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An accelerated hybrid projection method with a self-adaptive step-size sequence for solving split common fixed point problems

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This paper attempts to focus on the split common fixed point problem for demicontractive mappings. We give an accelerated hybrid projection algorithm which combines the hybrid projection method and the inertial technique. The strong convergence theorems of this algorithm are obtained under mild conditions by a self-adaptive step-size sequence, which does not need prior knowledge of operator norms. Some numerical experiments in infinite Hilbert space are provided to illustrate the reliability and robustness of the algorithm and also to compare it with existing ones.

KEYWORDS:
self-adaptive step-size sequence, hybrid projection method, inertial technique, split common fixed point problem, demicontractive mapping

1 | INTRODUCTION

Let H be a Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$, $F(K)$ be the fixed point set of a mapping K. Recall the mapping $K : H \to H$ is said to be a strictly pseudo-contractive if there exists a constant $\eta \in [0,1)$ such that

$$\|Kz - Ky\|^2 \le \|z-y\|^2 + \eta\|z - Kz - (y - Ky)\|^2\,,\ \forall z, y \in H\,,$$

and is said to be a demicontractive if $F(K) \neq \emptyset$, there exists a constant $\eta \in (-\infty, 1)$ such that

$$\|Kz - p\|^2 \le \|z-p\|^2 + \eta\|Kz - z\|^2\,,\ \forall p \in F(K),\ z \in H\,,$$

or equivalently

$$\langle z-p, z-Kz\rangle \ge \frac{1-\eta}{2}\|z-Kz\|^2\,,\ \forall p \in F(K),\ z \in H\,.$$

Obviously, the strongly pseudo-contractive mapping with $F(K) \neq \emptyset$ is the demicontractive mapping. The opposite is contradictory. Both the demicontractive mapping and the strongly pseudo-contractive mapping were studied by many authors.[1,2,3,4] In addition, let H_1 and H_2 be Hilbert spaces, C and Q be two nonempty closed convex subsets of H_1 and H_2, respectively, $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (Censor and Elfving[5] introduced in 1994) is to find $z^* \in C$ such that $Az^* \in Q$. Further, let $K : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear mappings. The split common fixed point problem (Censor and Segal[6] introduced in 2009) is to find

$$z^* \in F(K)\ \text{ such that } Az^* \in F(S)\,. \tag{1}$$

If $K = P_C$ and $S = P_Q$, where $P_C : H_1 \to C$ and $P_Q : H_2 \to Q$ are metric projection mappings, then the split common fixed point problem is equivalent to the split feasibility problem. As we all know, z is a solution of problem (1) if and only if z is a solution of the fixed point equation $z = K(I - \mu A^{\mathrm{T}}(I-S)A)z$. Furthermore, Censor and Segal[6] proposed the following algorithm to solve problem (1) for two directed operators: $z_{n+1} = K(I - \mu A^{\mathrm{T}}(I - S)A)z_n$, where μ is a mild constant and A^{T} is the matrix transposition of A. And they proved the convergence of the sequence $\{z_n\}$ generated by this algorithm.

In the squel, the split feasibility problem and the split common fixed point problem have received widespread attention by many authors, such as, Moudafi,[7,8] Cui and Wang,[9] Eslamian,[10] Takahashi,[11] Suparatulatorn et al,[12] Vinh and Hoai,[13] Liu,[14] Zhou et al[15] and so on. It turns out that some studies only guaranteed weak convergence of the sequence. To fill this gap, some strong convergent results of the problem (1) were established by employing the Halpern algorithm and the viscosity algorithm, for instance, Boikanyo,[16] Kraikaew and Saejung,[17] and Wang et al.[18] Recently, Wang[19] proposed the new iterative algorithm to approximate the solution of the split common fixed point problem: $z_{n+1} = z_n - \mu_n[(I - K) + A^*(I - S)A)]z_n$, $\forall n \geq 1$, where $\{\mu_n\}$ is a self-adaptive step-size sequence and A^* is the adjoint operator of A. Meanwhile, the weak convergence and the strong convergence were obtained under some mild conditions.

In addition, in 2003, Nakajo and Takahashi[20] introduced the following hybrid projection method and guaranteed strong convergence of the solution of the fixed point problem.

$$\begin{cases} y_n = \beta_n z_n + (1 - \beta_n)K z_n, \\ C_n = \{u \in C : \|y_n - u\| \leqslant \|z_n - u\|\}, \\ Q_n = \{u \in C : \langle z_n - u, z_1 - z_n\rangle \geqslant 0\}, \\ z_{n+1} = P_{C_n \cap Q_n} z_1, \quad n \geq 1. \end{cases} \tag{2}$$

On the other hand, in 2001, Alvarez and Attouch[21] proposed an inertial proximal algorithm to study the inclusion problem of a maximal monotone operator:

$$z_{n+1} = J^T_{\lambda_n}(z_n + \alpha_n(z_n - z_{n-1})), \forall n \geq 1.$$

where the set-valued mapping $T : H \to 2^H$ is a maximal monotone operator and the function $J^T_{\lambda_n} = (I + \lambda_n T)^{-1}$ is the resolvent of T. The extrapolation term $\alpha_n(z_n - z_{n-1})$ takes into account an inertial effect of this algorithm. Under some mild conditions, most iterative algorithms by using this inertial effect have better convergence behavior for various problems, such as, the inclusion problem,[21,22], the variational inequality problem,[23,24] the fixed point problem,[25] and the references therein. Unfortunately, the selection of parameters related to the iterative algorithm usually causes interference in the process of approximate solution. Hence, the selection of parameters is also very meaningful in future research.

Based on the ideas of Wang,[19] Nakajo and Takahashi,[20] Alvarez and Attouch,[21] we propose a new inertial hybrid projection algorithm to approximate the solution of the problem (1) for demicontractive mappings by combining the inertial technique and the hybrid projection method. Simultaneously, we introduce a new self-adaptive step-size sequence, which does not need prior knowledge of operator norms. Under mild conditions, the corresponding strong convergence theorems are obtained in infinite Hilbert spaces. It is worth noting that such a self-adaptive step-size sequence guarantees the stability of the proposed algorithm. Finally, some numerical experiments in infinite Hilbert spaces are used to demonstrate the efficiency of our main results.

2 | PRELIMINARIES

For the convenience in the rest of this article, let C be a nonempty closed convex subset of a Hilbert space H. $\omega_w(z_n)$ denote the set of all weak cluster points of a sequence $\{z_n\}$, $\to$ and $\rightharpoonup$ represent strong convergence and weak convergence, respectively. Let P_C denote the metric projection from H onto C, i.e., $P_C z = \operatorname{argmin}_{y \in C} \|z - y\|$, $\forall z \in H$. The following property holds.

$$\langle P_C z - z, P_C z - y\rangle \leq 0 \Leftrightarrow \|y - P_C z\|^2 + \|z - P_C z\|^2 \leq \|z - y\|^2, \ \forall y \in C. \tag{3}$$

In addition, for any $z, y \in H$,

(I) $\|z + y\|^2 = \|z\|^2 + \|y\|^2 + 2\langle z, y\rangle \leq \|z\|^2 + 2\langle y, z + y\rangle$;

(II) $\|\alpha z + (1 - \alpha)y\|^2 = \alpha\|z\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|z - y\|^2$, $\forall \alpha \in \mathbb{R}$.

Definition 1. The mapping $K : H \to H$ and $F(K) \neq \emptyset$. $I - T$ is demiclosed at zero, if and only if, for any sequence $\{z_n\} \subset H$, satisfying $z_n \rightharpoonup z$ and $(I - K)z_n \to 0$, then $z \in F(K)$.

Definition 2. A Hilbert space H has Kadec-Klee property, that is, a sequence $\{z_n\} \subset H$ that satisfies $x_n \rightharpoonup z$ and $\|z_n\| \to \|z\|$, then $z_n \to z$.

Lemma 1 (Takahashi[11]). Let C be a closed convex subset of a Hilbert space H, and a mapping $K : C \to H$ be a demicontractive with $\eta \in (-\infty, 1)$. Then the fixed point set of K is closed and convex.

Lemma 2 (Zhou,[26] Marino and Xu[27]). Let C be a nonempty closed convex subset of a Hilbert space H. Let $K : C \to H$ be a strictly pseudo-contractive mapping with coefficient $\eta \in [0, 1)$. Then the fixed point set $F(K)$ is closed and convex, and $I - K$ is demiclosed at 0.

3 | MAIN RESULTS

In this section, a self-adaptive inertial hybrid projection algorithm is introduced to solve the split common fixed point problem for demecontractive mappings. For this purpose, we assume that the following conditions hold.

Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with the corresponding adjoint operator A^*. Let $K : H_1 \to H_1$ and $S : H_2 \to H_2$ be demicontractive mappings with coefficients $\eta_1 \in (-\infty, 1)$ and $\eta_2 \in (-\infty, 1)$, respectively. For any initial points $z_0, z_1 \in H_1$, the sequence $\{z_n\}$ generated by the following iterative process.

$$\begin{cases} w_n = z_n + \vartheta_n(z_n - z_{n-1}), \\ u_n = w_n - \mu_n \left[(I-K)w_n + A^*(I-S)Aw_n\right], \\ C_n = \{u \in H_1 : \|u_n - u\|^2 \le \|w_n - u\|^2 - \mu_n\theta_n\}, \\ Q_n = \{u \in H_1 : \langle z_n - z_1, z_n - u\rangle \le 0\}, \\ z_{n+1} = P_{C_n \cap Q_n} z_1, n \ge 1, \end{cases} \tag{4}$$

where

$$\theta_n = (1-\eta_1-2\mu_n)\|(I-K)w_n\|^2 + (1-\eta_2)\|(I-S)Aw_n\|^2 - 2\mu_n\|A^*(I-S)Aw_n\|^2.$$

Theorem 1. Assume that the solution set $\Omega = \{z^* : z^* \in F(K), Az^* \in F(S)\} \neq \emptyset$, $I - K$ and $I - S$ be demiclosed at 0. If the following conditions hold.

(C1) The sequence $\{\vartheta_n\}$ is bounded in $(-\infty, \infty)$;

(C2) If $(I-S)Aw_n \neq 0$, the stepsize $\mu_n = \sigma_n \min\{\frac{1-\eta_1}{2}, \frac{(1-\eta_2)\|(I-S)Aw_n\|^2}{2\|A^*(I-S)Aw_n\|^2}\}$ with $\sigma_n \in (0,1)$. Otherwise, $\mu_n = \sigma_n(1-\eta_1)/2$.

Then the iterative sequence $\{z_n\}$ generated by proposed algorithm (4) converges strongly to $\hat{z} = P_\Omega z_1 \in \Omega$.

Proof. **Step 1** Firstly, we show that $\Omega \subset C_n \cap Q_n$.

It is obvious that $C_n \cap Q_n$ is closed convex set, that is, $P_{C_n \cap Q_n} z_1$ is well defined. Put any $z \in \Omega$, i.e., $z \in F(K)$ and $Az \in F(S)$. From the definition of μ_n,

$$\theta_n = (1-\eta_1-2\mu_n)\|(I-K)w_n\|^2 + (1-\eta_2)\|(I-S)Aw_n\|^2 - 2\mu_n\|A^*(I-S)Aw_n\|^2 \ge 0\,. \tag{5}$$

By algorithm (4), we have

$$\begin{aligned} \|u_n - z\|^2 &= \|w_n - z\|^2 - 2\mu_n\langle (I-K)w_n + A^*(I-S)Aw_n, w_n - z\rangle \\ &\quad + \mu_n^2\|(I-K)w_n + A^*(I-S)Aw_n\|^2 \\ &= \|w_n - z\|^2 - 2\mu_n\langle (I-K)w_n, w_n - z\rangle - 2\mu_n\langle (I-S)Aw_n, Aw_n - Az\rangle \\ &\quad + \mu_n^2\|(I-K)w_n + A^*(I-S)Aw_n\|^2 \\ &\le \|w_n - z\|^2 - \mu_n(1-\eta_1)\|(I-K)w_n\|^2 - \mu_n(1-\eta_2)\|(I-S)Aw_n\|^2 \\ &\quad + 2\mu_n^2\|(I-K)w_n\|^2 + 2\mu_n^2\|A^*(I-S)Aw_n\|^2 \\ &= \|w_n - z\|^2 - \mu_n(1-\eta_1-2\mu_n)\|(I-K)w_n\|^2 \\ &\quad - \mu_n\left[(1-\eta_2)\|(I-S)Aw_n\|^2 - 2\mu_n\|A^*(I-S)Aw_n\|^2\right] \\ &= \|w_n - z\|^2 - \mu_n\theta_n\,,\ n \ge 1\,. \end{aligned} \tag{6}$$

This implies that $\Omega \subset C_n$.

On the other hand,

$$Q_1 = \{u \in H_1 \,:\, \langle z_1 - z_1, z_1 - u\rangle \le 0\} = H_1,$$

this means that $\Omega \subset Q_1$. Suppose $\Omega \subset Q_m$ for some $m \in \mathbb{N}$, we have $\Omega \subset C_m \cap Q_m$. Using $z_{m+1} = P_{C_m \cap Q_m} z_1$ and the projection property, we get that

$$\langle z_{m+1} - z_1, z_{m+1} - u\rangle \le 0\,,\ \forall u \in C_m \cap Q_m,$$

and

$$\langle z_{m+1} - z_1, z_{m+1} - z\rangle \le 0\,,\ \forall z \in \Omega.$$

This implies that $\Omega \subset Q_{m+1}$. By induction, we have $\Omega \subset Q_n$. Hence, $\Omega \subset C_n \cap Q_n$.

Step 2 We show that the sequence $\{z_n\}$ is bounded and $\lim_{n\to\infty} \|z_n - z_{n+1}\| = 0$.

From Lemma 1, $F(K)$ and $F(S)$ are closed convex sets, which means that Ω is a nonempty closed convex set. So, there exists a point $\hat{z} = P_\Omega z_1 \in \Omega$. By virtue of $z_{n+1} = P_{C_n \cap Q_n} z_1$, $\Omega \subset C_n \cap Q_n$ and the formula (3), we have

$$\|z_1 - z_{n+1}\| \le \|z_1 - \hat{z}\|.$$

Besides, we have that $\{\|z_1 - z_n\|\}$ is bounded, i.e., the sequence $\{z_n\}$ is bounded. Using the definition of Q_n and $z_{n+1} = P_{C_n \cap Q_n} z_1 \in Q_n$, we know that

$$z_n = P_{Q_n} z_1 \text{ and } \|z_1 - z_n\| \le \|z_1 - z_{n+1}\|,$$

which implies that $\{\|z_1 - z_n\|\}$ is bounded and nondecreasing. Furthermore, $\lim_{n\to\infty} \|z_1 - z_n\|$ exists. In addition, it follows from (3) that

$$\|z_n - z_{n+1}\|^2 \le \|z_1 - z_{n+1}\|^2 - \|z_1 - z_n\|^2.$$

Thus, we have $\lim_{n\to\infty} \|z_n - z_{n+1}\| = 0$.

Step 3 We show that the sequence $\{z_n\}$ converges strongly to $\hat{z} = P_\Omega z_1$.

From algorithm (4) and the condition (C1),

$$\|w_n - z_n\| = \vartheta_n \|z_n - z_{n-1}\| \to 0 \text{ as } n \to \infty.$$

By the boundedness of $\{z_n\}$, there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \rightharpoonup p$, for any $p \in \omega_w(z_n)$. This implies that $w_{n_j} \rightharpoonup p$. In addition, A is a bounded linear operator, we have $Aw_{n_j} \rightharpoonup Ap$. Next,

$$\|u_n - z_{n+1}\|^2 \le \|w_n - z_{n+1}\|^2 - \mu_n \theta_n \le \|w_n - z_{n+1}\|^2,$$

and

$$\begin{aligned}
\|u_n - z_n\| &\le \|u_n - z_{n+1}\| + \|z_{n+1} - z_n\| \\
&\le \|w_n - z_{n+1}\| + \|z_{n+1} - z_n\| \\
&\le \|w_n - z_n\| + 2\|z_{n+1} - z_n\| \to 0, \ \text{ as } \ n \to \infty.
\end{aligned}$$

These imply that $\{w_n\}$ and $\{u_n\}$ are bounded. Besides, it follows from formulas (5) and (6) that

$$\begin{aligned}
\mu_n \theta_n &\le \|w_n - z\|^2 - \|u_n - z\|^2 \\
&\le (\|w_n - z\| - \|u_n - z\|)(\|w_n - z\| + \|u_n - z\|) \\
&\le \|w_n - u_n\|(\|w_n - z\| + \|u_n - z\|) \\
&\le (\|w_n - z_n\| + \|z_n - u_n\|)(\|w_n - z\| + \|u_n - z\|) \to 0,\ n \to \infty.
\end{aligned}$$

When $(I - S)Aw_n \ne 0$, it follows from the definition of θ_n that

$$\lim_{n\to\infty} \|(I - K)w_n\| = \lim_{n\to\infty} \|(I - S)Aw_n\| = 0.$$

Since $I - K$ and $I - S$ are demiclosed at 0, we have that $p \in F(K)$ and $Ap \in F(S)$, i.e., $p \in \Omega$. From $z_n = P_{Q_n} z_1$, $\hat{z} = P_\Omega z_1$ and the weak lower semicontinuity of the norm, we obtain

$$\|\hat{z} - z_1\| \le \|p - z_1\| \le \liminf_{j\to\infty} \|z_{n_j} - z_1\| \le \limsup_{j\to\infty} \|z_{n_j} - z_1\| \le \|\hat{z} - z_1\|,\ \forall p \in \omega_w(z_n),$$

which implies that $\lim_{j\to\infty} \|z_{n_j} - z_1\| = \|\hat{z} - z_1\|$ and $p = \hat{z}$. By means of the Kadec-Klee property of Hilbert spaces, we have $\{z_{n_j}\}$ converges strongly to $\hat{z}$. Thus, the iterative sequence $\{z_n\}$ converges strongly to $\hat{z} = P_\Omega z_1$. $\square$

In addition, if K and S are two strictly pseudo-contractive mappings with $F(K) \neq \emptyset$ and $F(S) \neq \emptyset$, respectively, we have that K and S are demicontractive mappings. Further, it follows from Lemma 2 that the fixed point sets $F(K)$, $F(S)$ are closed and convex, and $I-K$, $I-S$ are demiclosed at 0. Therefore, we have the following corollary.

Corollary 1. Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with the corresponding adjoint operator A^*. Let $K : H_1 \to H_1$ and $S : H_2 \to H_2$ be strictly pseudo-contractive mappings with coefficients $\eta_1 \in [0,1)$ and $\eta_2 \in [0,1)$, respectively. Assume that $F(K) \neq \emptyset$ and $F(S) \neq \emptyset$. For any initial points $z_0, z_1 \in H_1$, the iterative sequence $\{z_n\}$ is generated by the following algorithm.

$$\begin{cases} w_n = z_n + \vartheta_n(z_n - z_{n-1}), \\ u_n = w_n - \mu_n \left[(I-K)w_n + A^*(I-S)Aw_n\right], \\ C_n = \{u \in H_1 : \|u_n - u\|^2 \leq \|w_n - u\|^2 - \mu_n\theta_n\}, \\ Q_n = \{u \in H_1 : \langle z_n - z_1, z_n - u\rangle \leq 0\}, \\ z_{n+1} = P_{C_n\cap Q_n} z_1, n \geq 1, \end{cases} \tag{7}$$

where

$$\theta_n = (1-\eta_1-2\mu_n)\|(I-K)w_n\|^2 + (1-\eta_2)\|(I-S)Aw_n\|^2 - 2\mu_n\|A^*(I-S)Aw_n\|^2.$$

If $(I-S)Aw_n \neq 0$, the stepsize $\mu_n = \sigma_n \min\{\frac{1-\eta_1}{2}, \frac{(1-\eta_2)\|(I-S)Aw_n\|^2}{2\|A^*(I-S)Aw_n\|^2}\}$ with $\sigma_n \in (0,1)$. Otherwise, $\mu_n = \sigma_n(1-\eta_1)/2$. The sequence $\{\vartheta_n\}$ is bounded in $(-\infty,\infty)$. Suppose that the solution set $\Omega = \{z^* : z^* \in F(K), Az^* \in F(S)\} \neq \emptyset$, then the iterative sequence $\{z_n\}$ generated by algorithm (7) converges strongly to $\hat{z} = P_\Omega z_1 \in \Omega$.

In particular, when the parameter $\{\vartheta_n\}$ is always equal to zero, we have the following corollary.

Corollary 2. Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with the corresponding adjoint operator A^*. Let $K : H_1 \to H_1$ and $S : H_2 \to H_2$ be demicontractive mappings with coefficients $\eta_1 \in (-\infty,1)$ and $\eta_2 \in (-\infty,1)$, respectively. Assume that $F(K) \neq \emptyset$ and $F(S) \neq \emptyset$. For any initial points $z_1 \in H_1$, the iterative sequence $\{z_n\}$ is generated by the following algorithm.

$$\begin{cases} u_n = z_n - \mu_n \left[(I-K)z_n + A^*(I-S)Az_n\right], \\ C_n = \{u \in H_1 : \|u_n - u\|^2 \leq \|z_n - u\|^2 - \mu_n\theta_n\}, \\ Q_n = \{u \in H_1 : \langle z_n - z_1, z_n - u\rangle \leq 0\}, \\ z_{n+1} = P_{C_n\cap Q_n} z_1, n \geq 1, \end{cases} \tag{8}$$

where

$$\theta_n = (1-\eta_1-2\mu_n)\|(I-K)z_n\|^2 + (1-\eta_2)\|(I-S)Az_n\|^2 - 2\mu_n\|A^*(I-S)Az_n\|^2.$$

If $(I-S)Az_n \neq 0$, the stepsize $\mu_n = \sigma_n \min\{\frac{1-\eta_1}{2}, \frac{(1-\eta_2)\|(I-S)Az_n\|^2}{2\|A^*(I-S)Az_n\|^2}\}$ with $\sigma_n \in (0,1)$. Otherwise, $\mu_n = \sigma_n(1-\eta_1)/2$. Suppose that the solution set $\Omega = \{z^* : z^* \in F(K), Az^* \in F(S)\} \neq \emptyset$, $I-K$ and $I-S$ be demiclosed at 0. Then the iterative sequence $\{z_n\}$ generated by algorithm (8) converges strongly to $\hat{z} = P_\Omega z_1 \in \Omega$.

4 | NUMERICAL EXAMPLES

In this section, all codes were written in Matlab R2018b, and ran on a Lenovo ideapad 720S with 1.6 GHz Intel Core i5 processor and 8GB of RAM. Firstly, some numerical examples in infinite Hilbert spaces are proposed to demonstrate the effectiveness and realization of convergence behavior of Algorithm 4. In addition, we introduce the following exists results and use them to compare the results in Algorithm 4.

Theorem 2 (Boikanyo[16]). Let H_1 and H_2 be Hilbert spaces. Let $K : H_1 \to H_1$ and $S : H_2 \to H_2$ be two demicontractive mappings with coefficients $k_1 \in (-\infty,1)$ and $k_2 \in (-\infty,1)$, respectively. Let $I-K$ and $I-S$ be demiclosed at 0. Let $A : H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*. The iterative sequence $\{z_n\}$ of the split common fixed point problem (1) is generated by the following iterative scheme.

$$\begin{cases} u_n = z_n - \beta_n A^*(I-S)Az_n, \\ z_{n+1} = \delta_n u + (1-\delta_n)((1-\omega)u_n + \omega K u_n), \ \forall n \geq 1, \end{cases} \tag{9}$$

where $\beta_n = \frac{(1-k_2)\|(I-S)Az_n\|^2}{2\|A^*(I-S)Az_n\|^2}$ with $Az_n \neq SAz_n$, otherwise, $\beta_n = 0$. Meanwhile, $\omega \in (0, 1-k_1)$ and $\delta_n \in (0,1)$ with $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\delta_n \to 0$ as $n \to \infty$. If the solution set Ω is nonempty, the iterative sequence $\{z_n\}$ converges strongly to a point $\hat{z} \in \Omega$.

Example 4.1. Let $H_1 = H_2 = R^3$.

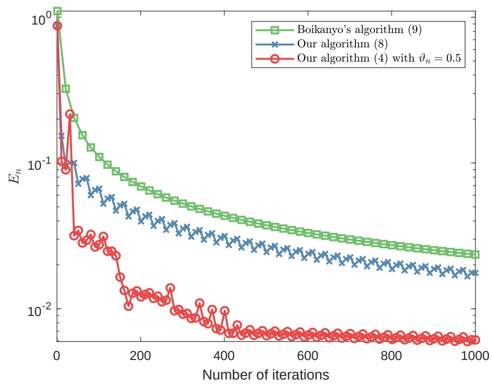
$$C = \{(z_1, z_2, z_3) \in H_1,\ z_2^2 + z_3^2 - 1 \leq 0\}$$

and

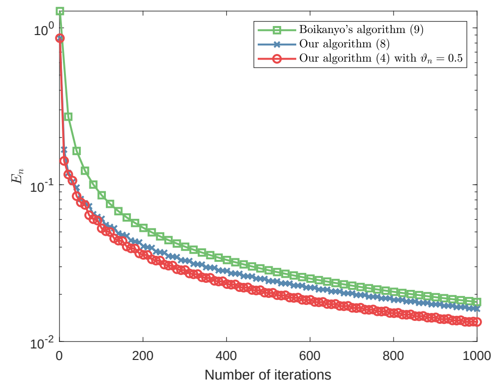
$$Q = \{(y_1, y_2, y_3) \in H_2,\ y_1^2 - y_2 + 5 \leq 0\}.$$

Let $K = P_C : H_1 \to C$ and $S = P_Q : H_2 \to Q$ be two metric projection mappings. Let the bounded nonlinear operator $A : H_1 \to H_2$ defined by $\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $z^* = (0, 1, 0)$ is a unique solution of split common fixed point problem (1).

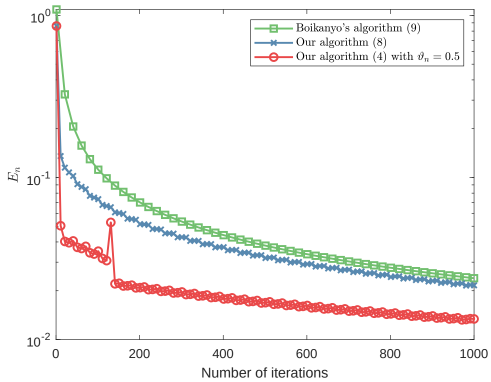
Next we give the relevant parameters in the iterative algorithms. In our algorithm (4), set $\vartheta_n = 0.5$ and $\sigma_n = 0.5$. In algorithm (9), set $\delta_n = \frac{1}{n+1}$, $u = z_0$ and $\omega = 0.5$. The error of the iterative algorithms is denoted by $E_n = \|z_n - z^*\|^2$. Take different initial points z_0, z_1 are generated randomly in MATLAB and maximum iteration 1000 as the stopping criterion. Our numerical results are shown in Figure 1 .



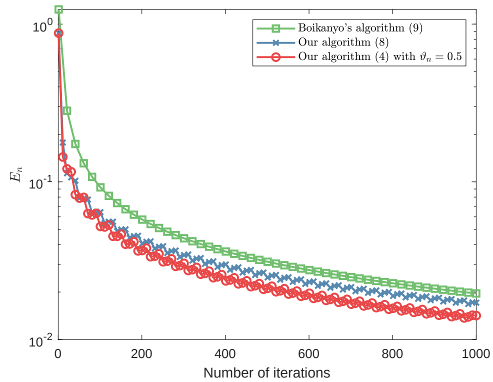
(a) Case I



(b) Case II



(c) Case III



(d) Case IV

FIGURE 1 Numerical results for Example 4.1

Example 4.2. Let $H_1 = H_2 = L_2([0, 2\pi])$ with the inner product $\langle z, y\rangle := \int_0^{2\pi} z(t)y(t)dt$ and with the norm which defined by $\|z\|_2 := \left(\int_0^{2\pi} |z(t)|^2 dt\right)^{\frac{1}{2}}$, $\forall z, y \in L_2([0, 2\pi])$. Further, we consider the following half-space

$$C = \left\{ z \in L_2([0, 2\pi]) \,\middle|\, \int_0^{2\pi} z(t)dt \leq 1 \right\} \text{ and } Q = \left\{ y \in L_2([0, 2\pi]) \,\middle|\, \int_0^{2\pi} |y(t) - \sin(t)|^2\, dt \leq 16 \right\}.$$

In addition, a linear continuous operator $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$, where $(Az)(t) := z(t)$. Then $(A^*z)(t) = z(t)$ and $\|A\| = 1$. We can also write the projections onto C and the projections onto Q as follows.

$$P_C(z) = \begin{cases} \frac{1-\int_0^{2\pi} z(t)dt}{4\pi^2} + z, & \int_0^{2\pi} z(t)dt > 1, \\ z, & \int_0^{2\pi} z(t)dt \leq 1. \end{cases}$$

$$P_Q(y) = \begin{cases} \sin + \frac{4}{\sqrt{\int_0^{2\pi} |y(t)-\sin(t)|^2 dt}}(y - \sin), & \int_0^{2\pi} |y(t) - \sin(t)|^2 dt > 16, \\ y, & \int_0^{2\pi} |y(t) - \sin(t)|^2 dt \leq 16. \end{cases}$$

Now, we solve the problem (1) where $K = P_C$ and $S = P_Q$. Choose different initial values z_0 and z_1. The error of the iterative algorithms is denoted by

$$E_n = \frac{1}{2}\left\|P_C\left(z_n\right) - z_n\right\|_2^2 + \frac{1}{2}\left\|P_Q\left(A\left(z_n\right)\right) - A\left(z_n\right)\right\|_2^2.$$

All parameters are the same as in Example 4.1. We take the error $E_n < 10^{-3}$ or maximum iteration 200 as the stopping criterion. All numerical results are shown in Table 1 and Figure 2 . In Table 1 , Iter. and Times(s) denote the number of iterations and the CPU time in seconds, respectively.

TABLE 1 Numerical results for Example 4.2

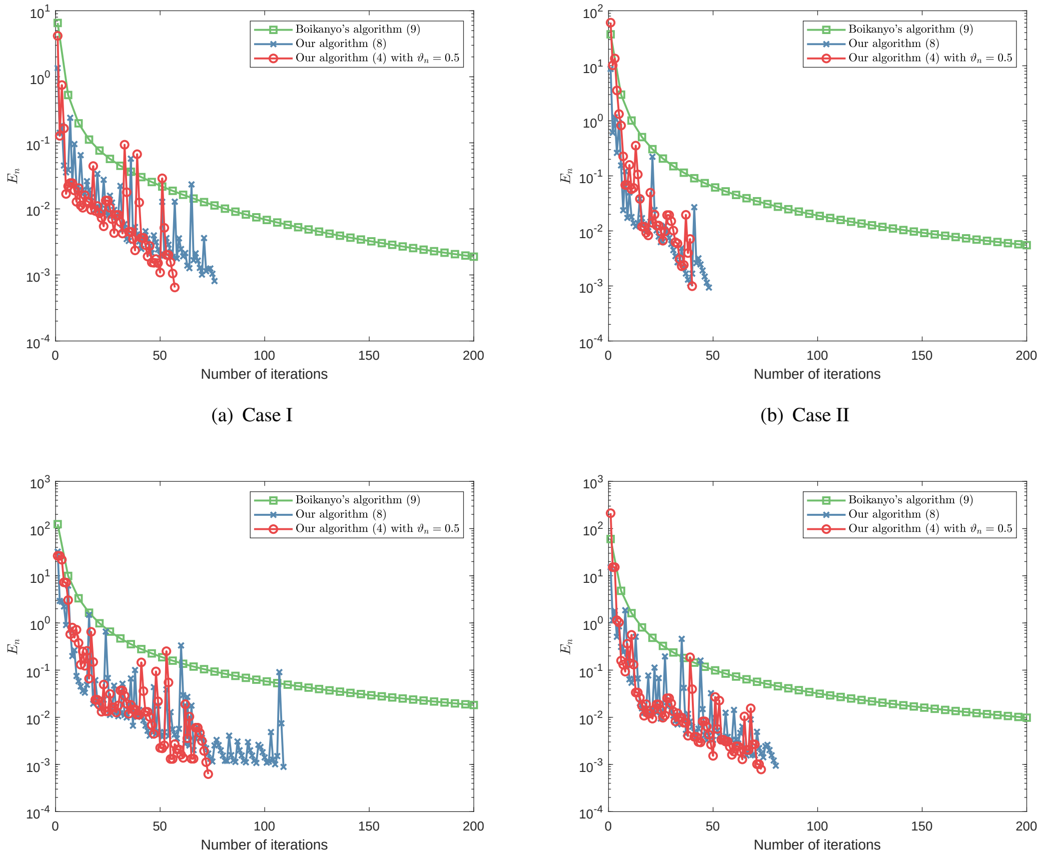
		our algorithm (4) with $\vartheta_n = 0.5$		our algorithm (8)		Boikanyo's algorithm (9)	
Cases	Initial values	Iter.	Time(s)	Iter.	Time(s)	Iter.	Time(s)
I	$z_0 = \sin(t), z_1 = \frac{t^2}{5}$	57	93.1586	76	119.7086	200	50.4863
II	$z_0 = t^2, z_1 = \frac{e^t}{20}$	40	110.9755	48	77.8733	200	61.2474
III	$z_0 = t^2, z_1 = \frac{2^t}{2}$	73	241.0647	109	225.5353	200	55.4393
IV	$z_0 = 2t^2, z_1 = \frac{t^3}{10}$	73	110.7121	80	145.8766	200	50.3234

Remark 1. (i) As show in Examples 4.1 and 4.2, we see that our algorithm (4) with the inertial term outperforms algorithm (8) and Boikanyo's algorithm (9) in the number of iterations. However, our algorithm (4) has no advantage in CPU time, because each time we need to calculate the projection onto C_n and Q_n.

(ii) Our proposed algorithm is consistent in the sense that the choice of initial points does not affect the required number of iterations needed to achieve desired results.

5 | CONCLUSION

The important conclusion is that we give an algorithm (i.e., algorithm (4)) to approximate the solution of the split common fixed point problem for demicontractive mappings by the inertial technique and the hybrid projection method in Section 3. For better convergence results, we introduce a new self-adaptive step-size sequence which does not need prior knowledge of operator norms. Furthermore, the corresponding strong convergence theorem (i.e., Theorem 1) is proven by such a self-adaptive step-size sequence. In addition, as we can see in Figures 1 and 2 , our results are effective in practice and improve the existing results.



(a) Case I (b) Case II

(c) Case III (d) Case IV

FIGURE 2 Convergence behavior of iteration error $\{E_n\}$ with different initial values for Example 4.2

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