The fractal dimension of pullback attractors for the 2D Navier-Stokes equations with delay

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Abstract

This paper is concerned with the bounded fractal and Hausdorff dimension of the pullback attractors for 2D non-autonomous incompressible Navier-Stokes equations with constant delay terms. Using the construction of trace formula with two bases for phase spaces of product flow, the upper boundedness of fractal dimension has been achieved.

Keywords: 2D Navier-Stokes equation; continuous delay; pullback attractors; fractal dimension.
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1 Introduction

The incompressible Navier-Stokes equations is the fundamental mathematical model to describe the conservation law for fluid flow, whose physical background can be founded in some literatures such as [20]. The rigorous mathematical analysis of three dimensional case goes back to Leray [21] and Hopf [16], which derived the so called Leray-Hopf weak solution, i.e., the global existence of weak solutions and local for strong solution. Moreover, the global existence and uniqueness of weak solution for two dimensional Navier-Stokes equations has been shown firstly by Ladyzhenskaya [18]. For the infinite-dimensional dynamic systems for 2D Navier-Stokes equations based on the weak and strong solutions, the existence and fractal dimension of global and pullback attractors can be referred in Constantin, Foias and Temam [10], Foias, Manley, Rosa and Temam [11], Ladyzhenskaya [19], Łukaszewicz and Kalita [25], Robinson [29], [30], Temam [33], Carvalho, Langa and Robinson [9], and literatures therein. Although there are fruitful results on dynamic systems for the 2D Navier-Stokes equations, the inertial manifold is still open.

The delay influence on differential equations was investigated in past decades which is also used in control theory and engineer especially from the mathematical analysis in physics, non-instant transmission phenomena and specially biological motivations, see Caraballo and Kiss [3], Caraballo, Marín-Rubio and Valero [4], [5], Hale and Lunel [15]. If the material in fluid flow is special, then the governing equations becomes 2D incompressible Navier-Stokes system with delay: continuous or distributed cases. For the well-posedness and dynamic systems for 2D Navier-Stokes flow with delay, we can refer to [1], [2], [6], [7], [8], [12], [13], [14], [24], [26], [27], [31] and some more generalized fluid flow model with delay in [23], [34]. The pullback dynamics for the 2D Navier-Stokes flow has been presented in above literatures, but the fractal dimension and robustness of pullback attractors have not been solved till now.

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The purpose for this paper is to investigated the finite fractal dimension of pullback attractors for 2D non-autonomous incompressible Navier-Stokes equations with continuous delay term in bounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t,u_t) + g(t,x), \quad x\in\Omega,\ t\in\mathbb{R}, \\ divu = 0, \quad x\in\Omega,\ t\in\mathbb{R}, \\ u(t,x)|_{\partial\Omega} = 0, \\ u_\tau(\theta,x) = u(\tau+\theta,x) = \phi(\theta,x),\ \theta\in[-h,0],\ x\in\Omega, \end{cases} \tag{1.1}$$

for $h > 0$. Here $u = (u^1(t,x), u^2(t,x))$ and $p = p(x,t)$ denote the unknown velocity field and pressure of fluid respectively, ν is the kinematic viscosity of the fluid, the nonlinear term $f(u_t, x)$ is the delay term, $g(t,x) \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ is the external force, $\phi(\theta,x)$ be the initial data, which contains $\phi(0,x) = \varphi(x) = u(t=\tau,x)$, u_t is defined as $u_t = u(t+\theta)$ with $\theta \in [-h,0]$.

Let X be a separable real Hilbert space, with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. Let $K \subset X$ be a non-empty compact subset and $\varepsilon > 0$, we denote $N_\varepsilon(K)$ to be the minimum number of open balls in X with radius ε which are necessary to cover K. The fractal dimension of K is defined as $\dim_F(K) = \limsup_{\varepsilon\to 0^+} \frac{\log(N_\varepsilon(K))}{-\log\varepsilon}$. Inspired by [9], [10], [11] and [33], we would prove the upper boundedness of pullback attractor for problem (1.1) with the constant delay. The main results and features of this paper are summarized as:

(1). Using the trace formula in [9], we can prove the finite fractal dimension of minimal family for pullback attractors $\mathcal{A}_{M_H}(t)$ for problem (1.1) in H. However, the phase space for global weak solution is $M_H = C_H \times H$, which contains $\mathcal{A}_{M_H}(t)$, the trace formula can not be used directly since the Banach spaces M_H and C_H is not a Hilbert space. To overcome this, we use two bases of the Hilbert space $\mathcal{M}_H = L^2(-h,0;H) \times H$ or $\tilde{\mathcal{M}}_H = L_H \times H$ (the choosing is decided by the time variable) which contains M_H instead to construct the linear operator for first variation equation which is quasi-differentiable and compact, then the basis with delay can be controlled, which leads to the upper boundedness of pullback attractors. This is a further result of [6], [7], [8], [12] and [13] partly, i.e., the continuous delay reduces to constant.

(2). However, the strategy used above is invalid for the 2D incompressible Navier-Stokes equations with variable continuous or distributed delays, such as $f(t,u_t) = f(t,u(t+s))$ or $f(t,u(t-\rho(t)))$ and $f = \int_{-h}^0 G(t,u(t+s))ds$. The difficulty is the delay basis can not be controlled, especially the eigenvalues for operator with delay is open, which is a challenging topic.

(3). The difference between classical and delay cases have the similar power for fractal and Hausdorff dimension of pullback attractors, which can be contracted with Carvalho, Langa and Robinson [9].

This paper is organized as follows. In Section 2, some preliminaries are given which will be used in sequel. Then we shall present the global well-posedness and pullback dynamic systems in Section 3. In Sections 4, we shall prove the finite dimension of pullback attractors for problem (1.1).

2 Preliminary

2.1 Some spaces

Denoting $E := \{u | u \in (C_0^\infty(\Omega))^2, divu = 0\}$, H is the closure of E in $(L^2(\Omega))^2$ topology, $|\cdot|_2$ and $(\cdot,\cdot)$ denote the norm and inner product in H respectively. V is the closure of the set E in $(H^1(\Omega))^2$ topology, $\|\cdot\|$ and $((\cdot,\cdot))$ denote the norm and inner product in V respectively. Clearly, $V \hookrightarrow H \equiv H' \hookrightarrow V'$, H' and V' are dual spaces of H and V respectively, where the injection is dense and continuous.The norm $\|\cdot\|_*$ and $\langle\cdot\rangle$ denote the norm in V' and the dual product between V and V', respectively.

P is the Helmholz-Leray orthogonal projection in $(L^2(\Omega))^2$ onto the space H, $A := -P\Delta$ is the Stokes operator, $\{\lambda_j\}_{j=1}^{\infty}$ $(0 < \lambda_1 \le \lambda_2 \le \cdots)$ is the eigenvalue of A with corresponding eigenfunctions $\{\omega_j\}_{j=1}^{\infty}$. We define the fractional operator A^s $(s \in \mathbb{C})$ (see [33]) as follows.

$$A^s u = \sum_{i=1}^{\infty} \lambda_j^s (u, \omega_j)\omega_j,\ u \in H,\ s \in \mathbb{C},\ j \in \mathbb{N}, \tag{2.1}$$

$$V^s = D(A^s) = \Big\{u \in H : A^s u \in H,\ \sum_{i=1}^{\infty} \lambda_i^{2s} |(u, \omega_i)|^2 < +\infty\Big\}, \tag{2.2}$$

$$|A^s u| = \Big(\sum_{i=1}^{\infty} \lambda_i^{2s} |(u, \omega_i)|^2\Big)^{1/2}, \tag{2.3}$$

where $D(A^s)$ denotes the domain of A^s with the inner product and the norm given by

$$(u, v)_{V^s} = (A^{\frac{s}{2}} u, A^{\frac{s}{2}} v), \qquad \|u\|_{V^s}^2 = (u, u)_{V^s}. \tag{2.4}$$

In particular, $V = V^1$ and $V^2 = W = (H^2(\Omega))^3 \bigcap (H_0^1(\Omega))^3$. In addition, the immersion

$$D(A^{\frac{s}{2}}) \hookrightarrow D(A^{\frac{r}{2}}), \quad s > r,$$

is continuous and

$$D(A^{\frac{s}{2}}) \hookrightarrow\hookrightarrow (L^{\frac{6}{(3-2s)}}(\Omega))^3, \quad 0 < s \le \tfrac{3}{2}, \tag{2.5}$$

is compact.

We can define the bilinear and trilinear form operators $B(\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ as

$$B(u, v) := P((u \cdot \nabla)v), \quad b(u, v, w) = (B(u, v), w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

here $B(u, v)$ is a linear continuous operator from V to V', and $b(u, v, w)$ satisfies $b(u, v, v) = 0$, $b(u, v, w) = -b(u, w, v)$ and

$$|b(u, v, w)| \le C|u|_2^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|v\| |w|_2^{\frac{1}{2}} \|w\|^{\frac{1}{2}},\ \forall\ u \in V,\ v \in V,\ w \in V. \tag{2.6}$$

For any $t \in (\tau, T)$, let u_t be a function defined on $(-h, 0)$ satisfying $u_t = u(t+s),\ s \in [-h, 0]$, we can extend to delay interval $u : (\tau - h, T) \to (L^2(\Omega))^2$. Based on this extension, we can introduce some phase space on delay interval $C_H = C([\tau - h, \tau]; H)$, $C_V = C([\tau - h, \tau]; V)$ as the Banach spaces with the norms

$$\|u\|_{C_H} = \sup_{\theta \in [-h,0]} |u(t + \theta)|_2,\ \|u\|_{C_V} = \sup_{\theta \in [-h,0]} \|u(t + \theta)\|$$

respectively.

The Lebesgue integrable spaces can be denote as $L_H^2 = L^2(\tau - h, \tau; H)$, $L_V^2 = L^2(\tau - h, \tau; V)$, and moreover $L_H^\infty = L^\infty(\tau - h, \tau; H)$, $L_V^\infty = L^\infty(\tau - h, \tau; V)$. The inner product and norm of $L_H^2 = L^2(\tau - h, \tau; H)$ is defined by

$$(u, v)_{L_H \times L_H} = \int_{\tau-h}^{\tau} (u(s), v(s))_{H \times H} ds \ \text{ and } \ \|u\|_{L_H} = \int_{\tau-h}^{\tau} \|u(s)\|_H ds$$

for $u, v \in L_H$.

2.2 Some retarded integral inequalities

In this part, we shall present some retarded integral inequalities from Li, Liu and Ju [22], which is useful to our estimate.

The literature [22] considers the following retarded integral inequalities:

$$y(t) \leq E(t,\tau)\|y_\tau\| + \int_\tau^t K_1(t,s)\|y_s\|ds + \int_t^\infty K_2(t,s)\|y_s\|ds + \rho, \quad \forall\, t \geq \tau \geq 0, \tag{2.7}$$

where E, K_1 and K_2 are non-negative measurable functions on $\mathbb{R}^2$, $\rho \geq 0$ denotes a constant. Let X be a Banach space with spatial variable, based on the retarded Banach space above, then we use $\|\cdot\|$ denotes the norm of space $C([-h,0];X)$ for some $h \geq 0$, $y(t) \geq 0$ is a continuous function defined on $C([-h,T];X)$, $y_t(s) = y(t+s)$ for $s \in [-h,0]$.

Some Notations and hypothesis:

Let $\mathcal{L}(E,K_1,K_2,\rho) = \{y \in C([-h,T];X) | y \geq 0$ and satisfies the inequality (2.7)$\}$, and

$$\kappa(K_1,K_2) = \sup_{t\geq\tau}\Big(\int_\tau^t K_1(t,s)ds + \int_t^\infty K_2(t,s)ds\Big).$$

We assume that

$$\lim_{t\to+\infty} E(t+s,s) = 0 \tag{2.8}$$

uniformly with respect to $s \in \mathbb{R}^+$. Moreover, we suppose $\kappa(K_1,K_2) < +\infty$.

Lemma 2.1 *Denote* $\vartheta = \sup_{t\geq s\geq\tau} E(t,s)$ *and* $\kappa = \kappa(K_1,K_2)$*, then we have the following estimates:*

(1) If $\kappa < 1$*, then for any* $R,\ \varepsilon > 0$*, there exists* $\tilde{T} > 0$ *such that*

$$\|y_t\| < \mu\rho + \varepsilon, \tag{2.9}$$

for $t > \tilde{T}$ *and all bounded functions* $y \in \mathcal{L}(E,K_1,K_2,\rho)$ *with* $\|y_0\| \leq R$*, where* $\mu = \frac{1}{1-\kappa}$*.*

(2) If $\kappa < \frac{1}{1+\vartheta}$*, then there exist* $M,\ \lambda > 0$ *which are independent on* ρ *such that*

$$\|y_t\| \leq M\|y_0\|e^{-\lambda t} + \gamma\rho, \quad t \geq \tau \tag{2.10}$$

for all bounded functions $y \in \mathcal{L}(E,K_1,K_2,\rho)$*, where* $\gamma = \frac{\mu+1}{1-\kappa c}$ *and* $c = \max\{\frac{\vartheta}{1-\kappa}, 1\}$*.*

(3) If $\kappa < \frac{1}{1+\vartheta}$*, then the solution reduces to trivial for the occasion* $\kappa c < 1$*.*

Proof. See Li, Liu and Ju [22]. □

Remark 2.2 *(The special case:* $K_2 = 0$*) Denote* $(K_1,K_2) = (K_1,0)$ *and let* $\vartheta,\ \kappa,\ \mu,\ \gamma$ *be the constants defined in Lemma 2.1. Then we have the similar estimates as in Lemma 2.1.*

3 Global Well-posedness and Pullback Dynamic Systems for (1.1)

In this section, we shall state the global well-posedness and pullback dynamic systems for process, which can be founded in [6, 7, 8], [12], [13], [14], [27].

3.1 Assumptions

(H-1) The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis:
(I-1) $f(x, y)$ is measurable with respect to x for any y and $f(x, 0) = 0$;
(I-2) $f(x, y)$ is Lipschitz continuous with respect to y for any x, i.e., there exists a constant $0 < C_f < 1$ small enough such that

$$|f(x, y_1) - f(x, y_2)| \le C_f |y_2 - y_1|, \quad \text{for any } y_1,\ y_2 \in \mathbb{R}. \tag{3.1}$$

(I-3) there exists $C_f > 0$, such that for all $\tau \le t$ and for all $u, v \in C([\tau - h, T]; H)$

$$\int_\tau^t |f(s, u_s) - f(s, v_s)|_2^2 ds \le C_g^2 \int_{\tau-h}^t |u(s) - v(s)|_2^2 ds. \tag{3.2}$$

By (I-2), there exist $L_f > 0$ and $\tilde{L}_f > 0$ such that for any $y_1,\ y_2 \in \mathbb{R}$,

$$|f(x, \eta_{1t}) - f(x, \eta_{2t})|_2 \le L_f \|\eta_1 - \eta_2\|_{C_H},\ \forall\ \eta_1,\ \eta_2 \in H, \tag{3.3}$$

$$\|f(x, \eta_{1t}) - f(x, \eta_{2t})\| \le L_f \|\eta_1 - \eta_2\|_{C_V},\ \forall\ \eta_1,\ \eta_2 \in V. \tag{3.4}$$

(H-2) The function $g(t, x) \in L^2_{loc}(\mathbb{R}; H)$ and there exists a $m > 0$ such that for any $t \in \mathbb{R}$,

$$\int_{-\infty}^t e^{ms} |g(s)|_2^2 ds < +\infty. \tag{3.5}$$

(H-3) *The parameter satisfies that $2\nu\lambda_1 - (\sigma + 2C_g) > 0$, λ_1 is the first eigenvalue of operator A.*

Remark 3.1 *The hypothesis* **(H-3)** *is equivalent to that the Lipschitz constant C_g and L_f is small enough, i.e., $C_g,\ L_f << 1$.*

Proof. Since here $\sigma > 0$ is an arbitrary fixed number and **(H-3)** holds for all bounded domains with regular boundary which determined the first eigenvalue λ_1 and viscosity ν, we can see the results obviously. □

Remark 3.2 *From the proof at Section 4, we can see that the Lipschitz constant small enough is essential to the finite boundedness of fractal dimension for pullback attractors, hence we assume that*

(H-3') *All the Lipschitz constants L_f, C_g and L are small enough.*

3.2 Global well-posedness for (1.1)

By the Helmholz-Leray projection, the system can transform to the abstract equivalent form

$$\begin{cases} \frac{d}{dt}u(t) + \nu A u + B(u(t)) = f(t, u_t) + g(t, x), \\ u(t) = \phi(t - \tau) \text{ for } t \in [\tau - h, \tau]. \end{cases} \tag{3.6}$$

• Global existence and uniqueness of weak solution:

Theorem 3.3 *(1) For the initial data $\phi \in C_H$, if $g \in L^2_{loc}(\mathbb{R}; V')$, f satisfies hypothesis* **(H-1)-(H-3)**, *then problem (3.6) possesses a weak solutions $u(t, x) \in C([\tau - h, +\infty); H)$.*

(2) If $g \in L^2_{loc}(\mathbb{R}; H)$, $\phi \in C_V$, **(H-1)-(H-3)** *hold, then the weak solution becomes strong, i.e., $u \in L^2(\tau, T; D(A)) \cap C([\tau - h, +\infty); V)$.*

Proof. See, e.g., Caraballo, Real [6]. □

3.3 Pullback dynamics for (1.1)

• Continuous process:

Lemma 3.4 *For the initial data $\phi \in C_H$, if $g \in L^2_{loc}(\mathbb{R}; V')$, f satisfies hypothesis* **(H-1)-(H-3)**, *then the weak solution of* (3.6) *generates a bi-parametric family of map, i.e., continuous process $\{U(t,s)\} : U(t,s)\phi = u_t(t,\tau;\phi)$ on C_H.*

Proof. See, e.g., Caraballo, Real [6]. □

Denoting $M_H = C_H \times H$, we have the following theorem.

Lemma 3.5 *For the initial data $\phi \in C_H$ and $\varphi \in H$, if $g \in L^2_{loc}(\mathbb{R}; V')$, f satisfies hypothesis* **(H-1)-(H-3)**, *then the weak solution of* (3.6) *generates a bi-parametric family of map, i.e., continuous process $\{S(t,s)\} : S(t,s)(\phi,\varphi) = (u_t(t,\tau;\phi,\varphi), u(t,\tau;\varphi))$ on M_H.*

Proof. See, e.g., Caraballo, Real [6]. □

• Pullback attracting:

Lemma 3.6 *(The pullback absorbing ball in C_H) For the initial data $(\phi,\varphi) \in C_H \times H$, if $g \in L^2_{loc}(\mathbb{R}; V')$, f satisfies hypothesis* **(H-1)-(H-3)**, *then the process $U(t,s)$ possesses a family of pullback absorbing balls $\{\mathcal{B}(t)\}$ in C_H with the center zero and radius $\hat{\rho}_H(t)$:*

$$\begin{aligned}\|S(t,t-s)(\phi,\varphi))\|_{C_H} &= \|u_t\|^2_{C_H}\\ &\le Ce^{-m(t-h)}\int_{-\infty}^{t} e^{ms}\|g(s)\|^2_{V'}ds + \hat{d}^2 e^{mh}(1+C_g)e^{-ms}\\ &= \hat{\rho}^2_H(t).\end{aligned} \tag{3.7}$$

Proof. See, e.g., Caraballo, Real [6] or García-Luengo, Marín-Rubio and Real [12], [13]. □

Lemma 3.7 *(The pullback absorbing ball in C_V) For the initial data $(\phi,\varphi) \in C_V \times V$, if $g \in L^2_{loc}(\mathbb{R}; H)$, f satisfies hypothesis* **(H-1)-(H-3)**, *then the process $S(t,s)$ possesses a family of pullback absorbing balls $\{\hat{\mathcal{B}}(t)\}$ in C_V with the center zero and radius $\hat{\rho}_V(t)$:*

$$\|S(t,t-s)(\phi,\varphi))\|^2_{C_V} = \max_{\theta\in[-h,0]}\|u(t+\theta,t-s,\phi,\varphi)\|^2 \le \hat{\rho}^2_V(t), \tag{3.8}$$

here

$$\hat{\rho}^2_V(t) = (a_3 + a_2)e^{a_1},$$

with $a_1 = \frac{C_1}{\nu^3}\hat{\rho}^2_H(t)\hat{I}_V$, $a_2 = \frac{\nu}{4}(\|g\|^2_{L^2_{loc}(\mathbb{R};H)} + L^2_g\hat{\rho}^2_H(t))$, and

$$\hat{I}^2_V = \frac{\lambda_1}{2\nu\lambda_1 - (\sigma + 2C_g)}\Big(\frac{\|g\|^2_{L^2_{loc}(\mathbb{R};H)}}{\sigma} + (1+hC_g)\hat{\rho}^2_H(t)\Big) = a_3.$$

Proof. See, e.g., Caraballo, Real [6] or García-Luengo, Marín-Rubio and Real [12], [13]. □

• Pullback attractors:

Theorem 3.8 *Assume that the initial data $(\phi,\varphi) \in C_H \times H$, if $g \in L^2_{loc}(\mathbb{R}; V')$, f satisfies hypothesis* **(H-1)-(H-3)**. *Then the process $\{S(\cdot,\cdot)\}$ generated by the weak solutions of problem possesses a bounded family of pullback attractors $\mathcal{A}_{M_H}(t)$ in M_H. Defining the projector $j : M_H \to C_H$, then $j(\mathcal{A}_{M_H}(t)) = \mathcal{A}_{C_H}(t)$ is the pullback attractor in C_H for $U(\cdot,\cdot)$.*

Proof. See, e.g., Caraballo and Real [6]. □

4 The finite fractal dimension of pullback attractors for (1.1) with delay

4.1 Fractal dimension of pullback attractors

The pullback attractors and its regularity for 2D incompressible Navier-Stokes equations with delays has been studied by Caraballo, Real [6, 7, 8], García-Luengo, Marín-Rubio and Real [12], [13], García-Luengo, Marín-Rubio and Planas [14], Marín-Rubio and Real [27], especially in [12, 13, 14, 27]. However, the fractal and Hausdorff dimension of attractors are unknown, which are our objective in this section.

• The global well-posedness of first variation equation
In this part, we will investigate the well-posedness of first variation equation corresponding to problem (1.1) as following

$$\begin{cases} \frac{d}{dt}U + \nu AU + B(u,U) + B(U,u) = f'(t,u_t)U_t, \\ U(\theta+h) = \xi(\theta,x) \ \text{ with } \ \theta \in [\tau-h,\tau] \ \text{ and } \ \text{ hence } \ U(\tau) = \xi(\tau,x) = \xi_0. \end{cases} \tag{4.1}$$

Setting $\mathbb{A}(t) = f'(t,u_t) : V \to H$ for every $t \in [\tau-h,+\infty)$ ($f(t,u_t)$ is Fréchet differentiable denote as $\mathbb{A}$) which is a bounded linear operator, i.e., $\mathbb{A}(t) \in \mathcal{L}(V,H)$, using the usual energy estimates, then problem (4.1) possesses a unique solution

$$U(t) \in L^\infty([\tau,T];H) \bigcap L^2([\tau,T];V), \ U_t \in ([\tau-h,T];H).$$

Thanks to $u(t,x) \in C([\tau-h,+\infty);H)$, it yields that

$$U(t) \in C([\tau,+\infty);H), \ U_t \in C([\tau-h,+\infty);H).$$

• The quasi-differentiability and compactness of linear operator $S(t,\tau)$
Defining the space $X_H = C([\tau-h,T];H) \times C([\tau,T];H)$, $\mathcal{M}_H = L^2(\tau-h,\tau;H) \times H$ and $M_H = C_H \times H$, then the quasi-differentiability of linear operator $\Lambda : M_H \to X_H$ can be shown as following theorem.

Theorem 4.1 *If the hypotheses* **(H)** *and* $\nu\lambda_1 - \frac{2L}{\nu\lambda_1} - \frac{8C_3}{\nu^2\lambda_1}\hat{\rho}_H^2 > 0$ *in below hold, and* $\xi \in C_H, \xi_0 \in D(A^{1/4})$*, then there exists a bounded linear operator* $\Lambda(t,s;\phi,\psi) : M_H \to X_H$ *such that*

(1) The function $\Lambda(t,s;\phi,\psi)(\xi,\xi_0) = (U_t,U)$ *is the first variation equation* (4.1)*;*

(2) Let $\mathcal{A}_{M_H}(t)$ *be the* $\mathcal{D}$*-pullback attractors of evolutionary process* $S(t,s)$ *to problem (3.6) in* M_H*, then the operator* $S(t,s)$ *is uniformly quasi-differentiable on* $\mathcal{A}_{M_H}(t)$*, i.e., for any initial data* (ϕ,ψ) *and* $(\phi_0,\psi_0) \in \mathcal{A}_{M_H}(t)$*, the linear operator* $\Lambda(t,s;\phi,\psi)$ *satisfies that for all* $t \geq s$*, the following convergence*

$$\sup_{\substack{(\phi,\psi) \\ (\phi_0,\psi_0)}\in\mathcal{A}} \ \sup_{\substack{\|\phi-\phi_0\|_{C_H}\leq\varepsilon \\ |\psi-\psi_0|_2\leq\varepsilon}} \frac{\|S(t,s)(\phi_0,\psi_0) - S(t,s)(\phi,\psi) - \Lambda(t,s;\phi,\psi)(\phi_0-\phi,\psi_0-\psi)\|_{M_H}}{\|(\phi_0-\phi,\psi_0-\psi)\|_{M_H}} \to 0$$

holds as ε *tends to* 0*;*

(3) The operator $\Lambda(t,s;\phi,\psi)$ *is compact for all* $t > s$*,* $s \in [\tau-h,+\infty)]$*.*

Proof. (1) By the well-posedness of first variational equation, it is easy to derive the result.

(2) Let $u(t)$ and $v(t)$ be two solutions to problem (3.6) with initial data

$$\begin{aligned}&u(\theta+h)=\phi(\theta,x),\ \ \theta\in[s-h,s],\\&u(s)=\psi(x)\end{aligned}$$

and

$$\begin{aligned}&v(\theta+h)=\phi_0(\theta,x),\ \ \theta\in[s-h,s],\\&v(s)=\psi_0(x)\end{aligned}$$

respectively, then $U(t,x)=\Lambda(t,s;\phi,\psi)(\phi_1-\phi,\psi_1-\psi)$ is the solution to problem (4.1) with initial data

$$(U(\theta+h,x),U(s,x))=(\phi_1-\phi,\psi_1-\psi),\quad \theta\in[s-h,s].$$

Denoting $w=v-u-U$ with initial data

$$w(\theta+h,x)=w(\theta)=0,\ \ \theta\in[s-h,s],$$

then w satisfies

$$\frac{d}{dt}w+\nu Aw+B(u,w)+B(w,u)+B(v-u,v-u)=f(t,v_t)-f(t,u_t)-f'(t,u_t)U_t. \quad (4.2)$$

By the Talor expansion, it follows that

$$\begin{aligned}f(t,v_t)-f(t,u_t)-f'(t,u_t)U_t &= f(t,v_t)-f(t,u_t)-f'(t,u_t)(v_t-u_t)+f'(t,u_t)w_t\\ &= f''(*)(v_t-u_t)^2+f'(t,u_t)w_t,\end{aligned}$$

where $*=\lambda v_t+(1-\lambda)u_t$ with $\lambda\in[0,1]$.

Multiplying (4.2) by w, denoting $z=v-u$, we derive that

$$\frac{1}{2}\frac{d}{dt}|w|_2^2+\nu\|w\|^2+b(w,u,w)+b(z,z,w)=(f''(*)(v_t-u_t)^2+f'(t,u_t)w_t,w),$$

which means

$$\frac{1}{2}\frac{d}{dt}|w|_2^2+\nu\|w\|^2 \quad\leq\quad b(w,u,w)+|b(z,z,w)|+L|z_t|_2^2|w|_2+L|w_t|_2|w|_2. \quad (4.3)$$

By the property of trilinear operator, we obtain that

$$|b(w,u,w)|=|b(w,w,u)| \quad\leq\quad C|w|_2^{1/2}\ \|w\|^{3/2}\ |u|_2^{1/2}\|u\|^{1/2}\leq\frac{\nu}{2}\|w\|^2+C|u|_2^2|w|_2^2 \quad (4.4)$$

and

$$|b(z,z,w)|=|b(z,w,z)| \quad\leq\quad C|z|_2\ \|w\|\ \|z\|\leq\frac{\nu}{2}\|w\|^2+C|z|_2^2\|z\|^2. \quad (4.5)$$

Thus it follows from (4.3)-(4.5) that

$$\frac{d}{dt}|w|_2^2\leq C_1\Big(|z|_2^4+\|z\|_{C_H}^4\Big)+C_2\Big(|w|_2^2+|w_t|_2^2\Big) \quad (4.6)$$

with $C_1=C_1(L,\nu,\lambda_1)$ and $C_2=C_2(|u|_2^2,L,\nu,\lambda_1)$.

Considering the resulting equation of difference for two solutions u and v, i.e., $z = u - v$ satisfies

$$z_t + \nu Az + B(u, z) + B(z, v) = f(t, u_t) - f(t, v_t), \tag{4.7}$$

multiplying (4.7) by z, it yields

$$\frac{1}{2}\frac{d}{dt}|z|_2^2 + \nu\|z\|^2 \le |b(z, v, z)| + |(f(t, u_t) - f(t, v_t), z)|, \tag{4.8}$$

and

$$|b(z, v, z)| = |b(z, z, v)| \le C|z|_2\|z\|\|v\| \le \frac{\nu}{2}\|z\|^2 + C\|v\|^2|z|_2^2 \tag{4.9}$$

and

$$|(f(t, u_t) - f(t, v_t), z)| \le \frac{\nu}{2}\|z\|^2 + L_f|u_t - v_t|_2^2, \tag{4.10}$$

hence

$$\begin{aligned} |z|_2^2 &\le |\psi - \psi_1|_2^2 + C|v|_2^2\int_\tau^t |z(s)|_2^2 ds + 2L_f\int_\tau^t |u_s - v_s|_2^2 ds \\ &\le \left(|\psi - \psi_1|_2^2 + 2hL_f|\phi - \phi_1|_{C_H}^2\right) + \int_\tau^t (C\|v\|^2 + 2L_f)|z(s)|_2^2 ds, \end{aligned} \tag{4.11}$$

which implies

$$|z|_2^2 \le \left(|\psi - \psi_1|_2^2 + 2hL_f\|\phi - \phi_1\|_{C_H}^2\right)e^{(C\int_\tau^T \|v\|^2 ds + 2L_f(T-\tau))} \tag{4.12}$$

and

$$\|z\|_{C_H}^2 \le \left(|\psi - \psi_1|_2^2 + 2hL_f\|\phi - \phi_1\|_{C_H}^2\right)e^{(C\int_{\tau-h}^t \|v\|^2 ds + 2L_f(T-\tau+h))}. \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.6), it yields

$$\frac{d}{dt}|w|_2^2 \le C\|(\phi - \phi_1, \psi - \psi_1)\|_{M_H}^4 e^{KT} + C(|w|_2^2 + |w_t|_2^2).$$

Noting the initial data of w, we know that

$$|w(t)|_2^2 \le C\|(\phi - \phi_1, \psi - \psi_1)\|_{M_H}^4 Te^{KT} + C\int_\tau^t (|w(s)|_2^2 + |w_s|_2^2)ds$$

and

$$\sup_{s\in[t-h,t]} |w(s)|_2^2 \le C\|(\phi - \phi_1, \psi - \psi_1)\|_{M_H}^4 Te^{KT} + C\int_\tau^t \sup_{r\in[s-h,s]} |w(r)|_2^2 dr$$

with $K = C|v|^2 + 2L_f$ is bounded.
Thus by the Gronwall inequality, we obtain that

$$\sup_{s\in[t-h,t]} |w(s)|_2^2 \le C\|(\phi - \phi_1, \psi - \psi_1)\|_{M_H}^4 Te^{KT}(1 + e^{KT}),$$

which means

$$|w(t)|_2^2 \le C(T)\|(\phi - \phi_1, \psi - \psi_1)\|_{M_H}^4. \tag{4.14}$$

From the expression of w, we conclude that

$$\begin{aligned}
&\frac{\|S(t,s)(\phi_0,\psi_0)-S(t,s)(\phi,\psi)-\Lambda(t,s;\phi,\psi)(\phi_0-\phi,\psi_0-\psi)\|_{M_H}}{\|(\phi_0-\phi,\psi_0-\psi)\|_{M_H}}\\
&\quad\le C(T)\|(\phi_0-\phi,\psi_0-\psi)\|^3_{X_V},
\end{aligned}$$

which implies that the uniformly differentiability of process $S(t,s)$ with respect to initial data on $\mathcal{A}_{M_H}(t)$.

(3) Multiplying (4.1) with U, we deduce that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|U|_2^2+\nu\|U\|^2 &\le |b(U,u,U)|+|(f'(t,u_t)U_t,U)|\\
&\le |b(U,u,U)|+\frac{L}{\nu\lambda_1}|U_t|_2^2+\frac{\nu}{4}\|U\|^2\\
&\le C_3|U|_2\|U\|\|u\|+\frac{L}{\nu\lambda_1}|U_t|_2^2+\frac{\nu}{4}\|U\|^2\\
&\le \frac{\nu}{4}\|U\|^2+\frac{4C_3}{\nu^2\lambda_1}\|u\|^2|U|_2^2+\frac{L}{\nu\lambda_1}|U_t|_2^2+\frac{\nu}{4}\|U\|^2, \qquad (4.15)
\end{aligned}$$

which implies

$$\frac{d}{dt}|U|_2^2+\nu\lambda_1|U|_2^2\le\frac{8C_3}{\nu^2\lambda_1}\|u\|^2|U|_2^2+\frac{2L}{\nu\lambda_1}|U_t|_2^2. \qquad (4.16)$$

Integrating (4.16) with time variable from τ to t, by the existence of pullback absorbing ball of u in H with radius $\hat{\rho}_H$, it yields

$$\begin{aligned}
&|U(t)|_2^2+\nu\lambda_1\int_\tau^t|U(r)|_2^2dr\\
\le\ &\frac{8C_3}{\nu^2\lambda_1}\hat{\rho}_H^2\int_\tau^t|U(s)|_2^2ds+|\xi_0|_2^2+\frac{2L}{\nu\lambda_1}\int_\tau^t|U_r|_2^2dr\\
\le\ &\Big(\frac{8C_3}{\nu^2\lambda_1}\hat{\rho}_H^2+\frac{2L}{\nu\lambda_1}\Big)\int_\tau^t|U(r)|_2^2dr+|\xi_0|_2^2+\frac{2L}{\nu\lambda_1}\|\phi\|_{C_H}^2. \qquad (4.17)
\end{aligned}$$

Based on the hypothesis as following

$$\nu\lambda_1-\frac{2L}{\nu\lambda_1}-\frac{8C_3}{\nu^2\lambda_1}\hat{\rho}_H^2>0, \qquad (4.18)$$

by the uniform Gronwall inequality, we derive that

$$|U(t)|_2^2\le e^{-(\nu\lambda_1-\frac{2L}{\nu\lambda_1}-\frac{8C_3}{\nu^2\lambda_1}\hat{\rho}_H^2)(t-\tau)}\Big[|\xi_0|_2^2+\frac{2L}{\nu\lambda_1}\|\phi\|_{C_H}^2\Big]=\tilde{\rho}_H, \qquad (4.19)$$

here $\tilde{\rho}_H$ is bounded.

Taking inner product of (4.1) by $A^{1/2}U$, noting the existence of pullback absorbing ball of u, we obtain

$$\begin{aligned}
&\frac{d}{dt}|A^{1/4}U|_2^2+2\nu|A^{3/4}U|_2^2\\
\le\ &2|b(u,U,A^{1/2}U)|+2|b(U,u,A^{1/2}U)|+2|(f'(t,u_t)U_t,A^{1/2}U)|\\
=\ &I_1+I_2+I_3.
\end{aligned}$$

Using the estimate $\|w\|_{L^4} \le C\|w\|_{D(A^{1/4})}$ and $|A^{1/2}w|_2 \le C|A^{1/4}u|_2^{1/2}|A^{3/4}w|_2^{1/2}$, it yields

$$\begin{aligned} I_1 &\le C|u|_2^{1/2}\|u\|^{1/2}|A^{1/2}U|_2\|A^{1/2}U\|_{L^4} \\ &\le C|u|_2^{1/2}\|u\|^{1/2}|A^{1/4}U|_2^{1/2}|A^{3/4}U|_2^{3/2} \\ &\le \frac{\nu}{3}|A^{3/4}U|_2^2 + \frac{C}{\nu}|u|_2^2\|u\|^2|A^{1/4}U|_2^2 \\ &\le \frac{\nu}{3}|A^{3/4}U|_2^2 + \frac{C}{\nu}\hat{\rho}_H^2(t)\|u\|^2|A^{1/4}U|_2^2, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} I_2 &\le C\|U\|_{L^4}|A^{1/2}u|_2\|A^{1/2}U|_{L^4} \\ &\le C|A^{1/4}U|_2|A^{1/2}u|_2|A^{3/4}U|_2 \\ &\le \frac{\nu}{3}|A^{3/4}U|_2^2 + \frac{C}{\nu}|A^{1/4}U|_2^2\|u\|^2, \end{aligned} \tag{4.21}$$

and

$$I_3 \le \frac{C}{\nu\lambda_1^{1/2}}|f'(t,u_t)U_t|_2^2 + \frac{\nu}{3}|A^{3/4}U|_2^2 \le \frac{CL}{\nu\lambda_1}|U_t|_2^2 + \frac{\nu}{3}|A^{3/4}U|_2^2, \tag{4.22}$$

which means

$$\frac{d}{dt}|A^{1/4}U|_2^2 + \nu|A^{3/4}U|_2^2 \le \frac{CL}{\nu\lambda_1}|U_t|_2^2 + \frac{C}{\nu}\hat{\rho}_H^2(t)\|u\|^2|A^{1/4}U|_2^2 + \frac{C}{\nu}|A^{1/4}U|_2^2\|u\|^2. \tag{4.23}$$

Integrating (4.23) with time variable from τ to t, it yields

$$\begin{aligned} &|A^{1/4}U|_2^2 + \nu\lambda_1^{1/2}\int_\tau^t |A^{1/4}U(r)|_2^2 dr \\ \le\ & \frac{CL}{\nu\lambda_1}\int_\tau^t |U_t(r)|_2^2 dr + |A^{1/4}\xi_0|_2^2 + \frac{C}{\nu}\int_\tau^t (\hat{\rho}_H^2 + 1)\|u\|^2|A^{1/4}U|_2^2 dr \\ \le\ & \frac{CL}{\nu\lambda_1}\|\xi\|_{C_H}^2 + |A^{1/4}\xi_0|_2^2 + \frac{C}{\nu}\int_\tau^t (\hat{\rho}_H^2 + 1 + \frac{L}{\lambda_1})\|u\|^2|A^{1/4}U|_2^2 dr. \end{aligned} \tag{4.24}$$

If we give a more hypotheses:

(H) there exists a $\delta > 0$ which will be determined later, such that we can denote

$$\kappa_\delta(t,s) = \left(\nu\lambda_1^{1/2} - \delta\right)(t-s) - \frac{C(\hat{\rho}_H^2 + 1 + \frac{L}{\lambda_1})}{\nu}\int_s^t \|u(r)\|^2 dr \tag{4.25}$$

which satisfies

$$\kappa_\delta(0,t) - \kappa_\delta(0,s) = -\kappa_\delta(t,s). \tag{4.26}$$

Denoting

$$\limsup_{\tau\to-\infty} \frac{1}{t-\tau}\int_\tau^t \|u(r)\|^2 dr = a_0 \in [0,+\infty) \tag{4.27}$$

for the investigation of uniform boundedness for operator Λ, there exists some $\delta > 0$, such that

$$\frac{C(\hat{\rho}_H^2 + 1 + \frac{L}{\lambda_1})a_0}{\nu} + \delta < \nu\lambda_1^{1/2}. \tag{4.28}$$

Then under the assumption (H), by the uniform Gronwall inequality, we can conclude that

$$|A^{1/4}U|_2^2 \le \Big[\frac{CL}{\nu\lambda_1}\|\xi\|_{C_H}^2 + |A^{1/4}\xi_0|_2^2\Big]e^{-\kappa_\delta(t,\tau)} = \tilde{\rho}_V^2, \tag{4.29}$$

here $\tilde{\rho}_V$ is bounded since $\nu\lambda_1 > 0$.

Using the similar uniform estimates as above, we can also derive that

$$\|A^{1/4}U\|_{C_H}^2 \le \tilde{\rho}_{C_H}^2, \tag{4.30}$$

here we omit the detail.

Since the embedding $D(A^{1/4}) \hookrightarrow\hookrightarrow H$ is compact, then the compactness of operator $\Lambda(t,s;\phi,\psi)$ holds. $\square$

Remark 4.2 *By the retard Gronwall inequality, we can derive a more delicate estimate than* (4.19) *from* (4.17)*, which is no need so strict restriction as in* (4.18)*, even the variable index in* (4.25)*.*

• The finite fractal dimension of pullback attractors by trace formula

In this subsection, since the Hausdorff dimension of attractor is not large than fractal dimension if they are finite, we shall use trace formula to estimate the fractal dimension only here.

Theorem 4.3 *Under the assumptions in Theorem 4.1 and (H-3'), the fractal dimensions of $\mathcal{A}_{M_H}(t)$ to problem (1.1) has finite dimension as*

(1) If $\frac{\pi n^2(\nu\lambda_1-1)}{2\lambda_1|\Omega|} > \frac{64\kappa(n)}{\nu^3\lambda_1}M$, then $dim_F(\mathcal{A}_{M_H}(t)) \le 2$.

(2) If $\frac{\pi n^2(\nu\lambda_1-1)}{2\lambda_1|\Omega|} \le \frac{64\kappa(n)}{\nu^3\lambda_1}M$, then $dim_F(\mathcal{A}_{M_H}) \le \frac{C\kappa(n)|\Omega|^{\frac{1}{2}}}{\nu^2}G + \hat{C}$, here $G = \frac{\|f\|_{L^\infty(-\infty,T^*;H)}}{\nu^2\lambda_1}$ is the generalized Grashof number for non-autonomous system.

Proof. • Step 1: The precompactness of the union of fiber for pullback attractors

From the existence of pullback attractors, for a fixed τ^*, $\bigcup \mathcal{A}_{M_H}(t)$ is precompact in M_H.

• Step 2: Extension of the first variation equation to abstract form with delay

Denoting $F(S(t,s)(\phi,\psi),t) = -\nu A - B(u,\cdot) - B(\cdot,u) + f(t,u_t)$, then $F(\cdot,t)$ is Gateaux differential in V at $(S(t,\tau)(\phi,\psi))$ which satisfies

$$F'(S(t,s)(\phi,\psi))U = -\nu AU - B(U(t,\tau)\psi, U) - B(U, U(t,\tau)\psi) + f(t,u_t)U_t \tag{4.31}$$

and $F'(S(t,s)(\phi,\psi),t) \in \mathcal{L}(V,V')$ is a continuous linear operator. Hence, the problem

$$\begin{cases} \frac{dU}{dt} = F'(S(t,\tau)(\phi,\psi),t)U, \quad (\phi,\psi) \in M_H, \\ (U(\theta+h,x), U(\tau,x)) = (\xi,\xi_0), \ \theta \in [\tau-h,\tau] \end{cases} \tag{4.32}$$

possesses a unique solution $(U_t, U(t)) \in X_H$.

Defining $\mathfrak{L} = -\nu A - B(u,\cdot) - B(\cdot,u)$, we can rewrite (4.1) as

$$\begin{aligned} \frac{d}{dt}U &= -\nu AU - B(u,U) - B(U,u) + f'(t,u_t)U_t \\ &= F'(S(t,\tau)(\phi,\psi),t)U \\ &= \mathfrak{L}U + f'(t,u_t)U_t. \end{aligned} \tag{4.33}$$

Denoting $(\xi, \xi_0) = (U_t, U(t)) = W(t)$, then the projections P_1 and P_2 can be defined as

$$P_1 W(t) = U_t, \quad P_2 W(t) = U(t).$$

Based on the well-posedness of (4.1), we define the operator $\mathbb{L} : D(\mathbb{L}) \subset X_H \to X_H$ as

$$\begin{aligned} D(\mathbb{L}) = \Big\{ &(\alpha, \beta) \in X_H | \alpha \textit{ is absolutely continuous in } [-h, 0], \\ &\frac{d\alpha}{dt} \in C([-h, 0]; H) \textit{ and } \beta = \alpha(0) \in D(\mathfrak{L}) \Big\}, \end{aligned} \tag{4.34}$$

hence

$$\mathbb{L}(\alpha, \beta) = (\frac{d\alpha}{dt}, \mathfrak{L}\beta) = (\dot{\alpha}, \mathfrak{L}\beta) \quad \textit{for} \quad (\alpha, \beta) \in D(\mathbb{L}) \tag{4.35}$$

with the domain $D(\mathbb{L})$ is dense in X_H.
From the definitions above, we can reformulate the first variation equation on X_H as

$$\begin{cases} \frac{dW}{dt} = \Xi W = \mathbb{L} W + (0, f'(t, u_t) P_1 W), \\ W(0) \in M_H, \end{cases} \tag{4.36}$$

thus, the first variation equation has been extends to delay form naturally which has the similar form as a ordinary differential equation.

• Step 3: The trace formula

For each $(\xi_1(t), \xi_{01}(t)),\ (\xi_2(t), \xi_{02}(t)),\ \cdots, (\xi_n(t), \xi_{0n}(t)) \in \mathcal{M}_H$, let $(U_{it}(t), U_i(t)) = \Lambda(t, s; \phi, \psi) \cdot (\xi_i, \xi_{0i})$ with $(\xi_i, \xi_{0i}) \in M_H$, and

$$\begin{aligned} &U_{1s}(s) = U_1(s+h, \tau; \xi_1),\ U_{2s}(s) = U_2(s+h, \tau; \xi_2),\ \cdots,\ U_{ns}(s) = U_n(s+h, \tau; \xi_n), \\ &U_1(s) = U_1(s, \tau; \xi_{01}),\ U_2(s) = U_2(s, \tau; \xi_{02}),\ \cdots,\ U_n(s) = U_n(s, \tau; \xi_{0n}), \end{aligned}$$

be the solution of problem (4.32) with initial data $(U_i(\theta+h), U_i(\tau)) = (\xi_i(\theta), \xi_{0i}))(i = 1, 2, \cdots, n)$ respectively, $Q_n(s)$ denotes the projection from $\mathcal{M}_H$ to the space

$$\text{span}\{(U_{1s}(s), U_1(s)),\ (U_{2s}(s), U_2(s)),\ \cdots,\ (U_{ns}(s), U_n(s))\}.$$

In addition, we denote

$$\hat{U}_1(s) = (U_{1s}(s), U_1(s)),\ \hat{U}_2(s) = (U_{2s}(s), U_2(s)), \cdots, \hat{U}_n(s) = (U_{ns}(s), U_n(s)) \tag{4.37}$$

and

$$\hat{\xi}_1 = (\xi_1, \xi_{01}),\ \hat{\xi}_2 = (\xi_2, \xi_{02}), \cdots, \hat{\xi}_n = (\xi_n, \xi_{0n}), \tag{4.38}$$

then by Lemma 4.19 in [9], it yields

$$\begin{aligned} &\|\hat{U}_1(t) \wedge \hat{U}_2(t) \wedge \cdots \wedge \hat{U}_n(t)\|_{\bigwedge^n(\mathcal{M}_H)} \\ = \ &\|\hat{\xi}_1 \wedge \hat{\xi}_2 \wedge \cdots \wedge \hat{\xi}_n\|_{\bigwedge^n(\mathcal{M}_H)} \exp\Big(\int_s^t Tr_n(F'(S(r, s)(\phi, \psi), r) \circ Q_n(r) dr) \Big) \\ = \ &\|\hat{\xi}_1 \wedge \hat{\xi}_2 \wedge \cdots \wedge \hat{\xi}_n\|_{\bigwedge^n(\mathcal{M}_H)} \exp\Big(\int_s^t Tr_n(\Xi(r) \circ Q_n(r) dr) \Big), \end{aligned} \tag{4.39}$$

here Tr denotes the trace.

• Step 4: The estimate of finite upper booundedness of pullback attractors
Since the product space $M_H \subset \mathcal{M}_H$, we set $\{\hat{\xi}_1(s) = (\xi_{1s}, \xi_1),\ \hat{\xi}_2(s) = (\xi_{2s}, \xi_2),\ \cdots,\ \hat{\xi}_n(s) = (\xi_{ns}, \xi_n)\}$ as an orthonormal basis for

$$\mathrm{span}\{(U_{1s}(s), U_1(s)),\ (U_{2s}(s), U_2(s)),\ \cdots,\ (U_{ns}(s), U_n(s))\},$$

then it follows that

$$\begin{aligned}
& Tr_n(\Xi(r) \circ Q_n(r)) \\
= & \sup_{\hat{\xi}_i \in \mathcal{M}_H, |\hat{\xi}_i| \le 1, i \le n} \Big(\sum_{i=1}^{n} \langle F'(S(r,s)(\phi,\psi), r)\hat{\xi}_i, \hat{\xi}_i \rangle \Big) \\
= & \sup_{\hat{\xi}_i \in \mathcal{M}_H, |\hat{\xi}_i| \le 1, i \le n} \sum_{i=1}^{n} \langle (\dot{\xi}_{ir}(r), \mathfrak{L}\xi_i(r)) + (0, f'(t, u_r)\xi_{ir}), (\xi_{ir}, \xi_i(r)) \rangle \\
= & \sum_{i=1}^{n} \Big[\int_{-h}^{0} \Big(\frac{d}{dr}\xi_i(r+\theta), \xi_i(r+\theta) \Big) dr + (\mathfrak{L}\xi_i(r), \xi_i(r)) + (f'(t, u_r)\xi_{ir}, \xi_i(r)) \Big] \\
= & \sum_{i=1}^{n} \Big\{ \frac{1}{2} \Big[|\xi_i(\theta)|_2^2 - |\xi_i(\theta - h)|_2^2 \Big] + (-\nu A\xi_i(r) - B(u, \xi_i(r)) - B(\xi_i(r), u), \xi_i(r)) \\
& + (f'(t, u_r)\xi_{ir}, \xi_i(r)) \Big\} \\
= & I_1 + I_2 + I_3.
\end{aligned} \tag{4.40}$$

Since $U_i(s) \in L^2(\tau, T; V)$, then $U_i(s) \in V$ for a.e. $s \ge \tau$, hence $\xi_i(s) \in V$ for a.e. $s \ge \tau$ and $i = 1, 2, \cdots, n$. Noting that $b(u, \xi_i(s), \xi_i(s)) = 0$, by the property of trilinear operator and the Lieb-Thirring inequality

$$\| \sum_{i=1}^{n} |\varphi_i(\tau)|^2 \|_{L^2}^2 \le \kappa(n) \sum_{i=1}^{n} \|D\varphi_i(\tau)\|_{L^2}^2,$$

it follows from Hölder's inequality and that

$$\begin{aligned}
\sum_{i=1}^{n} |b(\xi_i(r), u, \xi_i(r))| &\le C \int_{\Omega} |\nabla u| \sum_{i=1}^{n} |\xi_i(r)|^2 dr \\
&\le \frac{8}{\nu\kappa(n)} \|u\|^2 + \frac{\nu}{4} \sum_{i=1}^{n} \|D\xi_i(r)\|_{L^2}^2
\end{aligned} \tag{4.41}$$

and

$$I_2 \le -\nu \sum_{i=1}^{n} \|\xi_i(r)\|^2 + \frac{\nu}{4} \sum_{i=1}^{n} \|\xi_i(r)\|^2 + \frac{8}{\nu\kappa(n)} \|u\|^2. \tag{4.42}$$

Using the hypothesis, we have

$$\begin{aligned}
I_3 &= \sum_{i=1}^{n} (f'(u_t)\xi_{ir}, \xi_i(r)) \\
&\le \sum_{i=1}^{n} L \int_{\Omega} |\xi_{ir}| |\xi_i(r)| dr \\
&\le \frac{\nu}{4} \sum_{i=1}^{n} \|\xi_i(r)\|^2 + \frac{8L}{\nu\lambda_1} \sum_{i=1}^{n} |\xi_{ir}|_2^2.
\end{aligned} \tag{4.43}$$

Combining (4.40)-(4.43), by the hypothesis that $0 < L << 1$ such that $\frac{8L}{\nu\lambda_1} < \frac{\nu}{2}$, we can derive that

$$\begin{aligned}
& Tr_n(\Xi(r) \circ Q_n(r)) \\
= & \sum_{i=1}^{n} \frac{1}{2}\Big[|\xi_i(\theta)|_2^2 - |\xi_i(\theta - h)|_2^2\Big] - \frac{\nu}{2}\sum_{i=1}^{n} \|\xi_i(r)\|^2 + \frac{4\kappa(n)}{\nu}\|u\|^2 + \frac{8L}{\nu\lambda_1}\sum_{i=1}^{n}|\xi_{ir}|_2^2 \\
= & \sum_{i=1}^{n} \frac{1}{2}\Big[|\xi_i(\theta)|_2^2 - |\xi_i(\theta - h)|_2^2\Big] - \frac{\nu}{2}\sum_{i=1}^{n} \|\xi_i(r)\|^2 + \frac{4\kappa(n)}{\nu}\|U(s,\tau)u_0\|^2 + \frac{8L}{\nu\lambda_1}\sum_{i=1}^{n}|\xi_{ir}|_2^2 \\
\leq & \sum_{i=1}^{n} \frac{1}{2\lambda_1}\|\xi_i(r)\|^2 - \frac{\nu}{2}\sum_{i=1}^{n}\|\xi_i(r)\|^2 + \frac{4\kappa(n)}{\nu}\|U(s,\tau)u_0\|^2. \qquad (4.44)
\end{aligned}$$

Using the variational principle and $\displaystyle\sum_{i=1}^{n}\lambda_i \geq \frac{\pi n^2}{|\Omega|}$ from [17], choosing $-\frac{\nu}{2} + \frac{1}{2\lambda_1} < 0$, we obtain

$$\begin{aligned}
Tr_n(F'(U(s,\tau)v_0, s) \circ Q_n(s) \quad \leq & \quad -\frac{\nu\lambda_1 - 1}{2\lambda_1}\sum_{i=1}^{n}\|\xi_i(s)\|^2 + \frac{4\kappa(n)}{\nu}\|U(s,\tau)u_0\|^2 \\
\leq & \quad -\frac{\nu\lambda_1 - 1}{2\lambda_1}\sum_{i=1}^{n}\lambda_i + \frac{4\kappa(n)}{\nu}\|U(s,\tau)u_0\|^2 \\
\leq & \quad -\frac{\pi n^2(\nu\lambda_1 - 1)}{2\lambda_1|\Omega|} + \frac{4\kappa(n)}{\nu}\|U(s,\tau)u_0\|^2. \qquad (4.45)
\end{aligned}$$

Defining the averaging term

$$q_n = \sup_{t\in\mathbb{R}}\sup_{u_0\in\mathcal{A}(t)}\Big(\frac{1}{T}\int_{t-T}^{t} Tr_n(F'(U(s,\tau)u_0, s)\Big)ds, \qquad (4.46)$$

$$\hat{q}_n = \limsup_{T\to+\infty} q_n, \qquad (4.47)$$

we derive

$$q_n \leq -\frac{\pi n^2(\nu\lambda_1 - 1)}{2\lambda_1|\Omega|} + \frac{C}{\nu}\sup_{t\in\mathbb{R}}\sup_{u_0\in\mathcal{A}(t)}\Big(\frac{1}{T}\int_{t-T}^{t}\|U(s,\tau)u_0\|\Big)ds \qquad (4.48)$$

and

$$\hat{q}_n \leq -\frac{\pi n^2(\nu\lambda_1 - 1)}{2\lambda_1|\Omega|} + \frac{4\kappa(n)}{\nu}q, \qquad (4.49)$$

where $q = \limsup_{T\to+\infty}\sup_{t\in\mathbb{R}}\sup_{u_0\in\mathcal{A}(t)}\dfrac{1}{T}\displaystyle\int_{t-T}^{t}\|U(s,\tau)u_0\|^2 ds$.

Multiplying the equation (3.6) by u, using the hypotheses of delay term $f(\cdot,\cdot)$, using the estimate of trilinear operator, we derive

$$|u|_2^2 \leq \Big[|u_0|_2^2 + \frac{16C_g^2}{\nu\lambda_1}\|\phi\|_{C_H}^2\Big]e^{-(\nu\lambda_1 - \frac{16C_g^2}{\nu\lambda_1})(t-\tau)} + \frac{16e^{-(\nu\lambda_1 - \frac{16C_g^2}{\nu\lambda_1})(t-\tau)}}{\nu\lambda_1}\int_{\tau}^{t}|g(s)|_2^2 ds \qquad (4.50)$$

and

$$
\begin{aligned}
\nu \int_s^t \|v(r)\|^2 dr \quad \leq \quad & |u_0|_2^2 + \frac{16C_g^2}{\nu\lambda_1}\|\phi\|_{C_H}^2 + \frac{16}{\nu\lambda_1}\int_\tau^t |g(s)|_2^2 ds \\
& + \frac{16C_g^2}{\nu\lambda_1}\Big[|u_0|_2^2 + \frac{16C_g^2}{\nu\lambda_1}\|\phi\|_{C_H}^2\Big](t-\tau)e^{-(\nu\lambda_1 - \frac{16C_g^2}{\nu\lambda_1})(t-\tau)} \\
& + \frac{256C_g^2(t-\tau)e^{-(\nu\lambda_1 - \frac{16C_g^2}{\nu\lambda_1})(t-\tau)}}{\nu^2\lambda_1^2}\int_\tau^t |g(s)|_2^2 ds.
\end{aligned} \tag{4.51}
$$

Setting $s = t - T$ in (4.51), it follows

$$
q \leq \frac{16}{\nu^2\lambda_1}\lim_{T\to+\infty}\int_{t-T}^t |g(s)|_2^2 ds. \tag{4.52}
$$

Defining $M = \lim\limits_{T\to+\infty}\frac{1}{T}\int_{t-T}^t |g(r)|_2^2 dr$, then

$$
\hat{q}_n \leq -\frac{\pi n^2(\nu\lambda_1 - 1)}{2\lambda_1|\Omega|} + \frac{64\kappa(n)}{\nu^3\lambda_1}M. \tag{4.53}
$$

Case 1: If $\frac{\pi n^2(\nu\lambda_1-1)}{2\lambda_1|\Omega|} > \frac{64\kappa(n)}{\nu^3\lambda_1}M$, then by Lemma 4.19 in [9], we have $\dim_B(\mathcal{A}(t)) \leq 2$.

Case 2: Otherwise, by the theory in Temam [33], and Carvalho, Langa and Robinson [9], we obtain that the fractal and Hausdorff dimension of pullback attractors proceed as $\dim_F(\mathcal{A}(t)) \leq \frac{C|\Omega|^{\frac{1}{2}}}{\nu}q^{\frac{1}{2}} + \hat{C}$.

Denoting $\hat{M} = \limsup\limits_{T\to+\infty}\frac{1}{T}\int_{t-T}^t |g(r)|_2^2 dr \to \|g\|_{L^\infty(-\infty,T^*;H)}^2$, then we derive

$$
\begin{aligned}
\dim_F(\mathcal{A}_{M_H}(t)) \quad & \leq \quad \frac{C\kappa(n)|\Omega|^{\frac{1}{2}}}{\nu^4\lambda_1}M + \hat{C} \\
& \leq \quad \frac{C\kappa(n)|\Omega|^{\frac{1}{2}}}{\nu^4\lambda_1}\|f\|_{L^\infty(-\infty,T^*;H)} + \hat{C} \\
& \leq \quad \frac{C\kappa(n)|\Omega|^{\frac{1}{2}}}{\nu^2}\mathrm{G} + \hat{C},
\end{aligned} \tag{4.54}
$$

here $\mathrm{G} = \frac{\|f\|_{L^\infty(-\infty,T^*;H)}}{\nu^2\lambda_1}$. This completes the proof. □

In fact, Wang, Yang and Lu [35] has presented a sufficient condition when the pullback attractor reduces to a single trajectory as following

Theorem 4.4 *Assume that* $(\nu\lambda_1 - \frac{C_f^2}{\nu})(\frac{3\nu}{2\lambda_1} - \frac{4C_f^2}{\nu\lambda_1^2}) > 0$, *the initial data* $(\phi,\varphi) \in C_H \times H$, *if* $g \in L^2_{loc}(\mathbb{R}; V')$, *$f$ satisfies hypothesis above, then the pullback attractors* $\mathcal{A}_{C_H}$ *reduces to a single trajectory if* $G(t) \leq \sqrt{\frac{(\nu\lambda_1 - \frac{C_f^2}{\nu})(\frac{3\nu}{2\lambda_1} - \frac{4C_f^2}{\nu\lambda_1^2})}{16}}$, *here* $G^2(t) = \frac{\langle\|g\|_{V'}^2\rangle|_{\leq t}}{\nu^2\lambda_1}$, $\langle h\rangle|_{\leq t} = \limsup\limits_{s\to-\infty}\frac{1}{t-s}\int_s^t h(r)dr$.

Proof. See [35] for more detail. □

Remark 4.5 *The dynamics of 2D Navier-Stokes equation with delay has been investigated by Caraballo et al, especially the case of variable delay. Our results (Theorem 4.3) is an extended research of Caraballo and his coauthors' former existence of pullback attractors, which also gives a positive revision of [28]. However, since we choose two basis for phase space, it is invalid for the variable delay* $f(t, u(t-\rho(t)))$*, which is our next objective.*

Moreover, the small enough Lipschitz constant guarantee the delay basis has no influence in trace formula, one natural question is can we find a more weak condition?

4.2 Further research

In this paper, the fractal dimension for 2D Navier-Stokes model with constant delay in finite interval has been investigated, but what about the variable and distributed cases? Moreover, the stability and robustness of pullback attractors as finite delay disappears are still open.

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