On integral operators in weighted grand Lebesgue spaces of Banach-valued functions

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Abstract. The paper deals with boundedness problems of integral operators in weighted grand Bochner-Lebesgue spaces. We will treat both cases: when a weight function appears as a multiplier in the definition of the norm, or when it defines the absolute continuous measure of integration. Along with the diagonal case we deal with the off-diagonal case. To get the appropriate result for the Hardy-Littlewood maximal operator we rely on the reasonable bound of the sharp constant in the Buckley type theorem which is also derived in the paper.

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1 Introduction

The so-called L^p-maximal regularity plays a major role in Mathematical Analysis of PDEs. A result for the maximal L^p-regularity has recently been achieved with a Functional Analysis approach in [8] using the $\mathcal{R}$-boundedness of a family of integral operators (see also [23] for details). The results of this form obtained by Functional Analytic approach, until recently were only known in the Hilbert space case (for $p = 2$) (see [21, 22], and [35]).

The main goal of this paper is to introduce weighted grand Lebesgue spaces of functions valued in Banach spaces and to explore these spaces with respect to the boundedness of vector-valued integral operators, in particular, maximal functions, singular integrals and their commutators, fractional–type integrals, etc. The function spaces under our consideration are defined, generally speaking, on quasi-metric measure spaces with doubling measure.

It is known that the singular integrals are not extended to Banach-valued operators if the Banach space is arbitrary. Bourgain ([2] and [3]) and Burkholder ([4], [5]) showed that Hilbert transform is extended boundedly to $L^p(B)$, $1 < p < \infty$, if and only if the target Banach space B have the so-called UMD property i. e. the following inequality

$$\Big\| \sum_{k=1}^{n} \varepsilon_k d_k \Big\|_{L^p(B)} \le c \Big\| \sum_{k=1}^{n} d_k \Big\|_{L^p(B)}$$

holds, where $(d_k)_{k=1}^n$ is a martingale difference sequence in $L^p(B)$ and $\varepsilon_k = \pm 1$. The works of aforementioned authors on the UMD property had extremely important impact on Harmonic Analysis for Banach-space valued functions.

The above–mentioned statement makes for a convenient definition for UMD: A Banach space B has the UMD property if and only if the Hilbert transform extends to a bounded operator on $L^p(B)$, $1<p<\infty$.

Nowadays the concept of UMD has become the central notion in vector-valued Harmonic Analysis. There were established that a series of results from the classical and modern Littlewood-Paley and Calderón-Zygmund theories remain valid in the context of B-valued functions if and only if the Banach spaces have UMD properties. The theory of Banach-valued function spaces and integral operators in the context of UMD property was intensively studied in [1], [2], [3], [4], [5], [10], [11], [32], [33], [34], [36].

For scalar functions grand Lebesgue spaces $L^{p)}$ were introduced by T. Iwaniec and C. Sbordone in 1992 [12]. The theory of Iwaniec-Sbordone space is nowadays one of intensively developing directions in modern analysis. The necessity of introducing and studying these spaces grew because of their rather essential role in various fields. It turned out that in the theory of PDEs the grand Lebesgue spaces are appropriate to the existence and uniqueness solution, and, also, the regularity problem for various nonlinear differential equation. The boundedness problems for fundamental integral operators of Harmonic Analysis were intensively studied in [13], [14], [15], [16], [17], [18], [20], [24], [25], [26], [27], [28], [29], [30] (see also the monograph [31], Chapter 14 and references therein). In this paper the analogous problem is treated in Banach-valued weighted grand Lebesgue spaces.

2 Preliminaries

Let (X,d,μ) be a a quasi-metric measure space, i.e., X is an abstract set and d is a function $d: X\times X \mapsto [0,\infty)$ satisfying the conditions:

i) $d(x,y)=0$ if and only if $x=y$;

ii) $d(x,y)=d(y,x)$ for all $x,y\in X$;

iii) there exist a constant $\eta\geq 1$ such that for all $x,y,z\in X$,

$$d(x,y)\leq \eta[d(x,z)+d(z,y)].$$

A ball with center x and radius r in (X,d,μ) is denoted by $\mathcal{B}(x,r)$. It is also assumed that all balls are measurable with positive finite μ measure.

We say that μ is doubling measure if it satisfies the following condition: there is a positive constant A such that for all $x\in X$ and $r>0$,

$$\mu(\mathcal{B}(x,2r))\leq A\mu(\mathcal{B}(x,r)). \tag{1}$$

A quasi-metric measure space (X,d,μ) with doubling measure μ is called *a space of homogeneous type* (SHT shortly).

Given a Banach space B with a norm $\|\cdot\|$ and an almost everywhere positive locally integrable function (weight) $w: X\longrightarrow \mathbf{R}$, we denote by $L^p_w(X,B)$, $1<p<\infty$, the Bochner-Lebesgue space consisting of all B-valued strongly measurable functions f defined on X such that

$$\|f\|_{L^p_w(X,B)}=\left(\int\limits_X \|f(x)\|^p_B w(x)d\mu\right)^{1/p}<+\infty.$$

By slight abuse of notation in future sometimes $L^p_w(X,B)$ will be denoted by $L^p_w(B)$.

2.1 Weighted grand Bochner-Lebesgue spaces

On the base of the space $L^p_w(B)$ we introduce the weighted grand Bochner spaces. Let Φ_p, where $1<p<\infty$, be the collection of all nondecreasing, bounded functions $\varphi:(0,p-1]\longrightarrow \mathbf{R}$ such that $\varphi(0+)=0$.

Let X be a bounded set (i.e. it is contained in some ball $\mathcal{B}$) and let $1<p<\infty$ and $\varphi \in \Phi_p$. By $L_w^{p),\varphi}(X,B)$ (or simply $L_w^{p),\varphi}(B)$) we denote the set of all B-valued strongly measurable functions for which the norm

$$\|f\|_{L_w^{p),\varphi}(X,B)} = \sup_{0<\varepsilon<p-1} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon}(X,B)}$$

is finite.

Together with the space $L_w^{p),\varphi}(X,B)$ we consider the weighted grand Bochner-Lebesgue spaces in definition of which a weight function participates as a multiplier in norms.

The collection of all B-valued strongly measurable functions for which the norm

$$\|f\|_{\mathcal{L}_w^{p),\varphi}(X,B)} = \sup_{0<\varepsilon<p-1} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \|fw\|_{L^{p-\varepsilon}(X,B)}$$

is finite is denoted by $\mathcal{L}_w^{p),\varphi}(X,B)$.

It should be noticed that even for the classical case the spaces $L_w^{p),\varphi}$ and $\mathcal{L}_{w^{1/p}}^{p),\varphi}$ are different, while $\|f\|_{L_w^p} = \|fw^{1/p}\|_{L^p}$ (see [13] for the details).

Both these spaces $L_w^{p),\varphi}(X,B)$ and $\mathcal{L}_w^{p),\varphi}(X,B)$ are non-reflexive, non-separable Banach function spaces.

Denote by $\overline{\Phi}_\sigma$ the set of positive, measurable functions φ defined on $(0,\sigma)$, where $0<\sigma<p-1$, which are non-decreasing, bounded with the condition $\lim_{x\to 0}\varphi(x)=0$. It is noteworthy to mention that all the results of this paper are true also in the case of unbounded set X if the norms of the grand Lebesgue spaces are defined by the following way:

$$\|f\|_{L_w^{p),\varphi,\sigma}(X,B)} := \sup_{0<\varepsilon<\sigma} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon}(X,B)},$$

$$\|f\|_{\mathcal{L}_w^{p),\varphi,\sigma}(X,B)} := \sup_{0<\varepsilon<\sigma} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \|fw\|_{L^{p-\varepsilon}(X,B)},$$

where σ is a sufficiently small positive number depending on p. Notice that in these norms σ appears as an additional parameter.

If $\varphi(x)=x^\theta$, where $\theta>0$, then we denote:

$$L_w^{p),\varphi}(B) = L_w^{p),\theta}(B);\ \mathcal{L}_w^{p),\varphi}(B) = \mathcal{L}_w^{p),\theta}(B);\ L_w^{p),\varphi,\sigma}(B) = L_w^{p),\theta,\sigma}(B);\ \mathcal{L}_w^{p),\varphi,\sigma}(B) = \mathcal{L}_w^{p),\theta,\sigma}(B).$$

For structural properties and Köthe dual space for grand Bochner-Lebesgue spaces we refer to [19].

In the sequel we will essentially need the following definition.

A weight function w is said to be a weight of Muckenhoupt class $A_p(X)$ (or simply A_p) if there exists a constant C such that for all balls $\mathcal{B}\subset X$,

$$\left(\frac{1}{\mu\mathcal{B}}\int_{\mathcal{B}} w(t)d\mu(t)\right)\left(\frac{1}{\mu\mathcal{B}}\int_{\mathcal{B}} w^{1-p'}(t)d\mu(t)\right)^{p-1} \le C.$$

We will also need the following class of weights: let (X,d,μ) be a quasi–metric measure space with doubling measure. We say that μ-a.e. positive function u belongs to the Muckenhoupt–Wheeden class $\mathcal{A}_{p,q}(X)$ (or simply $\mathcal{A}_{p,q}$) if there is a positive constant C such that

$$\left(\frac{1}{\mu\mathcal{B}}\int_{\mathcal{B}} u^q(t)d\mu(t)\right)^{\frac{1}{q}}\left(\frac{1}{\mu\mathcal{B}}\int_{\mathcal{B}} u^{-p'}(t)d\mu(t)\right)^{\frac{1}{p'}} \le C.$$

It is easy to check that $u\in\mathcal{A}_{p,q}(X)$ if and only if $u^q\in A_{1+q/p'}(X)$.

2.2 Banach lattices

Definition 1. *A real Banach lattice B is a real vector space with a partial ordering $\leq$ such that*

(i) If $x \leq y$ for $x, y \in B$, then $x + y \leq y + z$ for all $z \in B$;

(ii) $\alpha x \geq 0$ for all $x \in X$ with $x \geq 0$ and $\alpha \in \mathbf{R}$ with $\alpha \geq 0$;

(iii) for all $x, y \in B$, there exists supremum $x \vee y$ and an infimum $x \wedge y$;

(iv) there is defined a norm $\|\cdot\|_B$ on B satisfying the condition if $|x| \leq |y|$, then $\|x\|_B \leq \|y\|_B$ for all $x, y \in X$, where $|x|$ is defined as follows: $|x| = x \vee -x$ for all $x \in B$.

If Banach lattice is considered on the space of measurable function $L^0(\Omega, \mu)$ defined on a measure space (Ω, Σ, μ), then $f \leq g$ means $f(\omega) \leq g(\omega)$ for μ- almost every $\omega \in \Omega$.

Definition 2. *Let (Ω, μ) be a measure space. Denote by $L^0(\Omega)$ the set of all measurable functions on Ω. A subspace B of $L^0(\Omega)$ equipped with a complete norm $\|\cdot\|_B$ is called a Banach function lattice over Ω if the following properties hold:*

(i) there exists $\varsigma \in B$ with $\varsigma > 0$ $\mu-$a.e.;

(ii) if $0 \leq f_n \uparrow f$ with a sequence $\{f_n\}_{n=1}^{\infty}$ in B, $f \in L^0(\Omega)$, and $\sup_{n \in \mathbf{N}} \|f_n\|_B < \infty$, then $f \in B$ and $\|f\|_B = \sup_{n \in \mathbf{N}} \|f_n\|_B$ (this property is called Fatou property);

(iii) if $f \in L^0(\Omega)$, $g \in B$, and $|f| \leq |g|$,then $f \in B$ and $\|f\|_B \leq \|g\|_B$.

Definition 3. *[23] Let B be a Banach lattice and let $1 \leq p, q \leq \infty$.*

(i) we say that B is $p-$convex if there exists a positive constant c such that

$$\Big\| \Big(\sum_{k=1}^{n} |x_k|^p \Big)^{1/p} \Big\|_B \leq c \Big(\sum_{k=1}^{n} \|x_k\|_B^p \Big)^{1/p}, \tag{2}$$

for all $x_1, \cdots, x_n \in B$, where under the symbol $\Big(\sum_{k=1}^{n} |x_k|^p \Big)^{1/p}$ we mean:

$$\Big(\sum_{k=1}^{n} |x_k|^p \Big)^{1/p} := \sup_{\|a\|_{\ell^{p'}} \leq 1} \sum_{k=1}^{n} a_k x_k, \ \ \frac{1}{p} + \frac{1}{p'} = 1.$$

(ii) we say that B is $q-$concave if the inverse of (2) holds.

The absolute value function $h \in B$ on Ω is defined by $|h|(\omega) = |h(\omega)|$, $\omega \in \Omega$. We identify a function $f \in L^p_w(X, B)$ defined on X with a function defined on the product $X \times \Omega$ by setting $f(x)(\omega) = f(x, \omega)$. We denote by $L^p_w(X, B) \otimes B$ the set of all vector-valued functions f of the type $f = \sum\limits_{j=1}^{k} f_j \otimes a_j$ for $f_j \in L^p_w(X, B)$, $a_j \in B$ and for some integer $k \geq 1$. This set is a dense subspace of $L^p_w(X, B)$ for $1 \leq p < \infty$ and any weight w. Given an operator T in $L^p_w(X, B)$, we define its extension $\overline{T}$ to $L^p_w(X, B) \otimes E$ (see, e.g., [32]) by

$$\overline{T} f(x, \omega) = T(f(\cdot, \omega))(x), \quad (x, \omega) \in X \times \Omega.$$

For further properties of Banach lattices and Bochner–Lebesgue integrals we refer, e.g., to [23], [10], [32].

2.3 Lattice Hardy–Littlewood maximal operator

Let B be a Banach lattice and let $\mathcal{F}$ be a finite collection of dyadic cubes in $\mathbf{R}^n$. We introduce the lattice Hardy–Littlewood maximal function (see e.g., [23], [9]). For any $f \in L^1_{loc}(\mathbf{R}^n, B)$ first we define dyadic lattice maximal function

$$\widetilde{M}_{\mathcal{F}} f(x) = \sup \langle |f| \rangle_Q \chi_Q(x), \ \ x \in \mathbf{R}^n,$$

in which the modulus and supremum are taken in the Banach lattice B, and the symbol $\langle g \rangle_Q$ denotes the integral average of a real-valued function g.

Definition 4. *A Banach lattice B has the Hardy–Littlewood property (HL property), if there is $r \in (1,\infty)$ such that for all finite collection of cubes $\mathcal{F}$ and all $f \in L^r(\mathbf{R}^n;B)$,*

$$\|\widetilde{M}_{\mathcal{F}} f\|_{L^r(\mathbf{R}^n;B)} \le C_{B,r,n}\|f\|_{L^r(\mathbf{R}^n;B)} \tag{3}$$

with a constant $C_{B,r,n} > 0$.

If we need to mention that B has the HL property with estimate (3), then we say that B has the HL_r property.

Remark 1. It is known (see, e.g., [23]) that if a Banach lattice B has HL property, then B is $q-$convex for some $q \in (1,\infty)$.

The extension of $\widetilde{M}_{\mathcal{F}} f$ to infinite collections of cubes $\mathcal{F}$ may be not well-defined, as it is a priori not clear whether this supremum exists. Moreover, $\widetilde{M}_{\mathcal{F}} f$ might not be strongly μ-measurable. On those Banach lattices, where the supremum is well-defined, the dyadic lattice Hardy–Littlewood maximal function for a dyadic system $\mathcal{D}$ is defined as

$$\widetilde{M}_{\mathcal{D}} f = \sup_{Q\in\mathcal{D}} \langle |f|\rangle_Q \chi_Q$$

and the lattice Hardy–Littlewood maximal function as

$$\widetilde{M} f = \sup_{Q\subset\mathbf{R}^n} \langle |f|\rangle_Q \chi_Q,$$

where the supremum is taken over all cubes in $\mathbf{R}^n$ with sides parallel to the axis.

If B is a Banach function lattice and $f : \mathbf{R}^n \to B$ is a locally integrable function, then it is clear that $\widetilde{M} f(x)$ is a function of ω given by the formula:

$$\widetilde{M} f(x)(\omega) = \sup_{Q\ni x} \frac{1}{|Q|}\int\limits_Q |f(y,\omega)|dy, \quad x \in \mathbf{R}^n.$$

Now we introduce the following definition:

Definition 5. *Let a Banach lattice B have HL_r property. We say that B has $\overline{HL}_r$ property if the following holds for the constant $C_{B,r,n}$ in (3):*

$$\sup_{0<\varepsilon<\sigma} C_{B,r-\varepsilon,n} < \infty \tag{4}$$

for some constant $\sigma \in (0, r-1)$.

Example 1. Let $B = L^s(\Omega)$, $s \in (1,\infty)$. Then B has $\overline{HL}_s$ property.

Indeed, by using the value of the bound of the norm of the Hardy–Littlewood maximal operator $\|M_{\mathcal{D}}\|_{L^s(\mathbf{R}^n)}$ in Lebesgue spaces

$$\|\widetilde{M}_{\mathcal{D}}\|_{L^s(\mathbf{R}^n)} \le s', \quad s' = \frac{s}{s-1},$$

(see e.g., [23], Sec. 5.1) and Fubini's theorem, we find that

$$\begin{aligned}
&\|\widetilde{M}_{\mathcal{F}} f\|_{L^s\left(\mathbf{R}^n, L^s(\Omega)\right)} = \|\|\widetilde{M}_{\mathcal{F}} f(\cdot,\omega)\|_{L^s(\Omega)}\|_{L^s(\mathbf{R}^n)} \\
&= \left(\int\limits_\Omega\int\limits_{\mathbf{R}^n} (\widetilde{M}_{\mathcal{F}} f(\cdot,\omega)(x))^s dx d\mu\right)^{1/s} \\
&\le s'\left(\int\limits_\Omega\int\limits_{\mathbf{R}^n} (|f(x,\omega)|)^s dx d\mu\right)^{1/s} = s'\|f\|_{L^s\left(\mathbf{R}^n, L^s(\Omega)\right)}.
\end{aligned}$$

Thus,

$$\|\widetilde{M}_{\mathcal{F}} f\|_{L^s\left(\mathbf{R}^n, L^s(\Omega)\right)} \le s'\|f\|_{L^s\left(\mathbf{R}^n, L^s(\Omega)\right)}$$

which means that $B = L^s(\Omega)$ has $\overline{HL}_s$ property.

Lemma 1. *Let B be a Banach lattice with the $\overline{HL}_r$ property, $1<r<\infty$. Then $\widetilde{M}_{\mathcal{F}}$ has weak $(1,1)$ type with the bound for which the estimate*

$$\|\widetilde{M}_{\mathcal{F}}\|_{1,\infty}\le C(B,r,n)$$

holds with a constant $C(B,r,n)$ satisfying condition (4) for some positive constant σ.

Proof. Suppose that B has $\overline{HL}_r$ property. Then (3) holds together with condition (4). For the same r we have (see [23], P.102)

$$\|\widetilde{M}_{\mathcal{F}}\|_{1,\infty}\le 2^r\|\widetilde{M}_{\mathcal{F}}\|_r^r+1. \tag{5}$$

Since B satisfies $\overline{HL}_r$ property we are done.

The next statement is Buckley-type theorem (see [6] for the classical case) for lattice Hardy–Littlewood maximal functions.

Proposition 1. *Let B be a $q-$convex Banach lattice for some $q\ge 1$. Assume that B has the $\overline{HL}_r$ property for some $r\in(1,\infty)$ and that $\widetilde{M}f$ is well-defined for any $f\in L^1_{loc}(\mathbf{R}^n;B)$. Then there exists a positive constant $C_{B,p,q,r,n}$ such that for any $p\in(1,\infty)$, for all $w\in A_p$ and $f\in L^p_w(\mathbf{R}^n,B)$,*

$$\|\widetilde{M}f\|_{L^p_w(\mathbf{R}^n,B)}\le C_{B,p,q,r,n}[w]_{A_p}^{\max\{\frac{1}{p-1},\frac{1}{q}\}}\|f\|_{L^p_w(\mathbf{R}^n,B)}, \tag{6}$$

where the constant $C_{B,p,q,r,n}$ satisfies the condition

$$\sup_{0<\varepsilon<\sigma} C_{B,p-\varepsilon,q,r,n}<\infty \tag{7}$$

for some small positive constant σ.

Remark 2. If $p=r$ in the previous theorem, then we denote $C_{B,p,q,r,n}$ by $C_{B,p,q,n}$, and consequently, estimate (7) reads as follows:

$$\sup_{0<\varepsilon<\sigma} C_{B,p-\varepsilon,q,n}<\infty.$$

Remark 3. It was shown in [23] (see Theorem 5.6.4) that the exponent of $[w]_{A_p}$ is sharp in the sense that we can not replace it with smaller one.

For the next statement we refer e.g., to [23]

Lemma 2. *Let B be a Banach lattice. If B is q_0- convex, $1<q_0<\infty$, then it is $q-$ convex for all $1<q<q_0<\infty$. Moreover, for the appropriate convexity constants we have*

$$M^{(q)}(B)\le M^{(q_0)}(B).$$

Proof of Proposition 1 is given in [23] (p. 100) but we are interested in the value of the constant $C_{B,p,q,r,n}$ in (6) which enables us to conclude that (7) holds.

Taking Lemma 2 into account, without the loss of generality we can assume that $p\neq q$. Suppose also that $r\neq p$. Let $\mathcal{F}$ be a finite collection of cubes. Taking (5) into account we have the estimate

$$\|\widetilde{M}_{\mathcal{F}}\|_{1,\infty}\le 2^q\|\widetilde{M}_{\mathcal{F}}\|_q^q+1. \tag{8}$$

In fact, this estimate enables us to conclude that $\widetilde{M}_{\mathcal{F}}$ is of weak $(1,1)$ type with a bound $\|\widetilde{M}_{\mathcal{F}}\|_{1,\infty}$ independent of $\mathcal{F}$. Furthermore, the following pointwise inequality holds (see [23], pp.101-103)

$$\|\widetilde{M}f(x)\|_B\le 2M^{(q)}(B)\|\widetilde{M}_{\mathcal{F}}\|_{1,\infty}\overline{A}_{q,\mathcal{S}}\Big(\|f\|_B\Big)(x),$$

where $M^{(q)}(B)$ is the $q-$convexity constant of the Banach lattice B, the constant $\|\widetilde{M}_{\mathcal{F}}\|_{1,\infty}$ does not depend on p and the operator $\overline{A}_{q,\mathcal{S}}$ is defined as follows:

$$\overline{A}_{q,\mathcal{S}}f = \left(\sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q^q \chi_Q\right)^{1/q}$$

with sparse family of dyadic intervals $\mathcal{S}$.

Since the estimate (see [23], p.98)

$$\|\overline{A}_{q,\mathcal{S}}f\|_{L_w^{q+1}(\mathbf{R}^n)} \le C_q[w]_{A_{q+1}}^{1/q}\|f\|_{L_w^{q+1}(\mathbf{R}^n)},$$

holds with the constant

$$C_q = 2^{1/q}\Big(\frac{q+1}{q}\Big)^{(q+1)/q}, \tag{9}$$

we conclude that by the sharp variant of Rubio de Francia's weighted extrapolation theorem (see Theorem 3.1 of [7]),

$$\|\overline{A}_{q,\mathcal{S}}f\|_{L_w^p(\mathbf{R}^n)} \le \overline{C}_{p,q}[w]^{\max\{\frac{1}{p-1},\frac{1}{q}\}}\|f\|_{L_w^p(\mathbf{R}^n)},$$

where

$$\overline{C}_{p,q} = C_q \max\left\{2^{q+1-p}(p')^{q+1-p}, 2^{\frac{p-q-1}{p-1}}p^{\frac{p-q-1}{p-1}}\right\}, \tag{10}$$

with C_q defined by (9).

Summarizing these estimates we find that

$$\begin{aligned}\|\widetilde{M}f\|_{L_w^p(\mathbf{R}^n,B)} &\le 2M^{(q)}(B)\|\widetilde{M}_{\mathcal{F}}\|_{1,\infty}\|\overline{A}_{q,\mathcal{S}}\|_{L_w^p(B)}\\ &\le 2M^{(q)}(B)\|\widetilde{M}_{\mathcal{F}}\|_{1,\infty}\overline{C}_{p,q}[w]^{\max\{\frac{1}{p-1},\frac{1}{q},\}}\|f\|_{L_w^p(\mathbf{R}^n,B)}\end{aligned}$$

with $\overline{C}_{p,q}$ defined by (10). Thus, we have the desired result.

If $r = p$, then we have the same conclusion by the assumption that B has $\overline{HL}_p$ property and Lemma 1.

The statement has been proved.

3 Main results

In this section we formulate the main results of this paper.

Let G be a bounded open set in $\mathbf{R}^n$ and let B be a Banach lattice. For $f \in L^1(G,B)$, we define

$$\widetilde{M}_G f(x) = \sup_{Q\subset\mathbf{R}^n}\left(\frac{1}{|Q|}\int\limits_{G\cap Q}|f(y)|dy\right)\chi_{G\cap Q}(x),$$

where as before, Q denotes cube in $\mathbf{R}^n$ with sides parallel to the coordinate axes.

Theorem 1. *Let G be a bounded open set in $\mathbf{R}^n$. Assume that a Banach function lattice B has the $\overline{HL}_r$ property for some $r\in(1,\infty)$ and that $\widetilde{M}_G f$ is well-defined for any $f\in L^1_{loc}(G;B)$. Suppose that w is a weight function on $\mathbf{R}^n$. Then the operator $\widetilde{M}_G$ is bounded in $L_w^{p),\varphi}(G,B)$ for all $p\in(1,\infty)$, $\varphi\in\Phi_p$ and $w\in A_p(\mathbf{R}^n)$.*

The next statements are formulated for a quasi-metric measure space with doubling measure (X,d,μ).

Theorem 2. *Let (X,d,μ) be an SHT with bounded X. Then every linear operator K bounded in $L_w^r(B)$ for arbitrary $r\in(1,\infty)$ and $w\in A_r(X)$, is also bounded in $L_w^{p),\varphi}(X,B)$ for every $p\in(1,\infty)$, $\varphi\in\Phi_p$ and $w\in A_p(X)$.*

Theorem 3. *Let (X,d,μ) be an SHT with bounded X. Every linear operator K bounded in $L^r_{w^r}(X,B)$ for arbitrary $r\in(1,\infty)$ and for any $w^r\in A_r(X)$, is also bounded in $\mathcal{L}^{p),\varphi}_w(X,B)$ for every $p\in(1,\infty)$, all $\varphi\in\Phi_p$ and $w^p\in A_p(X)$.*

The next statement treats with the off-diagonal case.

Theorem 4. *Let (X,d,μ) be an SHT, where X is bounded, and let $1<p<q<\infty$. We set $\frac{1}{p}-\frac{1}{q}=\alpha$. Let w be a weight function on X. Suppose that for the linear operator K the following inequality*

$$\|K(fw^\alpha)\|_{L^{q_0}_w(X,B)}\le C\|f\|_{L^{p_0}_w(X,B)}$$

holds for all p_0 and q_0 satisfying the condition $1<p_0<q_0<\infty$ and $\frac{1}{p_0}-\frac{1}{q_0}=\alpha$, and all $w\in A_{1+q_0/(p_0)'}(X)$, where a positive constant C does not depend on f. Then, for every $\varphi\in\Phi_p$ and ψ defined by

$$\psi(\varepsilon):=\varphi\Big(p-\frac{q-\varepsilon}{1+\alpha(q-\varepsilon)}\Big)^{\alpha(q-\varepsilon)+1},\quad 0<\varepsilon<q-1$$

and all $w\in A_{1+q/p'}(X)$, the inequality

$$\|K(fw^\alpha)\|_{L^{q),\psi}_w(X,B)}\le C\|f\|_{L^{p),\varphi}_w(X,B)}$$

holds with a positive constant C independent of f.

The next statement concerns the case when a weight function appears as a multiplier in the norm of grand Lebesgue space.

Theorem 5. *Let (X,d,μ) be an SHT with bounded X, and let $1<p<q<\infty$. We set $\frac{1}{p}-\frac{1}{q}=\alpha$. Let u be a $\mu-$a.e. positive function on X. Suppose that for the linear operator K the following inequality*

$$\|uKf\|_{L^{q_0}(X,B)}\le C\|uf\|_{L^{p_0}(X,B)}$$

holds for all p_0 and q_0 satisfying the condition $1<p_0<q_0<\infty$ and $\frac{1}{p_0}-\frac{1}{q_0}=\alpha$, and all $u\in\mathcal{A}_{p_0,q_0}(X)$, where a positive constant C does not depend on f. Then, for every $\varphi\in\Phi_p$ and ψ defined by

$$\psi(\varepsilon):=\varphi\Big(p-\frac{q-\varepsilon}{1+\alpha(q-\varepsilon)}\Big)^{\alpha(q-\varepsilon)+1},\quad 0<\varepsilon<q-1$$

and all $u\in\mathcal{A}_{p,q}(X)$, the inequality

$$\|uKf\|_{L^{q),\psi}(X,B)}\le C\|uf\|_{L^{p),\varphi}(X,B)}$$

holds with a positive constant C independent of f.

Corollary 1. *Suppose that (X,d,μ) be an SHT, where X is bounded. Let $1<p<q<\infty$. We set $\frac{1}{p}-\frac{1}{q}=\alpha$. Let w be a weight function on X. Suppose that for the linear operator K the following inequality*

$$\|K(fw^\alpha)\|_{L^{q_0}_w(X,B)}\le C\|f\|_{L^{p_0}_w(X,B)}$$

holds for all p_0 and q_0 satisfying the condition $1<p_0<q_0<\infty$ and $\frac{1}{p_0}-\frac{1}{q_0}=\alpha$, and all $w\in A_{1+q_0/(p_0)'}(X)$, where a positive constant C does not depend on f. Then the inequality

$$\|K(fw^\alpha)\|_{L^{q),q\theta/p}_w(X,B)}\le C\|f\|_{L^{p),\theta}_w(X,B)}$$

holds and all $w\in A_{1+q/p'}(X)$ with a positive constant C independent of f.

Corollary 2. *Suppose that (X, d, μ) be an SHT with bounded X, and that $1 < p < q < \infty$. We set $\frac{1}{p} - \frac{1}{q} = \alpha$. Let u be a μ- almost everywhere positive function on X. Suppose that for the linear operator K the following inequality*

$$\|uKf\|_{L^{q_0}(X,B)} \le C\|uf\|_{L^{p_0}(X,B)}$$

holds for all p_0 and q_0 satisfying the condition $1 < p_0 < q_0 < \infty$ and $\frac{1}{p_0} - \frac{1}{q_0} = \alpha$, and all $w \in \mathcal{A}_{p_0,q_0}(X)$, where a positive constant C does not depend on f. Then the inequality

$$\|uKf\|_{L^{q),q\theta/p}(X,B)} \le C\|uf\|_{L^{p),\theta}(X,B)}$$

holds and all $w \in \mathcal{A}_{p,q}(X)$ with a positive constant C independent of f.

4 Proofs of the main results

This section is devoted to the proofs of the main results.

4.1 Proof of Theorem 1

Let $1 < p < \infty$. Suppose also that B has $\overline{HL}_r$ property for some $r \in (1, \infty)$. Taking Remark 1 into account we have that B is $q-$convex for some $q \in (1, \infty)$. Thus, taking Remark 2 and Lemma 2 into account, without loss of generality we can assume that $q \ne p$ and $r \ne p$. Let $w \in A_p$. Then by the openness property of the A_p class, there is a positive constant σ such that $w \in A_{p-\sigma}$. By the monotonicity property of A_p classes we have that $w \in A_{p-\varepsilon}$ for $0 < \varepsilon < \sigma$. Applying Proposition 1 we have the estimate

$$\|\widetilde{M}f\|_{L^{p-\varepsilon}_w(B)} \le C_{B,p-\varepsilon,q,r,n}[w]_{A_{p-\varepsilon}}^{\max\{\frac{1}{p-\varepsilon-1},\frac{1}{q}\}}\|f\|_{L^{p-\varepsilon}_w(B)},$$

with the constant $C_{B,p-\varepsilon,q,r,n}$ satisfying (7). Consequently, by using the fact that $[w]_s \ge 1$ (this is a consequence of Lebesgue differentiation theorem) and monotonicity property of Muckenhoupt classes we find that

$$\varphi(\varepsilon)^{1/(p-\varepsilon)}\|\widetilde{M}f\|_{L^{p-\varepsilon}_w(B)} \le C_{B,p,\sigma,q,n}[w]_{A_{p-\sigma}}^{\max\{\frac{1}{p-\sigma-1},\frac{1}{q}\}}\|f\|_{L^{p),\varphi}_w(B)}, \quad 0 < \varepsilon < \sigma.$$

Thus,

$$I_1 = \sup_{0<r\le\sigma}(\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}}\|\widetilde{M}f\|_{L^{p-\varepsilon}_w(B)} \le c\|f\|_{L^{p),\varphi}_w(B)}. \tag{11}$$

Fix $\varepsilon \in (\sigma, p-1)$ so that $\frac{p-\sigma}{p-\varepsilon} > 1$. Using the Hölder inequality with respect to the exponent $(p-\sigma)/(p-\varepsilon)$ and observing that $\left(\frac{p-\sigma}{p-\varepsilon}\right)' = \frac{p-\sigma}{\varepsilon-\sigma}$ we have

$$\|\widetilde{M}f\|_{L^{p-\varepsilon}_w(B)} \le \|\widetilde{M}f\|_{L^{p-\sigma}_w(B)} \cdot (wG)^{(\varepsilon-\sigma)/[(p-\sigma)(p-\varepsilon)]}. \tag{12}$$

Further, since $\sigma < p-1$ and $\varepsilon \in (\sigma, p-1)$ we have

$$0 < \frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)} < \frac{p-1-\sigma}{p-\sigma}, \quad (p-1)\sigma^{-\frac{1}{p-\sigma}} > 1.$$

From the boundedness of $\widetilde{M}$ in $L^{p-\sigma}_w(B)$ and (12) we deduce (without loss of generality we can assume that $w(G) \ge 1$)

$$\begin{aligned}
I_2 &:= \sup_{\sigma<\varepsilon<p-1}[\varphi(\varepsilon)]^{\frac{1}{p-\varepsilon}}\|\widetilde{M}f\|_{L^{p-\varepsilon}_w(B)} \\
&\le \sup_{\sigma<\varepsilon<p-1}[\varphi(\varepsilon)]^{\frac{1}{p-\varepsilon}}[\varphi(\sigma)]^{-\frac{1}{p-\sigma}}\|\widetilde{M}f\|_{L^{p-\sigma}_w(B)}[w(G)]^{(\varepsilon-\sigma)/(p-\sigma)(p-\varepsilon)} \\
&\le c\sup_{\sigma<\varepsilon<p-1}[\varphi(\varepsilon)]^{\frac{1}{p-\varepsilon}}[\varphi(\sigma)]^{-\frac{1}{p-\sigma}}\|f\|_{L^{p),\varphi}_w(B)}[w(G)]^{(\varepsilon-\sigma)/(p-\sigma)(p-\varepsilon)} \\
&\le c\varphi(p-1)[\varphi(\sigma)]^{-\frac{1}{p-\sigma}}\|f\|_{L^{p),\varphi}_w(B)}[w(G)]^{(p-1-\sigma)/(p-\sigma)}.
\end{aligned}$$

Finally summarizing the estimates of I_1 and I_2 we conclude

$$\|\widetilde{M}f\|_{L^{p),\varphi}_w(B)} \leq C\|f\|_{L^{p),\varphi}_w(B)}$$

with a positive constant C independent of f.

Thus the theorem has been proved.

4.2 Proofs of Theorems 2 and 3

The vector space for all equivalence classes of strongly measurable functions from a σ-finite measure space (Ω, A, μ) into a Banach space B, identifying functions which are equal almost everywhere, is denoted by $L^0(\Omega, B)$. $L^0(\Omega, B)$ is a complete metric space.

Theorem A. [10]. *Let B_0 and B_1 be complex Banach spaces, let $1 \leq p_0, p_1; q_0, q_1 \leq \infty$, and let (Ω_0, A_0, μ_0) and (Ω_1, A_1, μ_1) be measure spaces. Let $T : L^{p_0}(\Omega_0, B_0) + L^{p_1}(\Omega_0, B_0) \longrightarrow L^0(\Omega_1, B_1)$ be a linear operator which maps $L^{p_0}(\Omega_0, B_0)$ into $L^{q_0}(\Omega_1, B_1)$ and $L^{p_1}(\Omega_0, B_0)$ into $L^{q_1}(\Omega_1, B_1)$.*

If

$$\|Tf\|_{L^{q_j}(\Omega_1,B_1)} \leq A_j\|f\|_{L^{p_j}(\Omega_0,B_0)} \quad \forall f \in L^{p_j}(\Omega_0, B_0), \quad (j = 0, 1)$$

then for all θ, $0 < \theta < 1$ the operator T maps $L^{p_\theta}(\Omega_0, B_0)$ into $L^{q_\theta}(\Omega_1, B_1)$ and, moreover,

$$\|Tf\|_{L^{q_\theta}(\Omega_1,B_1)} \leq A_0^{1-\theta} A_q^{\theta}\|f\|_{L^{p_\theta}(\Omega_0,B_0)} \quad \forall f \in L^{p_\theta}(\Omega_0, B_0),$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Proof of Theorem 2. Let $w \in A_p(X)$. Since the class $A_p(X)$ is open with respect to p, there exists some σ, such that $0 < \sigma < p - 1$ and $w \in A_{p-\sigma}(X)$. Applying interpolation Theorem A we infer that

$$\|Kf\|_{L^{p-\varepsilon}_w(B)} \leq c\|f\|_{L^{p-\varepsilon}_w(B)} \tag{13}$$

with a constant c independent on ε, $0 < \varepsilon \leq \sigma$. Consequently,

$$I_1 := \sup_{0<r\leq\sigma} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \|Kf\|_{L^{p-\varepsilon}_w(B)} \leq c\|f\|_{L^{p),\varphi}_w(B)} \tag{14}$$

The estimate for

$$I_2 := \sup_{\sigma<\varepsilon\leq p-1} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \|Kf\|_{L^{p-\varepsilon}_w}$$

is similar to that of I_2 from the proof of Theorem 1; therefore we omit the details.

Finally we have the desired result.

Proof of Theorem 3. By the condition of the Theorem $w^p \in A_p(X)$. Let us show that then there exists such σ, $0 < \sigma < p - 1$ that $w^{p-\sigma} \in A_{p-\sigma}(X)$. Since $w^p \in A_p(X)$ and $A_p(X)$ is open with respect to p, there exists such σ, $0 < \sigma < p - 1$ that $w^p \in A_{p-\sigma}(X)$. By the Jensen's inequality with the exponent $\frac{p}{p-\sigma}$ we get for all balls $\mathcal{B}$:

$$\left(\frac{1}{\mu\mathcal{B}}\int_{\mathcal{B}} w^{p-\sigma}(x)d\mu(x)\right)^{\frac{1}{p-\sigma}} \left(\frac{1}{\mu\mathcal{B}}\int_{\mathcal{B}} w^{(p-\sigma)(1-p-\sigma)'}(x)d\mu(x)\right)^{\frac{1}{(p-\sigma)'}}$$
$$\leq \left[\left(\frac{1}{\mu\mathcal{B}}\int_{\mathcal{B}} w^{p}(x)d\mu(x)\right) \left(\frac{1}{\mu\mathcal{B}}\int_{\mathcal{B}} w^{p(1-(p-\sigma))'}(x)d\mu(x)\right)^{p-\sigma-1}\right]^{\frac{1}{p}} \leq C.$$

Hence, $w^{p-\sigma} \in A_{p-\sigma}(\mu)$.

Now consider the operator

$$f \longrightarrow K_w f$$

where

$$K_w f = wK\left(\frac{f}{w}\right).$$

Note that the boundedness of K_w in $L^p(X,B)$ is equivalent to the boundedness of K in $L^p_{w^p}(X,B)$. On other hand, since $w^p \in A_p(\mu)$ and $w^{p-\sigma} \in A_{p-\sigma}(X)$ we have that by the condition of the theorem, the operator K is bounded in $L^p_{w^p}(X,B)$ and $L^{p-\sigma}_{w^{p-\sigma}}(X,B)$ simultaneously. Therefore the operator K_w is bounded in $L^p(X,B)$ and $L^{p-\sigma}(X,B)$. Taking interpolation Theorem A into account we conclude that for arbitrary $\varepsilon \in (0,\sigma)$,

$$\|K_w f\|_{L^{p-\varepsilon}(X,B)} \le c\|f\|_{L^{p-\varepsilon}(X,B)}$$

with some constant c independent of f and ε. Hence,

$$I_1 = \sup_{0<\varepsilon\le\sigma} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \|K_w f\|_{L^{p-\varepsilon}(X,B)} \le c \sup_{0<\varepsilon\le\sigma} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(X,B)} \le c\|f\|_{L^{p),\varphi}(X,B)}.$$

Further, repeating the arguments of the proof of Theorem 1, we find that

$$I_2 := \sup_{\sigma<\varepsilon\le p-1} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \|K_w f\|_{L^{p-\varepsilon}_w} \le c\|f\|_{L^{p),\varphi}(X,B)}.$$

We proved that the operator K_w in bounded in $L^{p),\varphi}(X,B)$; but it is equivalent to the boundedness of operator K in $\mathcal{L}^{p),\varphi}_w(X,B)$. Indeed, let $f = \frac{\psi}{w}$. Then we have

$$\begin{aligned}
\|Kf\|_{\mathcal{L}^{p),\varphi}_w(X,B)} &= \sup_{0<\varepsilon<p-1} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \left(\int_X (\|Kf(x)\|w(x))^{p-\varepsilon} d\mu(x)\right)^{\frac{1}{p-\varepsilon}} \\
&= \sup_{0<\varepsilon<p-1} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \left(\int_X \|K\left(\frac{\psi}{w}\right) w(x)\|^{p-\varepsilon} d\mu(x)\right)^{\frac{1}{p-\varepsilon}} \\
&\le c \sup_{0<\varepsilon<p-1} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \left(\int_X \|\psi(x)\|^{p-\varepsilon} d\mu(x)\right)^{\frac{1}{p-\varepsilon}} \\
&= c \sup_{0<\varepsilon<p-1} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \left(\int_X (\|f(x)\|w(x))^{p-\varepsilon} d\mu(x)\right)^{\frac{1}{p-\varepsilon}} \\
&= c\|f\|_{\mathcal{L}^{p),\varphi}_w(X,B)}.
\end{aligned}$$

4.3 Proofs of Theorems 4 and 5.

Proof of Theorem 4. First observe that $\psi \in \Phi_q$. Take σ such that $0 < \sigma < q-1$. Hölder's inequality and the fact that w is integrable on X yield the following inequality

$$\|K(fw^\alpha)\|_{L^{q-\varepsilon}_w(B)} \le \|K(fw^\alpha)\|_{L^{q-\sigma}_w(B)} w(X)^{\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}}$$

for $0 < \sigma < \varepsilon < q-1$. Hence,

$$\left\|K(fw^\alpha)\right\|_{L^{q),\psi(\cdot)}_w(B)} \le C \sup_{0<\varepsilon<\sigma} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \left\|K(fw^\alpha)\right\|_{L^{q-\varepsilon}_w(B)}, \tag{15}$$

where a positive constant C is independent of f.

Let $w \in A_{1+q/p'}$. Then by the openness property of Muckenhoupt classes, $w \in A_{1+(q/p')-s}(X)$ for some $s > 0$. Hence, there are positive constants σ_1 and σ_2 such that $w \in A_{1+\frac{q-\sigma_1}{(p-\sigma_2)'}}(X)$, and

$$\frac{1}{p-\sigma_2} - \frac{1}{q-\sigma_1} = \alpha. \tag{16}$$

By the assumption the operator

$$K_\alpha f = K(fw^\alpha)$$

is bounded from $L^p_w(B)$ to $L^q_w(B)$ and from $L^{p-\sigma_2}_w(B)$ to $L^{q-\sigma_1}_w(B)$.

By the Riesz-Thorin interpolation theorem (see Theorem A) we get that K_α is bounded from $L^{p-\eta}_w(B)$ to $L^{q-\varepsilon}_w(B)$ for η and ε satisfying

$$\frac{1}{p-\eta} = \frac{t}{p} + \frac{1-t}{p-\sigma_2}, \qquad \frac{1}{q-\varepsilon} = \frac{t}{q} + \frac{1-t}{q-\sigma_1}, \quad t \in [0,1].$$

Moreover,

$$\Big\|K_\alpha\Big\|_{L^{p-\eta}_w(B)\longrightarrow L^{q-\varepsilon}_w(B)} \le \Big\|K_\alpha\Big\|^t_{L^p_w(B)\longrightarrow L^q_w(B)} \Big\|K_\alpha\Big\|^{1-t}_{L^{p-\sigma_2}_w(B)\longrightarrow L^{q-\sigma_1}_w(B)}. \tag{17}$$

It is easy to see that

$$\begin{aligned}
&\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \frac{t}{p} - \frac{t}{q} + \frac{1-t}{p-\sigma_2} - \frac{1-t}{q-\sigma_1} \\
&= t\left[\frac{1}{p} - \frac{1}{q}\right] + (1-t)\left[\frac{1}{p-\sigma_2} - \frac{1}{q-\sigma_1}\right] \\
&= t\alpha + (1-t)\alpha = \alpha.
\end{aligned}$$

Hence,

$$\eta = p - \frac{q-\varepsilon}{1+\alpha(q-\varepsilon)}.$$

Consequently,

$$\begin{aligned}
&\sup_{0<\varepsilon<\sigma_1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|K(fw^\alpha)\|_{L^{q-\varepsilon}_w(B)} \le \sup_{0<\varepsilon<\sigma_1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|f\|_{L^{p-\eta}_w(B)} \\
&= \sup_{0<\eta<\sigma_2} \varphi(\eta)^{\frac{1}{p-\eta}} \|f\|_{L^{p-\eta}_w(B)}
\end{aligned}$$

Consequently, taking into account these estimates and (15) we find that

$$\begin{aligned}
&\Big\|K(fw^\alpha)\Big\|_{L^{q),\psi}_w(B)} \le C \sup_{0<\varepsilon<\sigma_1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \Big\|K(fw^\alpha)\Big\|_{L^{q-\varepsilon}_w(B)} \\
&\le C \sup_{0<\eta<\sigma_2} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|f\|_{L^{p-\eta}_w(B)} = C \sup_{0<\eta<\sigma_2} \varphi(\eta)^{\frac{1}{p-\eta}} \|f\|_{L^{p-\eta}_w(B)} \\
&\le C\|f\|_{L^{p),\varphi}_w(B)}.
\end{aligned}$$

Proof of Theorem 5 is similar to that of Theorem 4. We will check only the interpolation argument. Suppose that $u \in \mathcal{A}_{p,q}(X)$. Then $u^q \in A_{1+q/p'}(X)$ and, hence, by the openness property of Muckenhoupt classes, $u^q \in A_{1+q/p'-s}(X)$ for some small positive constant s. Consequently, we can choose small positive constants σ_1 and σ_2 such that that $u^q \in A_{1+\frac{q-\sigma_1}{(p-\sigma_2)'}}(X)$ and condition (16) is satisfied.

Further, applying Hölder's inequality with respect to the exponent $\frac{q}{q-\sigma_1}$ we see that $u^{q-\sigma_1} \in A_{1+\frac{q-\sigma_1}{(p-\sigma_2)'}}(X)$. Thus, $u \in \mathcal{A}_{p-\sigma_2,q-\sigma_1}(\mathrm{X})$.

From the assumption of the theorem we have that, $\widetilde{K}_u f = uK(f/u)$ is bounded from $L^p(X,B)$ to $L^q(X,B)$ and from $L^{p-\sigma_2}(X,B)$ to $L^{q-\sigma_1}(X,B)$. By virtue of the Riesz-Thorin interpolation Theorem A we find that $\widetilde{K}_u$ is bounded from $L^{p-\eta}(X,B)$ to $L^{q-\varepsilon}(X,B)$ for η and ε satisfying the condition:

$$\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha.$$

Since the estimate similar to (17) holds for the norm of the operator $\widetilde{K}_u$, arguing as in the proof of Theorem 4 (see also the proofs of other theorems) we conclude that we have the desired result.

The theorem has been proved.

Corollaries 1 and 2 are a direct consequences of Theorems 4 and 5 respectively. Indeed, keeping the notation of Theorem 4 observe that if $\varphi(x) = x^\theta$, then $\psi(x) \approx x^{q\theta/p}$ for small positive x.

5 Weighted norm inequalities for integral operators

Let B be a Banach lattice of real-valued measurable functions on a σ-finite measure space (Ω, ν). The following theorem was proved in [32].

Theorem B. *Let $\mathbf{T}$ be the torus and B be a UMD lattice. Let T be an operator bounded $L^p_w(\mathbf{T}, B)$ for all $w \in A_p(\mathbf{T})$, $1 < p < \infty$. Then $\overline{T}$ is bounded in $L^p(\mathbf{T}, B)$, $1 < p < \infty$.*

Applying this theorem and known results [13], [15], [20], we deduce the boundedness of the following operators in $L^{p),\varphi}_w(\mathbf{T}, B)$ for $w \in A_p(\overline{T})$ (see [32]):

i) The conjugate function

$$\widetilde{f}(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) ctg\, \frac{t-\theta}{2} d\theta;$$

ii) Carleson's maximal partial sum operator of Fourier series

$$Sf(t) = \sup_n |S_n f(t)| = \sup_n \left| \sum_{|k|\le n} \hat{f}(k) e^{fkt} \right|;$$

iii) The Cauchy singular integral operator

$$Cf(t) = p.v. \int_{\mathbf{R}} \frac{f(s)ds}{t - s + i(\varphi(t) - \varphi(s))};$$

where φ is a Lipschitz function in $\mathbf{R}$.

iv) Any convolution operator $Tf(x) = K * f(x)$ in $\mathbf{R}^n$ such that : $|K(x)| \le C|x|^{-n}$ and also

$$\int_{\varepsilon<|x|<R} |K(x)dx| \le C \quad (0 < \varepsilon < R < \infty)$$

$$\int_{|x|>R>|y|} |K(x-y) - K(x)| dx \le CR^{-\delta}$$

for some fixed $\delta > 0$.

6 More on singular integrals

In this section we will follow some definitions from [36]. We are going to discuss Banach-valued extensions of operators for functions defined on a homogeneous type space X and with values in a UMD Banach lattice.

Let $\mathcal{G}$ be a locally compact Hausdorff topological group with unit element e, H a compact subgroup of $\mathcal{G}$, and $\pi : \mathcal{G} \longrightarrow \mathcal{G}/H$ the canonical map. Let dg denote a left Haar measure on $\mathcal{G}$, which we assume to be normalized in the case of $\mathcal{G}$ compact. If A is a Borel subset of $\mathcal{G}$, we denote by $|A|$ the Haar measure of A. The homogeneous space $X = \mathcal{G}/H$ is the set of all left cosets $\pi(g) = gH$, $g \in \mathcal{G}$, equipped with the quotient topology. The Haar measure dg induces a measure μ on the Borel σ-algebra on X. For $f \in L^1(X)$,

$$\int\limits_X f(x)d\mu(x) = \int\limits_G f \circ \pi(g)dg.$$

The measure μ on X is invariant under the action of $\mathcal{G}$, that is, if $f \in L^1(X)$, $g \in \mathcal{G}$ and $R_g f(x) = f(g^{-1}x)$, then

$$\int\limits_X f(x)d\mu(x) = \int\limits_X R_g f(x)d\mu(x).$$

A quasi-distance on X is a map $d : X \times X \longrightarrow [0, \infty)$ satisfying conditions (i)- (iii) listed in the beginning of Section 2, and moreover,

iv) $d(gx, gy) = d(x, y)$ for all $g \in \mathcal{G}$, $x, y \in X$;

v) the balls $\mathcal{B}(x, l) = \{y \in X : \ d(x, y) < l\}$, $x \in X$, $l > 0$, are relatively compact and measurable, and the balls $\mathcal{B}(1, l)$, $l > 0$, form a basis of neighborhoods of $1 = \pi(e)$.

If μ satisfies doubling condition (1), then (X, d, μ) is called homogeneous type space.

We say that a linear operator T defined on $L^\infty_c(X)$ and with values in the space of all measurable functions, is a singular integral operator if the following conditions hold:

i) T has a bounded extension on $L^r(X)$ for some r, $1 < r \leq \infty$;

ii) there exists a kernel $K \in L^1_{loc}(X \times X \setminus \triangle)$, $\triangle = \{(x, x) : x \in X\}$, such that

$$Tf(x) = \int\limits_X K(x, y)f(y)d\mu(y)$$

for all $f \in L^\infty_c(X)$ and almost all $x \notin \ suppf$.

Let T be a singular integral operator with a kernel K. We say that K satisfies the condition (H_∞) if

$$|K(x, y) - K(x, 1)| \leq C \frac{d(y, 1)}{d(x, 1)\mu(\mathcal{B}(1, d(x, 1))}$$

whenever $d(x, 1) > 2d(y, 1)$, $1 = \pi(e)$.

Basing on Theorem 1.7 of [36] and Theorem 3.1 we infer the following statement:

Theorem 6. *Let B be a Banach lattice of real-valued measurable functions with the UMD property, let $1 < p < \infty$, $w \in A_p(X)$, and let T be a singular integral operator. Assume that the kernels $K(x, y)$ and $K(y, x)$ satisfy (H_∞) condition and $K(gx, gy) = K(x, y)$ for all $x, y \in X$, $g \in \mathcal{G}$. Then the operator T is bounded in $L^{p),\varphi}_w(B)$.*

Theorem 7. *Let B be a Banach space with the UMD property and with a normalized unconditional basis $(e_j)_{j\geq 1}$. Let $1 < p < \infty$, $w \in A_p(X)$, and let T be a singular integral operator. Assume that the kernels $K(x, y)$ and $K(y, x)$ satisfy (H_∞) condition and $K(gx, gy) = K(x, y)$ for all $x, y \in X$, $g \in \mathcal{G}$. Then for all $f = \sum\limits_j f_j e_j \in L^{p),\varphi}_w(B)$ the series $\sum\limits_j Tf_j e_j$ converges in $L^{p),\varphi}_w(B)$ and there exists a positive constant C_p such that*

$$\Big\| \sum_{j=1}^{\infty} Tf_j e_j \Big\|_{L^{p),\varphi}_w(B)} \leq C_p \Big\| \sum_{j=1}^{\infty} f_j e_j \Big\|_{L^{p),\varphi}_w(B)}.$$

7 The case of unbounded space

In this section we will assume that (X,d,μ) is a quasi-metric measure space with doubling measure μ such that X is unbounded, i.e., there is no ball containing X. Observe that in this case it might be happened that $w(X)=\infty$. Let σ be a number such that $0<\sigma<p-1$.

We will see that the statements proved above are also true for the case $\mu X=\infty$ if we replace the spaces $L^{p),\varphi}_w(B)$ by $L^{p),\varphi,\sigma}_w(B)$.

Let w be a weight on X and let and $p\in(1,\infty)$. Denote by $\sigma_{p,w}$ constant such that $w\in A_{p-\sigma_{p,w}}$ whenever $w\in A_p$. Because of the openness property of Muckenhoupt's A_p class, such a constant always exists.

Theorem 8. *Let $X=\mathbf{R}^n$, d be the Euclidean metric and let μ be the Lebesgue measure on $\mathbf{R}^n$. Suppose that $1<p<\infty$ and let $w\in A_p(\mathbf{R}^n)$. Assume that B has the $\overline{HL}_r$ property for some $r\in(1,\infty)$ and that $\widetilde{M}f$ is well-defined for any $f\in L^1_{loc}(\mathbf{R}^n;B)$. Then the operator $\mathcal{M}$ is bounded in $L^{p),\varphi,\sigma_{p,w}}_w(\mathbf{R}^n,B)$ for all $\varphi\in\overline{\Phi}_{\sigma_{p,w}}$.*

Proof of this statement is similar to that of Theorem 1. In this case we do not have the term I_2.

Analogously, we have the next statements:

Theorem 9. *Let (X,d,μ) be an SHT and let a linear operator K be bounded in $L^r_w(X,B)$ for arbitrary r, $1<r<\infty$, and $w\in A_r(X)$. Then K is bounded in $L^{p),\varphi,\sigma_{p,w}}_w(B)$ for every $p\in(1,\infty)$, all $w\in A_p(X)$ and $\varphi\in\overline{\Phi}_{\sigma_{p,w}}$.*

Theorem 10. *Let (X,d,μ) be an SHT. Suppose that a linear operator K is bounded in $L^r_w(X,B)$ for arbitrary r, $1<r<\infty$, and for any $w^r\in A_r(X)$. Then K is also bounded in $L^{\varphi,\sigma_{p,w}}_{w^p}(B)$ for every $p\in(1,\infty)$, for all $w\in A_p(X)$ and $\varphi\in\overline{\Phi}_{\sigma_{p,w}}$.*

To formulate the next statement we need to introduce the notation. Let $1<p<q<\infty$. Let us set $\frac{1}{p}-\frac{1}{q}=\alpha$. Let $w\in A_{1+q/p'}(X)$. We denote by σ and δ constants defined as follows:

$$w\in A_{1+\frac{q-\delta}{(p-\sigma)'}}(X),\quad \frac{1}{p-\sigma}-\frac{1}{q-\delta}=\alpha. \tag{18}$$

By the openness property of the Muckenhoupt's $A_p(X)$ weights such constants exist (see the proof of Theorem 4 for details).

Now we can formulate the next statement:

Theorem 11. *Let (X,d,μ) be an SHT. Suppose that $1<p<q<\infty$. We set $\frac{1}{p}-\frac{1}{q}=\alpha$. Suppose that for the linear operator K, the following inequality*

$$\|K(fw^\alpha)\|_{L^{q_0}_w(X,B)}\le C\|f\|_{L^{p_0}_w(X,B)}$$

holds for all p_0 and q_0 satisfying the condition $1<p_0<q_0<\infty$, $\frac{1}{p_0}-\frac{1}{q_0}=\alpha$, and for all $w\in A_{1+q_0/(p_0)'}(X)$, where a positive constant C does not depend on f. Then for all $w\in A_{1+q/p'}(X)$ and $\varphi\in\overline{\Phi}_\sigma$ the inequality

$$\|K(fw^\alpha)\|_{L^{q),\psi,\delta}_w(X,B)}\le C\|f\|_{L^{p),\varphi,\sigma}_w(X,B)}$$

holds with a positive constant C independent of f, where σ and δ are defined in (18), and ψ is given by

$$\psi(\varepsilon):=\varphi\Big(p-\frac{q-\varepsilon}{1+\alpha(q-\varepsilon)}\Big)^{\alpha(q-\varepsilon)+1}.$$

Theorem 11 can be obtained by using the arguments of the proof of Theorem 4. We only emphasize that since $\varphi\in\overline{\Phi}_\sigma$ we have that $\psi\in\overline{\Phi}_\delta$.

Corollary 3. *Let (X,d,μ) be an SHT. Suppose that $1<p<q<\infty$. We set $\frac{1}{p}-\frac{1}{q}=\alpha$. Suppose that for a linear operator K the following inequality*

$$\|K(fw^{\alpha})\|_{L_w^{q_0}(X,B)} \le C\|f\|_{L_w^{p_0}(X,B)}$$

holds for all p_0 and q_0 satisfying the condition $1<p_0<\infty$, $\frac{1}{p_0}-\frac{1}{q_0}=\alpha$, and for all $w\in A_{1+q_0/(p_0)'}(X)$, where a positive constant C does not depend on f. Then for all $\theta>0$ and $w\in A_{1+q/p'}(X)$ the inequality

$$\|K(fw^{\alpha})\|_{L_w^{q),q\theta/p,\delta}(X,B)} \le C\|f\|_{L_w^{p),\theta,\sigma}(X,B)}$$

holds with a positive constant C independent of f.

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