Asynchronous periodic sampling static consensus for second-order multi-agent systems with event-triggered mechanism

Qingquan Yang[a], Jing Li[a,*]

[a]School of Mathematics and Statistics, Xidian University, Xi'an, 710071,China

Abstract

In this paper, an asynchronous periodic sampling consensus method is proposed for second-order continuous-time multi-agent systems with event-triggered mechanism. Stochastic matrix theory is employed successfully to analyze the consensus of the closed-loop multi-agent systems. By appropriately choosing parameters of the proposed consensus control protocol, it is proved that states of all agents can reach consensus and the Zeno behaviour is excluded if the topology graph contains a directed spanning tree. Finally, a numerical simulation example is given to illustrate the advantages of the asynchronous periodic sampling consensus method.

Keywords: Multi-agent systems, asynchronous periodic sampling consensus, event-triggered mechanism.

1. Introduction

In the past decades, a large amount of attention is devoted to the consensus problems of multi-agent systems (see survey papers [1] and [2]). For the reason that traditional analog controllers are replaced widely by the digital ones, one of the main challenging problems in this filed is how to design the sampling consensus protocols. Comparing with the analog consensus scheme, the sampling consensus scheme has some advantages such as reducing the computing loads

*Corresponding author
Email address: xidianjing@126.com (Jing Li)

of controller and saving communication traffics among agents. Thus, lots of interesting works [3]-[8] on sampling consensus have been done at present stage.

According to sampling mechanism, to the best of our knowledge, the existing works on sampling consensus methods for multi-agent systems can be classified as two categories: synchronous ones and asynchronous ones. The synchronous sampling consensus is that the sampling instants are same for all agents [3]-[6]. In the asynchronous sampling consensus, the sampling instants are different from the others [7]-[8]. However, these works require each agent to communicate with its neighbors in all sampling instants.

To reduce unnecessary communication in limited bandwidth constraints, event-triggered communication schemes [9]-[12] are introduced into sampling consensus design of multi-agent systems, where communication occurs only when some detected conditions are satisfied. In [13]-[16], several event-triggered synchronous periodic sampling consensus schemes are developed. Then, these schemes are further extended to the asynchronous periodic sampling case [17]-[19]. In [17]-[18], the sampling period for each agent is same and edge event-driven techniques are studied. And the value of edge depends on the information of the corresponding two neighboring agents. Therefore, asynchronous means that the event-triggered action over each edge is independent of others. In [22], a novel distributed event-triggered sampling scheme, where agents exchange information via a limited communication medium, is investigated for second-order multi-agent systems. Event-based synchronization of linear dynamical networks is proposed in [23] which adopts synchronous sampling method. In [24], the authors investigate an event-triggered rendezvous control method for multiple two-wheeled mobile robots while the controllers are designed with equivalent event-checking periods and time-varying communication delays. For the multi-agent linear system in [25], the sampling periods and the triggering conditions of the input and output are asynchronous. In [19], asynchronous means that the sampling period and the event-triggered instant for each agent are different from others'. It is pointed out that the consensus scheme in [19] has more advantages such as flexible sampling periods. However, the consensus scheme in

[19] is developed just for first-order multi-agent systems and it is worth further exploring and generalization.

In this paper, we study the event-triggered asynchronous periodic sampling consensus for second-order multi-agent systems. It is nontrivial to extend the results from first-order multi-agent systems to second-order ones. For example, it is more difficult to design the parameters in consensus protocols, triggering conditions and sampling periods, for the reason that the closed-loop system matrix is more complex than one in the first-order case. Fortunately, we solve these difficulties by developing some new techniques. The main contributions of this paper are summarized as follows.

(i) This paper is the first study to focus on the event-triggered asynchronous periodic sampling consensus problem for second-order continuous-time multi-agent systems. This work further generalizes the classes of multi-agent systems to which event-triggered asynchronous periodic sampling consensus scheme can be applied.

(ii) By using some vital technics, it is ensured that the closed-loop system matrix is still stochastic, which plays the key role in proving the consensus of positions and velocities of multi-agent systems.

(iii) We get that the states of all agents achieve consensus with exponential rate and it is proved that the proposed event-triggered scheme is better than sample-data scheme from the theoretical perspective in this paper.

Notations: Let m be a nonzero constant, R denotes the set of real numbers, Z denotes the set of the nonnegative integers, R^N denotes N real vector space, $R^{N\times N}$ denotes $N\times N$ real matrix space, the symbol $\lfloor x\rfloor$ means floor function of the real number x and $V=\{1,2,\cdots,N\}$. $\mathbf{1}_n=(1,1,\cdots,1)^T\in R^n$.

2. Preliminaries

In this section, we will introduce some knowledge on algebraic graph theory and stochastic matrix theory that will be used in the following sections.

In this paper, the network topology of N agents is modeled as a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, in which $\mathcal{V} = \{v_1, v_2, ..., v_N\}$ is the set of the finite nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of the edges, and $\mathcal{A} = [a_{ij}] \in R^{N \times N}$ with $a_{ij} \geq 0$ and $a_{ii} = 0$ is the weighted adjacency matrix of $\mathcal{G}$. The neighbor set of node i is denoted with $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. Define the weighted Laplacian matrix as $\mathcal{L} = Q - \mathcal{A}$, where $Q = \text{diag}\{q_1, q_2, ..., q_N\}$, with $q_i = \Sigma_{j=1}^{N} a_{ij}$. The row sums of $\mathcal{L}$ are zero. i.e. $\mathcal{L} \cdot \mathbf{1}_N = \mathbf{0}$.

The following definitions are from [20]. A matrix A is called a nonnegative matrix if all its elements are equal to or greater than zero. A nonnegative matrix $A \in R^{n \times n}$ is said to be a row stochastic matrix, if it satisfies that $A \cdot \mathbf{1}_N = \mathbf{1}_N$. For a row stochastic matrix $A \in R^{N \times N}$, define the quantity $\mathcal{J}(A) = \frac{1}{2} \max\limits_{i,j} \sum\limits_{k} \mid a_{ik} - a_{jk} \mid = 1 - \min\limits_{i,j} \sum\limits_{k} \min\{a_{ik}, a_{jk}\}$. If $\mathcal{J}(A) < 1$, the matrix A is called scrambling.

The operator $\mathcal{D}$ for a vector $x = [x_1, \cdots, x_N]^T$ is defined as $\mathcal{D}(x) = \max\limits_{i \in \mathcal{V}} \{x_i\} - \min\limits_{i \in \mathcal{V}} \{x_i\}$.

Lemma 1. *[20] For arbitrary vectors v and w, $\mathcal{D}(v + w) \leq \mathcal{D}(v) + \mathcal{D}(w)$. For a arbitrary row stochastic matrices A and a vector, $\mathcal{D}(Av) \leq \mathcal{J}(A)\mathcal{D}(v)$. For arbitrary row stochastic matrices A and B, $\mathcal{J}(AB) \leq \mathcal{J}(A)\mathcal{J}(B)$.*

3. Problem description and consensus protocol design

3.1. Problem description

Consider a class of second-order multi-agent systems with N agents. The system dynamics of the ith agent is described as follows

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), \qquad i = 1, 2, \cdots, N, \end{cases} \tag{1}$$

where $x_i(t) \in R$ represents the position, $v_i(t) \in R$ is the velocity and $u_i(t) \in R$ is the control protocol to be designed below.

The objective of this paper is to design u_i for agent $i \in V$ such that the closed-loop multi-agent system satisfies

$$\begin{cases} \lim_{t\to\infty} x_i(t) - x_j(t) = 0, j \in V, \\ \lim_{t\to\infty} v_i(t) = 0 \end{cases} \tag{2}$$

for arbitrary initial conditions. That is, (1) achieves the static consensus of the positions.

In this paper, we consider the asynchronous sampling periodic setting. Let h_i be the sampling period of agent i, and define the sampling instants as $\mathcal{S}_i = \{kh_i\}_{k\in Z}$. Denote the event instants as $\mathcal{T}_i = \{t_k^i\}_{k\in Z}$ with $t_0^i = 0$, where the event instants are the time when agents communicate with others. Obviously, $\mathcal{T}_i \subseteq \mathcal{S}_i$ for all $i \in V$. According to the above illustration, we can define

$$\begin{cases} \hat{x}_i(t) = x_i(t_k^i), & \text{for } t \in [t_k^i, t_{k+1}^i), \\ \bar{x}_i(t) = x_i(kh_i), & \text{for } t \in [kh_i, kh_i + h_i), \\ \bar{v}_i(t) = v_i(kh_i). & \text{for } t \in [kh_i, kh_i + h_i). \end{cases}$$

Remark 1. Distributed problem solving depends extensively on agents being able to communicate shared data. Generally, the global information for each agent can not be obtained. Thus, it is meaningful that each agent is able to determine its own sampling period and detection parameters without knowledge of any global information.

3.2. Event-triggered asynchronous periodic sampling consensus protocol

In this subsection, to achieve the static consensus of positions shown in (2), we propose the following asynchronous periodic sampling consensus protocol with an event-triggered communication scheme

$$u_i(t) = -\gamma_i v_i(kh_i) + \frac{\rho_i}{d_i h_i} \Sigma_{j\in\mathcal{N}_i}(\hat{x}_j(kh_i) - \hat{x}_i(kh_i)) \tag{3}$$

for $t \in [kh_i, kh_i + h_i)$ and $kh_i \in \mathcal{S}_i$, where γ_i and ρ_i are the positive design parameters, and $d_i = q_i$. Especially, if $d_i = 0$, $u_i(t) \equiv 0$ for all t.

Remark 2. For (3), we do not design the term $\frac{\rho_i}{d_i h_i}\Sigma_{j\in\mathcal{N}_i}(\hat{v}_j(kh_i)-\hat{v}_i(kh_i))$. Thus, information exchanging can be reduced.

The measurement errors of positions between the sampling instants and their corresponding event-triggered instants are designed as follows

$$e_i(t)=\hat{x}_i(t)-\bar{x}_i(t), i\in V. \tag{4}$$

Design the event-triggering condition as follows

$$f_i(e_i(t),g_i(t))=g_i(t)-\mid e_i(t)\mid, t\in\mathcal{S}_i, \tag{5}$$

where $g_i(t)\geq 0$ is a time-varying threshold and it is given as

$$g_i(kh_i+h_i)=\beta_i g_i(kh_i), g_i(0)=\alpha_i \tag{6}$$

with $\alpha_i>0$, $0<\beta_i<1$, and $g_i(t)=g_i(kh_i)$ for $t\in[kh_i,kh_i+h_i)$.

Now, we can get a series of event instants

$$t^i_{k+1}=\inf\{t:t\in\mathcal{S}_i,t>t^i_k,f_i(e_i(t),g_i(t))<0\}. \tag{7}$$

Remark 3. The physical mechanism of the protocol is described as follows. For each agent, there exists a buffer recording the latest broadcasted state information from its neighbors, which works in such a way that the old data will be erased by the new data if they are from the same agent, and the other buffer records sampling instants of the states. Agent i may receive several broadcasted information from its neighbor j before the instant kh_i, but only the latest one is kept. Noted that if the current state satisfies (7), agent i will not update its value instantly until its sampling instants to read the data from the buffer. Agent i will read the data from the buffer until its next sampling instant occurs.

4. Main results

4.1. Building a precise discrete model

We discretize the second-order multi-agent systems. Choosing $t = kh_i + h_i$, we have

$$\begin{aligned} x_i((k+1)h_i) =& x_i(kh_i)+h_i v_i(kh_i)-\frac{1}{2}\gamma_i {h_i}^2 v_i(kh_i) \\ &+\frac{1}{2}\rho_i d_i^{-1} h_i \Sigma_{j\in\mathcal{N}_i}(\hat{x}_j(kh_i)-\hat{x}_i(kh_i)), \end{aligned} \tag{8}$$

$$\begin{aligned} v_i((k+1)h_i) =& v_i(kh_i)-\gamma_i h_i v_i(kh_i) \\ &+\rho_i d_i^{-1}\Sigma_{j\in\mathcal{N}_i}(\hat{x}_j(kh_i)-\hat{x}_i(kh_i)). \end{aligned} \tag{9}$$

The sampling instants of all the agents are collected together as a set $\cup_{i\in V}\cup_k kh_i$, and rewrite it by using $\{\tau_k\}_{k\in Z}$ in ascending order. That is, $\cup_{i\in V}\cup_k kh_i = \{\tau_k\}_{k\in Z}$. Then, we can rewrite (8) and (9) at the sampling instant τ_k as follows

$$\xi_i(k+1)=\begin{cases} \xi_i(k), & \text{if} \tau_{k+1}\notin\mathcal{S}_i; \\ \xi_i(k)+h_i\eta_i(k)-\frac{1}{2}\gamma_i {h_i}^2\eta_i(k) & \\ +\frac{1}{2}\rho_i d_i^{-1}h_i\Sigma_{j\in\mathcal{N}_i}(\hat{\xi}_j(k)-\hat{\xi}_i(k)), & \text{if}\,\tau_{k+1}\in\mathcal{S}_i; \end{cases} \tag{10}$$

and

$$\eta_i(k+1)=\begin{cases} \eta_i(k), & \text{if}\,\tau_{k+1}\notin\mathcal{S}_i; \\ \eta_i(k)-\gamma_i h_i\eta_i(k)+\rho_i d_i^{-1}\Sigma_{j\in\mathcal{N}_i}(\hat{\xi}_j(k)-\hat{\xi}_i(k)), & \\ \text{if}\,\tau_{k+1}\in\mathcal{S}_i, & \end{cases} \tag{11}$$

where

$$\hat{\xi}_i(k+1)=\begin{cases} \xi_i(k+1), & \text{if}\,\tau_{k+1}\in\mathcal{T}_i; \\ \hat{\xi}_i(k), & \text{if}\,\tau_{k+1}\notin\mathcal{T}_i, \end{cases}$$

where $\hat{\xi}_i(k)$ represents the position at event-triggered instants. They remain unchanged until they step into the next event instant. We have $\xi_i(k) = x_i(\tau_k)$ and $\eta_i(k) = v_i(\tau_k)$ if $\tau_k \in \mathcal{S}_i$; Otherwise, $\xi_i(k) \neq x_i(\tau_k)$ and $\eta_i(k) \neq v_i(k)$.

When $\tau_k \in \mathcal{S}_i$, the current measurement errors between the sampling states and the triggered states are defined as

$$\epsilon_i(k) = \hat{\xi}_i(k) - \xi_i(k), i \in V. \tag{12}$$

Substituting (12) into (10) and (11), we get

$$\begin{pmatrix} \xi(k+1) \\ \eta(k+1) \end{pmatrix} = A_k \begin{pmatrix} \xi(k) \\ \eta(k) \end{pmatrix} + B_k \epsilon(k),$$

where

$$A_k = \begin{pmatrix} I - \frac{1}{2}\rho D^{-1} H_k \mathcal{L}_k & H_k - \frac{1}{2}\gamma H_k^2 \\ -\rho D^{-1}\mathcal{L}_k & I - \gamma H_k \end{pmatrix},$$

$$B_k = \begin{pmatrix} -\frac{1}{2}\rho D^{-1} H_k \mathcal{L}_k \\ -\rho D^{-1}\mathcal{L}_k \end{pmatrix}$$

with $\rho = \mathrm{diag}\{\rho_1, \cdots, \rho_N\}$, $D = \mathrm{diag}\{d_1, \cdots, d_N\}$, $\gamma = \mathrm{diag}\{\gamma_1, \cdots, \gamma_N\}$ and $H = \mathrm{diag}\{h_1, \cdots, h_N\}$.

Remark 4. For the ith system, if $\tau_{k+1} \in \mathcal{S}_i$, the elements of ith row in matrix H_k and $\mathcal{L}_k$ are the same as matrix H and Laplacian matrix $\mathcal{L}$, respectively; Otherwise, the elements of the ith row in matrix H_k and $\mathcal{L}_k$ are zero. Obviously, it is possible that more than one agent is sampled at the instant τ_{k+1}.

By coordinate transformation, we have

$$\begin{pmatrix} \theta(k) \\ \phi(k) \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \begin{pmatrix} \xi(k) \\ \eta(k) \end{pmatrix},$$

and

$$\begin{aligned} \begin{pmatrix} \theta(k+1) \\ \phi(k+1) \end{pmatrix} = & \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} A_k \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} \theta(k) \\ \phi(k) \end{pmatrix} \\ & + \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} B_k \epsilon(k). \end{aligned}$$

Then, we can get the following closed-loop multi-agent systems

$$\begin{pmatrix} \theta(k+1) \\ \phi(k+1) \end{pmatrix} = E_k \begin{pmatrix} \theta(k) \\ \phi(k) \end{pmatrix} + F_k \epsilon(k), \tag{13}$$

where

$$E_k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, F_k = \begin{pmatrix} -\frac{1}{2}\rho D^{-1} H_k \mathcal{L}_k \\ -\frac{1}{2}\rho D^{-1} H_k \mathcal{L}_k - \rho D^{-1}\mathcal{L}_k \end{pmatrix},$$

$a = I - \frac{1}{2}\rho D^{-1} H_k \mathcal{L}_k - H_k + \frac{1}{2}\gamma H_k^2$, $b = H_k - \frac{1}{2}\gamma H_k^2$, $c = -H_k + \frac{1}{2}\gamma H_k^2 + \gamma H_k - \frac{1}{2}\rho D^{-1} H_k \mathcal{L}_k - \rho D^{-1}\mathcal{L}_k$ and $d = I + H_k - \frac{1}{2}\gamma H_k^2 - \gamma H_k$.

As shown in [19], the system matrix E_k plays a key role in the consensus analysis. The following lemma tells us that the matrix E_k can be a stochastic matrix if the positive control parameters γ_i, ρ_i and the sampling periods h_i are chosen appropriately, $i \in V$.

Lemma 2. *Consider the systems* (1)*, the consensus protocol* (3) *and the event-triggering condition* (5)*. There exist the design parameters* γ_i *and the sampling period* h_i *,* $i \in V$ *such that* E_k *is a stochastic matrix.*

Proof. See the Appendices.

Remark 5. Lemma 2 shows us the existence of control parameters for achieving consensus. However, from the proof we can see how to choose such parameters. For example, we choose $2\rho_i \leq \gamma_i \leq \frac{1}{2}$ and $0 < h_i \leq \frac{2+\rho_i-\sqrt{\Delta_1}}{2\gamma_i}$, where $\Delta_1 = (\rho_i+2)^2 - 8\gamma_i$, which ensures that E_k is stochastic matrix.

4.2. Consensus analysis

Define an integer $J = \sum_{i=1}^{N} \lceil \frac{H}{h_i} \rceil$, where $H = \max_{i \in V}\{h_i\}$. Based on [20], the following lemma is introduced.

Lemma 3. *[20] Assume that the graph contains a directed spanning tree and* E_k *is a stochastic matrix. There exists a constant* $0 < \mu < 1$ *such that* $\mathcal{J}(\Phi[lM + M, lM]) < \mu$ *for any finite integer* $M > (N-1)^2 J$ *and* $l \in Z$*, where* $\Phi[lM + M, lM] = \Phi[lM + M, j] = E_{lM+M-1} \cdots E_{lM} E_j$.

Now, we rewrite (13) as follows

$$\begin{pmatrix} \theta(lM+M) \\ \phi(lM+M) \end{pmatrix} = \Phi[lM+M, lM] \begin{pmatrix} \theta(lM) \\ \phi(lM) \end{pmatrix} + \sum_{j=lM}^{lM+M-1} \Phi[lM+M, j+1] F_j \epsilon(j), \tag{14}$$

where the transition matrix $\Phi[lM+M, j]$ is the same as the one in Lemma 3. For convenience, define $s(lM+M) = \begin{pmatrix} \theta(lM+M) \\ \phi(lM+M) \end{pmatrix}$. In what follows, we will give the main result of this paper.

Theorem 4. *Consider the system* (1) *with the consensus protocol* (3)*, and the event-triggering condition* (5)*. If the communication topology contains a directed spanning tree, for arbitrary* $\alpha_i > 0$ *and* $0 < \beta_i < 1$*, there exist the design parameters* γ_i, ρ_i *and the sampling periods* h_i*,* $i \in V$*, such that the states of all the agents reach static consensus given by (2).*

Proof. From the analysis above, we have

$$s(k) = \Phi[k, 0]s(0) + \sum_{j=0}^{k-1} \Phi[k, j+1] F_j \epsilon(j), \tag{15}$$

where $s(k) = [\theta^T(k), \phi^T(k)]^T$. Consider $\mathcal{D}(s(k))$ as a Lyapunov function candidate. Based on Lemmas 1 and 2, we know that E_k is a stochastic matrix if $\rho_i > 0$ and $\gamma_i > 0$ are chosen appropriately. Based on Lemma 3 and Theorem 9 in [19], we get that $\mathcal{D}(s(k))$ converges to 0 as $k \to \infty$. Then, $\mathcal{D}(\xi(k)) \to 0$ and $\mathcal{D}(\eta(k)) \to 0$ as $k \to \infty$.

In this position, we will discuss the convergence of $\eta_i(k)$. We claim that $\eta_i(k)$ converges to zero. In order to seek a contradiction, we assume that there are two cases: (1) $\eta_i(k)$ converges to a nonzero constant a, e.g., $a \neq 0$. (2) the value of $\eta_i(k)$ is always changing. As for the former case, for any small value $\varepsilon_1 > 0$, there exists a natural number N_1, such that for each natural number $k > N_1$, we have that $\mid \eta_i(k) \mid \leq a + \varepsilon_1$, $\mid \eta_i(k) - \eta_j(k) \mid < \varepsilon_1$, and $\mid \hat{\xi}_i(k) - \hat{\xi}_j(k) \mid < \varepsilon_1$, $i, j \in V$. From (11), for the situation that $\tau_{k+1} \in \mathcal{S}_i$ and $\tau_{k+1} \in \mathcal{S}_j$, it can be seen that

$$\begin{aligned}
&|\gamma_i h_i \eta_i(k) - \gamma_j h_j \eta_j(k)|\\
&\leq |-\eta_i(k+1) + \eta_j(k+1)|\\
&\quad + |-\eta_i(k) + \eta_j(k)|\\
&\quad + |\rho_i d_i^{-1} \Sigma_{j\in\mathcal{N}_i}(\hat{\xi}_j(k) - \hat{\xi}_i(k))|\\
&\leq 2\varepsilon_1 + (\rho_i + \rho_j)\varepsilon_1.
\end{aligned} \tag{16}$$

Because $\eta_i(k)$ converges to a constant a, we obtain

$$\begin{aligned}
&|\gamma_i h_i \eta_i(k) - \gamma_j h_j \eta_j(k)|\\
&= |\gamma_i h_i \eta_i(k) - \gamma_j h_j \eta_i(k)\\
&\quad + \gamma_j h_j \eta_i(k) - \gamma_j h_j \eta_j(k)|\\
&\geq |\gamma_i h_i - \gamma_j h_i| (|a| - \varepsilon_1) - |\gamma_j h_j| \varepsilon_1.
\end{aligned} \tag{17}$$

When $l \to \infty$, we have $\varepsilon_1 \to 0$. However, by comparing (22) with (23), we know that there appears the contradiction between (22) and (23). Therefore, the former assumption does not hold. As for the latter case, the proof is similar to the former, which also yields a contradiction. Hence, $\eta_i(k)$ converges to zero. Therefore, for arbitrary small positive number ε_2, there exists a natural number N_2, such that for each natural number $k > N_2$, we have that $|\eta_i(k)| < \varepsilon_2$, by the definition of $\eta_i(k)$, we have $|\eta_i(k)| < \varepsilon_2$. According to the definition of operator $\mathcal{D}$, for arbitrary small positive number ε_3, there exists a natural number N_3 such that for each natural number $k > N_3$, $|\xi_i(k) - \xi_j(k)| < \varepsilon_3$. By the definition of measurement error, for arbitrary small positive number ε_4, there exists a natural number N_4, such that $|\xi_i(k) - \hat{\xi}_i(k)| \leq g_i(\tau_k) < \varepsilon_4$, $i \in V$. Define $N = \max\{N_1, N_2, N_3, N_4\}$. For each $k > N$, we have

$$\begin{aligned}
|\xi_i(k) - \hat{\xi}_j(k)| &= |\xi_i(k) - \xi_j(k) + \xi_j(k) - \hat{\xi}_j(k)|\\
&\leq |\xi_i(k) - \xi_j(k)| + |\xi_j(k) - \hat{\xi}_j(k)|\\
&< \varepsilon_3 + \varepsilon_4,
\end{aligned} \tag{18}$$

From (24), it is easy to get that

$$\begin{aligned}
\mid \hat{\xi}_i(k) - \hat{\xi}_j(k) \mid =& \mid \hat{\xi}_i(k) - \xi_i(k) + \xi_i(k) - \hat{\xi}_j(k) \mid \\
\leq& \mid \hat{\xi}_i(k) - \xi_i(k) \mid + \mid \xi_i(k) - \hat{\xi}_j(k) \mid \\
\leq& \varepsilon_3 + 2\varepsilon_4.
\end{aligned}$$

When the time t is large enough, there exist $k > \max\{N_3, N_4\}$ and $l > \max\{N_1, N_2\}$ such that $\tau_k = lh_i$, $t \in [lh_i, lh_i + h_i)$ with $\tau_k \in \mathcal{S}_i$. It follows from (8) and (9) that

$$\begin{aligned}
\mid x_i(t) - \xi_i(k) \mid \leq& h_i \mid v_i(lh_i) \mid + \frac{1}{2}\gamma_i h_i^2 \mid v_i(lh_i) \mid \\
&+ \frac{\rho_i h_i}{2d_i} \sum_{j \in \mathcal{N}_i} \mid \hat{x}_j(lh_i) - \hat{x}_i(lh_i) \mid \\
\leq& (h_i + \frac{1}{2}\gamma_i h_i^2)\varepsilon_2 + \frac{1}{2}\rho_i h_i(\varepsilon_3 + 2\varepsilon_4).
\end{aligned}$$

For any large enough t, we have

$$\begin{aligned}
\mid x_i(t) - x_j(t) \mid \leq& \mid x_i(t) - \xi_i(k) \mid + \mid \xi_j(k) - x_j(t) \mid + \mid \xi_i(k) - \xi_j(k) \mid \\
\leq& (h_i + h_j + \frac{1}{2}\gamma_i h_i^2 + \frac{1}{2}\gamma_i h_j^2)\varepsilon_2 \\
&+ \frac{1}{2}(\rho_i h_i + \rho_j h_j)(\varepsilon_3 + 2\varepsilon_4) + \varepsilon_3.
\end{aligned}$$

According to (9), it can be seen that

$$\begin{aligned}
|v_i(t)| \leq& |v_i(kh_i)| + \gamma_i h_i |v_i(kh_i)| + \rho_i d_i^{-1} h_i \Sigma_{j \in \mathcal{N}_i} |\hat{x}_j(kh_i) - \hat{x}_i(kh_i)| \\
\leq& (1 + \gamma_i h_i)\varepsilon_2 + \rho_i h_i(\varepsilon_3 + 2\varepsilon_4).
\end{aligned}$$

Obviously, h_i, ρ_i and γ_i, $i \in V$ are constants and $\gamma_i > 0$. When $N = \max\{N_1, N_2, N_3, N_3\} \to +\infty$, according to the above results, ε_2, ε_3 and ε_4 tend to zero. Thus, we get that $\mid x_i(t) - x_j(t) \mid \to 0$ and $v_i(t) \to 0$ as $t \to \infty$ for $i, j \in V$. The proof is completed.

4.3. Some important results

In this section, we show that there is no case where all sample-data points are event-triggered points, we are supposed to prove that the inter-event time has a positive lower bound τ_0 such that $t^i_{k+1} - t^i_k > \tau_0 > 0$, $k = 0, 1, \cdots,$ $i =, 1, 2, \cdots, N$.

For simplicity of presentation, denote $\ell_i = [c_1, \cdots, c_{i-1},\ c_i, c_{i+1}, \cdots, c_N]$ with $c_i = \gamma_i$, where $c_j = 0$, $j \in V/\{i\}$. Denote $\rho=$ diag$\{\rho_1, \cdots, \rho_N\}$, $D^{-1}=$ diag$\{d_1^{-1}, \cdots, d_N^{-1}\}$, $\gamma =$ diag$\{\gamma_1,\ \cdots, \gamma_N\}$ and $H^{-1} =$ diag$\{h_1^{-1}, \cdots, h_N^{-1}\}$. Define $x = [x_1, \cdots,\ x_N]^T$ and $v = [v_1, \cdots, v_N]^T$, then rewrite the second-order multi-agent systems (1) and (3), $i = 1, 2, \cdots, N$, as

$$\begin{cases} \dot{x}(t) = v(t), \\ \dot{v}(t) = -\Sigma_{j\in\mathcal{N}_i}\ell_j v(k^j_t h_j) - \rho D^{-1} H^{-1} \Sigma_{j\in\mathcal{N}_i}\mathcal{L}_j x(k^j_t h_j) \\ \qquad\qquad + \rho D^{-1} H^{-1} \mathcal{L} e(t), \end{cases}$$

where the jth row in matrix $\mathcal{L}_j$ is the same as Laplacian matrix $\mathcal{L}$, $e(t) = [e_1(t), \cdots, e_N(t)]^T$ and $t \in [k^j_t h_j, k^j_t h_j + h_j)$ with $k^j_t = \lfloor \frac{t}{h_j} \rfloor$.

Based on the Newton-Leibnitz formula, it is easy to obtain that

$$\chi(t - \tau_j(t)) = \chi(t) - \int_{t-\tau_j(t)}^{t} \dot{\chi}(s)ds.$$

Then, we arrive at

$$\begin{aligned} \dot{v}(t) =& -\gamma v(t) - \rho D^{-1} H^{-1} \mathcal{L} x(t) + \Sigma_{j\in\mathcal{N}_i}\ell_j \int_{t-\tau_j(t)}^{t} \dot{v}(s)ds \\ &+ \rho D^{-1} H^{-1} \Sigma_{j\in\mathcal{N}_i}\mathcal{L}_j \int_{t-\tau_j(t)}^{t} \dot{x}(s)ds + Be(t), \end{aligned}$$

where $\tau_j(t) = t - k^j_t h_j$. Choose a matrix $Q = [q_1, \cdots, q_{N-1}] \in R^{N\times(N-1)}$ such that $\frac{1}{\sqrt{N}}\mathbf{1}$ and the column vectors of Q form an orthonormal basis of R^N. And we have the following relationships, $Q^T\mathbf{1} = 0$, $Q^TQ = I_{N-1}$ and $QQ^T = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T$.

Define $\vec{x} = Q^T x$. Then, we have

$$\begin{cases} \dot{\vec{x}}(t) = Q^T v(t), \\ \dot{v}(t) = -\gamma v(t) - \rho D^{-1} H^{-1} \mathcal{L} Q \vec{x}(t) + \Sigma_{j\in\mathcal{N}_i}\ell_j \int_{t-\tau_j(t)}^{t} \dot{v}(s)ds \\ \qquad\qquad + \rho D^{-1} H^{-1} \Sigma_{j\in\mathcal{N}_i}\mathcal{L}_j Q \int_{t-\tau_j(t)}^{t} \dot{\vec{x}}(s)ds + Be(t), \end{cases}$$

Denote $y = (\vec{x}^T, v^T)^T$. Then, we have

$$\dot{y}(t) = Fy + \Sigma_{j\in\mathcal{N}_i} W_j \int_{t-\tau_j(t)}^{t} \dot{y}(s)ds + \mathcal{B}e(t), \tag{19}$$

and

$$\dot{y}(t) = My(t) - \Sigma_{j\in\mathcal{N}_i} W_j y(k_t^j) + \mathcal{B}e(t), \tag{20}$$

where

$$F = \begin{pmatrix} 0_{N\times(N-1)} & Q^T \\ -\rho D^{-1}H^{-1}\mathcal{L}Q & -\gamma \end{pmatrix}, W_j = \begin{pmatrix} 0_{N\times(N-1)} & 0_{(N-1)\times N} \\ \rho D^{-1}H^{-1}\mathcal{L}_j Q & \ell_j \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 0_{(N-1)\times 1} \\ B \end{pmatrix} \text{ and } M = \begin{pmatrix} 0_{N\times(N-1)} & Q^T \\ 0_{N\times(N-1)} & 0_{N\times N)} \end{pmatrix}.$$

We introduce a useful lemma as follows.

Lemma 5. *[21] For the matrix* $F^{'} = \begin{pmatrix} 0_{N\times(N-1)} & Q^T \\ -\mathcal{L}Q & -kI_N \end{pmatrix}$ *and k is a positive constant, assume that k satisfies that* $k > \max\limits_{\mu_i\in\Lambda^+(\mathcal{L})}\{\frac{Im(\mu_i)}{\sqrt{Re(\mu_i)}}\}$*, where μ_i is the ith eigenvalue of $\mathcal{L}$ and $\Lambda^+(\mathcal{L})$ is the set of the nonzero eigenvalues of $\mathcal{L}$. Then, $F^{'}$ is stable if and only if $\mathcal{G}$ contains a spanning tree.*

Lemma 6. *Take $\gamma_i = \gamma_j = r > 0$, $i, j \in \{1, 2, \cdots, N\}$. Based on the above lemma, we can get that if* $r > \max\limits_{\mu_i\in\Lambda^+(\rho D^{-1}H^{-1}\mathcal{L})}\{\frac{Im(\mu_i)}{\sqrt{Re(\mu_i)}}\}$*, F is stable if and only if $\mathcal{G}$ has a spanning tree in this paper, where μ_i is the ith eigenvalue of $\rho D^{-1}H^{-1}\mathcal{L}$ and $\Lambda^+(\rho D^{-1}H^{-1}\mathcal{L})$ is the set of the nonzero eigenvalues of $\rho D^{-1}H^{-1}\mathcal{L}$.*

PROOF. It is easy to be proved.

Lemma 7. *[22] Assume that all the eigenvalues of matrix $\mathcal{P}$ are in the open left half plane. Then there exist positive constants $\mu_1 > 1$ and $\mu_2 > 0$ such that*

$$\left\|e^{\mathcal{P}t}\right\| \leq \mu_1 e^{-\mu_2 t}, t \geq 0.$$

Theorem 8. *Consider the dynamical systems* (1)*. If the communication topology contains a directed spanning tree, for arbitrary $\alpha_i > 0$ and $0 < \beta_i < 1$, it is solved that there is no case where all sample-data points are event-triggered points.*

PROOF. See the Appendices.

Remark 6. Theorem 1 with the consensus protocol (3) proves that the positions achieve static consensus of (1). For the dynamic consensus situation, we can design a virtual leader described as follows

$$\begin{cases} \dot{x}_{N+1}(t) = v_d, \\ \dot{v}_{N+1}(t) = 0, \end{cases} \tag{21}$$

where v_d is a nonzero constant. Therefore, the topology contains $N+1$ agents. Let the leader be the $(N+1)$th agent. Thus, the adjacency matrix in the dynamic consensus situation is denoted as A_{N+1}. We define the positions and the velocities at the sampling instants, respectively. The consensus protocol is designed as

$$\begin{aligned} u_i(t) =& \frac{\sigma_1}{\Sigma_{j=1}^{N+1} a_{ij}} \Sigma_{j=1}^{N} a_{ij}(\hat{x}_j(t) - \hat{x}_i(t)) \\ &+ \frac{\sigma_1}{\Sigma_{j=1}^{N+1} a_{ij}} a_{i(N+1)}(x_{N+1}(k^{'} h_{N+1}) - x_i(kh_i)) \\ &+ \frac{\sigma_2}{\Sigma_{j=1}^{N+1} a_{ij}} \Sigma_{j=1}^{N} a_{ij}(\hat{v}_j(t) - \hat{v}_i(t)) \\ &+ \frac{\sigma_1}{\Sigma_{j=1}^{N+1} a_{ij}} a_{i(N+1)}(v_{N+1}(k^{'} h_{N+1}) - v_i(kh_i)), \end{aligned}$$

where a_{ij} is the (i, j) entry of the adjacency matrix A_{N+1}, $t \in [kh_i, kh_i + h_i)$, $k^{'} = \lfloor \frac{t}{h_N} \rfloor$, $kh_i \in \mathcal{S}_i$ and $k^{'} h_{N+1} \in \mathcal{S}_{N+1}$, where $\sigma_i > 0$, $i = 1, 2$. By using the Lyapunov's stability theorem, we can prove that there exist two positive constants ϵ and T, for any $t > T$, such that the closed loop system satisfies,

$$v_i(t) - v_j(t) < \epsilon \text{ and } x_i(t) - x_j(t) < \epsilon, t > T,$$

for arbitrary initial conditions. Thus, the states achieve bounded consensus.

5. Simulation example

In this section, a simulation example is given to verify the effectiveness of the proposed consensus scheme. Consider the multi-agent system (1) with 6 nodes, and the communication topology among nodes is shown in Fig. 1.

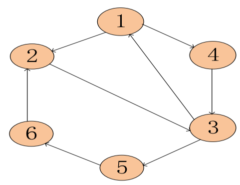

Figure 1: A graph contains a directed spanning tree.

In the simulation, we employ the consensus protocol (3) with the even-triggering function (6). Based on Remark 3, the triggering parameters in (3) are chosen as $\rho = 0.01 * \mathbf{1}_6$, $\gamma = \{0.12, 0.22, 0.32,\ 0.16, 0.09, 0.32\}$, and $H = [1.35, 1.395, 1.26, 1.44, 1.62, 1.26]$. The triggering parameters in (6) are selected as $\alpha_i = 10$ and $\beta_i = \exp(-0.03 \times h_i)$. The simulations results are shown in Figs.2-4 where the initial conditions are set to be $x(0) = [5, 4, 1, 7, 4, 1]$ and $v(0) = [2, 8, 5, 8, 10, 20]$.

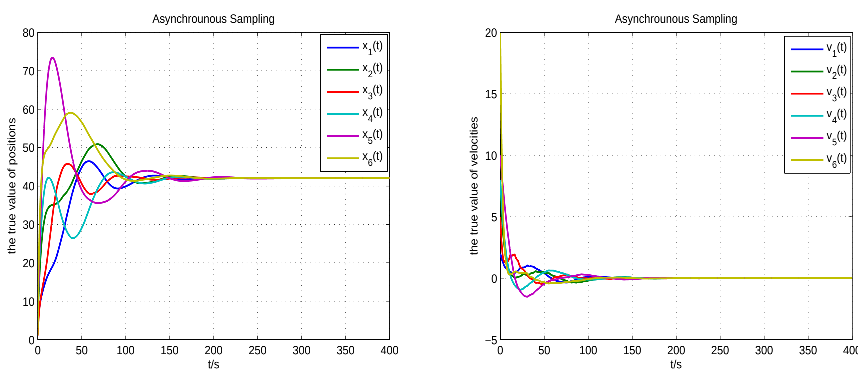

Figure 2: (a) Position curves of all agents; (b) Velocities curves of all agents.

It can be seen from Fig.2 that the positions of all agents quickly converge to a common constant and the velocities of all agents converge to zero. The triggering instants of all six agents are shown in Fig. 3. Fig. 4 depicts the trajectories of the measurement error $e(t)$ and the threshold $g(t)$.

The above simulation results are based on Theorem 1, which indeed verifies the effectiveness of the proposed consensus protocol.

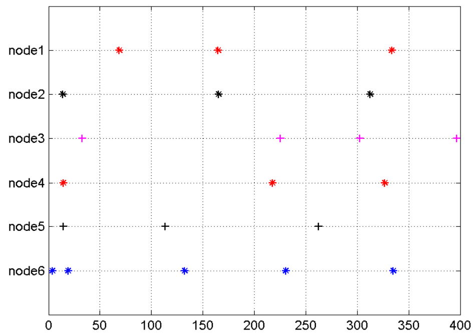

Figure 3: Event instants of all agents.

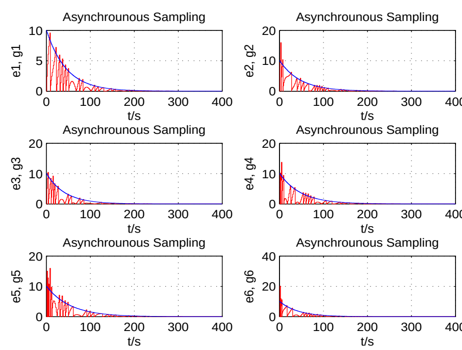

Figure 4: Trajectories of $e_i(t)$ and $g_i(t)$.

Remark 7. Because if the current state satisfies (7), agent i will not update its value instantly until its sampling instants to read the data from the buffer. Agent i will read the data from the buffer until its next sampling instant occurs. Thus, it can be seen that there are some instants that $e_i(t) > g_i(t)$ in Fig.4.

6. Conclusion

The asynchronous periodic sampling static consensus of second-order multi-agent systems is achieved in this paper by using event-triggered communication scheme. By comparing with the synchronous periodic sampling method, the asynchronous period method is more flexible and easier to be implemented.

Moreover, the use of event-triggered mechanism reduces the communication loads, which further saves the system energy.

7. Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant Nos. 61673308 and 61673014).

[1] Cao, Y., Yu, W., Ren, W., Chen, G.:'An overview of recent progress in the study of distributed multi-agent coordination', *IEEE Transactions on Industrial Informatics*, 2012, **9**, 1, pp. 427-438, 2012.

[2] Qin, J., Ma, Q., Shi, Y., Wang, L.:' Recent advances in consensus of multi-agent systems: A brief survey', *IEEE Transactions on Industrial Electronics*, 2017, **64**, 6, pp. 4972-4983

[3] Wang, S., Xie, D.:' Consensus of second-order multi-agent systems via sampled control: undirected fixed topology case', *IET Control Theory and Applications*, 2012, **6**, 7, pp. 893-899

[4] Chen, W., Li, X., Jiao, L.:' Quantized consensus of second-order continuous-time multi-agent systems with a directed topology via sampled data', *Automatica*, 2013, **49**, 7, pp. 2236-2242

[5] Tang, Z., Huang, t., Shao, J., Hu, J.:' Leader-following consensus for multi-agent systems via sampled-data control', *IET Control Theory and Applications*, 2011, **5**, pp. 1658-1665

[6] Wen, G., Duan, Z., Yu, W., Chen, G.:' Consensus of multi-agent systems with non-linear dynamics and sampled-data information: a delayed-input approach', *International Journal of Robust Nonlinear Control*, 2013, **23**, 6, pp. 602-619

[7] Ding, L., Zheng, W. X.:' Consensus tracking in heterogeneous nonlinear multi-agent networks with asynchronous sampled-data communication', *Systems & Control Letters*, 2016, **96**, 2, pp. 151-157

[8] Gao, Y., Zuo, M., Jiang, T., Du, J., Ma, J.:' Asynchronous consensus of multiple second-order agents with partial state information', *International Journal of Systems Science*, 2013, **44**, 5, pp. 966-977

[9] Dimarogonas, D. V., Frazzoli, E., Johansson, K. H.:' Distributed event-triggered control for multi-agent systems', *IEEE Transactions on Automatic Control*, 2012, **57**, 5, pp. 1291-1297

[10] Liu, K., Ji, Zh., Xie, G., Xu, R.:' Event-based broadcasting containment control for multi-agent systems under directed topology', *International Journal of Control*, 2016, **89**, 11, pp. 2360-2370

[11] Seyboth, G. S., Dimarogonas, D. V., Johansson, K. H.:' Event-based broadcasting for multi-agent average consensus', *Automatica*, 2013, **49**, 1, pp. 245-252

[12] Ding, L., Han, Q., Ge, X., Zhang, X.:' An overview of recent advances in event-triggered consensus of multiagent systems', *IEEE Transactions on Cybernetics*, 2018, **48**, 4, pp. 1110-1123

[13] Xia, H., Zheng, W., Shao, J.:' Event-triggered containment control for second-order multi-agent systems with sampled position data', *ISA Transactions*, 2017, 73, pp.91-99

[14] Guo, G., Ding, L., Han, Q.L.:' A distributed event-triggered transmission strategy for sampled-data consensus of multi-agent systems', *Automatica*, 2014, **50**, 5, pp. 1589-1596

[15] Liu, K., Ji, Zh.:' Consensus of multi-agent systems with time delay based on periodic sample and event hybrid control', *Neurocomputing*, 2017, **270**, pp. 11-17

[16] Xiao, F., Meng, X., Chen, T.:' Sampled-data consensus in switching networks of integrators based on edge events', *Automatica*, 2017, **75**, 1, pp. 214-220

[17] Cao, M., Xiao, F., Wang, L.:' Event-Based Second-Order Consensus Control for Multi-Agent Systems via Synchronous Periodic Event Detection', *IEEE Transactions on Automatic Control*, 2015, **60**, 9, pp. 2452-2457

[18] Cao, M., Xiao, F., Wang, L.:' Second-order consensus in time-delayed networks based on periodic edge-event driven control', *Systems and Control Letters*, 2016, **96**, pp. 37-44

[19] Meng, X., Xie, Soh, L, Y. C.:' Asynchronous periodic event-triggered consensus for multi-agent systems', *Automatica*, 2017, **84**, 1, pp. 214-220

[20] Hartfiel, D. J. Lecture notes in mathematics: Markov set-chains, Springer, 1998

[21] Tang, Zh., Huang, T., Shao, J., Hu, J.: 'Consensus of second-order multi-agent systems with nonuniform time-varying delays'. Neurocomputing, 2012, **97**, 15, pp. 410-414

[22] Zheng, D.: 'Linear System Theory'. Tsinghua University Press, Beijing, China, 2002

[23] Seyboth, G., Dimarogonas, D., Johansson, K.: 'Event-based broadcasting for multi-agent average consensus'. Automatica, 2013, **49**, pp. 245-252

8. Appendices

Proof of Lemma 2 According to the definition of the stochastic matrix, we can know that $a \geq 0, b \geq 0, c \geq 0, d \geq 0$ and $E_{ki1} + E_{ki2} + \cdots + E_{ki2n} = 1$. At the sampling instants $\tau_{k+1} \in \mathcal{S}_i$, the following inequalities hold

$$\begin{cases} 0 < \rho_i, \gamma_i h_i \leq 2, \\ \gamma_i h_i^2 - (2 + \rho_i) h_i + 2 \geq 0, \\ \gamma_i h_i^2 - (2 + \rho_i - 2\gamma_i) h_i - 2\rho_i \leq 0, \\ \gamma_i h_i^2 + (2\gamma_i - 2) h_i - 2 \leq 0. \end{cases} \tag{A.1}$$

Without loss of generality, we assume that $\rho_i > 0$, and $\gamma_i > 0$. From (A.1), we have

$$\begin{cases} (1) h_i \le \dfrac{2}{\gamma_i}; \\ (2) h_i \le \dfrac{2+\rho_i-\sqrt{\Delta_1}}{2\gamma_i} \text{ or } h_i \ge \dfrac{2+\rho_i+\sqrt{\Delta_1}}{2\gamma_i}; \\ (3) \dfrac{2+\rho_i-2\gamma_i-\sqrt{\Delta_2}}{2\gamma_i} \le h_i \le \dfrac{2+\rho_i-2\gamma_i+\sqrt{\Delta_2}}{2\gamma_i}; \\ (4) \dfrac{2-2\gamma_i-\sqrt{\Delta_3}}{2\gamma_i} \le h_i \le \dfrac{2-2\gamma_i+\sqrt{\Delta_3}}{2\gamma_i}, \end{cases}$$

where $\Delta_1 = (\rho_i+2)^2-8\gamma_i$, $\Delta_2 = (\rho_i+2-2\gamma_i)^2+8\gamma_i\rho_i$ and $\Delta_3 = 4\gamma_i^2+4$. One of the solutions is that if $2\rho_i \le \gamma_i \le \frac{1}{2}$, we obtain that $0 < h_i \le \frac{2+\rho_i-\sqrt{\Delta_1}}{2\gamma_i}$. Hence, E_k is a stochastic matrix and there exists an upper bound of the sampling period for each agent. The proof is completed. □

Proof of Theorem 2 First, we prove to that the states of the second-order multi-agent systems (1) achieve consensus with an exponential rate. By the numerical integration of (19), we have

$$y(t) = e^{Ft}y(0) + \int_0^t e^{F(t-\theta)}\{\mathcal{B}e(\theta) + \sum_{j=1}^{N} W_j \int_{\theta-\tau_j(\theta)}^{\theta} \dot{y}(s)ds\}d\theta.$$

It follows from the definitions of $g_i(t)$ that we can choose $\beta_i = e^{-\alpha h}$, where α and h are positive constants. Then, we can get that there exist positive constants μ_3 and ω such that

$$\|e_i(t)\| \le g_i(t) \le \mu_3 e^{-\omega t}, i \in V.$$

There is no loss of generality in assuming that the norm is 2-norm. Then, it can be seen that

$$\|e(t)\| \le \sqrt{N}\mu_3 e^{-\omega t}.$$

According to Lemma 5, all of the eigenvalues of the matrix F have negative real parts. It follows from Lemma 3 that there exist positive constants μ_1 and μ_2, where $\mu_2 > \omega$, such that

$$\|y(t)\| \leq \mu_1 e^{-\mu_2 t}\|\eta(0)\| + \mu_1 \int_0^t e^{-\omega(t-\theta)}\{\sqrt{N}\mu_3\|\mathcal{B}\|e^{-\mu_2\theta} + \|\sum_{j=1}^N W_j \int_{\theta-\tau_j(\theta)}^{\theta}[My(s) - \sum_{j=1}^N W_j y(s-\tau_j(s)) + \mathcal{B}e(s)]ds\|\}d\theta.$$

Denote $m_1 = \sqrt{N}\mu_3\|\mathcal{B}\|$, $m_2 = \|\sum_{j=1}^N W_j M\|$ and $m_4 = \sqrt{N}\|\sum_{j=1}^N W_j\|\mathcal{B}\|$. We have

$$\|y(t)\| \leq \mu_1 e^{-\mu_2 t}\|y(0)\| + \mu_1 \int_0^t e^{-\mu_2(t-\theta)}\{m_1 e^{-\omega\theta} + \int_{\theta-\tau_j(\theta)}^{\theta}[m_2\|y(s)\| + \|\sum_{j=1}^N W_j\|^2\|y(s-\tau_j(s))\| + m_4 e^{-\omega s}]ds\}d\theta. \tag{22}$$

Next, we can prove that there exist two constants $\lambda \in (0, \mu_2)$ and $\omega \in (0, \mu_2)$, where $\lambda > \omega$, satisfying

$$\frac{\alpha(m_2 + m_3 e^{\lambda d})(e^{\lambda d} - 1)}{\lambda(\mu_2 - \lambda)} < 1 \tag{23}$$

and

$$\delta = \frac{\mu_1\{m_1\omega + m_4(e^{\omega d} - 1)\}}{\omega(\mu_2 - \omega) - \mu_1(m_2 + m_3 e^{\omega d})(e^{\omega d} - 1)} > 0. \tag{24}$$

Then, the following inequality holds

$$\|y(t)\| < \mu_1\|y(0)\|e^{-\lambda t} + \delta e^{-\omega t} \triangleq \Delta(t), t \geq 0. \tag{25}$$

Therefore, the consensus for the states of all agents can be achieved.

Now construct a function

$$f(\lambda) = \alpha(m_2 + m_3 e^{\lambda d})(e^{\lambda d} - 1) - \lambda(\mu_2 - \lambda)$$

to prove (23) and (24). Clearly, we can obtain that $f(0) = 0$ and $\dot{f}(0) = \alpha(m_2 + m_3)d - \mu_2$. When d is bounded and $\lambda \in (0, \mu_2)$, we can get that $\dot{f}(0) < 0$. As a result, the inequality (23) holds. Similarly, (24) can be proved. We will show that (25) holds. In order to seek a contradiction, we assume that (25) does not hold. Hence, there exist some instants $t \geq 0$ such that the following inequality holds

$$\|y(t)\| \geq \Delta(t).$$

According to the continuity of $\eta(t)$ and $\omega(t)$, there exists a constant $t^* > 0$ such that

$$\begin{cases} \|y(t^*)\| = \mu_1 \|y(0)\| e^{-\lambda t^*} + \delta e^{-\omega t^*}, \\ \|y(t)\| < \Delta(t^*), 0 \leq t < t^*. \end{cases} \tag{26}$$

Denote $m_3 = \| \sum_{j=1}^{N} W_j \|^2 \|$. Obviously, $t - \tau_j(t)$ is bounded, $j \in V$. Thus, one can find a constant d which satisfies $\max_{j \in \mathcal{V}} |t - \tau_j(t)| \leq d$. Substitute (25) into (22)

$$\begin{aligned} \|y(t^*)\| <& \mu_1 e^{-\mu_2 t^*} \|y(0)\| + \mu_1 \int_0^{t^*} e^{-\mu_2 (t^* - \theta)} \{ m_1 e^{-\omega\theta} + \delta m_2 e^{-\omega s)} \\ &+ \int_{\theta - d}^{\theta} [\mu_1 m_2 \|y(0)\| e^{-\lambda s} + \mu_1 m_3 \|y(0)\| e^{-\lambda(s-d)} \\ &+ \delta m_3 e^{-\omega(s-d)} + m_4 e^{-\omega s}] ds \| \} d\theta. \end{aligned}$$

It is easy to be seen that

$$\begin{aligned} \|y(t^*)\| <& \mu_1 \|y(0)\| \{ e^{-\mu_2 t^*} + \mu_1 \int_0^{t^*} e^{-\mu_2 (t^* - \theta)} \int_{\theta - d}^{\theta} (m_3 e^{-\lambda s} + m_4 e^{-\lambda(s-d)} ds d\theta \} \\ &+ \mu_1 \int_0^{t^*} \{ m_1 e^{-\omega\theta} + \int_{\theta - d}^{\theta} (m_2 \delta e^{-\omega s} + \delta m_3 e^{-\omega(s-d)} + m_4 e^{-\omega s}) ds \} d\theta. \end{aligned}$$

By using the integral algorithm, the following inequality holds

$$\begin{aligned} \|y(t^*)\| <& \mu_1 \|y(0)\| \{ e^{-\mu_2 t^*} + (e^{-\lambda t^*} - e^{-\mu_2 t^*}) \frac{\mu_1 (m_2 + m_3 e^{\lambda d)(e^{\lambda d} - 1)}}{\lambda(r - \lambda)} \} \\ &+ \mu_1 \{ \frac{m_1}{\mu_2 - \omega} + \frac{m_2 \delta + m_3 \delta e^{\omega d} + m_4}{\omega(\mu_2 - \omega)} + (e^{\omega d} - 1) \frac{m_2 \delta + m_3 \delta e^{\omega d} + m_4}{\omega(\mu_2 - \omega)} \} \\ &\times (e^{-\omega t^*} - e^{-\mu_2 t^*}) \\ =& \mu_1 \|y(0)\| e^{-\lambda t^*} + \delta e^{-\omega t^*}. \end{aligned}$$

This result contradicts (26). That is, (25) holds. Therefore, we get that

$$\|v(t)\| < \mu_1 \|y(0)\| e^{-\lambda t} + \delta e^{-\omega t}, t \geq 0, \tag{27}$$

where $\lambda > \omega$.

In this position, we prove that the inter-event time has a positive lower bound τ_0 such that $t^i_{k+1} - t^i_k > \tau_0 > 0$, $k = 0, 1, \cdots$, $i =, 1, 2, \cdots, N$.

For $t \in [t^i_k, t^i_{k+1})$ and $t \in [lh_i, lh_i + h_i)$, where $t^i_k = lh_i$, based on the definition of $e_i(t)$, we have

$$|e_i(t)| = 0.$$

Because of the fact that the next event happens as soon as

$$f_i(e_i(t), g_i(t)) = g_i(t) - \mid e_i(t) \mid, t \in \mathcal{S}_i,$$

crosses zero, that is, it is not triggered before $\|e_i(t)\| = \mu_3 e^{-\omega t}$. Therefore, the event is not triggered in the interval $[lh_i, lh_i + h_i)$.

When $t \in [lh_i + h_i, lh_i + 2h_i)$, based on (11), we have

$$\begin{aligned} |e_i(t)| \leq & \int_{lh_i}^{lh_i + h_i} [\mu_1 \|y(0)\| e^{-\lambda s} + \delta e^{-\omega s}] ds \\ \leq & \int_{t^i_k}^{t} [\mu_5 e^{-\lambda t^i_k} + \delta e^{-\omega t^i_k}] ds \\ \leq & [\mu_5 e^{-\lambda t^i_k} + \delta e^{-\omega t^i_k}](t - t^i_k). \end{aligned}$$

Denote a lower bound on inter-event intervals as $\tau_i = t - t^i_k$. Then, we solve the following equation

$$[\mu_5 e^{(\omega - \lambda) t^i_k} + \delta](t - t^i_k) = \mu_3 e^{-\omega \tau_i}.$$

Because $0 < \omega < \lambda$, then we have that $\mu_5 e^{(\omega - \lambda) t^i_k} + \delta \leq \mu_5 + \delta$. For any $t^i_k > 0$, the solution $\tau_i(t^i_k)$ is greater than or equal to τ^*_i given by $[\mu_5 + \delta]\tau_i = \mu_3 e^{-\omega \tau_i}$. Similarly as in [23], we get that τ^*_i is strictly positive. As a result, all inter-event times are lower-bounded by a positive constant, i.e., $t^i_{k+1} - t^i_k > \tau_0 > 0$, $i = 1, 2, \cdots, N$.

As is stated in Remark 3 in this paper, for each agent i, it will not update its value instantly until its sampling instants to read the data from the buffer. And the agent will read the data from the buffer until its next sampling instant occurs. Thus, when we choose the sampling period h_i such that $h_i < \tau_0$, it is solved that there is no case where all sample-data points are event-triggered points. $\square$