widehat{h}_1(\mu^5)|^2 d\mu &\lesssim \int_{-\infty}^{\infty} \langle\mu\rangle^{2s+4} |\widehat{h}_1(z)|^2 dz \\
&\lesssim \int_{-\infty}^{\infty} \langle z\rangle^{\frac{2s+4}{5}} |\widehat{h}_1(z)|^2 dz \lesssim \|\chi h_1\|^2_{H_t^{\frac{s+2}{5}}(\mathbb{R})}.
\end{aligned}$$

Next, due to the continuity of operator $e^{t\partial_x^5}$, it suffices to show that the map

$$T(g) = \int_{-\infty}^{\infty} f(-\mu x e^{i\frac{\pi}{10}}) \widehat{g}(\mu) d\mu$$

is bounded from H^s to H^s. First, we consider $s=0$. Since

$$Tg(x) = \int_{-\infty}^{\infty} f(-\mu x e^{i\frac{\pi}{10}}) \widehat{g}(\mu) d\mu = \int_{-\infty}^{\infty} f(z) \widehat{g}(\frac{-z}{x e^{i\frac{\pi}{10}}}) dz,$$

then combining with variable changes and that f is a Schwarz function, we have

$$\begin{aligned}
\|Tg\|^2_{L^2_x} \lesssim \int |f(z)| \left\| \widehat{g}(\frac{-z}{x e^{i\frac{\pi}{10}}}) \frac{1}{x e^{i\frac{\pi}{10}}} \right\|^2_{L^2_x} dz &\lesssim \int_{\mathbb{R}} |f(z)| \int_0^{\infty} |\widehat{g}(\frac{-z}{x e^{i\frac{\pi}{10}}})|^2 \frac{1}{x^2 e^{i\frac{\pi}{5}}} dx dz \\
&\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |f(z)| |\widehat{g}(y)|^2 \frac{1}{z e^{i\frac{\pi}{10}}} dy dz \\
&\lesssim \|\widehat{g}\|^2_{L^2} \int_{\mathbb{R}} |f(z)| \frac{1}{z} dz \lesssim \|g\|^2_{L^2},
\end{aligned}$$

this completes the proof that $I \in C_t^0 H_x^s$ for $s = 0$.

When $s > 0$, for any $s \in \mathbb{N}$, we have

$$\partial_x^s Tg(x) = \int_0^\infty f^{(s)}(-\mu x e^{i\frac{\pi}{10}})(\mu e^{i\frac{\pi}{10}})^s \widehat{g}(\mu) d\mu,$$

so $\|\partial_x^s Tg(x)\|_{L^2}^2 \lesssim \|g\|_{H^s}^2$.

Next we will proof that $\eta W_0^t(0, h_1, h_2) \in C_x^0 H_t^{\frac{s+2}{5}}(\mathbb{R})$. We only give the proof of I. We will consider $\|f\|_{L_z^2}$ and $\|f\|_{H_z^1}$ separaterly.

$$\int_{\mathbb{R}} |f(-\mu x e^{i\frac{\pi}{10}})|^2 d\mu = \int_{|\mu| \leq 1} |f(-\mu x e^{i\frac{\pi}{10}})|^2 d\mu + \int_{|\mu|>1} |f(-\mu x e^{i\frac{\pi}{10}})|^2 d\mu,$$

since f is bounded, the first term is bounded. For the second term,

$$\int_{|\mu|>1} |f(-\mu x e^{i\frac{\pi}{10}})|^2 d\mu = \int_{|\frac{y}{x}|>1} |f(y)|^2 \frac{1}{-x e^{i\frac{\pi}{10}}} dy$$

which is bounded since f is a Schwarz function.

Next, we will also use the fact that f is a Schwarz function to obtain the bound of differential term, since

$$\frac{d}{d\mu} f(-\mu x e^{i\frac{\pi}{10}}) = f'(-\mu x e^{i\frac{\pi}{10}})(-x e^{i\frac{\pi}{10}}).$$

$\square$

Lemma 3.4. *(see [5]) For $s > -\frac{7}{4}$, with $b = b(s) < \frac{1}{2}$, we have*

$$\|\partial_x(uv)\|_{X^{s,-b}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}.$$

Lemma 3.5. *For $s > -\frac{2}{3}$, $a < \min\{-1-s, 2s+\frac{1}{2}\}$ and $\frac{1}{2} - b > 0$ sufficiently small, we have*

$$\|\partial_x(uvw)\|_{X^{s+a,-b}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}} \|w\|_{X^{s,b}}.$$

Proof. To prove

$$\|\partial_x(uvw)\|_{X^{s+a,-b}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}} \|w\|_{X^{s,b}},$$

by duality, it suffices to prove that

$$\iint \partial_x(uvw)\bar{\phi} dx dt \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}} \|w\|_{X^{s,b}} \|\bar{\phi}\|_{X^{-(s+a),b}} \tag{3.2}$$

for any $\phi \in X^{-(s+a),b}$.

Now we define

$$f(\xi,\tau)=\langle\xi\rangle^s\langle\tau-\xi^5\rangle^b\hat{u}(\xi,\tau),\qquad g(\xi,\tau)=\langle\xi\rangle^s\langle\tau-\xi^5\rangle^b\hat{v}(\xi,\tau),$$
$$h(\xi,\tau)=\langle\xi\rangle^s\langle\tau-\xi^5\rangle^b\hat{w}(\xi,\tau),\qquad r(\xi,\tau)=\langle\xi\rangle^{-(s+a)}\langle\tau-\xi^5\rangle^b\overline{\hat{\phi}(\xi,\tau)}.$$

Then the inequality (3.2) is equivalent to

$$\begin{aligned}&\left|\int M(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2)f(\xi_1,\tau_1)g(\xi_2-\xi_1,\tau_2-\tau_1)h(\xi-\xi_2,\tau-\tau_2)r(\xi,\tau)d\xi_1d\tau_1d\xi_2d\tau_2d\xi d\tau\right|\\&\lesssim\|f\|_{L^2_{\xi,\tau}}\|g\|_{L^2_{\xi,\tau}}\|h\|_{L^2_{\xi,\tau}}\|r\|_{L^2_{\xi,\tau}}\end{aligned}\tag{3.3}$$

where

$$M=\frac{\langle\xi_1\rangle^{-s}\langle\xi_2-\xi_1\rangle^{-s}\langle\xi-\xi_2\rangle^{-s}|\xi|\langle\xi\rangle^{a+s}}{\langle\tau_1-\xi_1^5\rangle^b\langle\tau_2-\tau_1-(\xi_2-\xi_1)^5\rangle^b\langle\tau-\tau_2-(\xi-\xi_2)^5\rangle^b\langle\tau-\xi^5\rangle^b}.$$

Using Cauchy-Schwarz and Young's inequality,

$$\text{LHS of (3.3)}\ \lesssim\sup_{\xi,\tau}\|M\|_{L^2_{\xi_1,\tau_1}}\|f\|_{L^2_{\xi,\tau}}\|g\|_{L^2_{\xi,\tau}}\|h\|_{L^2_{\xi,\tau}},$$

then it suffices to show that

$$\sup_{\xi,\tau}\iiiint_{\mathbb{R}^4}\frac{\langle\xi_1\rangle^{-2s}\langle\xi_2-\xi_1\rangle^{-2s}\langle\xi-\xi_2\rangle^{-2s}\langle\xi\rangle^2\langle\xi\rangle^{2(a+s)}d\xi_1d\xi_2d\tau_1d\tau_2}{\langle\tau_1-\xi_1^5\rangle^{2b}\langle\tau_2-\tau_1-(\xi_2-\xi_1)^5\rangle^{2b}\langle\tau-\tau_2-(\xi-\xi_2)^5\rangle^{2b}\langle\tau-\xi^5\rangle^{2b}}\lesssim 1.\tag{3.4}$$

Using the triangular inequality $\langle a\rangle\langle b\rangle\ge\langle a+b\rangle$, we have

$$\text{LHS of (3.4)}\ \lesssim\sup_{\xi}\iiint_{\mathbb{R}^3}\frac{\langle\xi_1\rangle^{-2s}\langle\xi_2-\xi_1\rangle^{-2s}\langle\xi-\xi_2\rangle^{-2s}\langle\xi\rangle^2\langle\xi\rangle^{2(a+s)}d\xi_1d\xi_2d\tau_2}{\langle\tau_2-(\xi_2-\xi_1)^5-\xi_1^5\rangle^{2b}\langle\tau_2+(\xi-\xi_2)^5-\xi^5\rangle^{2b}},$$

applying Lemma 2.4 in τ_2 integral, we are reduced to prove

$$\sup_{\xi}\iint_{\mathbb{R}^2}\frac{\langle\xi_1\rangle^{-2s}\langle\xi_2-\xi_1\rangle^{-2s}\langle\xi-\xi_2\rangle^{-2s}\langle\xi\rangle^2\langle\xi\rangle^{2(a+s)}d\xi_1d\xi_2}{\langle\xi_1^5+(\xi_2-\xi_1)^5+(\xi-\xi_2)^5-\xi^5\rangle^{1-}}\lesssim 1.\tag{3.5}$$

Let

$$\begin{aligned}G&=\xi^5-\xi_1^5-(\xi_2-\xi_1)^5-(\xi-\xi_2)^5\\&=\frac{5}{2}\xi_2(\xi_2-\xi_1)(\xi-\xi_2-\xi_1)(\xi_1^2+\xi^2+(\xi-\xi_2)^2+(\xi_2-\xi_1)^2),\end{aligned}$$

due to the symmetry of $\xi_1,\xi_2-\xi_1,\xi-\xi_2$, we may assume that $|\xi_1|\lesssim|\xi_2-\xi_1|\lesssim|\xi-\xi_2|$. We will discuss (3.5) in the following cases.

Case I: $|\xi_1| > 1$, $|\xi - \xi_2| \sim |\xi_2 - \xi_1| \sim |\xi| \sim |\xi_1|$. Then combining triangular inequality, $\langle G\rangle \sim \langle \xi_2\rangle\langle \xi_2 - \xi_1\rangle\langle \xi - \xi_1 - \xi_2\rangle\langle \xi\rangle^2 \geq \langle \xi\rangle^5$,

$$\text{LHS of (3.5)} \lesssim \sup_{\xi}\langle \xi\rangle^{2+2s+2a}\int_{\mathbb{R}}\langle \xi_1\rangle^{-4-6s}d\xi_1 \lesssim \sup_{\xi}\langle \xi\rangle^{-1-4s+2a} \lesssim 1,$$

provided that $a < 2s + \frac{1}{2}$.

Case II: $|\xi_2 - \xi_1| > 1$, $|\xi_1| \ll |\xi_2 - \xi_1| \sim |\xi - \xi_2| \sim |\xi|$.

Case II-a: $|\xi_1| \leq 1$, here $\langle G\rangle \sim \langle \xi\rangle^4\langle \xi_2\rangle$ Then

$$\text{LHS of (3.5)} \lesssim \sup_{\xi}\langle \xi\rangle^{2+2s+2a-4s-4} \lesssim 1,$$

provided that $a < s + 1$.

Case II-b: $|\xi_1| > 1$. $|G| \sim |\xi|^5$. Then

$$\text{LHS of (3.5)} \lesssim \sup_{\xi}\langle \xi\rangle^{2+2s+2a-4s-5}\int_{\mathbb{R}}\langle \xi_1\rangle^{-2s}d\xi_1$$

If $s < 0$, then $\langle \xi_1\rangle^{-2s} \lesssim \langle \xi\rangle^{-2s}$.

$$\text{LHS of (3.5)} \lesssim \sup_{\xi}\langle \xi\rangle^{2a-4s-3} \lesssim 1$$

provided $a < 2s + \frac{3}{2}$.

If $s > 0$, then

$$\text{LHS of (3.5)} \lesssim \int_{\mathbb{R}}\langle \xi_1\rangle^{-3-4s+2a}d\xi_1 \lesssim 1$$

provided $a < \min\{2s+1, s+\frac{3}{2}\}$.

Case III: $|\xi_1| > 1$, $|\xi| \ll |\xi_1| \sim |\xi - \xi_2|$, then $\langle G\rangle \sim \langle \xi_1\rangle^5$.

Case III-a: $|\xi| > 1$.

$$\text{LHS of (3.5)} \lesssim \sup_{\xi}\langle \xi\rangle^{2a+2s+2}\int_{\mathbb{R}}\langle \xi_1\rangle^{-6s-5}d\xi_1 \lesssim 1,$$

provided that $a < -1 - s$ and $s > -\frac{2}{3}$ or $-s - 1 < a < 2s + 1$.

Case III-b: $|\xi| \leq 1$. Then

$$\text{LHS of (3.5)} \lesssim \int_{\mathbb{R}}\langle \xi_1\rangle^{-5-6s}d\xi_1 \lesssim 1$$

provided $s > -\frac{2}{3}$.

□

Lemma 3.6. *For $\frac{1}{2}-b>0$ sufficiently small, we have*

$$\left\|\eta(t)\int_0^t W_R^{t-t'}Gdt'\right\|_{L_x^\infty H_t^{\frac{s+2}{5}}}+\left\|\eta(t)\left[\int_0^t W_R^{t-t'}Gdt'\right]_x\right\|_{L_x^\infty H_t^{\frac{s+1}{5}}}$$
$$\lesssim \|G\|_{X^{s,-b}}+\left\|\int_{\mathbb{R}}\chi_R(\xi,\tau)\langle\tau-\xi^5\rangle^{\frac{s-3}{5}}|\widehat{G}(\xi,\tau)|d\xi\right\|_{L_\tau^2}, \qquad s>-1,$$

where $R=\{|\tau|\gg|\xi|^5\}\cup\{|\xi|\lesssim 1\}$.

Proof. We only consider the equation at $x=0$.

$$\begin{aligned}\int_0^t W_R^{t-t'}Gdt' &= \int_0^t\int_{\mathbb{R}} e^{i(t-t')\xi^5}\mathcal{F}_x(G)(\xi,t')d\xi dt'\\ &= \int_0^t\int_{\mathbb{R}} e^{i(t-t')\xi^5}\int_{\mathbb{R}} e^{i\tau t'}\widehat{G}(\xi,\tau)d\tau d\xi dt'\end{aligned}$$

and

$$\int_0^t e^{it'(\tau-\xi^5)}dt'=\frac{e^{it'(\tau-\xi^5)}}{i(\tau-\xi^5)}\Big|_0^t=\frac{e^{it(\tau-\xi^5)}-1}{i(\tau-\xi^5)}.$$

Then, we will bound

$$\iint_{\mathbb{R}^2} e^{it\xi^5}\frac{e^{it(\tau-\xi^5)}-1}{i(\tau-\xi^5)}\widehat{G}(\xi,\tau)d\tau d\xi=\iint_{\mathbb{R}^2}\frac{e^{it\tau}-e^{it\xi^5}}{i(\tau-\xi^5)}\widehat{G}(\xi,\tau)d\tau d\xi.$$

On the other hand,

$$\begin{aligned}\eta(t)\int_0^t W_R^{t-t'}Gdt' &= \eta(t)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{it\tau}-e^{it\xi^5}}{i(\tau-\xi^5)}\widehat{G}(\xi,\tau)\psi(\tau-\xi^5)d\tau d\xi\\ &\quad+\eta(t)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{it\tau}}{i(\tau-\xi^5)}\widehat{G}(\xi,\tau)\psi^C(\tau-\xi^5)d\tau d\xi\\ &\quad-\eta(t)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{it\xi^5}}{i(\tau-\xi^5)}\widehat{G}(\xi,\tau)\psi^C(\tau-\xi^5)d\tau d\xi\\ &= A+B+C,\end{aligned}$$

where $\psi\in C^\infty$, $\psi=1$ in $[-1,1]$, $\psi=0$ outside $[-2,2]$, $\psi^C=1-\psi$.

By Taylor expanding, we have

$$\frac{e^{it\tau}-e^{it\xi^5}}{i(\tau-\xi^5)}=e^{it\tau}\sum_{k=1}^\infty\frac{e^{it(\xi^5-\tau)}t^k}{k!}[i(\tau-\xi^5)]^{k-1}.$$

Then,

$$
\begin{aligned}
\|A\|_{H_t^{\frac{s+2}{5}}} &\lesssim \left|\sum_{k=1}^{\infty} \frac{\|\eta(t)t^k\|_{H^1}}{k!}\right| \left\| \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it\tau}(\tau-\xi^5)^{k-1}\psi(\tau-\xi^5)\widehat{G}(\xi,\tau)d\xi d\tau \right\|_{H_t^{\frac{s+2}{5}}} \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left\| \langle\tau\rangle^{\frac{s+2}{5}} \int_{\mathbb{R}} (\tau-\xi^5)^{k-1}\psi(\tau-\xi^5)\widehat{G}(\xi,\tau)d\xi \right\|_{L^2_\tau} \\
&\lesssim \left\| \langle\tau\rangle^{\frac{2(s+2)}{5}} \int_{\mathbb{R}} \psi(\tau-\xi^5)\widehat{G}(\xi,\tau)d\xi \right\|_{L^2_\tau} \\
&\lesssim \left[\int_{\mathbb{R}} \langle\tau\rangle^{\frac{2(s+2)}{5}} \left(\int_{|\tau-\xi^5|<1} \langle\xi\rangle^{-2s} d\xi \right) \left(\int_{|\tau-\xi^5|<1} \langle\xi\rangle^{2s} \widehat{G}(\xi,\tau) d\xi \right) d\tau \right]^{\frac{1}{2}} \\
&\lesssim \sup_{\tau} \left[\langle\tau\rangle^{\frac{2(s+2)}{5}} \int_{|\tau-\xi^5|<1} \langle\xi\rangle^{-2s} d\xi \right]^{\frac{1}{2}} \left[\int_{|\tau-\xi^5|<1} \langle\xi\rangle^{2s} \widehat{G}(\xi,\tau) d\xi \right]^{\frac{1}{2}} \lesssim \|G\|_{X^{s,-b}},
\end{aligned}
$$

where we have used that

$$
\langle\tau\rangle^{\frac{2(s+2)}{5}} \int_{|\tau-\xi^5|<1} \langle\xi\rangle^{-2s} d\xi \lesssim \begin{cases} 1, & |\tau| \le 1, \\ \langle\tau\rangle^{\frac{2s+4}{5}} \int_{|\tau-\xi^5|<1} \langle\tau\rangle^{-\frac{2s}{5}} \tau^{-\frac{4}{5}} d\tau, & |\tau| > 1. \end{cases}
$$

For the second inequality, we have used that $\tau \sim \xi^5$.

For the derivative terms, we have

$$
\begin{aligned}
\eta(t) \int_0^t W_R^{t-t'} G_x dt' &= \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{i\xi(e^{it\tau} - e^{it\xi^5})}{i(\tau-\xi^5)} \widehat{G}(\xi,\tau)\psi(\tau-\xi^5) d\tau d\xi \\
&+ \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{i\xi e^{it\tau}}{i(\tau-\xi^5)} \widehat{G}(\xi,\tau)\psi^C(\tau-\xi^5) d\tau d\xi \\
&- \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{i\xi e^{it\xi^5}}{i(\tau-\xi^5)} \widehat{G}(\xi,\tau)\psi^C(\tau-\xi^5) d\tau d\xi \\
&= \tilde{A} + \tilde{B} + \tilde{C}.
\end{aligned}
$$

By the same way, we have $\|\tilde{A}\|_{H_t^{\frac{s+1}{5}}} \lesssim \|G\|_{X^{s,-b}}$.

For the term B,

$$
\langle\tau\rangle \lesssim \chi_R(\xi,\tau)\langle\tau-\xi^5\rangle + \langle\xi\rangle^5.
$$

We obtain

$$\begin{aligned}\|B\|_{H_t^{\frac{s+2}{5}}} &\lesssim \left\| \langle\tau\rangle^{\frac{s+2}{5}} \int_{\mathbb{R}} \frac{1}{i(\tau-\xi^5)} \widehat{G}(\xi,\tau) d\xi \right\|_{L^2_\tau} \\ &\lesssim \left\| \langle\tau\rangle^{\frac{s+2}{5}} \int_{\mathbb{R}} \psi(\tau-\xi^5) \widehat{G}(\xi,\tau) d\xi \right\|_{L^2_\tau} \\ &\lesssim \left\| \int_{\mathbb{R}} \chi_R \langle\tau-\xi^5\rangle^{\frac{s-3}{5}} \widehat{G}(\xi,\tau) d\xi \right\|_{L^2_\tau} + \left\| \int_{\mathbb{R}} \frac{|\xi|^{s+2}}{i(\tau-\xi^5)} \widehat{G}(\xi,\tau) d\xi \right\|_{L^2_\tau},\end{aligned}$$

the second term is bounded by $\|G\|_{X^{s,-b}}$ if

$$\sup_\tau \int_{\mathbb{R}} \frac{\xi^4}{\langle\tau-\xi^5\rangle^{2-2b}} d\xi < \infty,$$

which holds provided that $b < \frac{1}{2}$.

For $s \le \frac{1}{2}$, we consider

$$\begin{aligned}&\left\| \langle\tau\rangle^{\frac{s+2}{5}} \int_{\mathbb{R}} \frac{1}{\langle\tau-\xi^5\rangle} |\widehat{G}(\xi,\tau)| d\xi \right\|_{L^2_\tau} \\ &\lesssim \left[\int_{\mathbb{R}} \langle\tau\rangle^{\frac{2(s+2)}{5}} \left(\int_{\mathbb{R}} \frac{1}{\langle\tau-\xi^5\rangle^{2-2b}\langle\xi\rangle^{2s}} d\xi \right) \left(\int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s}|\widehat{G}(\xi,\tau)|^2}{\langle\tau-\xi^5\rangle^{2b}} d\xi \right) d\tau \right]^{\frac{1}{2}} \\ &\lesssim \sup_\tau \langle\tau\rangle^{\frac{2(s+2)}{5}} \left(\int_{\mathbb{R}} \frac{1}{\langle\tau-\xi^5\rangle^{2-2b}\langle\xi\rangle^{2s}} d\xi \right) \|G\|_{X^{s,-b}}.\end{aligned}$$

To bound the last term, we take variable changes $z = \xi^5$, then the above inequality have the bound

$$\begin{aligned}\sup_\tau \langle\tau\rangle^{\frac{2(s+2)}{5}} \|G\|_{X^{s,-b}} \int_{\mathbb{R}} \frac{1}{\langle\tau-z\rangle^{2-2b}\langle z\rangle^{\frac{2(s+2)}{5}}} dz &\lesssim \|G\|_{X^{s,-b}} \sup_\tau \langle\tau\rangle^{\frac{2(s+2)}{5} - \min\{2-2b, \frac{2(s+2)}{5}\}} \\ &\lesssim \|G\|_{X^{s,-b}}\end{aligned}$$

provided that $b < \frac{1}{2}$ and $-2 \le s \le \frac{1}{2}$.

Next, we consider the derivative term $\tilde{B}$. If $s \ge -1$,

$$\begin{aligned}\|\tilde{B}\|_{H_t^{\frac{s+1}{5}}} &\lesssim \left\| \langle\tau\rangle^{\frac{s+1}{5}} \int_{\mathbb{R}} \frac{|\xi|}{\langle\tau-\xi^5\rangle} |\widehat{G}(\xi,\tau)| d\xi \right\|_{L^2_\tau} \\ &\lesssim \left\| \int_{\mathbb{R}} \chi_R(\xi,\tau) \langle\tau-\xi^5\rangle^{\frac{s-3}{5}} |\widehat{G}(\xi,\tau)| d\xi \right\|_{L^2_\tau} + \left\| \int_{\mathbb{R}} \frac{|\xi|^{s+2}}{\langle\tau-\xi^5\rangle} |\widehat{G}(\xi,\tau)| d\xi \right\|_{L^2_\tau},\end{aligned}$$

for $b < \frac{1}{2}$, we have the bound

$$\|G\|_{X^{s,-b}} + \left\| \int_{\mathbb{R}} \chi_R(\xi,\tau) \langle\tau-\xi^5\rangle^{\frac{s-3}{5}} |\widehat{G}(\xi,\tau)| d\xi \right\|_{L^2_\tau}.$$

For $s \leq -1$, we consider

$$\left\| \langle \tau \rangle^{\frac{s+1}{5}} \int_{\mathbb{R}} \frac{|\xi|}{\langle \tau - \xi^5 \rangle} |\widehat{G}(\xi,\tau)| d\xi \right\|_{L^2_\tau}.$$

On the region $|\tau| \ll |\xi|^5$ and $|\xi| \gtrsim 1$, we obtain the bound

$$\left\| \langle \tau \rangle^{\frac{s-4}{5}} \int_{\mathbb{R}} \chi_Q(\xi,\tau) |\xi|^6 |\widehat{G}(\xi,\tau)| d\xi \right\|_{L^2_\tau}$$

where $Q = \{|\tau| \ll |\xi|^5\} \cap \{|\xi| \gtrsim 1\}$.

On the region $|\tau| \gtrsim |\xi|^5$ or $|\xi| \lesssim 1$, we have

$$\begin{aligned} \|\tilde{B}\|_{H_t^{\frac{s+1}{5}}} &\lesssim \left\| \langle \tau \rangle^{\frac{s+1}{5}} \int_{\mathbb{R}} \frac{|\xi|}{\langle \tau - \xi^5 \rangle} |\widehat{G}(\xi,\tau)| d\xi \right\|_{L^2_\tau} \\ &\lesssim \left\| \int_{\mathbb{R}} \frac{|\xi|^{s+2}}{\langle \tau - \xi^5 \rangle} |\widehat{G}(\xi,\tau)| d\xi \right\|_{L^2_\tau} \lesssim \|G\|_{X^{s,-b}}. \end{aligned}$$

For C, firstly we consider $|\xi| \leq 1$.

$$\begin{aligned} &\left\| \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{it\xi^5}}{i(\tau - \xi^5)} \widehat{G}(\xi,\tau) \psi^C(\tau - \xi^5) d\tau d\xi \right\|_{H_t^{\frac{s+2}{5}}} \\ &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|\eta(t) e^{it\xi^5}\|_{H_t^{\frac{s+2}{5}}}}{i(\tau - \xi^5)} \widehat{G}(\xi,\tau) \psi^C(\tau - \xi^5) d\tau d\xi \\ &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\chi_{[-1,1]}(\xi)}{\langle \tau - \xi^5 \rangle} |\widehat{G}(\xi,\tau)| d\xi d\tau \\ &\lesssim \|G\|_{X^{s,-b}} \left\| \frac{\chi_{[-1,1]}(\xi)}{\langle \tau - \xi^5 \rangle^{1-b}} \right\|_{L^2_{\tau,\xi}} \lesssim \|G\|_{X^{s,-b}}. \end{aligned}$$

When $|\xi| \geq 1$, make variable changes $z = \xi^5$, then

$$\begin{aligned} &\left\| \eta(t) \iint_{|\xi| \geq 1} \frac{e^{it\xi^5}}{i(\tau - \xi^5)} \widehat{G}(\xi,\tau) \psi^C(\tau - \xi^5) d\tau d\xi \right\|_{H_t^{\frac{s+2}{5}}} \\ &\lesssim \left\| \langle z \rangle^{\frac{s-2}{5}} \int_{\mathbb{R}} \frac{\widehat{G}(\xi(z),\tau)}{\langle \tau - z \rangle} d\tau \right\|_{L^2_{|z| \geq 1}} \\ &\lesssim \left(\int_{|z| \geq 1} \langle z \rangle^{\frac{2(s-2)}{5}} \langle z \rangle^{-\frac{2s}{5}} \int_{\mathbb{R}} \frac{|\widehat{G}(\xi(z),\tau)|^2}{\langle \tau - z \rangle^{2b}} \langle z \rangle^{\frac{2s}{5}} d\tau dz \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_{|\xi^5| \geq 1} \langle \xi \rangle^{2(s-2)} \langle \xi \rangle^{-2s} \int_{\mathbb{R}} \frac{|\widehat{G}(\xi,\tau)|^2}{\langle \tau - \xi^5 \rangle^{2b}} \langle \xi \rangle^{2s} d\tau d\xi \right)^{\frac{1}{2}} \\ &\lesssim \int_{|\xi^5| \geq 1} \langle \xi \rangle^{-4} dz \|G\|_{X^{s,-b}} \lesssim \|G\|_{X^{s,-b}}. \end{aligned}$$

Similarly we get $\|\tilde{C}\|_{H_t^{\frac{s+1}{5}}} \lesssim \|G\|_{X^{s,-b}}$.

$\square$

Lemma 3.7. *Let R be the set $\{|\tau| \gg |\xi|^5\} \cup \{|\xi| \lesssim 1\}$. For $-\frac{1}{2} < s+a < 2$ and $a < 2s + \frac{3}{2}$, $2 < s+a < \frac{9}{2}$ and $0 < a < \min\{\frac{3}{2}, \frac{7-2s}{4}\}$, we have*

$$\left\| \int_{\mathbb{R}} \chi_R(\xi,\tau) \langle \tau - \xi^5 \rangle^{\frac{s+a-2}{5}} |\widehat{uvw}(\xi,\tau)| d\xi \right\|_{L^2_\tau} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}} \|w\|_{X^{s,b}}, \tag{3.6}$$

for any $u, v, w \in X^{s,b}$.

Proof. To obtain (3.6), it suffices to show that

$$\left\| \iiint \iint \frac{\chi_R(\xi,\tau)\langle \tau - \xi^5 \rangle^{\frac{s+a-2}{5}} |f(\xi_1,\tau_1) g(\xi_2-\xi_1, \tau_2-\tau_1) h(\xi-\xi_2, \tau-\tau_2)| d\tau_1 d\tau_2 d\xi_1 d\xi_2 d\xi}{\langle \xi_1 \rangle^s \langle \xi_2 - \xi_1 \rangle^s \langle \xi - \xi_2 \rangle^s \langle \tau_1 - \xi_1^5 \rangle^b \langle (\tau_2 - \tau_1) - (\xi_2 - \xi_1)^5 \rangle^b \langle (\tau - \tau_2) - (\xi - \xi_2)^5 \rangle^b} \right\|_{L^2_\tau}$$
$$\lesssim \|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|h\|_{L^2_{\xi,\tau}}. \tag{3.7}$$

Using Cauchy-Schwarz in τ and Young's inequality, we have

$$\text{LHS of (3.7)} \lesssim \left(\sup_\tau \iiint \iint_{\mathbb{R}^5} \chi_R(\xi,\tau) M_0^2 d\tau_1 d\tau_2 d\xi_1 d\xi_2 d\xi \right)^{\frac{1}{2}} \|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|h\|_{L^2_{\xi,\tau}},$$

where

$$M_0 = \frac{\langle \tau - \xi^5 \rangle^{\frac{s+a-2}{5}}}{\langle \xi_1 \rangle^s \langle \xi_2 - \xi_1 \rangle^s \langle \xi - \xi_2 \rangle^s \langle \tau_1 - \xi_1^5 \rangle^b \langle (\tau_2 - \tau_1) - (\xi_2 - \xi_1)^5 \rangle^b \langle (\tau - \tau_2) - (\xi - \xi_2)^5 \rangle^b}.$$

It suffices to show that

$$\sup_\tau \iiint \iint_{\mathbb{R}^5} \chi_R(\xi,\tau) M_0^2 d\tau_1 d\tau_2 d\xi_1 d\xi_2 d\xi \lesssim 1. \tag{3.8}$$

Using Lemma 2.4 in the τ_1 and τ_2 integral, we have

$$\begin{aligned} &\text{LHS of (3.8)} \\ &\lesssim \sup_\tau \iiint_{\mathbb{R}^3} \frac{\chi_R(\xi,\tau) \langle \tau - \xi^5 \rangle^{\frac{2(s+a-2)}{5}}}{\langle \xi_1 \rangle^{2s} \langle \xi_2 - \xi_1 \rangle^{2s} \langle \xi - \xi_2 \rangle^{2s} \langle \tau - (\xi_1^5 + (\xi_2 - \xi_1)^5 + +(\xi - \xi_2)^5) \rangle^{2b}} d\xi_1 d\xi_2 d\xi. \end{aligned} \tag{3.9}$$

When $2 \le s+a < \frac{9}{2}$, we use the inequality

$$\langle \tau - \xi^5 \rangle \lesssim \langle \tau - \xi_1^5 - (\xi - \xi_2)^5 - (\xi_2 - \xi_1)^5 \rangle \langle \xi_1 \rangle^5 \langle \xi \rangle^5 \langle \xi - \xi_2 \rangle^5 \langle \xi_2 - \xi_1 \rangle^5$$

and we get the inequality

$$\text{LHS of (3.8)} \lesssim \iiint_{\mathbb{R}^3} \langle\xi\rangle^{2(s+a-2)}\langle\xi_1\rangle^{2(a-2)}\langle\xi_2-\xi_1\rangle^{2(a-2)}\langle\xi-\xi_2\rangle^{2(a-2)} d\xi d\xi_1 d\xi_2.$$

We use Lemma 2.4 in the ξ_1 integral. Notice that $\beta = 2(2-a) > 1$, so $a < \frac{3}{2}$, then

$$\int_{\mathbb{R}} \langle\xi_1\rangle^{2(a-2)}\langle\xi_2-\xi_1\rangle^{2(a-2)} d\xi_1 \lesssim \langle\xi_2\rangle^{2(a-2)},$$

then use Lemma 2.4 in the ξ_2 integral, thus

$$\text{RHS of(3.9)} \lesssim \int_{\mathbb{R}} \langle\xi\rangle^{2(s+a-2)}\langle\xi\rangle^{2(a-2)} d\xi \lesssim 1$$

provided that $a < \frac{7-2s}{4}$.

When $-\frac{1}{2} < s + a < 2$, we use triangular inequality to get

$$\begin{aligned} &\text{RHS of (3.9)} \\ &\lesssim \sup_{\tau} \iiint_{\mathbb{R}^3} \frac{\chi_R(\xi,\tau) d\xi_1 d\xi_2 d\xi}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_2\rangle^{2s}\langle\xi_2-\xi_1\rangle^{2s}\langle\xi^5-(\xi_1^5+(\xi-\xi_2)^5+(\xi_2-\xi_1)^5)\rangle^{\frac{2(3-s-a)}{5}}} \\ &\lesssim \sup_{\tau} \iiint_{\mathbb{R}^3} \frac{\chi_R(\xi,\tau) d\xi_1 d\xi_2 d\xi}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_2\rangle^{2s}\langle\xi_2-\xi_1\rangle^{2s}\langle G\rangle^{\frac{2(3-s-a)}{5}}}, \end{aligned} \tag{3.10}$$

Due to the symmetry of ξ_1, $\xi-\xi_2$, $\xi_2-\xi_1$, we may assume that $|\xi_1| \lesssim |\xi_2-\xi_1| \lesssim |\xi-\xi_2|$ without loss of generality.

Case I: $|\xi_1| > 1$, $|\xi-\xi_2| \sim |\xi_2-\xi_1| \sim |\xi| \sim |\xi_1|$. Then combining triangular inequality, $\langle G\rangle \sim \langle\xi_2\rangle\langle\xi_2-\xi_1\rangle\langle\xi-\xi_1-\xi_2\rangle\langle\xi\rangle^2 \geq \langle\xi\rangle^5$,

$$\text{RHS of (3.10)} \lesssim \int_{\mathbb{R}} \langle\xi\rangle^{-2(2-s-a)-6s} d\xi,$$

provided that $a < 2s + \frac{3}{2}$.

Case II: $|\xi_2-\xi_1| > 1$, $|\xi_1| \ll |\xi_2-\xi_1| \sim |\xi-\xi_2| \sim |\xi|$. Then $|G| \sim |\xi|^5$. Then

$$\text{RHS of (3.10)} \lesssim \int_{\mathbb{R}} \langle\xi\rangle^{-2(2-s-a)-4s} \frac{d\xi d\xi_1}{\langle\xi_1\rangle^{2s}}.$$

If $s < 0$, then $\langle\xi_1\rangle^{-2s} \lesssim \langle\xi\rangle^{-2s}$.

$$\text{RHS of (3.10)} \lesssim \int_{\mathbb{R}} \langle\xi\rangle^{-2(2-s-a)-6s} d\xi \lesssim 1$$

provided $a < 2s + \frac{3}{2}$.

If $s > 0$, then

$$\text{RHS of (3.10)} \lesssim \int_{\mathbb{R}} \langle \xi_1 \rangle^{-2(2-s-a)-6s} d\xi_1 \lesssim 1$$

provided $a < 2s + \frac{3}{2}$.

Case III: $|\xi_1| > 1$, $|\xi| \ll |\xi_1| \sim |\xi - \xi_2|$, then $|G| \sim |\xi_1|^5$. Then

$$\text{RHS of (3.10)} \lesssim \int_{\mathbb{R}} \langle \xi_1 \rangle^{-6s-2(2-s-a)} d\xi_1 \lesssim 1,$$

provided that $a < 2s + \frac{3}{2}$.

□

4 Proof of Theorem 1.3

For discussing the contraction theory, we will proof the map Φ defined in (1.4) has a unique fixed point in $X^{s,b}$. Let ϕ^* be extension of ϕ such that $\|\phi^*\|_{H^s(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R}^+)}$. Recall that

$$\Phi(u(x,t)) = \eta(\frac{t}{T}) W_R^t \phi^*(x) + \eta(\frac{t}{T}) W_0^t(0, h_1 - p_1 - q_1, h_2 - p_2 - q_2) + \eta(\frac{t}{T}) \int_0^t W_R^{t-t'} G(u) dt', \quad (4.1)$$

where $G(u), p_i, q_i$ have been defined before. We will use the above results to bound the three terms in $\Phi(u(x,t))$. We will use Lemma 2.1 to obtain

$$\|\eta(t/T) W_R^t \phi^*\|_{X^{s,b}} \lesssim \|\phi^*\|_{H^s(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R}^+)}.$$

For the Duhamel term, we apply Lemma 2.2 and Lemmas 3.4-3.5 to obtain

$$\left\| \eta(\frac{t}{T}) \int_0^t W_R^{t-t'} G(u) dt' \right\|_{X^{s,b}} \lesssim T^{1-2b} \|(u^3)_x\|_{X^{s,-b}} \lesssim T^{1-2b} \| \|u\|^3_{X^{s,b}}.$$

Finally, for the second term, we apply Lemmas 3.2 and 2.3 to obtain

$$\begin{aligned}
&\|\eta(\frac{t}{T}) W_0^t(0, h_1 - p_1 - q_1, h_2 - p_2 - q_2)\|_{X^{s,b}} \\
&\lesssim \|\chi(h_1 - p_1 - q_1)\|_{H_t^{\frac{s+2}{5}}(\mathbb{R})} + \|\chi(h_2 - p_2 - q_2)\|_{H_t^{\frac{s+1}{5}}(\mathbb{R})} \\
&\lesssim \|h_1 - p_1\|_{H_t^{\frac{s+2}{5}}(\mathbb{R}^+)} + \|q_1\|_{H_t^{\frac{s+2}{5}}(\mathbb{R}^+)} + \|h_2 - p_2\|_{H_t^{\frac{s+1}{5}}(\mathbb{R}^+)} + \|q_2\|_{H_t^{\frac{s+1}{5}}(\mathbb{R}^+)}.
\end{aligned}$$

By Kato smoothing and Lemma 3.1 we have

$$\|p_1\|_{H_t^{\frac{s+2}{5}}(\mathbb{R})} + \|p_2\|_{H_t^{\frac{s+1}{5}}(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R}^+)},$$

for the q_i terms, we apply Lemmas 3.4-3.7 to obtain

$$\|q_1\|_{H_t^{\frac{s+2}{5}}(\mathbb{R})} + \|q_2\|_{H_t^{\frac{s+1}{5}}(\mathbb{R})} \lesssim T^{\frac{1}{2}-b-}\|u\|_{X^{s,b}}^3.$$

Combining those results, we have

$$\|\Phi(u)\|_{X^{s,b}} \lesssim \|\phi\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H_t^{\frac{s+2}{5}}(\mathbb{R}^+)} + \|h_2\|_{H_t^{\frac{s+1}{5}}(\mathbb{R}^+)} + T^{\frac{1}{2}-b-}\|u\|_{X^{s,b}}^3.$$

Then we have that $\Phi(u)$ has a fixed point u if

$$T = T(\|\phi\|_{H_x^s(\mathbb{R}^+)}, \|h_1\|_{H_t^{\frac{s+2}{5}}(\mathbb{R}^+)}, \|h_2\|_{H_t^{\frac{s+1}{5}}(\mathbb{R}^+)})$$

sufficiently small.

Next, we prove continuity in H^s. For the W_0^t term, it follows from Lemma 3.3. For the Duhamel terms, it follows from that $X^{s,b} \subset C_t^0 H_x^s$ for $b > \frac{1}{2}$,. In fact the solution $u \in C_x^0 H^t_{\frac{s+2}{5}}$ follows from Lemmas 3.4-3.5 for the W_0^t term. For the linear flow on $\mathbb{R}$, we have from Kato smoothing and Lemma 3.1. For the Duhamel term, we get the continuity from Lemmas 3.6-3.7.

5 Uniqueness

It is not clear if different extensions of initial data produce the same solution on $\mathbb{R}^+$, we start with a proof of uniqueness in the case $s > \frac{1}{2}$. The uniqueness of solution (under additional assumptions) will follow from an approximation argument.

Lemma 5.1. *[5] Let $s \in \mathbb{R}$. For $0 < b < \frac{1}{2} < 1 - b$, we have*

$$\|\eta(t)\int_0^t W_R^{t-t'} G(u)dt'\|_{C(\mathbb{R}_t; H^s(\mathbb{R}_x))} \lesssim \|G\|_{X^{s,-b}}.$$

Lemma 5.2. *If $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, and if $u \in L^p(\mathbb{R}^n)$, $v \in L^q(\mathbb{R}^n)$, then $u * v \in L^r(\mathbb{R}^n)$ and*

$$\|u * v\|_{L^r(\mathbb{R}^n)} \le C(p, q, r, n)\|u\|_{L^p(\mathbb{R}^n)}\|v\|_{L^q(\mathbb{R}^n)}.$$

Proof of uniqueness. u, v be the solution we obtained above, noting that

$$u - v = \Phi(u) - \Phi(v) = \int_0^t W_R^{t-t'}(G(u) - G(v))dt'.$$

Then

$$\|\Phi(u) - \Phi(v)\|_{C_t^0 H_x^s} = \left\|\int_0^t W_R^{t-t'}(G(u) - G(v))dt'\right\|_{C_t^0 H_x^s} \lesssim \|G(u) - G(v)\|_{X^{s,-b}}.$$

Next, we will estimate the nonlinear terms. Since

$$u^3 - v^3 = (u-v)^3 + 3uv(u-v),$$

combing Lemmas 3.4, 3.5 we obtain

$$\begin{aligned}
\|(u^3-v^3)_x\|_{X^{s,-b}} &\lesssim \|\partial_x(u-v)^3\|_{X^{s,-b}} + \|\partial_x(uv(u-v))\|_{X^{s,-b}} \\
&\lesssim \|u-v\|^3_{X^{s,b}} + \|uv\|_{X^{s,b}}\|u-v\|_{X^{s,b}} \\
&\lesssim \|u-v\|^3_{X^{s,b}} + \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}\|u-v\|_{X^{s,b}},
\end{aligned}$$

when $s > \frac{1}{2}$. Where we have used that

$$\|uv\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}.$$

Indeed,

$$\begin{aligned}
\|uv\|_{X^{s,b}} = \left\| \langle\xi\rangle^s \|e^{it\xi^5}\widehat{uv}(\xi,t)\|_{H^b_t} \right\|_{L^2_\xi} &= \left\| \langle\xi\rangle^s \|e^{it\xi^5}(\widehat{u} * \widehat{v})(\xi,t)\|_{H^b_t} \right\|_{L^2_\xi} \\
&\lesssim \left\| \langle\xi\rangle^s \|e^{it\xi^5}\widehat{u}(\xi,t)\|_{H^b_t} \right\|_{L^2_\xi} \left\| \langle\xi\rangle^s \|e^{it\xi^5}\widehat{v}(\xi,t)\|_{H^b_t} \right\|_{L^2_\xi} \\
&\lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}},
\end{aligned}$$

here we have used the Lemma 5.2 and that H^s is the Hardy space when $s > \frac{1}{2}$.

Hence

$$\|\Phi(u) - \Phi(v)\|_{C^0_t H^s_x} \to 0$$

as $u \to v$ when $s > \frac{1}{2}$. The case of $s < \frac{1}{2}$ will follow from an approximation argument.

Thus, local Lipschitz continuity of the data-to-solution map has been established, and we established the uniqueness.

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