INTRINSIC DECAY RATES FOR THE ENERGY OF A SINGULAR NONLOCAL VISCOELASTIC SYSTEM

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Abstract. This work deals with intrinsic decay rates for the energy of an initial boundary value problem with a nonlocal boundary condition for a system of nonlinear singular viscoelastic equations. We prove the intrinsic decay rates for the energy of a singular one-dimensional viscoelastic system with a nonlinear source term and nonlocal boundary condition of relaxation kernels described by the inequality $g_i'(t) \leq -H(g_i(t))$, $(i = 1, 2)$ for all $t \geq 0$, with H convex.

1. Introduction

In fluid dynamics, the decay of solutions of problem has attracted much attention and challenge among physicists and mathematicians. In this paper, we study the intrinsic decay rates for the energy of the following system

$$\left\{\begin{array}{l} u_{tt} - \dfrac{1}{x}(xu_x)_x + \displaystyle\int_0^t g_1(t-s)\dfrac{1}{x}(xu_x(x,s))_x\,ds = |v|^{q+1}|u|^{p-1}u, \text{ in } Q, \\ \\ v_{tt} - \dfrac{1}{x}(xv_x)_x + \displaystyle\int_0^t g_2(t-s)\dfrac{1}{x}(xv_x(x,s))_x\,ds = |u|^{p+1}|v|^{q-1}v, \text{ in } Q, \end{array}\right. \tag{1.1}$$

with initial data

$$\left\{\begin{array}{lll} u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & x \in (0,\alpha), \\ \\ v(x,0) = v_0(x), & v_t(x,0) = v_1(x), & x \in (0,\alpha), \end{array}\right. \tag{1.2}$$

and nonlocal boundary condition

$$u(\alpha,t) = v(\alpha,t) = 0, \ \int_0^\alpha xu(x,t)dx = \int_0^\alpha xv(x,t)dx = 0, \tag{1.3}$$

where $Q := (0,\alpha) \times (0,T)$, $\alpha < \infty$, $T < \infty$, p, $q > 1$. It is assumed that the kernels g_1 and g_2 meet certain conditions to be determined later, and $u_0(x)$, $v_0(x)$, $u_1(x)$ and $v_1(x)$ are given functions. The convolution term $\int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds$ and $\int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x\,ds$ reflects the memory effects of materials due to viscoelasticity. Here the convolution kernel g_1 and g_2 satisfies proper conditions exhibiting "memory character" which will be explained later. This type of problems

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arises in viscoelasticity and in systems governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear singular Boltzmann's model. Over the past few decades, the uninterrupted, mixed problems of a wide range of partial and differential equations have been rewarded and the reason for this great concern is that these problems are particularly inspired by physics and physical science.

Generally speaking, nonlocal boundary conditions can be encountered in many scientific domains and are widely applied in heat transmission theory, and many engineering models population dynamics, and control theory, and chemical engineering, and medical science, and chemical reaction diffusion, and thermo elasticity, and heat conduction processes, and biological processes. See in this regard the works by Cahlon and Shi [1], Mesloub and Lekrine [2], Ewing and Lin [3], Shi [4], Choi and Chan [5], Cannon [6], Capasso-Kunisch [7], Yurchuk [8], Shi and Shilor [9], Ionkin and Moiseev [10], Kamynin [11], Mesloub [12, 13], Ionkin [14], Mesloub and Messaoudi [15, 16], Kartynnik [17], Pulkina [18, 19], Mesloub and Bouziani [20, 21].

The motivation of our work is due to some results regarding the following research papers:

Mesloub, S.; Mesloub, F. [22] studied the solvability of a mixed nonlocal problem for a nonlinear singular viscoelastic equation

$$\begin{cases} u_{tt}\left(t\right)-\dfrac{1}{x}\left(xu_{x}\left(t\right)\right)_{x}+\displaystyle\int_{0}^{t}g\left(t-s\right)\dfrac{1}{x}\left(xu_{x}\left(x,s\right)\right)_{x}ds+au_{t}\left(t\right) \\ =f\left(x,t,u_{x},u\right),\ \text{ in } Q, \\ u(x,0)=u_{0}(x),\ u_{t}(x,0)=u_{1}\left(x\right),\ x\in\left(0,1\right), \\ u_{x}\left(1,t\right)=0,\ \displaystyle\int_{0}^{1}xu\left(x,t\right)dx=0\ \ t\in\left[0,T\right], \end{cases}$$

where $Q:=(0,1)\times(0,T)$, $a>0$ and for the relaxation function $g\left(t\right)$, we assume that $g:\mathbb{R}_{+}\longrightarrow\mathbb{R}_{+}$ is a bounded C^{2} function such that

$$g\left(s\right)\geq 0,\ g'\left(s\right)\leq 0 \text{ and } \int_{0}^{\infty}g\left(s\right)ds<1.$$

Shuntang WU. [23] studied the blow-up of solutions for a singular nonlocal viscoelastic equation

$$\begin{cases} u_{tt}\left(t\right)-\dfrac{1}{x}\left(xu_{x}\left(t\right)\right)_{x}+\displaystyle\int_{0}^{t}g\left(t-s\right)\dfrac{1}{x}\left(xu_{x}\left(x,s\right)\right)_{x}ds=\left|u\right|^{p-2}u,\ \text{ in } Q, \\ u(x,0)=u_{0}(x),\ u_{t}(x,0)=u_{1}\left(x\right),\ x\in\left(0,\alpha\right), \\ u\left(\alpha,t\right)=0,\ \displaystyle\int_{0}^{\alpha}xu\left(x,t\right)dx=0\ \ t\in\left[0,T\right], \end{cases}$$

where $Q:=(0,\alpha)\times(0,T)$, $\alpha<\infty$, $T<\infty$, $p>2$, and $g:\mathbb{R}^{+}\to\mathbb{R}^{+}$ represents the kernel of the memory term which is specified later (See [23]).

Liu, W.; Sun, Y and Li, G. [24] studied the on decay and blow-up of solutions for a singular nonlocal viscoelastic problem with a nonlinear source term

$$\begin{cases} u_{tt}(t) - \frac{1}{x}(xu_x(t))_x + \int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds + au_t = |u|^{p-2}u, \text{ in } Q, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in (0,l), \\ u(l,t) = 0,\ \int_0^l xu(x,t)\,dx = 0, \quad t \in [0,T], \end{cases}$$

where $Q := (0,l) \times (0,T)$, $a \geq 0$, $l < \infty$, $T < \infty$, $p > 2$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ represents the kernel of the memory term which is specified later (See [24]).

Zarai, A.; Draifia, A & Boulaaras, S. [25] studied the global existence and decay of solutions of a singular nonlocal viscoelastic system

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x ds + u_t = |v|^{q+1}|u|^{p-1}u, \text{ in } Q, \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x ds + v_t = |u|^{p+1}|v|^{q-1}v, \text{ in } Q, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in (0,\alpha), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ x \in (0,\alpha), \\ u(\alpha,t) = v(\alpha,t) = 0,\ \int_0^\alpha xu(x,t)\,dx = \int_0^\alpha xv(x,t)\,dx = 0, \end{cases}$$

where $Q := (0,\alpha) \times (0,T)$, $\alpha < \infty$, $T < \infty$, $p, q > 1$, et $g_1(.)$, $g_2(.) : \mathbb{R}^+ \to \mathbb{R}^+$, given functions (See [25]).

Marcelo M. Irena Lasiecka and Claudete M. Webler. [28] studied the intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density. Cavalcanti et al. [32] studied the well-posedness and the optimal decay rate estimates of the energy associated with the following non-linear viscoelastic equation with strong damping. Wu [29] studied the general decay of energy for a viscoelastic equation with damping and source terms. Mu and Ma [31] studied the system of nonlinear wave equations with Balakrishnan–Taylor damping. M. M. Cavalcanti, V. N. Domingos Cavalcanti, I. Lasiecka and F. A. Falcao Nascimento. [33] studied the intrinsic decay rate estimates for the wave equation with competing viscoelastic and frictional dissipative effects. I. Lasiecka, S. A. Messaoudi and M. I. Mustafa. [34] studied the note on intrinsic decay rates for abstract wave equations with memory. I. Lasiecka and X. Wang. [35] studied the intrinsic decay rate estimates for semilinear abstract second order equations with memory. Hao and Cai [30] studied the uniform decay of solutions for the coupled viscoelastic wave equations. Cavalcanti M. Filho VND. Cavalcanti JSP. Soriano JA. [36] studied the existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping. For more results in this direction, see $[37-40]$.

However, $[1-41]$, did not study the intrinsic decay rates for the energy of system $(1.1)-(1.3)$. Motivated by the above research, we will consider the intrinsic decay rates for the energy of relaxation kernels described by the inequality $g_i'(t) \leq -H(g_i(t)),\ (i=1,2)$ for all $t \geq 0$, of the model $(1.1)-(1.3)$ in this paper.

The outline of the paper is as follows. In the second section we present some basic concepts, establish some useful inequalities, which will be used for the remaining of the present paper, and state the local existence theorem, we define the energy $E(t)$ associated to $(1.1)-(1.3)$ and show that it is a non-increasing function of t. Finally, in section 3, we prove the intrinsic decay rates for the energy of the posed system.

2. **Preliminaries**

In this section, we introduce some functional spaces, establish some useful inequalities, which will be used for the remaining of the present paper, and state the local existence theorem, we define the energy $E(t)$ associated to $(1.1)-(1.3)$ and show that it is a non-increasing function of t. Let $L_x^p = L_x^p((0,\alpha))$ be the weighted Banach space equipped with the norm

$$\|u\|_{L_x^p(0,\alpha)} := \left(\int_0^\alpha x\,|u|^p\,dx\right)^{\frac{1}{p}}.$$

$H = L_x^2((0,\alpha))$ be, in particular, the Hilbert space of square integral functions having the finite norm

$$\|u\|_H := \left(\int_0^\alpha x u^2 dx\right)^{\frac{1}{2}}.$$

$V := V_x^{1.1}((0,\alpha))$ be the Hilbert space equipped with the norm

$$\|u\|_V := \left(\|u\|_H^2 + \|u_x\|_H^2\right)^{\dfrac{1}{2}},$$

and

$$V_0 := \{u \in V \text{ such that } u(\alpha)=0\}.$$

We give some useful inequalities:

- **Cauchy–Schwarz inequality**. If $f,\ g \in L^2(0,\alpha)$, then
$$\left(\int_0^\alpha f(t)\,g(t)\,dt\right)^2 \leq \|f(t)\|_{L^2(0,\alpha)}^2 \times \|g(t)\|_{L^2(0,\alpha)}^2.$$
- ε–**Cauchy inequality**. For all $\alpha,\ \beta \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+^*$, we have
$$|\alpha\beta| \leq \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2.$$
- **Jensen's inequality**. Let F be a convex increasing function on $[\alpha,b]$, let $f:\Omega \longmapsto [\alpha,b]$, and let h be an integrable function such that $h(x) \geq 0$ and $\displaystyle\int_\Omega h(x)\,dx = h_0 > 0$. Then, we have
$$\int_\Omega F^{-1}(f(x))\,h(x)\,dx \leq h_0 F^{-1}\left[h_0^{-1}\int_\Omega f(x)\,h(x)\,dx\right].$$

- **Poincaré-type Inequality.** For any u in V_0 we have

$$\int_0^{\alpha} xu^2(x)dx \leq C_p \int_0^{\alpha} x(u_x(x))^2 dx,$$

and

$$V_0 := \{u \in V \text{ such that } u(\alpha) = 0\}.$$

We now state the result of local existence, which has been given in [22].

Theorem 1. [22]*Suppose that* $2 < p,\ q < 3$ *and*

$$g_i(0) > 0, \quad 1 - \int_0^{\infty} g_i(s)ds = l_i > 0, \quad (i = 1, 2).$$

Then, for any $(u_0, v_0) \in V_0^2$ *and* $(v_1, v_2) \in H^2$, *system* $(1.1) - (1.3)$ *has a unique local solution*

$$u, v \in C(0, t_*; V_0) \cap C^1(0, t_*; H),$$

for $t_* > 0$ *small enough.*

For the relation function g_1 and g_2 is a $C^1(\mathbb{R}_+, \mathbb{R}_+)$, we give the following assumptions
(**A1**) $g_1(0) > 0,\ g_2(0) > 0$ and

$$1 - \int_0^{\infty} g_i(s)ds = l_i > 0, \quad (i = 1, 2). \tag{2.1}$$

(**A2**) $g_i'(t) \leq -H(g_i(t)),\ (i = 1, 2)$ for all $t \geq 0$, where $H \in C^1(\mathbb{R}_+)$ which $H(0) = 0$ is a given strictly increasing and convex function. Moreover

$$H \in C^2(0, \infty) \text{ and } \lim \inf_{x \to 0^+} \left\{x^2 H''(x) - xH'(x) + H(x)\right\} \geq 0.$$

(**A3**) With reference to the function H introduced above, let $y(t)$ be the solution of the ODE

$$y'(t) + H(y(t)) = 0, \quad y(0) = g(0) > 0.$$

(**A4**) We assume that there exists $\alpha_0 \in [0, 1)$ such that $y^{1-\alpha_0} \in L_1(1, \infty)$.

Remark 1. *The assumptions* $1 - \displaystyle\int_0^{\infty} g_i(s)ds > 0,\ (i = 1, 2)$ *is necessary to guarantee the hyperbolicity of the system* $(1.1) - (1.3)$.

In order to formulate the long-time behavior results, we recall the binary notation

$$\begin{cases} (g \circ w)(t) := x\displaystyle\int_0^t g(t-s)\left|w(x,s) - w(x,t)\right|^2 ds, \\[2ex] (g \diamond w)(t) := x\displaystyle\int_0^t g(t-s)(w(t) - w(s))\, ds. \end{cases} \tag{2.2}$$

We define the corresponding energy functional by

$$
\begin{aligned}
E(t) \quad : \quad = & \left(\frac{p+1}{2}\right)\|u_t\|^2_{L^2_\rho(0,\alpha)} + \left(\frac{q+1}{2}\right)\|v_t\|^2_{L^2_\rho(0,\alpha)} \\
& + \left(\frac{p+1}{2}\right)\left(1-\int_0^t g_1(s)ds\right)\|u_x\|^2_{L^2_\rho(0,\alpha)} \\
& + \left(\frac{q+1}{2}\right)\left(1-\int_0^t g_2(s)ds\right)\|v_x\|^2_{L^2_\rho(0,\alpha)} - \int_0^\alpha x\,|u|^{p+1}\,|v|^{q+1}\,dx \\
& + \left(\frac{p+1}{2}\right)\int_0^\alpha (g_1\circ u_x)(t)\,dx + \left(\frac{q+1}{2}\right)\int_0^\alpha (g_2\circ v_x)(t)\,dx. \qquad (2.3)
\end{aligned}
$$

Lemma 1. *Let (u,v) be the solution of system $(1.1)-(1.3)$ then $E(t)$, is a no increasing function, that is $\forall t \geq 0$*

$$
\begin{aligned}
\frac{d}{dt}\{E(t)\} \quad = & \quad \left(\frac{p+1}{2}\right)\int_0^\alpha (g_1'\circ u_x)(t)dx - \left(\frac{p+1}{2}\right)g_1(t)\|u_x\|^2_{L^2_\rho(0,\alpha)} \\
& + \left(\frac{q+1}{2}\right)\int_0^\alpha (g_2'\circ v_x)(t)dx - \left(\frac{q+1}{2}\right)g_2(t)\|v_x\|^2_{L^2_\rho(0,\alpha)} \\
\leq & \quad 0. \qquad (2.4)
\end{aligned}
$$

Proof. Multiplying the $1^{\text{ére}}$ term in (1.1) by $(p+1)xu_t$, and the $2^{\text{éme}}$ term in (1.1) by $(q+1)xv_t$ integrating over $(0,\alpha)$, summing up, we obtain

$$
\begin{aligned}
& (p+1)\int_0^\alpha xu_{tt}u_t dx - (p+1)\int_0^\alpha (xu_x)_x u_t dx \\
& +(p+1)\int_0^\alpha \left(\int_0^t g_1(t-s)(xu_x(x,s))_x ds\right)u_t(x,t)\,dx \\
& +(q+1)\int_0^\alpha xv_{tt}v_t dx - (q+1)\int_0^\alpha (xv_x)_x v_t dx \\
& +(q+1)\int_0^\alpha \left(\int_0^t g_2(t-s)(xv_x(x,s))_x ds\right)v_t(x,t)\,dx \\
= \quad & (p+1)\int_0^\alpha x\,|v|^{q+1}\,|u|^{p-1}\,u_t\,\mathrm{u}dx + (q+1)\int_0^\alpha x\,|u|^{p+1}\,|v|^{q-1}\,v_t v dx. \qquad (2.5)
\end{aligned}
$$

By direct calculations, we get

$$
(p+1)\int_0^\alpha xu_{tt}u_t dx = \left(\frac{p+1}{2}\right)\frac{d}{dt}\left\{\|u_t\|^2_{L^2_\rho(0,\alpha)}\right\}, \qquad (2.6)
$$

and

$$
(q+1)\int_0^\alpha xv_{tt}v_t dx = \left(\frac{q+1}{2}\right)\frac{d}{dt}\left\{\|v_t\|^2_{L^2_\rho(0,\alpha)}\right\}. \qquad (2.7)
$$

And by using integration by parts, we have

$$
-(p+1)\int_0^\alpha (xu_x(x,t))_x u_t(x,t)\,dx = \left(\frac{p+1}{2}\right)\frac{d}{dt}\left\{\|u_x\|^2_{L^2_\rho(0,\alpha)}\right\}, \qquad (2.8)
$$

and

$$
-(q+1)\int_0^\alpha (xv_x(x,t))_x v_t(x,t)\,dx = \left(\frac{q+1}{2}\right)\frac{d}{dt}\left\{\|v_x\|^2_{L^2_\rho(0,\alpha)}\right\}. \qquad (2.9)
$$

By direct calculations, we get

$$\begin{aligned} & (p+1)\int_0^\alpha x\,|v|^{q+1}\,|u|^{p-1}\,u_t\,\mathrm{u}dx+(q+1)\int_0^\alpha x\,|u|^{p+1}\,|v|^{q-1}\,v_t v dx \\ = {} & \frac{d}{dt}\left\{\int_0^\alpha x\,|v|^{q+1}\,|u|^{p+1}\,dx\right\}. \end{aligned} \tag{2.10}$$

By using integrant par party, we have

$$\begin{aligned} & (p+1)\int_0^\alpha\left(\int_0^t g_1(t-s)\,(xu_x(x,s))_x\,ds\right)u_t(x,t)\,dx \\ = {} & \left(\frac{p+1}{2}\right)\frac{d}{dt}\left\{\int_0^\alpha (g_1\circ u_x)(t)\,dx-\left(\int_0^t g_1(s)\,ds\right)\|u_x\|^2_{L^2_\rho(0,\alpha)}\right\} \\ & -\left(\frac{p+1}{2}\right)\int_0^\alpha (g_1'\circ u_x)(t)\,dx+\left(\frac{p+1}{2}\right)g_1(t)\,\|u_x\|^2_{L^2_\rho(0,\alpha)}, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} & (q+1)\int_0^\alpha\left(\int_0^t g_2(t-s)\,(xv_x(x,s))_x\,ds\right)v_t(x,t)\,dx \\ = {} & \left(\frac{q+1}{2}\right)\frac{d}{dt}\left\{\int_0^\alpha (g_2\circ v_x)(t)\,dx-\left(\int_0^t g_2(s)\,ds\right)\|v_x\|^2_{L^2_\rho(0,\alpha)}\right\} \\ & -\left(\frac{q+1}{2}\right)\int_0^\alpha (g_2'\circ v_x)(t)\,dx+\left(\frac{q+1}{2}\right)g_2(t)\,\|v_x\|^2_{L^2_\rho(0,\alpha)}. \end{aligned} \tag{2.12}$$

By replacement (2.6) – (2.12) into (2.5), then we get

$$\begin{aligned} & \frac{d}{dt}\left\{\left(\frac{p+1}{2}\right)\|u_t\|^2_{L^2_\rho(0,\alpha)}+\left(\frac{q+1}{2}\right)\|v_t\|^2_{L^2_\rho(0,\alpha)}\right. \\ & +\left(\frac{p+1}{2}\right)\left(1-\int_0^t g_1(s)\,ds\right)\|u_x\|^2_{L^2_\rho(0,\alpha)} \\ & +\left(\frac{q+1}{2}\right)\left(1-\int_0^t g_2(s)\,ds\right)\|v_x\|^2_{L^2_\rho(0,\alpha)} \\ & +\left(\frac{p+1}{2}\right)\int_0^\alpha (g_1\circ u_x)(t)\,dx+\left(\frac{q+1}{2}\right)\int_0^\alpha (g_2\circ v_x)(t)\,dx \\ & \left.-\int_0^\alpha x\,|v|^{q+1}\,|u|^{p+1}\,dx\right\} \\ = {} & \left(\frac{p+1}{2}\right)\int_0^\alpha (g_1'\circ u_x)(t)\,dx-\left(\frac{p+1}{2}\right)g_1(t)\,\|u_x\|^2_{L^2_\rho(0,\alpha)} \\ & +\left(\frac{q+1}{2}\right)\int_0^\alpha (g_2'\circ v_x)(t)\,dx-\left(\frac{q+1}{2}\right)g_2(t)\,\|v_x\|^2_{L^2_\rho(0,\alpha)}, \end{aligned} \tag{2.13}$$

by using (2.3) in (2.13), we get (2.4).
Then the Proof the lemma is complete. □

3. **Decay of Solutions**

In this section, we prove the intrinsic decay rates for the energy of the posed system. Now, we are in a position to state our main result.

Lemma 2. *Let us assume that* $(\mathbf{A1})-(\mathbf{A4})$ *are the place. Then, there exists a positive constant* $T_0>0$ *such that*

$$E\left((n+1)T\right)+\tilde{H}\left(C_{15}^{-1}E\left((n+1)T\right)\right)\leq E\left(nT\right),\quad n=1,2,3...$$

for all $T>T_0$ *and all* $n\in N$, *where* $\tilde{H}$ *is given in* (3.54) *and* C_{15} *is given in* (3.57).

Proof. For this purpose, a by now standard procedure is to multiplying the $1^{\text{ére}}$ term in (1.1) by the viscoelastic multiplier

$$\left(g_1\diamond u\right)(t)=x\int_0^t g_1\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right)ds,$$

and the $2^{\text{éme}}$ term in (1.1) by the viscoelastic multiplier

$$\left(g_2\diamond v\right)(t)=x\int_0^t g_2\left(t-s\right)\left(v\left(t\right)-v\left(s\right)\right)ds,$$

and integrating over $\Omega\times(nT,(n+1)T)$, summing up, we obtain

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(u_{tt},\left(g_1\diamond u\right)(t)\right)_{L^2(0,\alpha)}dt-\int_{nT}^{(n+1)T}\left(\frac{1}{x}(xu_x)_x,\left(g_1\diamond u\right)(t)\right)_{L^2(0,\alpha)}dt\\
&+\int_{nT}^{(n+1)T}\left(\int_0^t g_1\left(t-s\right)\frac{1}{x}(xu_x(x,s))_x ds,\left(g_1\diamond u\right)(t)\right)_{L^2(0,\alpha)}dt\\
&+\int_{nT}^{(n+1)T}\left(v_{tt},\left(g_2\diamond v\right)(t)\right)_{L^2(0,\alpha)}dt\\
&-\int_{nT}^{(n+1)T}\left(\frac{1}{x}(xv_x)_x,\left(g_2\diamond v\right)(t)\right)_{L^2(0,\alpha)}dt\\
&+\int_{nT}^{(n+1)T}\left(\int_0^t g_2\left(t-s\right)\frac{1}{x}(xv_x(x,s))_x ds,\left(g_2\diamond v\right)(t)\right)_{L^2(0,\alpha)}dt\\
=&\int_{nT}^{(n+1)T}\left(|v|^{q+1}|u|^{p-1}u,\left(g_1\diamond u\right)(t)\right)_{L^2(0,\alpha)}dt\\
&+\int_{nT}^{(n+1)T}\left(|u|^{p+1}|v|^{q-1}v,\left(g_2\diamond v\right)(t)\right)_{L^2(0,\alpha)}dt.
\end{aligned}\tag{3.1}$$

By direct calculations, we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(u_{tt},\left(g_1\diamond u\right)(t)\right)_{L^2(0,\alpha)}dt\\
=&\left.\left(u_t\left(t\right),\int_0^t g_1\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right)ds\right)_{L^2_\rho(0,\alpha)}\right|_{nT}^{(n+1)T}\\
&-\int_{nT}^{(n+1)T}\left(u_t,\int_0^t g_1'\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right)ds\right)_{L^2_\rho(0,\alpha)}dt\\
&-\int_{nT}^{(n+1)T}\left(\int_0^t g_1\left(s\right)ds\right)\left\|u_t\right\|^2_{L^2_\rho(0,\alpha)}dt,
\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}
&\int_{nT}^{(n+1)T} \left(v_{tt}, (g_2 \diamond v)(t)\right)_{L^2(0,\alpha)} dt \\
= &\left(v_t(t), \int_0^t g_2(t-s)(v(t)-v(s))\,ds\right)_{L^2_\rho(0,\alpha)}\Bigg|_{nT}^{(n+1)T} \\
&- \int_{nT}^{(n+1)T} \left(v_t, \int_0^t g_2'(t-s)(v(t)-v(s))\,ds\right)_{L^2_\rho(0,\alpha)} dt \\
&- \int_{nT}^{(n+1)T} \left(\int_0^t g_2(s)\,ds\right) \|v_t\|^2_{L^2_\rho(0,\alpha)}\,dt. \qquad (3.3)
\end{aligned}$$

And by using integration by parts, we have

$$\begin{aligned}
&- \int_{nT}^{(n+1)T} \left(\frac{1}{x}(xu_x)_x, (g_1 \diamond u)(t)\right)_{L^2(0,\alpha)} dt \\
= &\int_{nT}^{(n+1)T} \left(u_x(x,t), \int_0^t g_1(t-s)(u_x(t)-u_x(s))\,ds\right)_{L^2_\rho(0,\alpha)} dt, \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
&- \int_{nT}^{(n+1)T} \left(\frac{1}{x}(xv_x)_x, (g_2 \diamond v)(t)\right)_{L^2(0,\alpha)} dt \\
= &\int_{nT}^{(n+1)T} \left(v_x(x,t), \int_0^t g_2(t-s)(v_x(t)-v_x(s))\,ds\right)_{L^2_\rho(0,\alpha)} dt, \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{nT}^{(n+1)T} \left(\int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds, (g_1 \diamond u)(t)\right)_{L^2(0,\alpha)} dt \\
= &\int_{nT}^{(n+1)T} \left\|\int_0^t g_1(t-s)(u_x(t)-u_x(s))\,ds\right\|^2_{L^2_\rho(0,\alpha)} dt \\
&- \int_{nT}^{(n+1)T} \left(\int_0^t g_1(t-s)u_x(t)\,ds, \int_0^t g_1(t-s)(u_x(t)-u_x(s))\,ds\right)_{L^2_\rho(0,\alpha)} dt, \qquad (3.6)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{nT}^{(n+1)T} \left(\int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x\,ds, (g_2 \diamond v)(t)\right)_{L^2(0,\alpha)} dt \\
= &\int_{nT}^{(n+1)T} \left\|\int_0^t g_2(t-s)(v_x(t)-v_x(s))\,ds\right\|^2_{L^2_\rho(0,\alpha)} dt \\
&- \int_{nT}^{(n+1)T} \left(\int_0^t g_2(t-s)v_x(t)\,ds, \int_0^t g_2(t-s)(v_x(t)-v_x(s))\,ds\right)_{L^2_\rho(0,\alpha)} dt. \qquad (3.7)
\end{aligned}$$

By replacement of $(3.2)-(3.7)$ into (3.1), we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(\int_0^t g_1(s)\,ds\right)\|u_t\|^2_{L^2_\rho(0,\alpha)}\,dt \\
&+\int_{nT}^{(n+1)T}\left(\int_0^t g_2(s)\,ds\right)\|v_t\|^2_{L^2_\rho(0,\alpha)}\,dt \\
=\;&\left(u_t(t),\int_0^t g_1(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(0,\alpha)}\Bigg|_{nT}^{(n+1)T} \\
&+\left(v_t(t),\int_0^t g_2(t-s)(v(t)-v(s))\,ds\right)_{L^2_\rho(0,\alpha)}\Bigg|_{nT}^{(n+1)T} \\
&-\int_{nT}^{(n+1)T}\left(u_t,\int_0^t g_1'(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(0,\alpha)}dt \\
&-\int_{nT}^{(n+1)T}\left(v_t,\int_0^t g_2'(t-s)(v(t)-v(s))\,ds\right)_{L^2_\rho(0,\alpha)}dt \\
&+\int_{nT}^{(n+1)T}\left(u_x,\int_0^t g_1(t-s)(u_x(t)-u_x(s))\,ds\right)_{L^2_\rho(0,\alpha)}dt \\
&+\int_{nT}^{(n+1)T}\left(v_x,\int_0^t g_2(t-s)(v_x(t)-v_x(s))\,ds\right)_{L^2_\rho(0,\alpha)}dt \\
&+\int_{nT}^{(n+1)T}\left\|\int_0^t g_1(t-s)(u_x(t)-u_x(s))\,ds\right\|^2_{L^2_\rho(0,\alpha)}dt \\
&+\int_{nT}^{(n+1)T}\left\|\int_0^t g_2(t-s)(v_x(t)-v_x(s))\,ds\right\|^2_{L^2_\rho(0,\alpha)}dt \\
&-\int_{nT}^{(n+1)T}\left(\int_0^t g_1(t-s)u_x(t)\,ds,\int_0^t g_1(t-s)(u_x(t)-u_x(s))\,ds\right)_{L^2_\rho(0,\alpha)}dt \\
&-\int_{nT}^{(n+1)T}\left(\int_0^t g_2(t-s)v_x(t)\,ds,\int_0^t g_2(t-s)(v_x(t)-v_x(s))\,ds\right)_{L^2_\rho(0,\alpha)}dt \\
&-\int_{nT}^{(n+1)T}\left(|v|^{q+1}|u|^{p-1}u,(g_1\diamond u)(t)\right)_{L^2(0,\alpha)}dt \\
&-\int_{nT}^{(n+1)T}\left(|u|^{p+1}|v|^{q-1}v,(g_2\diamond v)(t)\right)_{L^2(0,\alpha)}dt \\
=\;&J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8 \\
&+J_9+J_{10}+J_{11}+J_{12}. \qquad (3.8)
\end{aligned}$$

Estimate for $|J_1|$, where

$$\begin{aligned} J_1 \quad : \quad &= \left(u_t\left((n+1)T\right), \int_0^{(n+1)T} g_1\left((n+1)T-s\right)\left(u\left((n+1)T\right)-u\left(s\right)\right)ds \right)_{L^2_\rho(0,\alpha)} \\ &\quad - \left(u_t\left(nT\right), \int_0^{nT} g_1\left(nT-s\right)\left(u\left(nT\right)-u\left(s\right)\right)ds \right)_{L^2_\rho(0,\alpha)}. \end{aligned}$$

Now, let $m \in N$ be an arbitrary, natural number. By using Young's inequality (for $\varepsilon = 1$) and Poincaré-type inequality, (2.2) and (2.3), we get

$$\begin{aligned} &\left(u_t\left(mT\right), \int_0^{mT} g_1\left(mT-s\right)\left(u\left(mT\right)-u\left(s\right)\right)ds \right)_{L^2_\rho(0,\alpha)} \\ &\leq \left(\frac{1}{p+1}\right)\left\{\left(\int_0^{mT} g_1\left(s\right)ds\right)+C_p\right\}E(mT), \end{aligned} \tag{3.9}$$

then by using (3.9), we get

$$|J_1| \leq C_1\left[E((n+1)T)+E(nT)\right], \tag{3.10}$$

where

$$C_1 := \left(\frac{1}{p+1}\right)\left\{\|g_1\|_{L^1(0,\infty)}+C_p\right\} > 0.$$

Similarly, we obtain

$$|J_2| \leq C_2\left[E((n+1)T)+E(nT)\right], \tag{3.11}$$

where

$$C_2 := \left(\frac{1}{q+1}\right)\left\{\|g_2\|_{L^1(0,\infty)}+C'_p\right\} > 0.$$

Estimate for $|J_3|$. By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_1}{2}\right)$, we get

$$\begin{aligned} |J_3| \quad &\leq \quad \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t\|^2_{L^2_\rho(0,\alpha)}\,dt \\ &\quad + \frac{1}{4\varepsilon_1}\int_{nT}^{(n+1)T}\left\|\int_0^t g'_1\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right)ds\right\|^2_{L^2_\rho(0,\alpha)} dt. \end{aligned} \tag{3.12}$$

By using Poincaré-type inequality, Cauchy-Schwarz inequality and (2.2), we get

$$\begin{aligned} &\int_{nT}^{(n+1)T}\left\|\int_0^t g'_1\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right)ds\right\|^2_{L^2_\rho(0,\alpha)} dt \\ &\leq \quad -C_p g_1\left(0\right)\int_{nT}^{(n+1)T}\int_0^\alpha \left(g'_1 \circ u_x\right)\left(t\right)dxdt. \end{aligned} \tag{3.13}$$

By replacement (3.13) into (3.12), we get

$$|J_3| \leq \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t\|^2_{L^2_\rho(0,\alpha)}\,dt - \frac{C_p g_1\left(0\right)}{4\varepsilon_1}\int_{nT}^{(n+1)T}\int_0^\alpha \left(g'_1 \circ u_x\right)\left(t\right)dxdt. \tag{3.14}$$

Similarly, we obtain

$$|J_4| \leq \varepsilon_2 \int_{nT}^{(n+1)T} \|v_t\|^2_{L^2_\rho(0,\alpha)}\,dt - \frac{C'_p g_2\left(0\right)}{4\varepsilon_2}\int_{nT}^{(n+1)T}\int_0^\alpha \left(g'_2 \circ v_x\right)\left(t\right)dxdt. \tag{3.15}$$

Estimate for $|J_5|$. By using Young's inequality $\left(\text{for } \varepsilon = \dfrac{\varepsilon_3}{2}\right)$, we get

$$
\begin{aligned}
|J_5| \quad &\leq \quad \varepsilon_3 \int_{nT}^{(n+1)T} \|u_x\|^2_{L^2_\rho(0,\alpha)}\, dt \\
&\quad + \frac{1}{4\varepsilon_3} \int_{nT}^{(n+1)T} \left\| \int_0^t g_1(t-s)(u_x(t) - u_x(s))\, ds \right\|^2_{L^2_\rho(0,\alpha)} dt, \qquad (3.16)
\end{aligned}
$$

by using Cauchy-Schwarz inequality and (2.2), we get

$$
\begin{aligned}
&\left\| \int_0^t g_1(t-s)(u_x(t) - u_x(s))\, ds \right\|^2_{L^2_\rho(0,\alpha)} \\
&\leq \quad \|g_1\|_{L^1(0,\infty)} \int_0^\alpha (g_1 \circ u_x)(t)\, dx, \qquad (3.17)
\end{aligned}
$$

by replacement (3.17) into (3.16), we get

$$
\begin{aligned}
|J_5| \quad &\leq \quad \varepsilon_3 \int_{nT}^{(n+1)T} \|u_x\|^2_{L^2_\rho(0,\alpha)}\, dt \\
&\quad + \frac{1}{4\varepsilon_3} \|g_1\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^\alpha (g_1 \circ u_x)(t)\, dxdt. \qquad (3.18)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
|J_6| \quad &\leq \quad \varepsilon_3 \int_{nT}^{(n+1)T} \|v_x\|^2_{L^2_\rho(0,\alpha)}\, dt \\
&\quad + \frac{1}{4\varepsilon_3} \|g_2\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^\alpha (g_2 \circ v_x)(t)\, dxdt. \qquad (3.19)
\end{aligned}
$$

Estimate $|J_7|$. By using (3.17), we get

$$
|J_7| \leq \|g_1\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^\alpha (g_1 \circ u_x)(t)\, dxdt. \qquad (3.20)
$$

Similarly, we obtain

$$
|J_8| \leq \|g_2\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^\alpha (g_2 \circ v_x)(t)\, dxdt. \qquad (3.21)
$$

Now, estimate J_9. By Young's inequality $\left(\text{for } \varepsilon = \dfrac{\varepsilon_4}{2}\right)$, Cauchy-Schwarz inequality and (3.17), we get

$$
\begin{aligned}
|J_9| \quad &\leq \quad \varepsilon_4 \|g_1\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \left(\int_0^t g_1(s)\, ds \right) \|u_x\|^2_{L^2_\rho(0,\alpha)}\, dt \\
&\quad + \frac{1}{4\varepsilon_4} \|g_1\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^\alpha (g_1 \circ u_x)(t)\, dxdt. \qquad (3.22)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
|J_{10}| \quad &\leq \quad \varepsilon_4 \|g_2\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \left(\int_0^t g_2(s)\, ds \right) \|v_x\|^2_{L^2_\rho(0,\alpha)}\, dt \\
&\quad + \frac{1}{4\varepsilon_4} \|g_2\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^\alpha (g_2 \circ v_x)(t)\, dxdt. \qquad (3.23)
\end{aligned}
$$

Now, estimate $|J_{11}|$. By using Young's inequality $\left(\text{for } \varepsilon = \dfrac{\varepsilon_3}{2}\right)$, Poincaré-type inequality and $\|u_x\|^2_{L^2_\rho(0,\alpha)} \leq \dfrac{2}{(p-1)l_1}E(0)$, $\|v_x\|^2_{L^2_\rho(0,\alpha)} \leq \dfrac{2}{(q-1)l_2}E(0)$ and (3.17) , we get

$$\begin{aligned} |J_{11}| \quad &\leq \quad c_1\varepsilon_3 \int_{nT}^{(n+1)T} \|u_x\|^2_{L^2_\rho(0,\alpha)}\, dt + c_1'\varepsilon_3 \int_{nT}^{(n+1)T} \|v_x\|^2_{L^2_\rho(0,\alpha)}\, dt \\ &\quad + \frac{C_p}{4\varepsilon_3} \|g_1\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^\alpha (g_1 \circ u_x)(t) dx dt, \end{aligned} \tag{3.24}$$

where

$$\begin{cases} c_1 := \dfrac{C_p}{4}\left(\dfrac{2}{(p-1)l_1}E(0)\right)^{2p-1} > 0 \\ c_1' := \dfrac{C_p'}{4}\left(\dfrac{2}{(q-1)l_2}E(0)\right)^{2q+1} > 0. \end{cases}$$

Similarly, we get

$$\begin{aligned} |J_{12}| \quad &\leq \quad c_2\varepsilon_3 \int_{nT}^{(n+1)T} \|u_x\|^2_{L^2_\rho(0,\alpha)}\, dt + c_2'\varepsilon_3 \int_{nT}^{(n+1)T} \|v_x\|^2_{L^2_\rho(0,\alpha)}\, dt \\ &\quad + \frac{C_p'}{4\varepsilon_3} \|g_2\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^\alpha \left(g_2 \circ v_x\right)(t)\, dx dt, \end{aligned} \tag{3.25}$$

where

$$\begin{cases} c_2 := \dfrac{C_p}{2}\left(\dfrac{2}{(p-1)l_1}E(0)\right)^{2p+1} > 0 \\ c_2' := \dfrac{C_p'}{2}\left(\dfrac{2}{(q-1)l_2}E(0)\right)^{2q-1} > 0. \end{cases}$$

Combining (3.10), (3.11), (3.14), (3.15) and $(3.18)-(3.25)$ into (3.8), and recalling that $\|g_1\|_{L^1(0,\infty)} < 1$ and $\|g_2\|_{L^1(0,\infty)} < 1$, we write

$$\begin{aligned}
& \int_{nT}^{(n+1)T} \left(\int_0^t g_1(s)\,ds \right) \|u_t\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& + \int_{nT}^{(n+1)T} \left(\int_0^t g_2(s)\,ds \right) \|v_t\|^2_{L^2_\rho(0,\alpha)}\,dt \\
\leq\ & (C_1 + C_2)\left[E((n+1)T) + E(nT)\right] \\
& + \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t\|^2_{L^2_\rho(0,\alpha)}\,dt + \varepsilon_2 \int_{nT}^{(n+1)T} \|v_t\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& - \frac{C_p g_1(0)}{4\varepsilon_1} \int_{nT}^{(n+1)T} \int_0^\alpha (g_1' \circ u_x)(t)\,dxdt \\
& - \frac{C_p' g_2(0)}{4\varepsilon_2} \int_{nT}^{(n+1)T} \int_0^\alpha (g_2' \circ v_x)(t)\,dxdt \\
& + \varepsilon_3 (1 + c_1 + c_2) \int_{nT}^{(n+1)T} \|u_x\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& + \varepsilon_3 (1 + c_1' + c_2') \int_{nT}^{(n+1)T} \|v_x\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& + \left(\frac{(1 + C_p)}{4\varepsilon_3} + 1 + \frac{1}{4\varepsilon_4} \right) \int_{nT}^{(n+1)T} \int_0^\alpha (g_1 \circ u_x)(t)\,dxdt \\
& + \left(\frac{(1 + C_p')}{4\varepsilon_3} + 1 + \frac{1}{4\varepsilon_4} \right) \int_{nT}^{(n+1)T} \int_0^\alpha (g_2 \circ v_x)(t)\,dxdt \\
& + \varepsilon_4 \int_{nT}^{(n+1)T} \left(\int_0^t g_1(s)\,ds \right) \|u_x\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& + \varepsilon_4 \int_{nT}^{(n+1)T} \left(\int_0^t g_2(s)\,ds \right) \|v_x\|^2_{L^2_\rho(0,\alpha)}\,dt.
\end{aligned} \tag{3.26}$$

Since $g_1(0) > 0$ and $g_2(0) > 0$, we can select a points $t_1 < T$ with t_1 close to zero such that for all $t \geq t_1$

$$\begin{cases} \displaystyle\int_0^t g_1(s)\,ds \geq t_1 g_1(t_1) := c_0 > 0, \\[2ex] \displaystyle\int_0^t g_2(s)\,ds \geq t_1 g_2(t_1) := c_0' > 0. \end{cases}$$

Then (3.26) is equivalent

$$
\begin{aligned}
& \int_{nT}^{(n+1)T} \{t_1 g_1(t_1) - \varepsilon_1\} \|u_t\|^2_{L^2_\rho(0,\alpha)} \, dt \\
& + \int_{nT}^{(n+1)T} \{t_1 g_2(t_1) - \varepsilon_2\} \|v_t\|^2_{L^2_\rho(0,\alpha)} \, dt \\
\leq \; & (C_1 + C_2) \left[E((n+1)T) + E(nT) \right] \\
& - \frac{C_p g_1(0)}{4\varepsilon_1} \int_{nT}^{(n+1)T} \int_0^\alpha \left(g_1' \circ u_x \right)(t) \, dx dt \\
& - \frac{C_p' g_2(0)}{4\varepsilon_2} \int_{nT}^{(n+1)T} \int_0^\alpha \left(g_2' \circ v_x \right)(t) \, dx dt \\
& + \varepsilon_3 \left(1 + c_1 + c_2\right) \int_{nT}^{(n+1)T} \|u_x\|^2_{L^2_\rho(0,\alpha)} \, dt \\
& + \varepsilon_3 \left(1 + c_1' + c_2'\right) \int_{nT}^{(n+1)T} \|v_x\|^2_{L^2_\rho(0,\alpha)} \, dt \\
& + \left(\frac{(1 + C_p)}{4\varepsilon_3} + 1 + \frac{1}{4\varepsilon_4} \right) \int_{nT}^{(n+1)T} \int_0^\alpha \left(g_1 \circ u_x \right)(t) \, dx dt \\
& + \left(\frac{(1 + C_p')}{4\varepsilon_3} + 1 + \frac{1}{4\varepsilon_4} \right) \int_{nT}^{(n+1)T} \int_0^\alpha \left(g_2 \circ v_x \right)(t) \, dx dt \\
& + \varepsilon_4 \int_{nT}^{(n+1)T} \left(\int_0^t g_1(s) \, ds \right) \|u_x\|^2_{L^2_\rho(0,\alpha)} \, dt \\
& + \varepsilon_4 \int_{nT}^{(n+1)T} \left(\int_0^t g_2(s) \, ds \right) \|v_x\|^2_{L^2_\rho(0,\alpha)} \, dt. \qquad (3.27)
\end{aligned}
$$

Now, multiplying $1^{\text{ére}}$ term in (1.1) by xu and the $2^{\text{éme}}$ term in (1.1) by xv and integrating over $\Omega \times (nT, (n+1)T)$, summing up, we obtain

$$
\begin{aligned}
& \int_{nT}^{(n+1)T} \left(u_{tt}(t), u(t) \right)_{L^2_\rho(0,\alpha)} dt - \int_{nT}^{(n+1)T} \left((x u_x)_x, u(t) \right)_{L^2(0,\alpha)} dt \\
& + \int_{nT}^{(n+1)T} \left(\int_0^t g_1(t-s) \left(x u_x(x,s) \right)_x ds, u(t) \right)_{L^2(0,\alpha)} dt \\
& + \int_{nT}^{(n+1)T} \left(v_{tt}(t), v(t) \right)_{L^2_\rho(0,\alpha)} dt - \int_{nT}^{(n+1)T} \left((x v_x)_x, v(t) \right)_{L^2(0,\alpha)} dt \\
& + \int_{nT}^{(n+1)T} \left(\int_0^t g_2(t-s) \left(x v_x(x,s) \right)_x ds, v(t) \right)_{L^2(0,\alpha)} dt \\
= \; & \int_{nT}^{(n+1)T} \left(|v|^{q+1} |u|^{p-1} u(t), u(t) \right)_{L^2_\rho(0,\alpha)} dt \\
& + \int_{nT}^{(n+1)T} \left(|u|^{p+1} |v|^{q-1} v(t), v(t) \right)_{L^2_\rho(0,\alpha)} dt. \qquad (3.28)
\end{aligned}
$$

By using

$$
u_{tt}(t) \, u(t) = \frac{d}{dt} \left\{ u_t(t) \, u(t) \right\} - u_t^2(t),
$$

we get

$$\begin{aligned}&\int_{nT}^{(n+1)T}\left(u_{tt}\left(t\right),u\left(t\right)\right)_{L_{\rho}^{2}(0,\alpha)}dt\\ =\;&\left(u_{t}\left(t\right),u\left(t\right)\right)_{L_{\rho}^{2}(0,\alpha)}\Big|_{nT}^{(n+1)T}-\int_{nT}^{(n+1)T}\left\Vert u_{t}\right\Vert_{L_{\rho}^{2}(0,\alpha)}^{2}dt.\end{aligned}\tag{3.29}$$

Similarly, we obtain

$$\begin{aligned}&\int_{nT}^{(n+1)T}\left(v_{tt}\left(t\right),v\left(t\right)\right)_{L_{\rho}^{2}(0,\alpha)}dt\\ =\;&\left(v_{t}\left(t\right),v\left(t\right)\right)_{L_{\rho}^{2}(0,\alpha)}\Big|_{nT}^{(n+1)T}-\int_{nT}^{(n+1)T}\left\Vert v_{t}\right\Vert_{L_{\rho}^{2}(0,\alpha)}^{2}dt.\end{aligned}\tag{3.30}$$

And by using integration by parts, we have

$$-\int_{nT}^{(n+1)T}\left((xu_{x})_{x},u\left(t\right)\right)_{L^{2}(0,\alpha)}dt=\int_{nT}^{(n+1)T}\left\Vert u_{x}\right\Vert_{L_{\rho}^{2}(0,\alpha)}^{2}dt,\tag{3.31}$$

and

$$-\int_{nT}^{(n+1)T}\left((xv_{x})_{x},v\left(t\right)\right)_{L^{2}(0,\alpha)}dt=\int_{nT}^{(n+1)T}\left\Vert v_{x}\right\Vert_{L_{\rho}^{2}(0,\alpha)}^{2}dt,\tag{3.32}$$

and

$$\begin{aligned}&\int_{nT}^{(n+1)T}\left(\int_{0}^{t}g_{1}\left(t-s\right)\left(xu_{x}(x,s)\right)_{x}ds,u\left(x,t\right)\right)_{L^{2}(0,\alpha)}dt\\ =\;&\int_{nT}^{(n+1)T}\int_{0}^{t}g_{1}\left(t-s\right)\left(\left(u_{x}\left(x,t\right)-u_{x}(x,s)\right),u_{x}\left(x,t\right)\right)_{L_{\rho}^{2}(0,\alpha)}dsdt\\ &-\int_{nT}^{(n+1)T}\left(\int_{0}^{t}g_{1}\left(s\right)ds\right)\left\Vert u_{x}\right\Vert_{L_{\rho}^{2}(0,\alpha)}^{2}dt,\end{aligned}\tag{3.33}$$

and

$$\begin{aligned}&\int_{nT}^{(n+1)T}\left(\int_{0}^{t}g_{2}\left(t-s\right)\left(xv_{x}(x,s)\right)_{x}ds,v\left(x,t\right)\right)_{L^{2}(0,\alpha)}dt\\ =\;&\int_{nT}^{(n+1)T}\int_{0}^{t}g_{2}\left(t-s\right)\left(\left(v_{x}\left(x,t\right)-v_{x}(x,s)\right),v_{x}\left(x,t\right)\right)_{L_{\rho}^{2}(0,\alpha)}dsdt\\ &-\int_{nT}^{(n+1)T}\left(\int_{0}^{t}g_{2}\left(s\right)ds\right)\left\Vert v_{x}\right\Vert_{L_{\rho}^{2}(0,\alpha)}^{2}dt.\end{aligned}\tag{3.34}$$

By direct calculations, we get

$$\begin{aligned}&\int_{nT}^{(n+1)T}\left(\left|v\right|^{q+1}\left|u\right|^{p-1}u\left(t\right),u\left(t\right)\right)_{L_{\rho}^{2}(0,\alpha)}dt\\ &+\int_{nT}^{(n+1)T}\left(\left|u\right|^{p+1}\left|v\right|^{q-1}v\left(t\right),v\left(t\right)\right)_{L_{\rho}^{2}(0,\alpha)}dt\\ =\;&2\int_{nT}^{(n+1)T}\left(\int_{0}^{\alpha}x\left|v\right|^{q+1}\left|u\right|^{p+1}dx\right)dt.\end{aligned}\tag{3.35}$$

By replacement of $(3.29)-(3.35)$ into (3.28), we get

$$
\begin{aligned}
& -2\int_{nT}^{(n+1)T}\left(\int_0^{\alpha} x\,|v|^{q+1}\,|u|^{p+1}\,dx\right)dt \\
& -\int_{nT}^{(n+1)T}\|u_t\|^2_{L^2_\rho(0,\alpha)}\,dt-\int_{nT}^{(n+1)T}\|v_t\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& +\int_{nT}^{(n+1)T}\|u_x\|^2_{L^2_\rho(0,\alpha)}\,dt+\int_{nT}^{(n+1)T}\|v_x\|^2_{L^2_\rho(0,\alpha)}\,dt \\
= & \left(u_t(t),u(t)\right)_{L^2_\rho(0,\alpha)}\Big|_{nT}^{(n+1)T}+\left(v_t(t),v(t)\right)_{L^2_\rho(0,\alpha)}\Big|_{nT}^{(n+1)T} \\
& -\int_{nT}^{(n+1)T}\int_0^t g_1(t-s)\left(\left(u_x(x,t)-u_x(x,s)\right),u_x(x,t)\right)_{L^2_\rho(0,\alpha)}dsdt \\
& -\int_{nT}^{(n+1)T}\int_0^t g_2(t-s)\left(\left(v_x(x,t)-v_x(x,s)\right),v_x(x,t)\right)_{L^2_\rho(0,\alpha)}dsdt \\
& +\int_{nT}^{(n+1)T}\left(\int_0^t g_1(s)\,ds\right)\|u_x\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& +\int_{nT}^{(n+1)T}\left(\int_0^t g_2(s)\,ds\right)\|v_x\|^2_{L^2_\rho(0,\alpha)}\,dt.
\end{aligned}
\tag{3.36}
$$

To estimate the term

$$
\begin{aligned}
I_1 \ &: = \left(u_t(t),u(t)\right)_{L^2_\rho(0,\alpha)}\Big|_{nT}^{(n+1)T} \\
&: = \left(u_t((n+1)T),u((n+1)T)\right)_{L^2_\rho(0,\alpha)}-\left(u_t(nT),u(nT)\right)_{L^2_\rho(0,\alpha)},
\end{aligned}
$$

by using Young's inequality for $(\varepsilon=1)$, Poincaré-type inequality, $\|u_t\|^2_{L^2_\rho(0,\alpha)}\leq\dfrac{2}{(p+1)}E(t)$ and $\|u_x\|^2_{L^2_\rho(0,\alpha)}\leq\dfrac{2}{(p+1)\,l_1}E(t)$, we get

$$
\left(u_t(x,t),u(x,t)\right)_{L^2_\rho(0,\alpha)}\leq\frac{1}{(p+1)}\left(1+\frac{C_p}{l_1}\right)E(t),
\tag{3.37}
$$

then

$$
|I_1|\leq C_3\left\{E((n+1)T)+E(nT)\right\},
\tag{3.38}
$$

where

$$
C_3:=\frac{1}{(p+1)}\left(1+\frac{C_p}{l_1}\right)>0.
$$

Similarly to estimate the term

$$
\begin{aligned}
I_2 \ &: = \left(v_t(t),v(t)\right)_{L^2_\rho(0,\alpha)}\Big|_{nT}^{(n+1)T} \\
&: = \left(v_t((n+1)T),v((n+1)T)\right)_{L^2_\rho(0,\alpha)}-\left(v_t(nT),v(nT)\right)_{L^2_\rho(0,\alpha)},
\end{aligned}
$$

we get

$$
|I_2|\leq C_4\left\{E((n+1)T)+E(nT)\right\},
\tag{3.39}
$$

where $C_4 := \frac{1}{(q+1)}\left(1+\frac{C_p'}{l_2}\right) > 0$.
To estimate the term

$$I_3 := -\int_{nT}^{(n+1)T}\int_0^t g_1(t-s)\left(\left(u_x(x,t)-u_x(x,s)\right),u_x(x,t)\right)_{L^2_\rho(0,\alpha)}dsdt.$$

By using Young's inequality $\left(\text{for } \varepsilon=\dfrac{\varepsilon_5}{2}\right)$ and (2.4), we get

$$\begin{aligned}|I_3| \leq{} & \frac{1}{4\varepsilon_5}\int_{nT}^{(n+1)T}\int_0^\alpha (g_1\circ u_x)(t)\,dxdt \\ & +\varepsilon_5\int_{nT}^{(n+1)T}\left(\int_0^t g_1(s)\,ds\right)\|u_x\|^2_{L^2_\rho(0,\alpha)}dt.\end{aligned}\tag{3.40}$$

Similarly to estimate the term

$$I_4 := -\int_{nT}^{(n+1)T}\int_0^t g_2(t-s)\left(\left(v_x(x,t)-v_x(x,s)\right),v_x(x,t)\right)_{L^2_\rho(0,\alpha)}dsdt,$$

we get

$$\begin{aligned}|I_4| \leq{} & \frac{1}{4\varepsilon_5}\int_{nT}^{(n+1)T}\int_0^\alpha (g_2\circ v_x)(t)\,dxdt \\ & +\varepsilon_5\int_{nT}^{(n+1)T}\left(\int_0^t g_2(s)\,ds\right)\|v_x\|^2_{L^2_\rho(0,\alpha)}dt.\end{aligned}\tag{3.41}$$

By replacement of $(3.38)-(3.41)$ into (3.36), we get

$$\begin{aligned}& -2\int_{nT}^{(n+1)T}\left(\int_0^\alpha x|v|^{q+1}|u|^{p+1}dx\right)dt \\ & -\int_{nT}^{(n+1)T}\|u_t\|^2_{L^2_\rho(0,\alpha)}dt-\int_{nT}^{(n+1)T}\|v_t\|^2_{L^2_\rho(0,\alpha)}dt \\ & +\int_{nT}^{(n+1)T}\|u_x\|^2_{L^2_\rho(0,\alpha)}dt+\int_{nT}^{(n+1)T}\|v_x\|^2_{L^2_\rho(0,\alpha)}dt \\ \leq{} & (C_3+C_4)\left\{E((n+1)T)+E(nT)\right\} \\ & +\frac{1}{4\varepsilon_5}\int_{nT}^{(n+1)T}\int_0^\alpha (g_1\circ u_x)(t)\,dxdt+\frac{1}{4\varepsilon_5}\int_{nT}^{(n+1)T}\int_0^\alpha (g_2\circ v_x)(t)\,dxdt \\ & +(\varepsilon_5+1)\int_{nT}^{(n+1)T}\left(\int_0^t g_1(s)\,ds\right)\|u_x\|^2_{L^2_\rho(0,\alpha)}dt \\ & +(\varepsilon_5+1)\int_{nT}^{(n+1)T}\left(\int_0^t g_2(s)\,ds\right)\|v_x\|^2_{L^2_\rho(0,\alpha)}dt.\end{aligned}\tag{3.42}$$

Multiplying (3.27) by γ_1 and (3.42) by γ_2 and combining suitably, we get

$$
\begin{aligned}
& \left[\gamma_1\left\{t_1 g_1(t_1)-\varepsilon_1\right\}-\gamma_2\right]\int_{nT}^{(n+1)T}\|u_t\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& +\left[\gamma_1\left\{t_1 g_2(t_1)-\varepsilon_2\right\}-\gamma_2\right]\int_{nT}^{(n+1)T}\|v_t\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& +\gamma_2\int_{nT}^{(n+1)T}\|u_x\|^2_{L^2_\rho(0,\alpha)}\,dt+\gamma_2\int_{nT}^{(n+1)T}\|v_x\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& -2\gamma_2\int_{nT}^{(n+1)T}\left(\int_0^\alpha x\,|v|^{q+1}\,|u|^{p+1}\,dx\right)dt \\
\leq\; & \left\{\gamma_1\left(C_1+C_2\right)+\gamma_2\left(C_3+C_4\right)\right\}\left[E((n+1)T)+E(nT)\right] \\
& -\gamma_1\frac{C_p g_1(0)}{4\varepsilon_1}\int_{nT}^{(n+1)T}\int_0^\alpha\left(g_1'\circ u_x\right)(t)\,dxdt \\
& -\gamma_1\frac{C_p' g_2(0)}{4\varepsilon_2}\int_{nT}^{(n+1)T}\int_0^\alpha\left(g_2'\circ v_x\right)(t)\,dxdt \\
& +\gamma_1\varepsilon_3\left(1+c_1+c_2\right)\int_{nT}^{(n+1)T}\|u_x\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& +\gamma_1\varepsilon_3\left(1+c_1'+c_2'\right)\int_{nT}^{(n+1)T}\|v_x\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& +\left\{\gamma_1\left(\frac{(1+C_p)}{4\varepsilon_3}+1+\frac{1}{4\varepsilon_4}\right)+\frac{\gamma_2}{4\varepsilon_5}\right\}\int_{nT}^{(n+1)T}\int_0^\alpha\left(g_1\circ u_x\right)(t)\,dxdt \\
& +\left\{\gamma_1\left(\frac{(1+C_p')}{4\varepsilon_3}+1+\frac{1}{4\varepsilon_4}\right)+\frac{\gamma_2}{4\varepsilon_5}\right\}\int_{nT}^{(n+1)T}\int_0^\alpha\left(g_2\circ v_x\right)(t)\,dxdt \\
& +\left\{\gamma_1\varepsilon_4+\gamma_2\left(\varepsilon_5+1\right)\right\}\int_{nT}^{(n+1)T}\left(\int_0^t g_1(s)\,ds\right)\|u_x\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& +\left\{\gamma_1\varepsilon_4+\gamma_2\left(\varepsilon_5+1\right)\right\}\int_{nT}^{(n+1)T}\left(\int_0^t g_2(s)\,ds\right)\|v_x\|^2_{L^2_\rho(0,\alpha)}\,dt. \qquad (3.43)
\end{aligned}
$$

Let $\varepsilon_1 := \frac{t_1 g_1(t_1)}{2}$, $\varepsilon_2 := \frac{t_1 g_2(t_1)}{2}$, $\gamma_2 := 1$, $\max\left\{\frac{4}{t_1 g_1(t_1)}, \frac{4}{t_1 g_2(t_1)}\right\} \leq \gamma_1$, $\varepsilon_3 := \frac{3\varepsilon}{\gamma_1 \max\left\{(1+c_1+c_2),\left(1+c_1'+c_2'\right)\right\}}$, $\varepsilon_4 := \frac{\varepsilon}{\gamma_1}$, $\varepsilon_5 := 2\varepsilon$, into (3.43), we get

$$\begin{aligned}
& \int_{nT}^{(n+1)T} \|u_t\|^2_{L^2_\rho(0,\alpha)}\, dt + \int_{nT}^{(n+1)T} \|v_t\|^2_{L^2_\rho(0,\alpha)}\, dt \\
& + \int_{nT}^{(n+1)T} \|u_x\|^2_{L^2_\rho(0,\alpha)}\, dt + \int_{nT}^{(n+1)T} \|v_x\|^2_{L^2_\rho(0,\alpha)}\, dt \\
& -2 \int_{nT}^{(n+1)T} \left(\int_0^\alpha x\, |v|^{q+1} |u|^{p+1}\, dx\right) dt \\
\leq\; & C_5 \left[E((n+1)T) + E(nT)\right] \\
& -C_6 \int_{nT}^{(n+1)T} \int_0^\alpha \left(g_1' \circ u_x\right)(t)\, dxdt - C_7 \int_{nT}^{(n+1)T} \int_0^\alpha \left(g_2' \circ v_x\right)(t)\, dxdt \\
& +3\varepsilon \int_{nT}^{(n+1)T} \|u_x\|^2_{L^2_\rho(0,\alpha)}\, dt + 3\varepsilon \int_{nT}^{(n+1)T} \|v_x\|^2_{L^2_\rho(0,\alpha)}\, dt \\
& +C_8 \int_{nT}^{(n+1)T} \int_0^\alpha \left(g_1 \circ u_x\right)(t)\, dxdt + C_9 \int_{nT}^{(n+1)T} \int_0^\alpha \left(g_2 \circ v_x\right)(t)\, dxdt \\
& + (3\varepsilon + 1) \int_{nT}^{(n+1)T} \left(\int_0^t g_1(s)\, ds\right) \|u_x\|^2_{L^2_\rho(0,\alpha)}\, dt \\
& + (3\varepsilon + 1) \int_{nT}^{(n+1)T} \left(\int_0^t g_2(s)\, ds\right) \|v_x\|^2_{L^2_\rho(0,\alpha)}\, dt,
\end{aligned} \tag{3.44}$$

where $C_5 := \left\{\gamma_1 (C_1 + C_2) + (C_3 + C_4)\right\} > 0$, $C_6 := \gamma_1 \frac{C_p g_1(0)}{2t_1 g_1(t_1)} > 0$, $C_7 := \gamma_1 \frac{C_p' g_2(0)}{2t_1 g_2(t_1)} > 0$, and

$$\begin{cases}
C_8 := \left\{\gamma_1 \left(\frac{(1+C_p)\gamma_1 \max\left\{(1+c_1+c_2),\left(1+c_1'+c_2'\right)\right\}}{12\varepsilon} + 1 + \frac{\gamma_1}{4\varepsilon}\right) + \frac{1}{8\varepsilon}\right\} > 0, \\
C_9 := \left\{\gamma_1 \left(\frac{\left(1+C_p'\right)\gamma_1 \max\left\{(1+c_1+c_2),\left(1+c_1'+c_2'\right)\right\}}{12\varepsilon} + 1 + \frac{\gamma_1}{4\varepsilon}\right) + \frac{1}{8\varepsilon}\right\} > 0.
\end{cases}$$

Adding and subtracting in (3.44) the term

$$\begin{cases}
-\int_{nT}^{(n+1)T} \int_0^\alpha \left(\int_0^t g_1(s)\, ds\right) |u_x|^2\, dxdt \quad \text{and} \quad \int_{nT}^{(n+1)T} \int_0^\alpha a(x) \left(g_1 \circ u_x\right)(t)\, dxdt, \\
\text{and} \\
-\int_{nT}^{(n+1)T} \int_0^\alpha \left(\int_0^t g_2(s)\, ds\right) |v_x|^2\, dxdt \quad \text{and} \quad \int_{nT}^{(n+1)T} \int_0^\alpha a(x) \left(g_2 \circ v_x\right)(t)\, dxdt,
\end{cases}$$

in order to recover the energy $E(t)$, we obtain

$$
\begin{aligned}
& (1-3\varepsilon)\int_{nT}^{(n+1)T}\int_0^\alpha \left(1-\int_0^t g_1(s)\,ds\right)|u_x|^2\,dxdt \\
& +(1-3\varepsilon)\int_{nT}^{(n+1)T}\int_0^\alpha \left(1-\int_0^t g_2(s)\,ds\right)|v_x|^2\,dxdt \\
& +\int_{nT}^{(n+1)T}\|u_t\|^2_{L^2_\rho(0,\alpha)}\,dt+\int_{nT}^{(n+1)T}\|v_t\|^2_{L^2_\rho(0,\alpha)}\,dt \\
& +\int_{nT}^{(n+1)T}\int_0^\alpha (g_1\circ u_x)(t)\,dxdt+\int_{nT}^{(n+1)T}\int_0^\alpha (g_2\circ v_x)(t)\,dxdt \\
& -2\int_{nT}^{(n+1)T}\left(\int_0^\alpha x\,|v|^{q+1}\,|u|^{p+1}\,dx\right)dt \\
\leq\; & C_5\left[E((n+1)T)+E(nT)\right] \\
& +C_8\int_{nT}^{(n+1)T}\int_0^\alpha (g_1\circ u_x)(t)\,dxdt+C_8\int_{nT}^{(n+1)T}\int_0^\alpha k_1\left(-g_1'\circ u_x\right)(t)\,dxdt \\
& +C_9\int_{nT}^{(n+1)T}\int_0^\alpha (g_2\circ v_x)(t)\,dxdt+C_9\int_{nT}^{(n+1)T}\int_0^\alpha k_2\left(-g_2'\circ v_x\right)(t)\,dxdt, \qquad (3.45)
\end{aligned}
$$

where $k_1 := \dfrac{C_6}{C_8} > 0$ and $k_2 := \dfrac{C_7}{C_9} > 0$. From (3.45), choosing ε sufficiently small, $k_1 > 0,\ k_2 > 0$ and T large enough and using

$$
\begin{aligned}
& \alpha_1\Big\{\|u_t\|^2_{L^2_\rho(0,\alpha)}+\|v_t\|^2_{L^2_\rho(0,\alpha)}+\|u_x\|^2_{L^2_\rho(0,\alpha)}+\|v_x\|^2_{L^2_\rho(0,\alpha)} \\
& -\int_0^\alpha x\,|u|^{p+1}\,|v|^{q+1}\,dx+\int_0^\alpha (g_1\circ u_x)(t)\,dx+\int_0^\alpha (g_2\circ v_x)(t)\,dx\Big\} \\
\leq\; & E(t)\leq \alpha_2\Big\{\|u_t\|^2_{L^2_\rho(0,\alpha)}+\|v_t\|^2_{L^2_\rho(0,\alpha)}+\|u_x\|^2_{L^2_\rho(0,\alpha)}+\|v_x\|^2_{L^2_\rho(0,\alpha)} \\
& -\int_0^\alpha x\,|u|^{p+1}\,|v|^{q+1}\,dx+\int_0^\alpha (g_1\circ u_x)(t)\,dx+\int_0^\alpha (g_2\circ v_x)(t)\,dx\Big\},
\end{aligned}
$$

into (3.45), we get

$$
\begin{aligned}
& \int_{nT}^{(n+1)T} E(t)\,dt \\
\leq\; & C_{10}\left[E((n+1)T)+E(nT)\right] \\
& +C_{11}\int_{nT}^{(n+1)T}\int_0^\alpha (g_1\circ u_x)(t)\,dxdt+C_{11}\int_{nT}^{(n+1)T}\int_0^\alpha k_1\left(-g_1'\circ u_x\right)(t)\,dxdt \\
& +C_{12}\int_{nT}^{(n+1)T}\int_0^\alpha (g_2\circ v_x)(t)\,dxdt+C_{12}\int_{nT}^{(n+1)T}\int_0^\alpha k_2\left(-g_2'\circ v_x\right)(t)\,dxdt. \qquad (3.46)
\end{aligned}
$$

In the last step, we need to relate the viscoelastic energy to the viscoelastic damping. In the case when the relaxation function obeys a linear equation, this relation is straightforward and is expressed by a suitable multiplication. However, in the case of general decays, additional arguments are used. Here, we follow [41]. From the

(**A2**) made on the viscoelastic kernel $g_1,\ g_2$ and from [41, **Lemma 4**] we obtain

$$\left\{\begin{array}{ll} (g_1\circ u_x)(t)\le \hat{H}_\alpha^{-1}(-g_1'\circ u_x)(t), & t\in[nT,(n+1)T], \\ & \text{and} \\ (g_2\circ v_x)(t)\le \hat{H}_\alpha^{-1}(-g_2'\circ v_x)(t), & t\in[nT,(n+1)T], \end{array}\right. \tag{3.47}$$

where $\hat{H}_\alpha$ is a rescaling of H_α with

$$H_\alpha(s)=\alpha s^{1-\frac{1}{\alpha}}H\left(s^{\frac{1}{\alpha}}\right),$$

and $\alpha\in(0,1)$ is such that

$$\left\{\begin{array}{l} \sup_{t>0}\int_0^t g_1^{1-\alpha}(t-s)\left\|u_x(t)-u_x(s)\right\|^2 ds<\infty, \\ \text{and} \\ \sup_{t>0}\int_0^t g_1^{1-\alpha}(t-s)\left\|v_x(t)-v_x(s)\right\|^2 ds<\infty. \end{array}\right.$$

From (**A2**) it is clear that $\alpha\ge\alpha_0$. The main point, however, is that the argument can be reiterated (based on [41, **Lemma 8**] leading to $\alpha=1$). This allows us to replace H_α, the function in (3.46), by the original function $\hat{H}$ which is a rescaling of $H(s)$. This means that $\hat{H}=cH\left(\frac{C}{s}\right)$ for some $c,\ C>0$. Now, from (3.46) and taking (3.47) into account, we deduce that

$$\begin{aligned} &\int_{nT}^{(n+1)T}E(t)\,dt \\ \le\ & C_{10}\left[E((n+1)T)+E(nT)\right] \\ &+C_{11}\int_{nT}^{(n+1)T}\int_0^\alpha\left[\hat{H}_\alpha^{-1}+k_1\right](-g_1'\circ u_x)(t)\,dxdt \\ &+C_{12}\int_{nT}^{(n+1)T}\int_0^\alpha\left[\hat{H}_\alpha^{-1}+k_2\right](-g_2'\circ v_x)(t)\,dxdt. \end{aligned} \tag{3.48}$$

Next, we shall employ the following version of **Jensen's inequality** applied to measures and convex functions F. We shall use (3.48) in order to bring the functions H in front of the integrals. Let us denote $\alpha_0:=\alpha$. We note that the function $\hat{H}^{-1}+k_1$ is concave. Let $F^{-1}=\hat{H}^{-1}+k_1,\ f(x)=(-g_1'\circ\nabla u)(t),\ h(x)=T,\ h_0=T\alpha_0$ and $h_0^{-1}=\alpha_0^{-1}T^{-1}$, thus, we have

$$\begin{aligned} &\int_{nT}^{(n+1)T}\int_0^\alpha\left[\hat{H}_\alpha^{-1}+k_1\right](-g_1'\circ u_x)(t)\,dxdt \\ \le\ & \alpha_0 T\left[\hat{H}_\alpha^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_0^\alpha(-g_1'\circ u_x)(t)\,dxdt\right]. \end{aligned} \tag{3.49}$$

Similarly, we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\int_0^{\alpha}\left[\hat{H}_{\alpha}^{-1}+k_2\right](-g_2'\circ v_x)(t)\,dxdt \\
(3.50)\qquad \leq\ & \alpha_0 T\left[\hat{H}_{\alpha}^{-1}+k_2\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_0^{\alpha}(-g_2'\circ v_x)(t)\,dxdt\right].
\end{aligned}$$

On the other hand, from the identity (2.4) for the energy, we can write

$$\begin{aligned}
& E((n+1)T)-E(nT) \\
=\ & \left(\frac{p+1}{2}\right)\int_{nT}^{(n+1)T}\left\{\int_0^{\alpha}(g_1'\circ u_x)(t)dx-g_1(t)\|u_x\|^2_{L^2_{\rho}(0,\alpha)}\right\}dt \\
& +\left(\frac{q+1}{2}\right)\int_{nT}^{(n+1)T}\left\{\int_0^{\alpha}(g_2'\circ v_x)(t)dx-g_2(t)\|v_x\|^2_{L^2_{\rho}(0,\alpha)}\right\}dt \\
=\ & -\int_{nT}^{(n+1)T}D(t)\,dt,
\end{aligned}$$

where

$$\begin{aligned}
D(t)\ :\ =\ & -\left(\frac{p+1}{2}\right)\left\{\int_0^{\alpha}(g_1'\circ u_x)(t)dx-g_1(t)\|u_x\|^2_{L^2_{\rho}(0,\alpha)}\right\} \\
(3.51)\qquad & -\left(\frac{q+1}{2}\right)\left\{\int_0^{\alpha}(g_2'\circ v_x)(t)dx-g_2(t)\|v_x\|^2_{L^2_{\rho}(0,\alpha)}\right\}.
\end{aligned}$$

En replacement (3.49) and (3.50) into (3.48) and using

$$E(nT)=E((n+1)T)+\int_{nT}^{(n+1)T}D(t)\,dt, \tag{3.52}$$

we get

$$\begin{aligned}
& \int_{nT}^{(n+1)T}E(t)\,dt \\
\leq\ & C_{10}\left[2E((n+1)T)+\int_{nT}^{(n+1)T}D(t)\,dt\right] \\
& +C_{11}\alpha_0 T\left[\hat{H}_{\alpha}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_0^{\alpha}(-g_1'\circ u_x)(t)\,dxdt\right] \\
& +C_{12}\alpha_0 T\left[\hat{H}_{\alpha}^{-1}+k_2\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_0^{\alpha}(-g_2'\circ v_x)(t)\,dxdt\right] \\
\leq\ & C_{10}\left[2E((n+1)T)+\int_{nT}^{(n+1)T}D(t)\,dt\right] \\
& +C_{13}B\left[\int_{nT}^{(n+1)T}\int_0^{\alpha}(-g_1'\circ u_x)(t)\,dxdt+\int_{nT}^{(n+1)T}\int_0^{\alpha}(-g_2'\circ v_x)(t)\,dxdt\right],
\end{aligned}$$

where $C_{13} := \max\{C_{11}, C_{12}\} > 0$ and $B := \max\left\{\left[\hat{H}_\alpha^{-1} + k_1\right], \left[\hat{H}_\alpha^{-1} + k_2\right]\right\}$, and by using (3.51), we get

$$\begin{aligned} & \int_{nT}^{(n+1)T} \int_0^\alpha \left(-g_1' \circ u_x\right)(t)\, dxdt + \int_{nT}^{(n+1)T} \int_0^\alpha \left(-g_2' \circ v_x\right)(t)\, dxdt \\ \leq \quad & K_3 \int_{nT}^{(n+1)T} D(t)\, dt, \end{aligned}$$

where

$$K_3 := \frac{2}{\min\{p+1, q+1\}} > 0,$$

thus, we get

$$\begin{aligned} & \int_{nT}^{(n+1)T} E(t)\, dt \\ \leq \quad & C_{10}\left[2E((n+1)T) + \int_{nT}^{(n+1)T} D(t)\, dt\right] + C_{13}K_3 B\left[\int_{nT}^{(n+1)T} D(t)\, dt\right] \\ \leq \quad & 2C_{10}E((n+1)T) + C_{14}\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\, dt\right], \end{aligned} \tag{3.53}$$

where

$$\begin{cases} C_{14} := \max\{C_{10}, C_{13}K_3\} > 0, \\ \tilde{H} := [1+B]^{-1}. \end{cases} \tag{3.54}$$

In particular integrating $\frac{d}{dt}\{E(t)\} \leq 0$ from t to $(n+1)T$ yields

$$E((n+1)T) \leq E(t) \quad \text{for all } (n+1)T \geq t, \tag{3.55}$$

integrating (3.55) from nT to $(n+1)T$ yields

$$\begin{aligned} \int_{nT}^{(n+1)T} E(t)\, dt \quad & \geq \quad \int_{nT}^{(n+1)T} E((n+1)T)\, dt \\ & = \quad \int_{nT}^{(n+1)T} dt E((n+1)T) \\ & = \quad TE((n+1)T), \end{aligned} \tag{3.56}$$

by replacement (3.56) into (3.53), we get

$$TE((n+1)T) \leq 2C_{10}E((n+1)T) + C_{14}\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\, dt\right],$$

then

$$(T - 2C_{10})E((n+1)T) \leq C_{14}\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\, dt\right].$$

For T large enough, where C_{10} is a positive constant, which implies that

$$E((n+1)T) \leq C_{15}\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\, dt\right],$$

where

$$C_{15} := \frac{C_{14}}{(T - 2C_{10})} > 0, \tag{3.57}$$

which gives that

$$\tilde{H}\left(C_{15}^{-1} E\left((n+1)T\right)\right) \leq \int_{nT}^{(n+1)T} D(t)\,dt, \tag{3.58}$$

by using (3.52) into (3.58), we get

$$\tilde{H}\left(C_{15}^{-1} E\left((n+1)T\right)\right) \leq E(nT) - E\left((n+1)T\right),$$

from the above we have

$$E\left((n+1)T\right) + \tilde{H}\left(C_{15}^{-1} E\left((n+1)T\right)\right) \leq E(nT), \quad n = 1, 2, 3....$$

This completes the proof. □

Lemma 3. *Let p be a positive, increasing function such that $p(0) = 0$. Since p is increasing, we can define an increasing function q, $q(x) \equiv x - (I+p)^{-1}(x)$. Consider a sequence F_n of positive numbers which satisfies*

$$F_{m+1} + p(F_{m+1}) \leq F_m. \tag{3.59}$$

Then $F_m \leq S(m)$ where $S(t)$ is a solution of the differential equation

$$\frac{d}{dt}\{S(t)\} + q(S(t)) = 0, \quad S(0) = F_0. \tag{3.60}$$

Moreover, if $p(x) > 0$ for $x > 0$ then $\lim_{t \longrightarrow \infty} S(t) = 0$.

Proof. Proof of the **Lemma 3** use the proof retraction. Assume $F_m \leq S(m)$ and prove that $F_{m+1} \leq S(m+1)$.
Inequality (3.59) is equivalent to

$$(I+p) F_{m+1} \leq F_m,$$

and since $(I+p)^{-1}$ is monotone increasing, $F_{m+1} \leq (I+p)^{-1} F_m$, and using $(I+p)^{-1} F_m = (I-q) F_m$, we get

$$\begin{aligned} F_{m+1} &\leq (I-q) F_m \\ &= F_m - q(F_m). \end{aligned} \tag{3.61}$$

On the other hand, since q is an increasing function, the solution $S(t)$ of equation (3.60) is described by a nonlinear contraction.
In particular integrating $\frac{d}{dt}\{S(t)\} \leq 0$ from m to τ yields

$$S(\tau) \leq S(m) \quad \text{for all } t \geq \tau. \tag{3.62}$$

Integrating equation (3.60) from m to $(m+1)$ yields

$$S(m+1) - S(m) + \int_m^{m+1} q(S(\tau))\,d\tau = 0. \tag{3.63}$$

Since q is increasing, by using (3.62) we obtain for all $m \leq \tau \leq m+1$

$$\begin{aligned} \int_m^{m+1} q\left(S\left(\tau\right)\right) d\tau &\leq \int_m^{m+1} q\left(S\left(m\right)\right) d\tau \\ &= q\left(S\left(m\right)\right) \int_m^{m+1} d\tau \\ &= q\left(S\left(m\right)\right), \end{aligned}$$

then

$$-\int_m^{m+1} q\left(S\left(\tau\right)\right) d\tau \geq -q\left(S\left(m\right)\right), \quad \text{for all } m \leq \tau \leq m+1, \tag{3.64}$$

by replacement (3.64) into (3.63) and using the inductive assumption $F_m \leq S\left(m\right)$, we get

$$\begin{aligned} S\left(m+1\right) &\geq S\left(m\right) - q\left(S\left(m\right)\right) \\ &= \left(I-q\right) S\left(m\right) \\ &\geq \left(I-q\right) F_m \\ &= F_m - q\left(F\left(m\right)\right), \end{aligned} \tag{3.65}$$

comparing (3.61) with (3.65) yields

$$S\left(m+1\right) \geq F_{m+1}.$$

This completes the proof. □

Theorem 2. *Let us assume that* $(\mathbf{A1}) - (\mathbf{A4})$ *ar the place. Then there exist positive constants* $c_1,\ c_2$ *and* T_0 *such that the solution of problem* $(1.1) - (1.3)$ *satisfies* $E\left(t\right) \leq s\left(t\right)$, *where* $s\left(t\right)$ *verifies the ODE*

$$s_t + \hat{H}\left(s\right) = 0,\ \ s\left(0\right) = E\left(0\right),\ \ t \geq T_0 > 0,$$

with $\hat{H}\left(s\right) = c_1 H\left(c_2 s\right).$

Proof. Thus, we are in a position to apply the result of **Lemma 2** with

$$F_m \equiv E\left(mt\right), \quad F_0 \equiv E\left(0\right).$$

This yields

$$E\left(mT\right) \leq S\left(m\right), \quad m = 0, 1, 2, 3....$$

Setting $t = mT + \tau$ and recalling the evolution property gives

$$E\left(t\right) \leq E\left(mT\right) \leq S\left(m\right) \leq S\left(\frac{t-\tau}{T}\right) \leq S\left(\frac{t}{T} - 1\right).$$

This completes the proof. □

References

[1] Cahlon, D.; Shi, P.: Stepwise stability for the heat equation with a nonlocal constraint. SIAM J Numer Anal. 1995; $32 : 571 - 593$.

[2] Mesloub, S.; Lekrine, N.: On a nonlocal hyperbolic mixed problem. Acta Sci Math (Szeged). 2004; $70 : 65 - 75$.

[3] Ewing, R.; Lin, T.: A class of parameter estimation techniques for fluid flow in porous media. AdvWater Resour. 1991; $14 : 89 - 97$.

[4] Shi, P.: Weak solution to an evolution problem with a non local constraint. SIAM J Math Anal. 24(N 1):$46 - 58$.

[5] Choi, Y.; Chan, K.: A parabolic equation with nonlocal boundary conditions arising from electrochemistry. Nonlinear Anal. 1992; $18 : 317 - 331$.
[6] Cannon, R.: The solution of heat equation subject to the specification of energy. Q Appl Math. 1963; $21 : 155 - 160$.
[7] Capasso, V.; Kunisch, K.: A reaction-diffusion system arising in modeling man-environment diseases. Quart Appl Math. 1988; $46 : 431 - 449$.
[8] Yurchuk, N.: Mixed problem with an integral condition for certain parabolic equations. Differ Uravn. 1986; $22(19) : 2117 - 2126$.
[9] Shi, P.; Shilor, M.: Design of contact patterns in one dimensional thermoelasticity, in Theoretical aspects of industrial design. Philadelphia: SIAM; 1992.
[10] Ionkin, N.; Moiseev, EI.: A problem for the heat conduction equation with two-point boundary condition. Differ Uravn. 1979; $15(7) : 1284 - 1295$.
[11] Kamynin, L.: Aboundary-value problem in the theory of heat conduction with non-classical boundary conditions. (Russian) Ž Vyc isl Mat i Mat Fiz. 1964;4:1006–1024. MR 0171085.
[12] Mesloub, S.: On a singular two dimensional nonlinear evolution equation with non local conditions. Nonlinear Anal. 2008; $68 : 2594 - 2607$.
[13] Mesloub, S.: A nonlinear nonlocal mixed problem for a second order parabolic equation. J Math Anal Appl. 2006; $316 : 189 - 209$.
[14] Ionkin, N.: Solution of boundary value problem in heat conduction theory with nonclassical boundary conditions. Differ Uravn. 1977; $13(2) : 1177 - 1182$.
[15] Mesloub, S.; Messaoudi, S.: Global existence, decay, and blow up of solutions of a singular nonlocal viscoelastic problem. Acta Appl Math. 2010; $110 : 705 - 724$.
[16] Mesloub, S.; Messaoudi, S.: A non local mixed semilinear problem for second order hyperbolic equations. Electron J Differ Equ. 2003; $2003(30) : 1 - 17$.
[17] Kartynnik, A.: Three-point boundary value problem with an integral space-variable condition for a second order parabolic equation. Differ Equ. 1990; $26 : 1160 - 1162$.
[18] Pulkina, L.: A nonlocal problem with integral conditions for hyperbolic equations. Electron J Differ Equ. 1999; $45 : 1 - 6$.
[19] Pulkina, L.: On solvability in L2 of nonlocal problem with integral conditions for a hyperbolic equation. Differ Uravn. 2000; 2.
[20] Mesloub, S.; Bouziani, A.: Problème mixte avec conditions aux limites intégrales pour une classe d'équations paraboliques bidimensionnelles. Bull. Classe Sci. Acad. R. Belg. 1998; $6 : 59 - 69$.
[21] Mesloub, S.; Bouziani; A.: Mixed problem with a weighted integral condition for a parabolic equation with Bessel operator. J Appl Math Stoch Anal. 2002; $15(3) : 291 - 300$.
[22] Mesloub, S.; Mesloub, F.: Solvability of a mixed nonlocal problem for a nonlinear singular viscoelastic equation. Acta Appl Math $(2010)110 : 109 - 129$.
[23] Wu, S.: Blow-up of solutions for a singular nonlocal viscoelastic equation. J Partial Differ Equ. 2011; $24(2) : 140 - 149$.
[24] Liu, W.; Sun, Y and Li, G.: On decay and blow-up of solutions for a singular nonlocal viscoelastic problem with a nonlinear source term. Math. AP 18 Mar 2013.
[25] Draifia, A.; Zarai, A.; Boulaaras, S.: Global existence and decay of solutions of a singular nonlocal viscoelastic system. Springer-Verlag Italia S.r.l., part of Springer Nature 2018.
[26] Zarai, A.; Draifia, A.& Boulaaras, S.: Blow-up of solutions for a system of nonlocal singular viscoelastic equations. Applicable Analysis 2017.
[27] Boulaaras, S.; Draifia, A.; Alnegga, M.: Polynomial decay rate for kirchhoff type in viscoelasticity with logarithmic nonlinearity and not necessarily decreasing kernel. Symmetry 2019.
[28] Marcelo, M. Cavalcanti, Valéria N. Domingos Cavalcanti, Irena Lasiecka. Claudete M. Webler.: Intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density. Adv. Nonlinear Anal. 2016; aop.
[29] Wu, S.: General decay of energy for a viscoelastic equation with damping and source terms, Taiwan. J. Math. 16 (2012), $113 - 128$.
[30] Hao, J.; Cai, L.: Uniform decay of solutions for coupled viscoelastic wave equations. Electronic Journal of Di erential Equations, Vol. 2016 (2016), No. 72, pp. $1 - 11$.
[31] Mu, C.; Ma, J.: On a system of nonlinear wave equations with Balakrishnan–Taylor damping. Z. Angew. Math. Phys. 65 (2014), $91 - 113$.

[32] Cavalcanti, M.; Domingos Cavalcanti, V; Ferreira, J.: Existence and uniform decay for nonlinear viscoelastic equation with strong damping, Math. Meth. Appl. Sci. 24 (2001) , 1043−1053.
[33] M. M. Cavalcanti, V. N. Domingos Cavalcanti, I. Lasiecka and F. A. Falcoo Nascimento, Intrinsic decay rate estimates for the wave equation with competing viscoelastic and frictional dissipative e¤ects, Discrete Contin. Dyn. Syst. Ser. B 19 (2014), no. 7, 1987-2012.
[34] I. Lasiecka, S. A. Messaoudi and M. I. Mustafa, Note on intrinsic decay rates for abstract wave equations with memory, J. Math. Phys. 54 (2013), Article ID 031504.
[35] I. Lasiecka and X. Wang, Intrinsic decay rate estimates for semilinear abstract second order equations with memory, in: New Prospects in Direct, Inverse and Control Problems for Evolution Equations, Springer INdAM Ser. 10, Springer, Cham (2014), 271-303.
[36] Cavalcanti M, Filho VND, Cavalcanti JSP, Soriano JA. Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping. Di¤erential Integral Equ. 2001; 14 : 85-116.
[37] M.M. Cavalcanti, M. Aassila, J.A. Soriano, Asymptotic stability and energy decay rates for solutions of the wave equation with memory in a star-shaped domain, SIAM J. Control Opt. 38 (5) (2000) 1581 − 1602.
[38] W.J. Hrusa, Global existence and asymptotic stability for a nonlinear hyperbolic Volterra equation with large initial data, SIAM J. Math. Anal. 16 (1985) 110 − 134
[39] M. Renardy, Coercive estimates and existence of solutions for a model of one-dimensional viscoelasticity with a nonintegrable memory function, J. Integral Eqs. Appl. 1 (1988) 7 − 16.
[40] Q. Tiehu, Asymptotic behavior of a class of abstract integrodi¤erential equations and applications, J. Math. Anal. Appl. 233 (1999) 130 − 147.
[41] C. M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Ration. Mech. Anal. 37 (1970), 297 − 308.

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