A Mixed Finite Element Method for Solving Coupled Wave Equation of Kirchhoff Type with Nonlinear Boundary Damping and Memory Term

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Abstract

In this paper, we deal with the numerical approximation of the coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory term. Since the equation is a nonlinear equation, the Raviart-Thomas mixed finite element method is one of the most suitable techniques to obtain the approximated solution. In this paper, we will show that using the Raviart-Thomas method the optimal convergence order of the scheme can be achieved. To that end, we prove the necessary lemmas and the main theorem. Finally, the efficiency of the method is certified by numerical examples.

Keywords: Nonlinear wave equation, semi-discretization, Raviart-Thomas mixed finite element, convergence.
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1. Introduction

Mixed finite element method is one of the most useful methods to solve second-order differential equations; therefore, this method has been considered in lots of research works. For instance, in [1], a mixed finite element Galerkin method is analyzed for a strongly damped wave equation. Liu et al. [2], presented an H^1-Galerkin mixed finite element method for a class of second-order Schrodinger equation. Liu et al. [3], considered a new numerical scheme based on the H^1-Galerkin mixed finite element method for a class of second-order pseudo-hyperbolic equations. In [4] the authors presented two splitting mixed finite element schemes for the pseudo-hyperbolic equation. In that work, a mixed finite element method for approximating the solution of nearly incompressible elasticity and Stokes equations was presented as well.

The author in [5] presented a new weak Galerkin mixed finite element method for the Helmholtz equation with large wave numbers. In [6], the space-time discretization using a combination of mixed finite element method (Raviart-Thomas) and a finite difference scheme for the time discretization was applied to a wave equation. A splitting

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positive definite mixed finite element method used for second-order viscoelasticity wave equation in [7]. The authors of [8] presented an H^1-Galerkin mixed finite element methods for parabolic partial integro-differential equations. In [1], a mixed finite element Galerkin method and a second-order implicit-time discretization scheme were presented to solve a strongly damped wave equation. In [9], an extended Raviart–Thomas mixed finite element method was applied to approximate the solution of damped Boussinesq equation. In [10], the solution of a viscoelasticity wave equation using mixed finite element approximations were studied. In [11], an H^1-Galerkin mixed finite element method for parabolic partial differential equations was applied. In [12], the authors presented a priori error estimates for mixed finite element displacement formulations of the acoustic wave equation. In [13], some mixed finite element methods, explicit and implicit in time, for a fourth-order wave equation were studied. The authors in [14]] applied Galerkin mixed finite element methods to approximate the solution of a class of second-order pseudo-hyperbolic equations. In [15], a new family of quadrangular (two dimensional) or cubic (three dimensional) mixed finite elements for the approximation of elastic wave equation was used. In [16], a new mixed finite element weak Galerkin (WG) method for the second order elliptic equation was introduced. The more interested readers can see [17, 18, 19] for more details. This paper is concerned with the Raviart-Thomas mixed finite element method for the following coupled Kirchhoff type wave equation with nonlinear boundary damping and memory term [20, 21, 22]:

$$U_{tt}(x,t) - \left(1 + \|\nabla U\|_\Omega^2 + \|\nabla V\|_\Omega^2\right)\Delta U(x,t) - \Delta U_t(x,t) = f(x,t) \qquad x \in \Omega, \quad t \in [0,T], \tag{1a}$$

$$V_{tt}(x,t) - \left(1 + \|\nabla U\|_\Omega^2 + \|\nabla V\|_\Omega^2\right)\Delta V(x,t) - \Delta V_t(x,t) = f(x,t) \qquad x \in \Omega, \quad t \in [0,T], \tag{1b}$$

$$U = V = \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 \qquad \text{on } \Sigma_1 = \Gamma_1 \times [0,T], \tag{1c}$$

$$\left(1 + \|\nabla U\|_\Omega^2 + \|\nabla V\|_\Omega^2\right)\frac{\partial U}{\partial \nu} + \frac{\partial U_t}{\partial \nu} + U + U_t + g(t)|U_t|^p U_t = g * |U|^\gamma U \qquad \text{on } \Sigma_0 = \Gamma_0 \times [0,T], \tag{1d}$$

$$\left(1 + \|\nabla U\|_\Omega^2 + \|\nabla V\|_\Omega^2\right)\frac{\partial V}{\partial \nu} + \frac{\partial V_t}{\partial \nu} + V + V_t + g(t)|V_t|^p V_t = g * |V|^\gamma V \qquad \text{on}, \Sigma_0 = \Gamma_0 \times [0,T], \tag{1e}$$

$$U(x,0) = U_0(x) \qquad U_t(x,0) = U_1(x) \qquad x \in \Omega, \tag{1f}$$

$$V(x,0) = V_0(x) \qquad V_t(x,0) = V_1(x) \qquad x \in \Omega, \tag{1g}$$

where U and V represent the transverse displacements, Ω is a polygon domain of $\mathbb{R}^2$ with a boundary $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and Γ_0 and Γ_1 have positive measures. Here, (1a) has its origin in the mathematical description of small amplitude vibrations of an elastic string [22]. Furthermore, we define $\frac{\partial U}{\partial \nu} := \nabla U \cdot \nu$ where ν is the unit outer normal vector pointing towards Ω and

$$g * u(t) := \int_0^t g(t-r)u(r)dr, \quad \|\nabla U\|_\Omega^2 := \sum_{i=1}^2 \int_\Omega \left|\frac{\partial U}{\partial x_i}(x)\right|^2 dx,$$

and $0 \le g \le \alpha_1$ (a constant), $\gamma > 0$, $p \ge \gamma$. We have the following assumptions on the kernel g of the memory term

- $\frac{\partial g}{\partial t} \in L^\infty(0,\infty) \cap L^1(0,\infty)$,
- $g(t) \ge 0 \quad t \ge t_0$, where t_0 is a constant,

- $g(t) \leq g'(t) \leq -m_2 g(t) \quad \forall t \in [0, t_0]$,

- $g(0) = 0 \quad |g'(t)| \leq m_2 g(t) \quad \forall t \in [0, t_0]$,

for some $m_0, m_1, m_2 > 0$, $m_1 > 2(\gamma + 2)$ and $1 - \int_0^\infty g(r)\mathrm{d}r$. The function f satisfies the mentioned conditions of [21] which guaranties its meaningfulness. Let us define $W := \{u \in H^1(\Omega) |\ u = 0 \text{ on } \Gamma_1\}$ and assume U_0, U_1, V_0 and V_1 belong to $H^{3/2}(\Omega) \cap W$ should satisfy the following assumptions:

$$\left(1 + \|\nabla U_0\|_\Omega^2 + \|\nabla V_0\|_\Omega^2\right)\Delta U_0 + \Delta U_1 = \left(1 + \|\nabla U_0\|_\Omega^2 + \|\nabla V_0\|_\Omega^2\right)\Delta V_0 + \Delta V_1 \qquad \text{on } \Omega, \tag{2a}$$

$$U_0 = V_0 = \frac{\partial U_0}{\partial \nu} = \frac{\partial V_0}{\partial \nu} = 0 \qquad \text{on } \Omega, \tag{2b}$$

$$\left(1 + \|\nabla U_0\|_\Omega^2 + \|\nabla V_0\|_\Omega^2\right)\frac{\partial U_0}{\partial \nu} + \frac{\partial U_1}{\partial \nu} + U_0 + U_1 + g(t)|U_1|^p U_1 = 0 \qquad \text{on } \Gamma_0, \tag{2c}$$

$$\left(1 + \|\nabla U_0\|_\Omega^2 + \|\nabla V_0\|_\Omega^2\right)\frac{\partial V_0}{\partial \nu} + \frac{\partial V_1}{\partial \nu} + V_0 + V_1 + g(t)|V_1|^p V_1 = 0 \qquad \text{on } \Gamma_0. \tag{2d}$$

Now, we review the history and the background of (1). The first Kirchhoff equation was of the form

$$\rho h \frac{\partial^2 U}{\partial t^2} = \left(\rho_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial U}{\partial x}\right)^2 dx\right)\frac{\partial^2 U}{\partial x^2}, \qquad x \in [0, L], \quad t \geq 0, \tag{3}$$

where this equation extends the classical d 'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Eq. (3) is called the wave equation of the Kirchhoff type because Kirchhoff was the first one who introduced this equation in the study of oscillation of stretched strings and plates [23].
The equation

$$\rho h \frac{\partial^2 U}{\partial t^2} + C\frac{\partial U}{\partial t} + \left(\rho_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial U}{\partial x}\right)^2 dx\right)\frac{\partial^2 U}{\partial x^2} = q(x,t) \qquad x \in [0, L], \quad t \geq 0,$$

is a mathematical Kirchhoff model of nonlinear transverse vibration, neglecting the displacements along the string's axis and averaging tension N over its length L. In this equation, ρ is the density of the string material, E is the Young modulus of the latter, C is the viscous damping parameter, ρ_0 is the initial string tension value and $q(x,t)$ is the transverse load intensity [24]. Here, we consider the following equation [25]

$$\begin{cases} U_{tt} - \varphi\left(\|\nabla U\|_\Omega^2\right)\Delta U - a\Delta U_t = b|U|^{\beta-2}U & \text{in } \Omega \times (0,\infty), \\ U(x,t) = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \varphi\left(\|\nabla U\|_\Omega^2\right)\frac{\partial U}{\partial \nu} + a\frac{\partial U}{\partial \nu} = g(U_t) & \text{on } \Gamma_0 \times (0,\infty), \\ U(x,0) = U_1, \qquad U_t(x,0) = U_2 & \text{in } \Omega. \end{cases} \tag{4}$$

In the special case of (4), the dynamics of the moving string in Figure 1 can be described by

$$\rho h \frac{\partial^2 U}{\partial t^2} - a\frac{\partial^3 U}{\partial^2 x \partial t} = \left(\rho_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial U}{\partial x}\right)^2 dx\right)\frac{\partial^2 U}{\partial x^2} + f, \quad x \in [0, L], \ t \geq 0, \tag{5}$$

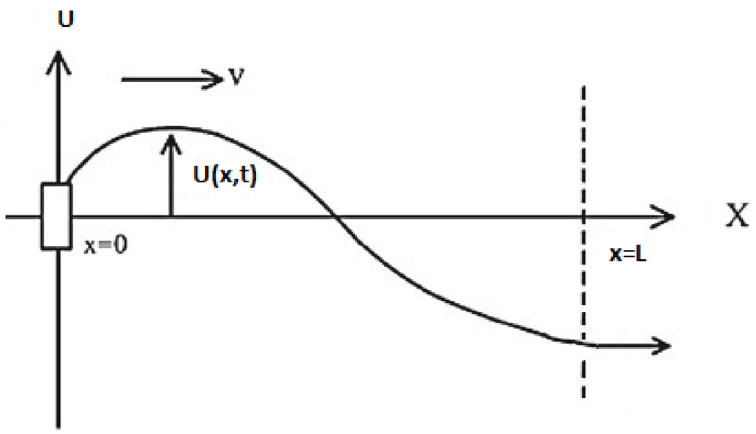

Figure 1: Schematic diagram of the axially moving Kirchhoff string.

where $U = U(x,t)$ is the lateral displacement at the space coordinate x and the time t. Additionally, E, ρ, h, ρ_0, a and f are the Young modulus of the latter, the mass density, the cross-section of the area, the length, the initial axial tension, the resistance modulus, respectively. Physically, Eq. (4) occurs in the study of vibrations of damped flexible space structures in a bounded domain in $\mathbb{R}^n$ .The term $a\,\Delta u_t$ is the internal material damping of Kelvin-Voigt type of the structure. The Kirchhoff-Carrier equations for a freely vibrating fixed-fixed string with two polarised displacements U_1 and U_2 described by [26, 27]

$$\begin{cases} \rho h \frac{\partial^2 U_1}{\partial t^2} - (\rho_0 + N)\,\frac{\partial^2 U_1}{\partial x^2} = 0, \quad x \in [0, L]\,, \ t \geq 0, \\ \rho h \frac{\partial^2 U_2}{\partial t^2} - (\rho_0 + N)\,\frac{\partial^2 U_2}{\partial x^2} = 0, \quad x \in [0, L]\,, \ t \geq 0, \end{cases} \tag{6}$$

where N is the axial tension created by the large amplitude motions and the coupling with the transverse motion and defined by

$$N = \frac{Eh}{2L} \int_0^L \left(\left(\frac{\partial U_1}{\partial x} \right)^2 + \left(\frac{\partial U_2}{\partial x} \right)^2 \right) dx. \tag{7}$$

In these equations ρ is density, E is Young's modulus, h is the crosse section, ρ_0 is the tension and L is the length.

In fact, Eq. (1) is a nonlinear PDE with a gradient term. In order to obtain the optimal convergence rate for the approximation technique, the efficient strategy is converting the equation into a system of two nonlinear first-order equations (using an auxiliary variable). Then, the Raviart-Thomas mixed finite element can be employed to solve the system.

The paper is organized as follows. In Section 2, the semi-discrete Raviart-Thomas mixed finite element method [28] for solving (1) is introduced in details. In Section 3, we prove the theorem about the convergence of semi-discretized Raviart-Thomas mixed finite element method for solving the equation and obtain the optimal degree of convergence. In Section 4, we demonstrate the accuracy of theoretical results using a numerical example. Finally, the conclusions are drawn in Section 5.

2. Mixed finite element method

In this section, we first mention the necessary definitions. Afterwards, we propose the Raviart-Thomas mixed finite element method for solving Eq. (1). It enables us to discuss convergence analysis of the method in the next sections.

Definition 2.1. Suppose $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$. For $1 \le p < \infty$, let

$$L^p(\Omega) := \left\{ u | \, u \text{ is measurable on } \Omega \text{ and } \|u\|_{L^p(\Omega)} < \infty \right\},$$

where $\|u\|_{L^p(\Omega)} := \left(\int\limits_{\Omega} |u(X)|^p dX \right)^{\frac{1}{p}}$. In particular, the space $L^2(\Omega)$ is a Hilbert space equipped with the inner product

$$(u, v)_\Omega = \int\limits_{\Omega} u(X)\, v(X)\, dX, \quad \forall u, v \in L^2(\Omega),$$

and norm

$$\|u\|_\Omega = \left(\int\limits_{\Omega} u^2(X) dX \right)^{\frac{1}{2}}, \quad \forall u \in L^2(\Omega), \quad X = (x_1, x_2, ..., x_d) \in \mathbb{R}^d.$$

Definition 2.2. [29, 30] Suppose $\Omega \subset \mathbb{R}^d$ and

$$L^1_{loc}(\Omega) := \left\{ f | \; f \in L^1(K) \quad \forall \, \text{compact } \; K \subset \text{ interior } \Omega \right\}. \tag{8}$$

Let k be a non-negative integer and $f \in L^1_{loc}(\Omega)$. Suppose that the weak derivatives $D^\alpha f$ [29], exist for all $|\alpha| \le k$. We define the Sobolev norm

$$\|f\|_{k,\Omega} := \left(\sum_{|\alpha| \le k} \|D^\alpha f\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}},$$

In this case, we define the Sobolev space

$$H^k(\Omega) := \left\{ f \in L^1_{loc}(\Omega) | \;\; \|f\|_{k,\Omega} < \infty \right\}.$$

$H^k(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v)_{k,\Omega} = \sum_{0 \le |m| \le k} \left(\int\limits_{\Omega} D^m u(X)\, D^m v(X) dX \right),$$

which induces the mentioned norm. Also $H_0^k(\Omega)$, $L^\infty\left([0,T]\,,\,H_0^k(\Omega)\right)$, $L^\infty\left([0,T]\,,\,L^2(\Omega)\right)$ and $L^\infty\left([0,T]\,,\,H^k(\Omega)\right)$ are define by

$$H_0^k(\Omega) := \left\{ f \in H^k(\Omega) \middle| \; f|_{\Gamma_1} = 0 \right\},$$

$$L^\infty\left([0,T]\,,\,H_0^k(\Omega)\right) := \left\{ f |\; f(.,t) \in H_0^k(\Omega) \;\; \& \; f(x,.) \in L^\infty([0,T]) \right\},$$

$$L^\infty\left([0,T]\,,\,L^2(\Omega)\right) := \left\{ f |\; f(.,t) \in L^2(\Omega) \;\; \& \;\; f(x,.) \in L^\infty([0,T]) \right\},$$

$$L^\infty\left([0,T]\,,\,H^k(\Omega)\right) := \left\{ f |\; f(.,t) \in H^k(\Omega) \;\; \& \;\; f(x,.) \in L^\infty([0,T]) \right\},$$

$$H(\mathrm{div},\Omega) := \left\{ q \in \left(L^2(\Omega)\right)^2 \mid \nabla \cdot q \in L^2(\Omega) \right\}$$

and we equip these spaces with the norms $\|\,.\,\|_{k,\Omega,0}$, $\|\,.\,\|_{\infty,k,\Omega,0}$, $\|\,.\,\|_{\infty,k,\Omega}$ and $\|\,.\,\|_{\infty,0,\Omega}$, respectively. Furthermore for $H(\mathrm{div},\Omega)$ we have

$$\|q\|_{H(\mathrm{div},\Omega)} = \left(\|q\|^2 + \|\nabla \cdot q\|^2\right)^{1/2} \tag{9}$$

Lemma 1. *Under the mentioned assumptions for Eq.* (1), *this equation has unique solution* $U,\; V : \Omega \to \mathbb{R}$ *such that* $U,\; V \in L^\infty\left([0,T]\,,\,H_0^1(\Omega)\right)$, $U_t,\; V_t \in L^\infty\left([0,T]\,,\,H_0^1(\Omega)\right)$ *and* $U_{tt},\; V_{tt} \in L^\infty\left([0,T]\,,\,L^2(\Omega)\right)$.
Proof. *See [20].*

Now, we present the Raiviart-Thomas mixed finite element method for solving Eq. (1) by introducing a variational formulation for this equation. For this purpose, suppose $X := L^2(\Omega)$, $\boldsymbol{M} := H(\mathrm{div},\Omega)$, $\boldsymbol{W} := \nabla U$ and $\boldsymbol{Z} := \nabla V$; therefore, we can write (1) in the following form:

$$U_{tt}(x,t) - \left(1 + \|\boldsymbol{W}\|_\Omega^2 + \|\boldsymbol{Z}\|_\Omega^2\right)\nabla \cdot \boldsymbol{W}(x,t) - \Delta U_t(x,t) = f(x,t) \qquad x \in \Omega,\;\; t \in [0,T], \tag{10a}$$

$$\boldsymbol{W} = \nabla U \qquad x \in \Omega,\;\; t \in [0,T], \tag{10b}$$

$$V_{tt}(x,t) - \left(1 + \|W\|_\Omega^2 + \|\boldsymbol{Z}\|_\Omega^2\right)\nabla \cdot \boldsymbol{Z}(x,t) - \Delta V_t(x,t) = f(x,t) \qquad x \in \Omega \;\; t \in [0,T], \tag{10c}$$

$$\boldsymbol{Z} = \nabla V \qquad x \in \Omega \;\; t \in [0,T], \tag{10d}$$

$$U = V = \boldsymbol{W} \cdot \nu = \boldsymbol{Z} \cdot \nu = 0 \qquad \text{on } \Sigma_1 = \Gamma_1 \times [0,T], \tag{10e}$$

$$\left(1 + \|\boldsymbol{W}\|_\Omega^2 + \|\boldsymbol{Z}\|_\Omega^2\right)\boldsymbol{W} \cdot \nu + \frac{\partial U_t}{\partial \nu} + U + U_t + g(t)|U_t|^p U_t = g * |U|^\gamma U \quad \text{on } \Sigma_0 = \Gamma_0 \times [0,T], \tag{10f}$$

$$\left(1 + \|\boldsymbol{W}\|_\Omega^2 + \|\boldsymbol{Z}\|_\Omega^2\right)\boldsymbol{Z} \cdot \nu + \frac{\partial V_t}{\partial \nu} + V + V_t + g(t)|V_t|^p V_t = g * |V|^\gamma V \quad \text{on } \Sigma_0 = \Gamma_0 \times [0,T], \tag{10g}$$

$$U(x,0) = U_0(x), \qquad U_t(x,0) = U_1(x), \qquad x \in \Omega, \tag{10h}$$

$$V(x,0) = V_0(x) \qquad V_t(x,0) = V_1(x) \qquad x \in \Omega, \tag{10i}$$

The mixed variational formulation of (1) based on (10) is given by:

$$\left(\frac{\partial^2}{\partial t^2}U,\varphi\right)_\Omega + \left(1+\|\boldsymbol{W}\|_\Omega^2+\|\boldsymbol{Z}\|_\Omega^2\right)(\boldsymbol{W},\nabla\varphi)_\Omega + \left(\nabla\frac{\partial}{\partial t}U,\nabla\varphi\right)_\Omega + (U,\varphi)_{\Gamma_0}$$
$$+\left(\frac{\partial}{\partial t}U,\varphi\right)_{\Gamma_0} + g(t)\left(\left|\frac{\partial}{\partial t}U\right|^p\frac{\partial}{\partial t}U,\varphi\right)_{\Gamma_0} = (g*|U|^\gamma U,\varphi)_{\Gamma_0} + (f,\varphi)_\Omega, \qquad \varphi\in X, \tag{11a}$$

$$(\boldsymbol{W},\boldsymbol{\psi})_\Omega = (\nabla U,\boldsymbol{\psi})_\Omega \qquad \boldsymbol{\psi}\in\boldsymbol{M}, \tag{11b}$$

$$\left(\frac{d^2}{dt^2}V,\varphi\right)_\Omega + \left(1+\|\boldsymbol{W}\|_\Omega^2+\|\boldsymbol{Z}\|_\Omega^2\right)(\boldsymbol{Z},\nabla\varphi)_\Omega + \left(\nabla\frac{\partial}{\partial t}V,\nabla\varphi\right)_\Omega + (V,\varphi)_{\Gamma_0}$$
$$+\left(\frac{\partial}{\partial t}V,\varphi\right)_{\Gamma_0} + g(t)\left(\left|\frac{\partial}{\partial t}V\right|^p\frac{d}{dt}V,\varphi\right)_{\Gamma_0} = (g*|V|^\gamma V,\varphi)_{\Gamma_0} + (f,\varphi)_\Omega \qquad \varphi\in X, \tag{11c}$$

$$(\boldsymbol{Z},\boldsymbol{\psi})_\Omega = (\nabla\boldsymbol{W},\boldsymbol{\psi})_\Omega \qquad \boldsymbol{\psi}\in\boldsymbol{M}, \tag{11d}$$

$$(U(0),\varphi)_\Omega = (U_0,\varphi)_\Omega \qquad \left(\left.\frac{\partial U}{\partial t}\right|_{t=0},\varphi\right)_\Omega = (U_1,\varphi)_\Omega \qquad \varphi\in X, \tag{11e}$$

$$(V(0),\varphi)_\Omega = (V_0,\varphi)_\Omega \qquad \left(\left.\frac{\partial V}{\partial t}\right|_{t=0},\varphi\right)_\Omega = (V_1,\varphi)_\Omega \qquad \varphi\in X. \tag{11f}$$

The mixed finite element method for (1) is based on the weak formulation (11) and two finite element subspaces, i.e., $X_h \subset X$ and $\boldsymbol{M_h} \subset \boldsymbol{M}$ associated with a prescribed finite element partition $\mathcal{T}_h$ for the domain Ω. Let $\mathcal{T}_h$ be a partition of Ω into non-overlapping triangles such that no vertex of one triangle lies in the interior of an edge of another triangle. (i.e., $\Omega = \bigcup_{K\in\mathcal{T}_h} K$) and h denotes the maximum diameter of the partition. Let the finite dimensional subspace $X_h \in X$ and $\boldsymbol{M_h} \in \boldsymbol{M}$ be the Raviart-Thomas-Nedlec spaces [31, 32] of order $k+1$ where $k \geq 0$.

Finally, based on variational formulation (11) and the mentioned finite element spaces we define the Raviart-Thomas mixed finite element method for (1):

find $(u_h, v_h)\times(\boldsymbol{w_h},\boldsymbol{z_h}) \in (X_h, X_h)\times(\boldsymbol{M_h},\boldsymbol{M_h})$ such that

$$\left(\frac{\partial^2}{\partial t^2}u_h,\varphi\right)_\Omega+\left(1+\|\boldsymbol{w_h}\|_\Omega^2+\|\boldsymbol{z_h}\|_\Omega^2\right)(\boldsymbol{w_h},\nabla\varphi)_\Omega+\left(\nabla\frac{\partial}{\partial t}u_h,\nabla\varphi\right)_\Omega+(u_h,\varphi)_{\Gamma_0}$$
$$+\left(\frac{\partial}{\partial t}u_h,\varphi\right)_{\Gamma_0}+g(t)\left(\left|\frac{\partial}{\partial t}u_h\right|^p\frac{\partial}{\partial t}u_h,\varphi\right)_{\Gamma_0}=(g*|u_h|^\gamma u_h,\varphi)_{\Gamma_0}+(f,\varphi)_\Omega \qquad \varphi\in X_h, \tag{12a}$$

$$(\boldsymbol{w_h},\boldsymbol{\psi})_\Omega=(\nabla u_h,\boldsymbol{\psi})_\Omega \qquad \boldsymbol{\psi}\in\boldsymbol{M_h}, \tag{12b}$$

$$\left(\frac{d^2}{dt^2}v_h,\varphi\right)_\Omega+\left(1+\|\boldsymbol{w_h}\|_\Omega^2+\|\boldsymbol{z_h}\|_\Omega^2\right)(\boldsymbol{z_h},\nabla\varphi)_\Omega+\left(\nabla\frac{\partial}{\partial t}v_h,\nabla\varphi\right)_\Omega+(v_h,\varphi)_{\Gamma_0}$$
$$+\left(\frac{\partial}{\partial t}v_h,\varphi\right)_{\Gamma_0}+g(t)\left(\left|\frac{\partial}{\partial t}v_h\right|^p\frac{d}{dt}v_h,\varphi\right)_{\Gamma_0}=(g*|v_h|^\gamma v_h,\varphi)_{\Gamma_0}+(f,\varphi)_\Omega \qquad \varphi\in X_h, \tag{12c}$$

$$(\boldsymbol{z_h},\boldsymbol{\psi})_\Omega=(\nabla\boldsymbol{w_h},\boldsymbol{\psi})_\Omega \qquad \boldsymbol{\psi}\in\boldsymbol{M_h}, \tag{12d}$$

$$(u_h(0),\varphi)_\Omega=(U_0,\varphi)_\Omega \quad \left(\left.\frac{\partial u_h}{\partial t}\right|_{t=0},\varphi\right)_\Omega=(U_1,\varphi)_\Omega \qquad \varphi\in X_h, \tag{12e}$$

$$(v_h(0),\varphi)_\Omega=(V_0,\varphi)_\Omega \quad \left(\left.\frac{\partial v_h}{\partial t}\right|_{t=0},\varphi\right)_\Omega=(V_1,\varphi)_\Omega \qquad \varphi\in X_h. \tag{12f}$$

The main theorem of the paper is given here. It derives the error estimation of the semi-discretized Raviart-Thomas method for solving (1), i.e., Eq. (11).

Theorem 2. ***Convergence.*** *Under Assumption 1, the following error estimation holds*

$$\|U(t)-u_h(t)\|_\Omega^2+\|V(t)-v_h(t)\|_\Omega^2+\|\boldsymbol{W}(t)-\boldsymbol{w_h}(t)\|_\Omega^2+\|\boldsymbol{Z}(t)-\boldsymbol{z_h}(t)\|_\Omega^2\le Ch^{2l} \qquad 1\le l\le k.$$

The mentioned theorem will be proved in Section 3. Before we start to prove the theorem, some necessary lemmas should be presented and proved. Before that, in this section we review some necessary lemmas which will be used in the next section.

Lemma 3. *(The Gronwall Lemma) [30] Let $\varphi\in C([0,t_1])$ and $\varphi\in C^1((0,t_1))$, if there exists a constant $\alpha\in\mathbb{R}$ and a continuous function g_1 such that for $t\in(0,t_1)$, φ satisfies the inequality*

$$\frac{d}{dt}\varphi(t)\le\alpha\varphi(t)+g_1(t), \tag{13}$$

or

$$\varphi(t)\le\varphi(0)+\int_0^t[\alpha\varphi(s)+g_1(s)]ds, \tag{14}$$

then

$$\varphi(t)\le e^{\alpha t}\varphi(0)+\int_0^t g_1(s)e^{\alpha(t-s)}ds.$$

Lemma 4. *Suppose for the function $f \in C([0,T] \times \overline{\Omega})$ exists positive constants m_f and M_f such that [33, 34]:*

$$m_f(x-s) \leq f(t,x)x - f(t,s)s \leq M_f(x-s), \quad x \geq 0,\ s \geq 0,\ t \geq 0,$$

then there exist $C_1 \geq C_2 > 0$, such that for vectors $u,\ v \in \mathbb{R}$ and $t \in [0,T]$ we have

$$|f(t,|v|)\,v - f(t,|u|)\,u| \leq C_1\,|v-u|\,,$$

$$C_2|v-u|^2 \leq (f(t,|v|)\,v - f(t,|u|)\,u)\,.\,(u-v)\,.$$

Lemma 5. *There exist projection operators $\Pi_h \times R_h : X \times \boldsymbol{M} \to X_h \times \boldsymbol{M_h}$ such that [35, 36]*

1. *The operator $\Pi_h : X \to X_h$ is a L^2-projection, i.e.,*

$$(\nabla \cdot \boldsymbol{w_h}, u - \Pi_h u)_\Omega = 0 \qquad \forall \boldsymbol{w_h} \in \boldsymbol{M_h}, \quad u \in L^2(\Omega), \tag{15}$$

$$\|u - \Pi_h u\|_{L^2(\Omega)} \leq ch^l \|u\|_{H^l(\Omega)} \qquad 0 \leq l \leq k+1. \tag{16}$$

2. *The operator $R_h : \boldsymbol{M} \to \boldsymbol{M_h}$ with $divR_h = \Pi_h div$ satisfies*

$$(\nabla \cdot (\boldsymbol{w} - \mathcal{R}_h \boldsymbol{w})\,, \varphi)_\Omega = 0 \qquad \forall \varphi \in X_h \qquad \boldsymbol{w} \in \boldsymbol{M}, \tag{17}$$

$$\|\boldsymbol{w} - R_h \boldsymbol{w}\|_\Omega \leq ch^l \|\boldsymbol{w}\|_{l,\Omega} \qquad 0 \leq l \leq k+1, \tag{18}$$

$$\|\nabla \cdot (\boldsymbol{w} - \Pi_h \boldsymbol{w})\|_\Omega \leq ch^l \|\nabla \cdot \boldsymbol{w}\|_{l,\Omega} \qquad 0 \leq l \leq k+1. \tag{19}$$

3. Proof of the convergence theorem

The main purpose of this section is to prove a theorem about the convergence order of the scheme which mentioned in Section 2. Hence, for the sake of simplicity and preventing the complexity of the proof, we separate it into some lemmas. Afterwards, we employ the lemmas to prove the main theorem. From now on, we make the following regularity assumption:

Assumptions 1. *We suppose that $U(t)$, $V(t)$, $\boldsymbol{W}(t)$, $\boldsymbol{Z}(t) \in H^l(\Omega)$, $1 \leq l \leq k+1$, are the exact solutions of* (10) *and $u_h(t)$, $v_h(t) \in X_h$ and $\boldsymbol{w_h}$, $\boldsymbol{z_h} \in \boldsymbol{M_h}$ are the approximate solutions of* (10) *that are obtained by solving* (12)*. Additionally, we assume that $U_0, U_1, V_0,\ V_1 \in H^l(\Omega)$, $1 \leq l \leq k+1$.*

Let us define the error terms of U, V, $\boldsymbol{W}$ and $\boldsymbol{Z}$ as

$$e_{1,h} := u_h - \Pi_h U \qquad e_{2,h} := v_h - \Pi_h V,$$

$$e_{3,h} := \boldsymbol{w_h} - R_h \boldsymbol{W}, \qquad e_{4,h} := \boldsymbol{z_h} - R_h \boldsymbol{Z}.$$

By subtracting (11a–11d) from (12a–12d), the following results are obtained

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial t^2}e_{1,h},\varphi\right)_\Omega+\left(1+\|\boldsymbol{w_h}\|_\Omega^2+\|\boldsymbol{z_h}\|_\Omega^2\right)(\boldsymbol{w_h},\nabla\varphi)_\Omega-\left(1+\|\boldsymbol{W}\|_\Omega^2+\|\boldsymbol{Z}\|_\Omega^2\right)(\boldsymbol{W},\nabla\varphi)_\Omega\\
&\quad+\left(\nabla\frac{\partial}{\partial t}e_{1,h},\nabla\varphi\right)_\Omega+(e_{1,h},\varphi)_{\Gamma_0}+\left(\frac{\partial}{\partial t}e_{1,h},\varphi\right)_{\Gamma_0}+g(t)\left(\left|\frac{\partial}{\partial t}u_h\right|^p\frac{\partial}{\partial t}u_h,\varphi\right)_{\Gamma_0}-g(t)\left(\left|\frac{\partial}{\partial t}U\right|^p\frac{\partial}{\partial t}U,\varphi\right)_{\Gamma_0}\\
&\quad=(g*|u_h|^\gamma u_h,\varphi)_{\Gamma_0}-(g*|U|^\gamma U,\varphi)_{\Gamma_0}+\left(\frac{\partial^2}{\partial t^2}\left(U-\Pi_hU\right),\varphi\right)_\Omega+\left(\nabla\frac{\partial}{\partial t}\left(U-\Pi_hU\right),\nabla\varphi\right)_\Omega\\
&\quad+(U-\Pi_hU,\varphi)_{\Gamma_0}+\left(\frac{\partial}{\partial t}\left(U-\Pi_hU\right),\varphi\right)_{\Gamma_0},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial t^2}e_{2,h},\varphi\right)_\Omega+\left(1+\|\boldsymbol{w_h}\|_\Omega^2+\|\boldsymbol{z_h}\|_\Omega^2\right)(\boldsymbol{z_h},\nabla\varphi)_\Omega-\left(1+\|\boldsymbol{W}\|_\Omega^2+\|\boldsymbol{Z}\|_\Omega^2\right)(\boldsymbol{Z},\nabla\varphi)_\Omega\\
&\quad+\left(\nabla\frac{\partial}{\partial t}e_{2,h},\nabla\varphi\right)_\Omega+(e_{2,h},\varphi)_{\Gamma_0}+\left(\frac{\partial}{\partial t}e_{2,h},\varphi\right)_{\Gamma_0}+g(t)\left(\left|\frac{\partial}{\partial t}v_h\right|^p\frac{\partial}{\partial t}v_h,\varphi\right)_{\Gamma_0}-g(t)\left(\left|\frac{\partial}{\partial t}V\right|^p\frac{\partial}{\partial t}V,\varphi\right)_{\Gamma_0}\\
&\quad=(g*|v_h|^\gamma v_h,\varphi)_{\Gamma_0}-(g*|V|^\gamma V,\varphi)_{\Gamma_0}+\left(\frac{\partial^2}{\partial t^2}\left(V-\Pi_hV\right),\varphi\right)_\Omega+\left(\nabla\frac{\partial}{\partial t}\left(V-\Pi_hV\right),\nabla\varphi\right)_\Omega\\
&\quad+(V-\Pi_hV,\varphi)_{\Gamma_0}+\left(\frac{\partial}{\partial t}\left(V-\Pi_hV\right),\varphi\right)_{\Gamma_0},
\end{aligned}
\tag{21}
$$

$$
(e_{3,h},\boldsymbol{\psi})_\Omega=(\nabla e_{1,h},\boldsymbol{\psi})_\Omega+\left(\left(\Pi_hU-U\right),\nabla\cdot\boldsymbol{\psi}\right)_\Omega+(\boldsymbol{W}-R_h\boldsymbol{W},\boldsymbol{\psi})_\Omega,
\tag{22}
$$

$$
(e_{4,h},\boldsymbol{\psi})_\Omega=(\nabla e_{2,h},\boldsymbol{\psi})_\Omega+\left(\left(\Pi_hV-V\right),\nabla\cdot\boldsymbol{\psi}\right)_\Omega+(\boldsymbol{Z}-R_h\boldsymbol{Z},\boldsymbol{\psi})_\Omega.
\tag{23}
$$

Plugging $\varphi=\frac{\partial}{\partial t}e_{1,h}$ into (20) and $\varphi=\frac{\partial}{\partial t}e_{2,h}$ into (21) as well as adding these equations together yield

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial t^2}e_{1,h},\frac{\partial}{\partial t}e_{1,h}\right)_\Omega+\left(\frac{\partial^2}{\partial t^2}e_{2,h},\frac{\partial}{\partial t}e_{2,h}\right)_\Omega+\left(\nabla\frac{\partial}{\partial t}e_{1,h},\nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega+\left(\nabla\frac{\partial}{\partial t}e_{2,h},\nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega+\left(e_{1,h},\frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}\\
&\quad+\left(e_{2,h},\frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}+\left(\frac{\partial}{\partial t}e_{1,h},\frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}+\left(\frac{\partial}{\partial t}e_{2,h},\frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}+L_1=L_2+L_3+L_4+L_5+L_6,
\end{aligned}
\tag{24}
$$

where

$$L_1 := \left(1 + \|\boldsymbol{w_h}\|_\Omega^2 + \|\boldsymbol{z_h}\|_\Omega^2\right)\left(\boldsymbol{w_h} + \boldsymbol{z_h}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega - \left(1 + \|\boldsymbol{W}\|_\Omega^2 + \|\boldsymbol{Z}\|_\Omega^2\right)\left(\boldsymbol{W} + \boldsymbol{Z}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega.$$

$$L_2 := g(t)\left(\left(\left|\frac{\partial}{\partial t}u_h\right|^p\frac{\partial}{\partial t}u_h, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}U\right|^p\frac{\partial}{\partial t}U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}\right) + g(t)\left(\left(\left|\frac{\partial}{\partial t}v_h\right|^p\frac{\partial}{\partial t}v_h, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}V\right|^p\frac{\partial}{\partial t}V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}\right),$$

$$L_3 := \left(g * |u_h|^\gamma u_h, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} - \left(g * |\Pi_h U|^\gamma \Pi_h U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} + \left(g * |\Pi_h U|^\gamma \Pi_h U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}\left(g * |U|^\gamma U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}$$

$$- \left(g * |v_h|^\gamma v_h, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} - \left(g * |\Pi_h V|^\gamma \Pi_h V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}\left(g * |\Pi_h V|^\gamma \Pi_h V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} - \left(g * |V|^\gamma V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0},$$

$$L_4 := \left(\frac{\partial^2}{\partial t^2}\left(U - \Pi_h U\right), \frac{\partial}{\partial t}e_{1,h}\right)_\Omega + \left(\frac{\partial^2}{\partial t^2}\left(V - \Pi_h V\right), \frac{\partial}{\partial t}e_{2,h}\right)_\Omega,$$

$$L_5 := \left(\nabla\frac{\partial}{\partial t}\left(U - \Pi_h U\right), \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega + \left(\nabla\frac{\partial}{\partial t}\left(V - \Pi_h V\right), \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega,$$

$$L_6 := \left(U - \Pi_h U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} + \left(V - \Pi_h V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} + \left(\frac{\partial}{\partial t}\left(U - \Pi_h U\right), \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} + \left(\frac{\partial}{\partial t}\left(V - \Pi_h V\right), \frac{\partial}{\partial t}e_{2,h}\right)$$

To obtain the mentioned error bound of Theorem 1 the challenging part is dealing with L_1 and L_2. The bound of L_1 will be proved in Lemma 9 where Lemma 7 and Lemma 8 are used employed for it. Regarding L_2, the main proof will be given in Lemma 10. Furthermore, the following lemma will be used in the proof process of Lemma 7.

Lemma 6. *Using the above notations we have*

$$\frac{d}{dt}\left\|e_{3,h}\right\|_\Omega^2 = \left(e_{3,h}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega + \left(\frac{\partial}{\partial t}\left(\boldsymbol{W} - R_h\boldsymbol{W}\right), e_{3,h}\right)_\Omega, \tag{25}$$

and

$$\frac{d}{dt}\left\|e_{4,h}\right\|_\Omega^2 = \left(e_{4,h}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega + \left(\frac{\partial}{\partial t}\left(\boldsymbol{Z} - R_h\boldsymbol{Z}\right), e_{4,h}\right)_\Omega. \tag{26}$$

Proof. *By taking derivative with respect to t from* (22) *and* (23)*, and putting* $\boldsymbol{\psi} = e_{3,h}$ *and* $\boldsymbol{\psi} = e_{4,h}$ *into these equations, respectively yields*

$$\frac{d}{dt}\left\|e_{3,h}\right\|_\Omega^2 = \left(e_{3,h}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega + \left(\frac{\partial}{\partial t}\left(\boldsymbol{W} - R_h\boldsymbol{W}\right), e_{3,h}\right)_\Omega + \left(\frac{\partial}{\partial t}\left(\Pi_h U - U\right), \nabla\cdot e_{3,h}\right)_\Omega, \tag{27}$$

and

$$\frac{d}{dt}\left\|e_{4,h}\right\|_\Omega^2 = \left(e_{4,h}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega + \left(\frac{\partial}{\partial t}\left(\boldsymbol{Z} - R_h\boldsymbol{Z}\right), e_{4,h}\right)_\Omega + \left(\frac{\partial}{\partial t}\left(\Pi_h V - V\right), \nabla\cdot e_{4,h}\right)_\Omega, \tag{28}$$

as well as using (17) *we conclude*

$$\left(\frac{\partial}{\partial t}\left(\Pi_h U - U\right), \nabla\cdot e_{3,h}\right)_\Omega = 0, \tag{29}$$

$$\left(\frac{\partial}{\partial t}\left(\Pi_h V - V\right), \nabla\cdot e_{4,h}\right)_\Omega = 0. \tag{30}$$

Now combining (27)-(30)*, the results can be easily obtained.*

Lemma 7. *Suppose that Assumption 1 holds and let*

$$L_{1,0} := \|\boldsymbol{w_h}\|_\Omega^2 \left(\boldsymbol{w_h}, \nabla \tfrac{\partial}{\partial t} e_{1,h}\right)_\Omega - \|R_h \boldsymbol{W}\|_\Omega^2 \left(R_h \boldsymbol{W}, \nabla \tfrac{\partial}{\partial t} e_{1,h}\right)_\Omega + \|\boldsymbol{z_h}\|_\Omega^2 \left(\boldsymbol{z_h}, \nabla \tfrac{\partial}{\partial t} e_{2,h}\right)_\Omega - \|R_h \boldsymbol{Z}\|_\Omega^2 \left(R_h \boldsymbol{Z}, \nabla \tfrac{\partial}{\partial t} e_{2,h}\right)_\Omega,$$

therefore, we can write

$$L_{1,0} \geq \frac{1}{2}\frac{d}{dt}\left\|e_{3,h}\right\|_\Omega^4 + \frac{1}{2}\frac{d}{dt}\left\|e_{4,h}\right\|_\Omega^4 + \boldsymbol{M}_0, \tag{31}$$

where

$$\begin{aligned} |M_0| \leq & \left(c_1 h^{2l} + c_2\right)\left(\left\|e_{3,h}\right\|_\Omega^2 + \left\|e_{3,h}\right\|_\Omega^4\right) + \left(c_3 h^{2l} + c_4\right)\left(\left\|e_{4,h}\right\|_\Omega^2 + \left\|e_{4,h}\right\|_\Omega^4\right) \\ & + \frac{1}{16}\left\|\nabla \frac{\partial}{\partial t} e_{1,h}\right\|_\Omega^2 + \frac{1}{16}\left\|\nabla \frac{\partial}{\partial t} e_{2,h}\right\|_\Omega^2 + c_5\left(h^{2l} + h^{6l}\right). \end{aligned} \tag{32}$$

Proof: *First of all, by adding and subtracting some terms to $L_{1,0}$, we have*

$$\begin{aligned} L_{1,0} = & \|\boldsymbol{w_h}\|_\Omega^2 \left(\boldsymbol{w_h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega - \|R_h \boldsymbol{W}\|_\Omega^2 \left(R_h \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \|\boldsymbol{w_h}\|_\Omega^2 \left(R_h \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega \\ & - \|\boldsymbol{w_h}\|_\Omega^2 \left(R_h \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \|R_h \boldsymbol{W}\|_\Omega^2 \left(\boldsymbol{w_h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right) - \|R_h \boldsymbol{W}\|_\Omega^2 \left(\boldsymbol{w_h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right) \\ & + \|\boldsymbol{z_h}\|_\Omega^2 \left(\boldsymbol{z_h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega - \|R_h \boldsymbol{Z}\|_\Omega^2 \left(R_h \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega + \|\boldsymbol{z_h}\|_\Omega^2 \left(R_h \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega \end{aligned} \tag{33}$$

$$- \|\boldsymbol{z_h}\|_\Omega^2 \left(R_h \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega + \|R_h \boldsymbol{Z}\|_\Omega^2 \left(\boldsymbol{z_h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right) - \|R_h \boldsymbol{Z}\|_\Omega^2 \left(\boldsymbol{z_h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right). \tag{34}$$

Rewording the above equation gives rise to

$$\begin{aligned} L_{1,0} = & \left(\|\boldsymbol{w_h}\|_\Omega^2 + \|R_h \boldsymbol{W}\|_\Omega^2\right)\left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega - \|R_h \boldsymbol{W}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega \\ & + \left(\|\boldsymbol{w_h}\|_\Omega^2 - \|R_h \boldsymbol{W}\|_\Omega^2\right)\left(R_h \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \left(\|\boldsymbol{z_h}\|_\Omega^2 + \|R_h \boldsymbol{Z}\|_\Omega^2\right)\left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega \\ & - \|R_h \boldsymbol{Z}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega + \left(\|\boldsymbol{z_h}\|_\Omega^2 - \|R_h \boldsymbol{Z}\|_\Omega^2\right)\left(R_h \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega, \end{aligned} \tag{35}$$

employing Lemma 6 and using the fact that

$$\left\|e_{3,h}\right\|_\Omega^2 \tfrac{d}{dt}\left\|e_{3,h}\right\|_\Omega^2 = \tfrac{1}{2}\tfrac{d}{dt}\left\|e_{3,h}\right\|_\Omega^4, \qquad \left\|e_{4,h}\right\|_\Omega^2 \tfrac{d}{dt}\left\|e_{4,h}\right\|_\Omega^2 = \tfrac{1}{2}\tfrac{d}{dt}\left\|e_{4,h}\right\|_\Omega^4,$$

we can obtain

$$\begin{aligned} L_{1,0} \geq & \frac{1}{2}\frac{d}{dt}\left\|e_{3,h}\right\|_\Omega^4 - \|R_h \boldsymbol{W}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \left(\|\boldsymbol{w_h}\|_\Omega^2 - \|R_h \boldsymbol{W}\|_\Omega^2\right)\left(R_h \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega \\ & + \frac{1}{2}\frac{d}{dt}\left\|e_{4,h}\right\|_\Omega^4 - \|R_h \boldsymbol{Z}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega + \left(\|\boldsymbol{z_h}\|_\Omega^2 - \|R_h \boldsymbol{Z}\|_\Omega^2\right)\left(R_h \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega \\ & + \left(\|R_h \boldsymbol{W}\|_\Omega^2 + \|\boldsymbol{w_h}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{W} - R_h \boldsymbol{W}\right), e_{3,h}\right)_\Omega + \left(\|R_h \boldsymbol{Z}\|_\Omega^2 + \|\boldsymbol{z_h}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{Z} - R_h \boldsymbol{Z}\right), e_{4,h}\right)_\Omega. \end{aligned} \tag{36}$$

Considering the above formula, we conclude

$$L_{1,0} \geq \frac{1}{2}\frac{d}{dt}\left\|e_{3,h}\right\|_\Omega^4 + \frac{1}{2}\frac{d}{dt}\left\|e_{4,h}\right\|_\Omega^4 + \boldsymbol{M}_0, \tag{37}$$

where

$$\begin{aligned}
\boldsymbol{M}_0 :=& \left(\|\boldsymbol{W}\|_\Omega^2 - \|R_h\boldsymbol{W}\|_\Omega^2\right)\left(e_{3,h}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega + \left(\|\boldsymbol{w_h}\|_\Omega^2 - \|R_h\boldsymbol{W}\|_\Omega^2\right)\left(R_h\boldsymbol{W} - \boldsymbol{W}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega \\
&+ \left(\|\boldsymbol{w_h}\|_\Omega^2 - \|R_h\boldsymbol{W}\|_\Omega^2\right)\left(\boldsymbol{W}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega - \|\boldsymbol{W}\|_\Omega^2\left(e_{3,h}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega \\
&+ \left(\|\boldsymbol{Z}\|_\Omega^2 - \|R_h\boldsymbol{Z}\|_\Omega^2\right)\left(e_{4,h}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega + \left(\|\boldsymbol{z_h}\|_\Omega^2 - \|R_h\boldsymbol{Z}\|_\Omega^2\right)\left(R_h\boldsymbol{Z} - \boldsymbol{Z}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega \\
&+ \left(\|\boldsymbol{z_h}\|_\Omega^2 - \|R_h\boldsymbol{Z}\|_\Omega^2\right)\left(\boldsymbol{Z}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega - \|\boldsymbol{Z}\|_\Omega^2\left(e_{4,h}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega \\
&+ \left(\|R_h\boldsymbol{W}\|_\Omega^2 + \|\boldsymbol{w_h}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{W} - R_h\boldsymbol{W}\right), e_{3,h}\right)_\Omega + \left(\|R_h\boldsymbol{Z}\|_\Omega^2 + \|\boldsymbol{z_h}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{Z} - R_h\boldsymbol{Z}\right), e_{4,h}\right)_\Omega \\
&+ \left(\|R_h\boldsymbol{W}\|_\Omega^2 + 2\,\|\boldsymbol{W}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{W} - R_h\boldsymbol{W}\right), e_{3,h}\right)_\Omega - \left(\|R_h\boldsymbol{W}\|_\Omega^2 + 2\,\|\boldsymbol{W}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{W} - R_h\boldsymbol{W}\right), e_{3,h}\right)_\Omega \\
&+ \left(\|R_h\boldsymbol{Z}\|_\Omega^2 + 2\,\|\boldsymbol{Z}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{Z} - R_h\boldsymbol{Z}\right), e_{4,h}\right)_\Omega - \left(\|R_h\boldsymbol{Z}\|_\Omega^2 + 2\,\|\boldsymbol{Z}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{Z} - R_h\boldsymbol{Z}\right), e_{4,h}\right)_\Omega.
\end{aligned} \tag{38}$$

Now applying the Cauchy-Schwarz and Young inequalities and Eq. (19) *yields*

$$\begin{aligned}
|\boldsymbol{M}_0| \leq& c_1h^{2l}\,\|\boldsymbol{W}\|_{l,\Omega}^2\left(\left\|e_{3,h}\right\|_\Omega^2 + \frac{1}{4}\left\|\nabla\frac{\partial}{\partial t}e_{1,h}\right\|_\Omega^2\right) + c_2h^{2l}\,\|\boldsymbol{W}\|_{l,\Omega}^2\left\|e_{3,h}\right\|_\Omega^4 + 8\,\|\boldsymbol{W}\|_\Omega^2\left\|e_{3,h}\right\|_\Omega^4 + 8\,\|\boldsymbol{W}\|_\Omega^4\left\|e_{3,h}\right\|_\Omega^2 \\
&+ \frac{1}{16}\left\|\nabla\frac{\partial}{\partial t}e_{1,h}\right\|_\Omega^2 + c_3h^{2l}\,\|\boldsymbol{Z}\|_{l,\Omega}^2\left(\left\|e_{4,h}\right\|_\Omega^2 + \frac{1}{4}\left\|\nabla\frac{\partial}{\partial t}e_{2,h}\right\|_\Omega^2\right) + c_4h^{2l}\,\|\boldsymbol{Z}\|_{l,\Omega}^2\left\|e_{4,h}\right\|_\Omega^4 + 8\,\|\boldsymbol{Z}\|_\Omega^4\left\|e_{4,h}\right\|_\Omega^2 \\
&+ 8\,\|\boldsymbol{Z}\|_\Omega^2\left\|e_{4,h}\right\|_\Omega^4 + \frac{1}{16}\left\|\nabla\frac{\partial}{\partial t}e_{2,h}\right\|_\Omega^2 + c_5h^{2l}\left\|e_{3,h}\right\|_\Omega^4\left\|\frac{\partial}{\partial t}\boldsymbol{W}\right\|_{l,\Omega}^2 + \frac{1}{4}\left\|e_{3,h}\right\|_\Omega^4 + 2c_6h^{6l}\,\|\boldsymbol{W}\|_{l,\Omega}^2\left\|\frac{\partial}{\partial t}\boldsymbol{W}\right\|_{l,\Omega}^2 \\
&+ c_7h^{2l}\left\|e_{3,h}\right\|_\Omega^2\|\boldsymbol{W}\|_{l,\Omega}^2 + c_8h^{2l}\left\|e_{4,h}\right\|_\Omega^4\left\|\frac{\partial}{\partial t}\boldsymbol{Z}\right\|_{l,\Omega}^2 + \frac{1}{4}\left\|e_{4,h}\right\|_\Omega^4 + 2c_9h^{6l}\,\|\boldsymbol{Z}\|_{l,\Omega}^2\left\|\frac{\partial}{\partial t}\boldsymbol{Z}\right\|_{l,\Omega}^2 + c_{10}h^{2l}\left\|e_{4,h}\right\|_\Omega^2\|\boldsymbol{Z}\|_{l,\Omega}^2,
\end{aligned}$$

and the proof is completed.

Lemma 8. *Suppose*

$$L_{1,2} := \|\boldsymbol{z_h}\|_\Omega^2\left(\boldsymbol{w_h}, \nabla\tfrac{\partial}{\partial t}e_{1,h}\right)_\Omega - \|R_h\boldsymbol{Z}\|_\Omega^2\left(R_h\boldsymbol{W}, \nabla\tfrac{\partial}{\partial t}e_{1,h}\right)_\Omega + \|\boldsymbol{w_h}\|_\Omega^2\left(\boldsymbol{z_h}, \nabla\tfrac{\partial}{\partial t}e_{2,h}\right)_\Omega - \|R_h\boldsymbol{W}\|_\Omega^2\left(R_h\boldsymbol{Z}, \nabla\tfrac{\partial}{\partial t}e_{2,h}\right)_\Omega,$$

therefore, under Assumption 1, we have

$$L_{1,2} = \frac{d}{dt}\left(\left\|e_{3,h}\right\|_\Omega^2\left\|e_{4,h}\right\|_\Omega^2\right) + L_{1,2,0} + L_{1,2,1}, \tag{39}$$

where

$$\left|L_{1,2,0}\right| \leq c_1h^{2l}\left\|e_{4,h}\right\|_\Omega^4 + c_2\left\|e_{4,h}\right\|_\Omega^2 + c_3h^{4l}\left\|e_{3,h}\right\|_\Omega^2 + c_4\left\|e_{3,h}\right\|_\Omega^2 + \frac{1}{16}\left\|\nabla\frac{\partial}{\partial t}e_{1,h}\right\|_\Omega^2, \tag{40}$$

and

$$\left|L_{1,2,1}\right| \leq \left(c_1 h^{2l} + c_2\right)\left(\left\|e_{3,h}\right\|_\Omega^4 + \left\|e_{3,h}\right\|_\Omega^2\right) + \left(c_3 h^{2l} + c_4\right)\left(\left\|e_{4,h}\right\|_\Omega^2 + \left\|e_{4,h}\right\|_\Omega^4\right)$$
$$+ \frac{1}{16}\left\|\nabla \frac{\partial}{\partial t} e_{1,h}\right\|_\Omega^2 + \frac{1}{16}\left\|\nabla \frac{\partial}{\partial t} e_{2,h}\right\|_\Omega^2 + c_5 h^{6l}. \tag{41}$$

Proof. *Using the definition of $L_{1,2}$ as well as adding and subtracting some terms yield*

$$\begin{aligned}
L_{1,2} =\ & \|\boldsymbol{z_h}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \|R_h \boldsymbol{Z}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \|\boldsymbol{z_h}\|_\Omega^2 \left(R_h \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega \\
& - \|R_h \boldsymbol{Z}\|_\Omega^2 \left(\boldsymbol{w_h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \|\boldsymbol{w_h}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega + \|R_h \boldsymbol{W}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega \\
& + \|\boldsymbol{w_h}\|_\Omega^2 \left(R_h \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega - \|R_h \boldsymbol{W}\|_\Omega^2 \left(\boldsymbol{z_h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega + \|R_h \boldsymbol{Z}\|_\Omega^2 \left(R_h \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega \\
& + \|R_h \boldsymbol{W}\|_\Omega^2 \left(R_h \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega - \|R_h \boldsymbol{Z}\|_\Omega^2 \left(R_h \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega - \|R_h \boldsymbol{W}\|_\Omega^2 \left(R_h \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega.
\end{aligned}$$

By substituting (27) *and* (28) *from Lemma 6, into the above formula, we arrive at*

$$\begin{aligned}
L_{1,2} =\ & \left(\|\boldsymbol{z_h}\|_\Omega^2 + \|R_h \boldsymbol{Z}\|_\Omega^2\right) \frac{d}{dt} \left\|e_{3,h}\right\|_\Omega^2 + \left(\|\boldsymbol{z_h}\|_\Omega^2 - \|R_h \boldsymbol{Z}\|_\Omega^2\right)\left(\left(R_h \boldsymbol{W} - \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega \right.\\
& \left. + \left(\boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega\right) - \left(\|R_h \boldsymbol{Z}\|_\Omega^2 - \|\boldsymbol{Z}\|_\Omega^2\right)\left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega - \|\boldsymbol{Z}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega \\
& + \left(\|\boldsymbol{w_h}\|_\Omega^2 + \|R_h \boldsymbol{W}\|_\Omega^2\right) \frac{d}{dt} \left\|e_{4,h}\right\|_\Omega^2 + \left(\|\boldsymbol{w_h}\|_\Omega^2 - \|R_h \boldsymbol{W}\|_\Omega^2\right)\left(\left(R_h \boldsymbol{Z} - \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega \right.\\
& \left. - \left(\boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega\right) - \left(\|R_h \boldsymbol{W}\|_\Omega^2 - \|\boldsymbol{W}\|_\Omega^2\right)\left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega - \|\boldsymbol{W}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega \\
& + \left(\|\boldsymbol{z_h}\|_\Omega^2 + \|R_h \boldsymbol{Z}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(R_h \boldsymbol{W} - \boldsymbol{W}\right), e_{3,h}\right)_\Omega + \left(\|\boldsymbol{w_h}\|_\Omega^2 + \|R_h \boldsymbol{W}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(R_h \boldsymbol{Z} - \boldsymbol{Z}\right), e_{4,h}\right)_\Omega \\
\geq\ & \left\|e_{4,h}\right\|_\Omega^2 \frac{d}{dt}\left\|e_{3,h}\right\|_\Omega^2 + \left\|e_{3,h}\right\|_\Omega^2 \frac{d}{dt}\left\|e_{4,h}\right\|_\Omega^2 + L_{1,2,0} + L_{1,2,1} \\
=\ & \frac{d}{dt}\left(\left\|e_{3,h}\right\|_\Omega^2 \left\|e_{4,h}\right\|_\Omega^2\right) + L_{1,2,0} + L_{1,2,1},
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
L_{1,2,0} :=\ & \left(\|\boldsymbol{z_h}\|_\Omega^2 - \|R_h \boldsymbol{Z}\|_\Omega^2\right)\left(\left(R_h \boldsymbol{W} - \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \left(\boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega\right) \\
& - \left(\|R_h \boldsymbol{Z}\|_\Omega^2 - \|\boldsymbol{Z}\|_\Omega^2\right)\left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega - \|\boldsymbol{Z}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega,
\end{aligned} \tag{43}$$

and

$$
\begin{aligned}
L_{1,2,1} &:= \left(\|\boldsymbol{w_h}\|_\Omega^2 - \|R_h\boldsymbol{W}\|_\Omega^2\right)\left(\left(R_h\boldsymbol{Z}-\boldsymbol{Z}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega + \left(\boldsymbol{Z}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega\right) - \left(\|R_h\boldsymbol{W}\|_\Omega^2 - \|\boldsymbol{W}\|_\Omega^2\right)\left(e_{4,h}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega \\
&+ \left(\|\boldsymbol{z_h}\|_\Omega^2 + \|R_h\boldsymbol{Z}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(R_h\boldsymbol{W}-\boldsymbol{W}\right), e_{3,h}\right)_\Omega + \left(\|\boldsymbol{w_h}\|_\Omega^2 + \|R_h\boldsymbol{W}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(R_h\boldsymbol{Z}-\boldsymbol{Z}\right), e_{4,h}\right)_\Omega \\
&+ \left(\|R_h\boldsymbol{W}\|_\Omega^2 + 2\,\|\boldsymbol{W}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{W}-R_h\boldsymbol{W}\right), e_{3,h}\right)_\Omega - \left(\|R_h\boldsymbol{W}\|_\Omega^2 + 2\,\|\boldsymbol{W}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{W}-R_h\boldsymbol{W}\right), e_{3,h}\right)_\Omega \\
&+ \left(\|R_h\boldsymbol{Z}\|_\Omega^2 + 2\,\|\boldsymbol{Z}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{Z}-R_h\boldsymbol{Z}\right), e_{4,h}\right)_\Omega - \left(\|R_h\boldsymbol{Z}\|_\Omega^2 + 2\,\|\boldsymbol{Z}\|_\Omega^2\right)\left(\frac{\partial}{\partial t}\left(\boldsymbol{Z}-R_h\boldsymbol{Z}\right), e_{4,h}\right)_\Omega \\
&- \|\boldsymbol{W}\|_\Omega^2\left(e_{4,h}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega. \qquad (44)
\end{aligned}
$$

Employing the Cauchy-Schwarz and Young inequalities and Eq. (19)*, we conclude*

$$
\left|L_{1,2,0}\right| \le c_1h^{2l}\,\|\boldsymbol{W}\|_{l,\Omega}^2\left\|e_{4,h}\right\|_\Omega^4 + 16\,\|\boldsymbol{W}\|_\Omega^2\left\|e_{4,h}\right\|_\Omega^2 + c_2h^{4l}\,\|\boldsymbol{Z}\|_{l,\Omega}^4\left\|e_{3,h}\right\|_\Omega^2 + 16\,\|\boldsymbol{Z}\|_\Omega^4\left\|e_{3,h}\right\|_\Omega^2 + \tfrac{1}{16}\left\|\nabla\tfrac{\partial}{\partial t}e_{1,h}\right\|_\Omega^2, \qquad (45)
$$

and

$$
\begin{aligned}
\left\|L_{1,2,1}\right\|_\Omega^2 &\le c_1h^{2l}\,\|\boldsymbol{Z}\|_{l,\Omega}^2\left\|e_{3,h}\right\|_\Omega^4 + 16\,\|\boldsymbol{Z}\|_\Omega^2\left\|e_{4,h}\right\|_\Omega^4 + c_3h^{4l}\,\|\boldsymbol{W}\|_{l,\Omega}^4\left\|e_{4,h}\right\|_\Omega^2 + 16\,\|\boldsymbol{Z}\|_\Omega^4\left\|e_{4,h}\right\|_\Omega^2 + \frac{1}{16}\left\|\nabla\frac{\partial}{\partial t}e_{2,h}\right\|_\Omega^2 \\
&+ c_3h^{2l}\left\|e_{3,h}\right\|_\Omega^4\left\|\frac{\partial}{\partial t}\boldsymbol{W}\right\|_{l,\Omega}^2 + \frac{1}{4}\left\|e_{3,h}\right\|_\Omega^4 + c_4h^{6l}\,\|\boldsymbol{W}\|_{l,\Omega}^2\left\|\frac{\partial}{\partial t}\boldsymbol{W}\right\|_{l,\Omega}^2 + c_5h^{2l}\left\|e_{3,h}\right\|_\Omega^2\|\boldsymbol{W}\|_{l,\Omega}^2 + c_6h^{2l}\left\|e_{4,h}\right\|_\Omega^4\left\|\frac{\partial}{\partial t}\boldsymbol{Z}\right\|_{l,\Omega}^2 \\
&+ \frac{1}{4}\left\|e_{4,h}\right\|_\Omega^4 + c_7h^{6l}\,\|\boldsymbol{Z}\|_{l,\Omega}^2\left\|\frac{\partial}{\partial t}\boldsymbol{Z}\right\|_{l,\Omega}^2 + c_8h^{2l}\left\|e_{4,h}\right\|_\Omega^2\|\boldsymbol{Z}\|_{l,\Omega}^2. \qquad (46)
\end{aligned}
$$

Considering (42), (45)*, and* (46) *we have completed the lemma.*

Lemma 9. *Under Assumption 1, we conclude*

$$
L_1 \ge \tfrac{d}{dt}\left\|e_{3,h}\right\|_\Omega^2 + \tfrac{d}{dt}\left\|e_{4,h}\right\|_\Omega^2 + \tfrac{1}{2}\tfrac{d}{dt}\left\|e_{3,h}\right\|_\Omega^4 + \tfrac{1}{2}\tfrac{d}{dt}\left\|e_{4,h}\right\|_\Omega^4 + \tfrac{d}{dt}\left(\left\|e_{3,h}\right\|_\Omega^2\left\|e_{4,h}\right\|_\Omega^2\right) + L_{1,1} + L_{1,2,0} + L_{1,2,1} + L_{1,3} + \boldsymbol{M}_0, \qquad (47)
$$

where

$$
\begin{aligned}
L_{1,1} &:= \|R_h\boldsymbol{W}\|_\Omega^2\left(R_h\boldsymbol{W}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega - \|\boldsymbol{W}\|_\Omega^2\left(\boldsymbol{W}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega + \|R_h\boldsymbol{W}\|_\Omega^2\left(R_h\boldsymbol{Z}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega - \|\boldsymbol{W}\|_\Omega^2\left(\boldsymbol{Z}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega, \\
L_{1,3} &:= \|R_h\boldsymbol{Z}\|_\Omega^2\left(R_h\boldsymbol{W}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega - \|\boldsymbol{Z}\|_\Omega^2\left(\boldsymbol{W}, \nabla\frac{\partial}{\partial t}e_{1,h}\right)_\Omega + \|R_h\boldsymbol{Z}\|_\Omega^2\left(R_h\boldsymbol{Z}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega - \|\boldsymbol{Z}\|_\Omega^2\left(\boldsymbol{Z}, \nabla\frac{\partial}{\partial t}e_{2,h}\right)_\Omega,
\end{aligned}
$$

also $L_{1,2,0}$ and $L_{1,2,1}$ have been defined in (43) *and* (44)*, respectively and $\boldsymbol{M}_0$ can be found in* (38)*. Moreover, we have*

$$
\left|L_{1,1}\right| \le c_1h^{2l} + c_2h^{4l} + \frac{1}{16}\left\|\nabla\frac{\partial}{\partial t}e_{1,h}\right\|_\Omega^2 + \frac{1}{16}\left\|\nabla\frac{\partial}{\partial t}e_{2,h}\right\|_\Omega^2, \qquad (48)
$$

and

$$
\left|L_{1,3}\right| \le c_1h^{2l} + c_2h^{4l} + c_3h^{6l} + \frac{1}{16}\left\|\nabla\frac{\partial}{\partial t}e_{1,h}\right\|_\Omega^2 + \frac{1}{16}\left\|\nabla\frac{\partial}{\partial t}e_{2,h}\right\|_\Omega^2. \qquad (49)
$$

Proof. *By rearranging L_1 we have*

$$L_1 = \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega + L_{1,0} + L_{1,1} + L_{1,2} + L_{1,3} + L_{1,4}, \tag{50}$$

where $L_{1,0}$ and $L_{1,2}$ have been defined in Lemmas 7 and 8, respectively and

$$L_{1,4} := \left(R_h \boldsymbol{W} - \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \left(R_h \boldsymbol{Z} - \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega.$$

Now from (50)*, Lemmas 6, 7 and 8 we conclude*

$$\begin{aligned} L_1 \geq\ & \frac{d}{dt} \left\|e_{3,h}\right\|_\Omega^2 + \frac{d}{dt} \left\|e_{4,h}\right\|_\Omega^2 + \frac{1}{2} \frac{d}{dt} \left\|e_{3,h}\right\|_\Omega^4 + \frac{1}{2} \frac{d}{dt} \left\|e_{4,h}\right\|_\Omega^4 + \frac{d}{dt} \left(\left\|e_{3,h}\right\|_\Omega^2 \left\|e_{4,h}\right\|_\Omega^2\right) \\ & + L_{1,1} + L_{1,2,0} + L_{1,2,1} + L_{1,3} + \boldsymbol{M}_0. \end{aligned} \tag{51}$$

Moreover, by adding and subtracting some terms, $L_{1,1}$ can be rewritten as

$$\begin{aligned} L_{1,1} =\ & \left(\|R_h \boldsymbol{W}\|_\Omega^2 - \|\boldsymbol{W}\|_\Omega^2\right) \left(R_h \boldsymbol{W} - \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \|\boldsymbol{W}\|_\Omega^2 \left(R_h \boldsymbol{W} - \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega \\ & + \left(\|R_h \boldsymbol{W}\|_\Omega^2 - \|\boldsymbol{W}\|_\Omega^2\right) \left(\boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \left(\|R_h \boldsymbol{W}\|_\Omega^2 - \|\boldsymbol{W}\|_\Omega^2\right) \left(R_h \boldsymbol{Z} - \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega \\ & + \|\boldsymbol{W}\|_\Omega^2 \left(R_h \boldsymbol{Z} - \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega + \left(\|R_h \boldsymbol{W}\|_\Omega^2 - \|\boldsymbol{W}\|_\Omega^2\right) \left(\boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega. \end{aligned}$$

Applying the Cauchy-Schwarz and Young inequalities and Eq. (19)*, we deduce that*

$$\begin{aligned} \left|L_{1,1}\right| \leq\ & \|R_h \boldsymbol{W} - \boldsymbol{W}\|_\Omega^2 \left(\|R_h \boldsymbol{W} - \boldsymbol{W}\|_\Omega^2 + \left\|\nabla \frac{\partial}{\partial t} e_{1,h}\right\|_\Omega^2\right) + 4 \|\boldsymbol{W}\|_\Omega^4 \|R_h \boldsymbol{W} - \boldsymbol{W}\|_\Omega^2 \\ & + \frac{1}{16} \left\|\nabla \frac{\partial}{\partial t} e_{1,h}\right\|_\Omega^2 + \|R_h \boldsymbol{W} - \boldsymbol{W}\|_\Omega^2 \left(\|\boldsymbol{W}\|_\Omega^2 + \frac{1}{4} \left\|\nabla \frac{\partial}{\partial t} e_{1,h}\right\|_\Omega^2\right) + \|R_h \boldsymbol{W} - \boldsymbol{W}\|_\Omega^2 \times \\ & \left(\|R_h \boldsymbol{Z} - \boldsymbol{Z}\|_\Omega^2 + \left\|\nabla \frac{\partial}{\partial t} e_{2,h}\right\|_\Omega^2\right) + 4 \|\boldsymbol{W}\|_\Omega^4 \|R_h \boldsymbol{Z} - \boldsymbol{Z}\|_\Omega^2 + \frac{1}{16} \left\|\nabla \frac{\partial}{\partial t} e_{2,h}\right\|_\Omega^2 \\ & + \|R_h \boldsymbol{W} - \boldsymbol{W}\|_\Omega^2 \left(\|\boldsymbol{Z}\|_\Omega^2 + \frac{1}{4} \left\|\nabla \frac{\partial}{\partial t} e_{2,h}\right\|_\Omega^2\right) \leq c_1 h^{4l} \|\boldsymbol{W}\|_{l,\Omega}^4 + c_1 h^{2l} \|\boldsymbol{W}\|_{l,\Omega}^2 \|\boldsymbol{W}\|_\Omega^4 \\ & + c_1 h^{2l} \|\boldsymbol{W}\|_{l,\Omega}^2 \|\boldsymbol{W}\|_\Omega^2 + c_2 h^{2l} \left\|\nabla \frac{\partial}{\partial t} e_{1,h}\right\|_\Omega^2 + \frac{1}{16} \left\|\nabla \frac{\partial}{\partial t} e_{1,h}\right\|_\Omega^2 + c_1 h^{4l} \|\boldsymbol{W}\|_{l,\Omega}^2 \|\boldsymbol{Z}\|_{l,\Omega}^2 \\ & + c_1 h^{2l} \|\boldsymbol{Z}\|_{l,\Omega}^2 \|\boldsymbol{W}\|_\Omega^4 + c_1 h^{2l} \|\boldsymbol{W}\|_{l,\Omega}^2 \|\boldsymbol{Z}\|_\Omega^4 + c_2 h^{2l} \left\|\nabla \frac{\partial}{\partial t} e_{2,h}\right\|_\Omega^2 + \frac{1}{16} \left\|\nabla \frac{\partial}{\partial t} e_{2,h}\right\|_\Omega^2. \end{aligned} \tag{52}$$

Again, by adding and subtracting some terms, $L_{1,3}$ can be written in the following form

$$\begin{aligned} L_{1,3} =\ & \left(\|R_h \boldsymbol{Z}\|_\Omega^2 - \|\boldsymbol{Z}\|_\Omega^2\right) \left(R_h \boldsymbol{W} - \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \left(\|R_h \boldsymbol{Z}\|_\Omega^2 - \|\boldsymbol{Z}\|_\Omega^2\right) \left(\boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega \\ & + \|\boldsymbol{Z}\|_\Omega^2 \left(R_h \boldsymbol{W} - \boldsymbol{W}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \left(\|R_h \boldsymbol{Z}\|_\Omega^2 - \|\boldsymbol{Z}\|_\Omega^2\right) \left(R_h \boldsymbol{Z} - \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega \\ & + \|\boldsymbol{Z}\|_\Omega^2 \left(R_h \boldsymbol{Z} - \boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega + \left(\|R_h \boldsymbol{Z}\|_\Omega^2 - \|\boldsymbol{Z}\|_\Omega^2\right) \left(\boldsymbol{Z}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega. \end{aligned}$$

Using the Cauchy-Schwarz and Young inequalities and Eq. (19)*, the following inequality can be deduced*

$$\left|L_{1,3}\right| \leq c_1 h^{6l} \|\boldsymbol{Z}\|_{l,\Omega}^4 \|\boldsymbol{W}\|_{l,\Omega}^2 + c_2 h^{4l} \|\boldsymbol{Z}\|_{l,\Omega}^4 \|\boldsymbol{W}\|_{\Omega}^2 + c_3 h^{4l} \|\boldsymbol{Z}\|_{\Omega}^4 \|\boldsymbol{W}\|_{l,\Omega}^2 + \frac{1}{16}\left\|\nabla \frac{\partial}{\partial t} e_{1,h}\right\|_{\Omega}^2$$

$$+ c_4 h^{6l} \|\boldsymbol{Z}\|_{l,\Omega}^6 + c_5 h^{2l} \|\boldsymbol{Z}\|_{l,\Omega}^2 \|\boldsymbol{Z}\|_{\Omega}^2 + c_6 h^{2l} \|\boldsymbol{Z}\|_{l,\Omega}^2 \|\boldsymbol{Z}\|_{\Omega}^4 + \frac{1}{16}\left\|\nabla \frac{\partial}{\partial t} e_{2,h}\right\|_{\Omega}^2. \tag{53}$$

Now considering (51)*,* (52) *and* (53)*, the proof is completed.*

The next lemma investigates the error bounds of the last two terms of (20) and (21).

Lemma 10. *Suppose Assumption1 holds, therefore, we conclude*

$$L_2 \geq c\left(\left\|\frac{\partial}{\partial t} e_{1,h}\right\|_{\Gamma_0}^2 + \left\|\frac{\partial}{\partial t} e_{2,h}\right\|_{\Gamma_0}^2\right) + L_{2,2}, \tag{54}$$

where

$$\left|L_{2,2}\right| \leq c_1 h^{2l} \left\|\frac{\partial}{\partial t} U\right\|_{2,\Omega}^2 + \left\|\frac{\partial}{\partial t} e_{1,h}\right\|_{\Omega}^2 + c_2 h^{2l}\left\|\frac{\partial}{\partial t} V\right\|_{l,\Omega}^2 + \left\|\frac{\partial}{\partial t} e_{2,h}\right\|_{\Omega}^2. \tag{55}$$

Proof. *It is easy to see that*

$$\begin{aligned} L_2 = {} & g(t)\left(\left(\left|\frac{\partial}{\partial t}u_h\right|^p \frac{\partial}{\partial t}u_h, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}\Pi_h U\right|^p \frac{\partial}{\partial t}\Pi_h U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}\right) \\ & + g(t)\left(\left(\left|\frac{\partial}{\partial t}\Pi_h U\right|^p \frac{\partial}{\partial t}\Pi_h U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}U\right|^p \frac{\partial}{\partial t}U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}\right) \\ & + g(t)\left(\left(\left|\frac{\partial}{\partial t}v_h\right|^p \frac{\partial}{\partial t}v_h, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}\Pi_h V\right|^p \frac{\partial}{\partial t}\Pi_h V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}\right) \\ & + g(t)\left(\left(\left|\frac{\partial}{\partial t}\Pi_h V\right|^p \frac{\partial}{\partial t}\Pi_h V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}V\right|^p \frac{\partial}{\partial t}V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}\right). \end{aligned}$$

Therefore we can write

$$L_2 = L_{2,1} + L_{2,2}, \tag{56}$$

where

$$\begin{aligned} L_{2,1} := {} & g(t)\left(\left(\left|\frac{\partial}{\partial t}u_h\right|^p \frac{\partial}{\partial t}u_h, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}\Pi_h U\right|^p \frac{\partial}{\partial t}\Pi_h U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}\right) \\ & + g(t)\left(\left(\left|\frac{\partial}{\partial t}v_h\right|^p \frac{\partial}{\partial t}v_h, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}\Pi_h V\right|^p \frac{\partial}{\partial t}\Pi_h V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}\right), \\ L_{2,2} := {} & g(t)\left(\left(\left|\frac{\partial}{\partial t}\Pi_h U\right|^p \frac{\partial}{\partial t}\Pi_h U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}U\right|^p \frac{\partial}{\partial t}U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}\right) \\ & + g(t)\left(\left(\left|\frac{\partial}{\partial t}\Pi_h V\right|^p \frac{\partial}{\partial t}\Pi_h V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}V\right|^p \frac{\partial}{\partial t}V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}\right). \end{aligned}$$

It follows by Lemma 4 that

$$\left(\left(\left|\frac{\partial}{\partial t}u_h\right|^p \frac{\partial}{\partial t}u_h, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}\Pi_h U\right|^p \frac{\partial}{\partial t}\Pi_h U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}\right) \geq c\left\|\frac{\partial}{\partial t}e_{1,h}\right\|_{\Gamma_0}^2, \tag{57}$$

and

$$\left(\left(\left|\frac{\partial}{\partial t}v_h\right|^p \frac{\partial}{\partial t}v_h, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} - \left(\left|\frac{\partial}{\partial t}\Pi_h V\right|^p \frac{\partial}{\partial t}\Pi_h V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}\right) \geq c\left\|\frac{\partial}{\partial t}e_{2,h}\right\|^2_{\Gamma_0}. \tag{58}$$

From (57) *and* (58) *we have*

$$L_2 \geq cg(t)\left(\left\|\frac{\partial}{\partial t}e_{1,h}\right\|^2_{\Gamma_0} + \left\|\frac{\partial}{\partial t}e_{2,h}\right\|^2_{\Gamma_0}\right) + L_{2,2}. \tag{59}$$

Finally, the Trace theorem and Cauchy-Schwarz inequality and Eq. (16) *imply that*

$$\left|L_{2,2}\right| \leq \alpha_2\left(c_9\left\|\frac{\partial}{\partial t}(\Pi_h U - U)\right\|^2_{\Omega} + c_2\left\|\frac{\partial}{\partial t}e_{1,h}\right\|^2_{\Omega}\right) + \alpha_2\left(c_9\left\|\frac{\partial}{\partial t}(\Pi_h V - V)\right\|^2_{\Omega} + c_2\left\|\frac{\partial}{\partial t}e_{2,h}\right\|^2_{\Omega}\right) \tag{60}$$

$$\leq c_{10}h^{2l}\left\|\frac{\partial}{\partial t}U\right\|^2_{l,\Omega} + \left\|\frac{\partial}{\partial t}e_{1,h}\right\|^2_{\Omega} + c_{10}h^{2l}\left\|\frac{\partial}{\partial t}V\right\|^2_{l,\Omega} + \left\|\frac{\partial}{\partial t}e_{2,h}\right\|^2_{\Omega}. \tag{61}$$

Now we return to the proof of the main theorem.

Proof of Theorem 2. Let us define

$$L_{3,1} := \left(g * |u_h|^\gamma u_h, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} - \left(g * |\Pi_h U|^\gamma \Pi_h U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}$$

$$+ \left(g * |v_h|^\gamma v_h, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} - \left(g * |\Pi_h V|^\gamma \Pi_h V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}, \tag{62}$$

and

$$L_{3,2} := \left(g * |\Pi_h U|^\gamma \Pi_h U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0} - \left(g * |U|^\gamma U, \frac{\partial}{\partial t}e_{1,h}\right)_{\Gamma_0}$$

$$+ \left(g * |\Pi_h V|^\gamma \Pi_h V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0} - \left(g * |V|^\gamma V, \frac{\partial}{\partial t}e_{2,h}\right)_{\Gamma_0}. \tag{63}$$

As a result, we can write

$$L_3 = L_{3,1} + L_{3,2}. \tag{64}$$

By applying the Cauchy-Schwarz and Young inequalities, Lemma 4 and the Trace theorem, we can see

$$\left|L_{3,1}\right| = \left|\int_0^t g(t-r)\left(\left(|u_h|^\gamma u_h - |\Pi_h U|^\gamma \Pi_h U\right)(r), \frac{\partial}{\partial t}e_{1,h}(t)\right)_{\Gamma_0} dr\right| + \left|\int_0^t g(t-r)\left(\left(|v_h|^\gamma v_h - |\Pi_h V|^\gamma \Pi_h V\right)(r), \frac{\partial}{\partial t}e_{2,h}(t)\right)_{\Gamma_0} dr\right|$$

$$\leq \int_0^t |g(t-r)|\left(\|g\|_\infty \left\|e_{1,h}(r)\right\|^2_{\Gamma_0} + \frac{1}{4\|g\|_\infty}\left\|\frac{\partial}{\partial t}e_{1,h}(t)\right\|^2_{\Gamma_0}\right) dr + \int_0^t |g(t-r)|\left(\|g\|_\infty \left\|e_{2,h}(r)\right\|^2_{\Gamma_0} dr + \frac{1}{4\|g\|_\infty}\left\|\frac{\partial}{\partial t}e_{2,h}\right\|^2_{\Gamma_0}\right)$$

$$\leq \left((\|g\|_\infty)^2 \int_0^t \left\|e_{1,h}(r)\right\|^2_{\Gamma_0} + \frac{1}{4}\left\|\frac{\partial}{\partial t}e_{1,h}(t)\right\|^2_{\Omega}\right) dr + \left((\|g\|_\infty)^2 \int_0^t \left\|e_{2,h}(r)\right\|^2_{\Gamma_0} dr + \frac{1}{4}\left\|\frac{\partial}{\partial t}e_{2,h}\right\|^2_{\Omega}\right) dr, \tag{65}$$

$$\left|L_{3,2}\right| \leq \left((\|g\|_\infty)^2 \int_0^t \|(\Pi_h U - U)(r)\|^2_{\Omega} + \frac{1}{4}\left\|\frac{\partial}{\partial t}e_{1,h}\right\|^2_{\Omega}\right) dr + \left((\|g\|_\infty)^2 \int_0^t \|(\Pi_h V - V)(r)\|^2_{\Omega} + \frac{1}{4}\left\|\frac{\partial}{\partial t}e_{2,h}\right\|^2_{\Omega}\right) dr$$

$$\leq c_{11}h^{2l}\int_0^t \|U(r)\|^2_{l,\Omega} dr + \frac{c}{4}\left\|\frac{\partial}{\partial t}e_{1,h}\right\|^2_{\Omega} + c_{11}h^{2l}\int_0^t \|V(r)\|^2_{l,\Omega} dr + \frac{c}{4}\left\|\frac{\partial}{\partial t}e_{2,h}\right\|^2_{\Omega}, \tag{66}$$

and

$$
\begin{aligned}
|L_4| &\leq \left\|\frac{\partial^2}{\partial t^2}\left(\Pi_h U - U\right)\right\|_{\Omega}^2 + \frac{1}{4}\left\|\frac{\partial}{\partial t} e_{1,h}\right\|_{\Omega}^2 + \left\|\frac{\partial^2}{\partial t^2}\left(\Pi_h V - V\right)\right\|_{\Omega}^2 + \frac{1}{4}\left\|\frac{\partial}{\partial t} e_{2,h}\right\|_{\Omega}^2 \\
&\leq c_{14} h^{2l}\left\|\frac{\partial^2}{\partial t^2} U\right\|_{k,\Omega}^2 + \frac{1}{4}\left\|\nabla\frac{\partial}{\partial t} e_{1,h}\right\|_{\Omega}^2 + c_{15} h^{2l}\left\|\frac{\partial^2}{\partial t^2} V\right\|_{k,\Omega}^2 + \frac{1}{4}\left\|\frac{\partial}{\partial t} e_{2,h}\right\|_{\Omega}^2 .
\end{aligned} \tag{67}
$$

Using Cauchy–Schwarz and Young inequalities and Lemma 4 will lead to

$$
\begin{aligned}
|L_5| &\leq \left\|\nabla\frac{\partial}{\partial t}\left(\Pi_h U - U\right)\right\|_{\Omega}^2 + \frac{1}{4}\left\|\nabla\frac{\partial}{\partial t} e_{1,h}\right\|_{\Omega}^2 + \left\|\nabla\frac{\partial}{\partial t}\left(\Pi_h V - V\right)\right\|_{\Omega}^2 + \frac{1}{4}\left\|\nabla\frac{\partial}{\partial t} e_{2,h}\right\|_{\Omega}^2 \\
&\leq c_{16} h^{2l}\left\|\nabla\frac{\partial}{\partial t} U\right\|_{l,\Omega}^2 + \frac{1}{4}\left\|\nabla\frac{\partial}{\partial t} e_{1,h}\right\|_{\Omega}^2 + c_{17} h^{2l}\left\|\nabla\frac{\partial}{\partial t} V\right\|_{l,\Omega}^2 + \frac{1}{4}\left\|\nabla\frac{\partial}{\partial t} e_{2,h}\right\|_{\Omega}^2,
\end{aligned} \tag{68}
$$

and

$$
\begin{aligned}
|L_6| &\leq \|U - \Pi_h U\|_{\Omega}^2 + \left\|\frac{\partial}{\partial t}\left(U - \Pi_h U\right)\right\|_{\Omega}^2 + \frac{1}{2}\left\|\nabla\frac{\partial}{\partial t} e_{1,h}\right\|_{\Omega}^2 + \|V - \Pi_h V\|_{\Omega}^2 + \left\|\frac{\partial}{\partial t}\left(V - \Pi_h V\right)\right\|_{\Omega}^2 + \frac{1}{2}\left\|\nabla\frac{\partial}{\partial t} e_{2,h}\right\|_{\Omega}^2 \\
&\leq h^{2l}\|U\|_{l,\Omega}^2 + h^{2l}\left\|\frac{\partial}{\partial t} U\right\|_{l,\Omega}^2 + \frac{1}{2}\left\|\nabla\frac{\partial}{\partial t} e_{1,h}\right\|_{\Omega}^2 + h^{2l}\|V\|_{l,\Omega}^2 + h^{2l}\left\|\frac{\partial}{\partial t} V\right\|_{l,\Omega}^2 + \frac{1}{2}\left\|\nabla\frac{\partial}{\partial t} e_{2,h}\right\|_{\Omega}^2 .
\end{aligned} \tag{69}
$$

Now using (24) and Lemmas 9 and 10 gives rise to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{\partial}{\partial t} e_{1,h}\right\|_{\Omega}^2 + \left\|\frac{\partial}{\partial t} e_{2,h}\right\|_{\Omega}^2\right) + \left(\left\|\nabla\frac{\partial}{\partial t} e_{1,h}\right\|_{\Omega}^2 + \left\|\nabla\frac{\partial}{\partial t} e_{2,h}\right\|_{\Omega}^2\right) + \frac{1}{2}\frac{d}{dt}\left(\left\|e_{1,h}\right\|_{\Gamma_0}^2 + \left\|e_{2,h}\right\|_{\Gamma_0}^2\right) \\
&\qquad + (1 + g(t))\left(\left\|\frac{\partial}{\partial t} e_{1,h}\right\|_{\Gamma_0}^2 + \left\|\frac{\partial}{\partial t} e_{2,h}\right\|_{\Gamma_0}^2\right) + \frac{1}{2}\frac{d}{dt}\left(\left\|e_{3,h}\right\|_{\Omega}^2 \left\|e_{4,h}\right\|_{\Omega}^2\right) \\
&\qquad + \frac{1}{2}\frac{d}{dt}\left(\left\|e_{3,h}\right\|_{\Omega}^4 + \left\|e_{4,h}\right\|_{\Omega}^4\right) + \left\|e_{1,h}\right\|_{\Gamma_0}^2 + \left\|e_{2,h}\right\|_{\Gamma_0}^2 \frac{d}{dt}\left(\left\|e_{3,h}\right\|_{\Omega}^2 \left\|e_{4,h}\right\|_{\Omega}^2\right) \\
&\qquad \leq \left\|e_{1,h}\right\|_{\Gamma_0}^2 + \left\|e_{2,h}\right\|_{\Gamma_0}^2 + \left|L_{1,1}\right| + \left|L_{1,2,0}\right| + \left|L_{1,2,1}\right| + \left|L_{1,3}\right| \\
&\qquad + |\boldsymbol{M}_0| + \left|L_{2,2}\right| + \left|L_{3,1}\right| + \left|L_{3,2}\right| + |L_4| + |L_5| + |L_6| .
\end{aligned} \tag{70}
$$

Employing Lemmas 7-10 and Eqs. (65)-(70) we infer that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{\partial}{\partial t}e_{1,h}\right\|_{\Omega}^{2}+\left\|\frac{\partial}{\partial t}e_{2,h}\right\|_{\Omega}^{2}\right)+\frac{7}{8}\left(\left\|\nabla\frac{\partial}{\partial t}e_{1,h}\right\|_{\Omega}^{2}+\left\|\nabla\frac{\partial}{\partial t}e_{2,h}\right\|_{\Omega}^{2}\right)+\frac{1}{2}\frac{d}{dt}\left(\left\|e_{1,h}\right\|_{\Gamma_0}^{2}+\left\|e_{2,h}\right\|_{\Gamma_0}^{2}\right)\\
&\qquad+(1+g(t))\left(\left\|\frac{\partial}{\partial t}e_{1,h}\right\|_{\Gamma_0}^{2}+\left\|\frac{\partial}{\partial t}e_{2,h}\right\|_{\Gamma_0}^{2}\right)+\frac{1}{2}\frac{d}{dt}\left(\left\|e_{3,h}\right\|_{\Omega}^{2}+\left\|e_{4,h}\right\|_{\Omega}^{2}\right)\\
&\qquad+\frac{1}{2}\frac{d}{dt}\left(\left\|e_{3,h}\right\|_{\Omega}^{4}+\left\|e_{4,h}\right\|_{\Omega}^{4}\right)+\left\|e_{1,h}\right\|_{\Gamma_0}^{2}+\left\|e_{2,h}\right\|_{\Gamma_0}^{2}+\frac{d}{dt}\left(\left\|e_{3,h}\right\|_{\Omega}^{2}\left\|e_{4,h}\right\|_{\Omega}^{2}\right)\\
&\qquad\leq\left\|e_{1,h}\right\|_{\Gamma_0}^{2}+\left\|e_{2,h}\right\|_{\Gamma_0}^{2}+c_{18}h^{2l}\left\|\nabla\frac{\partial}{\partial t}e_{1,h}\right\|_{\Omega}^{2}+c_{19}h^{2l}\|\boldsymbol{W}\|_{l,\Omega}^{2}\left\|e_{4,h}\right\|_{\Omega}^{4}\\
&\qquad+64\|\boldsymbol{W}\|_{l,\Omega}^{2}\left\|e_{3,h}\right\|_{\Omega}^{2}+c_{20}h^{4l}\|\boldsymbol{Z}\|_{l,\Omega}^{4}\left\|e_{3,h}\right\|_{\Omega}^{2}+64\|\boldsymbol{Z}\|_{\Omega}^{4}\left\|e_{3,h}\right\|_{\Omega}^{2}\\
&\qquad+c_{21}h^{2l}\|\boldsymbol{Z}\|_{l,\Omega}^{2}\left\|e_{4,h}\right\|_{\Omega}^{4}+64\|\boldsymbol{Z}\|_{\Omega}^{2}\left\|e_{4,h}\right\|_{\Omega}^{4}+c_{22}h^{4l}\|\boldsymbol{Z}\|_{l,\Omega}^{4}\left\|e_{4,h}\right\|_{\Omega}^{2}\\
&\qquad+64\|\boldsymbol{Z}\|_{l,\Omega}^{4}\left\|e_{4,h}\right\|_{\Omega}^{2}+\left((\|g\|_{\infty})^{2}\int_{0}^{t}\left\|e_{1,h}(r)\right\|_{\Omega}^{2}dr+\frac{5}{4}\left\|\frac{\partial}{\partial t}e_{1,h}\right\|_{\Omega}^{2}\right)\\
&\qquad+\left((\|g\|_{\infty})^{2}\int_{0}^{t}\left\|e_{2,h}(r)\right\|_{\Omega}^{2}dr+\frac{1}{4}\left\|\frac{\partial}{\partial t}e_{2,h}\right\|_{\Omega}^{2}\right)+c_{23}h^{2l}+c_{24}h^{4l}+c_{25}h^{6l}.
\end{aligned}
\tag{71}
$$

Now from Lemma 3, we have

$$
\begin{aligned}
&\frac{1}{2}\left(\left\|\frac{\partial}{\partial t}e_{1,h}\right\|_{\Omega}^{2}+\left\|\frac{\partial}{\partial t}e_{2,h}\right\|_{\Omega}^{2}\right)+\frac{3}{2}\left(\left\|e_{1,h}\right\|_{\Gamma_0}^{2}+\left\|e_{2,h}\right\|_{\Gamma_0}^{2}\right)+(1+g(t))\left(\left\|\frac{\partial}{\partial t}e_{1,h}\right\|_{\Gamma_0}^{2}+\left\|\frac{\partial}{\partial t}e_{2,h}\right\|_{\Gamma_0}^{2}\right)\\
&\qquad+\frac{1}{2}\left(\left\|e_{3,h}\right\|_{\Omega}^{2}+\left\|e_{4,h}\right\|_{\Omega}^{2}\right)+\frac{1}{2}\left(\left\|e_{3,h}\right\|_{\Omega}^{4}+\left\|e_{4,h}\right\|_{\Omega}^{4}\right)+\left(\left\|e_{3,h}\right\|_{\Omega}^{2}\left\|e_{4,h}\right\|_{\Omega}^{2}\right)\\
&\qquad\leq\left(\left\|\frac{\partial}{\partial t}e_{1,h}(0)\right\|_{\Omega}^{2}+\left\|\frac{\partial}{\partial t}e_{2,h}(0)\right\|_{\Omega}^{2}\right)+\frac{3}{2}\left(\left\|e_{1,h}(0)\right\|_{\Gamma_0}^{2}+\left\|e_{2,h}(0)\right\|_{\Gamma_0}^{2}\right)\\
&\qquad+(1+g(t))\left(\left\|\frac{\partial}{\partial t}e_{1,h}(0)\right\|_{\Gamma_0}^{2}+\left\|\frac{\partial}{\partial t}e_{2,h}(0)\right\|_{\Gamma_0}^{2}\right)+\frac{1}{2}\left(\left\|e_{3,h}(0)\right\|_{\Omega}^{4}+\left\|e_{4,h}(0)\right\|_{\Omega}^{4}\right)\\
&\qquad+\left(\left\|e_{3,h}(0)\right\|_{\Omega}^{2}\left\|e_{4,h}(0)\right\|_{\Omega}^{2}\right)+c_{26}h^{2l}+c_{27}h^{4l}+c_{28}h^{6l}.
\end{aligned}
\tag{72}
$$

Suppose V_h and $\boldsymbol{W}_h$ are (N_1+1)- and (N_2+1)- dimensional spaces, respectively. Also, assume $A=\left\{\varphi_j\right\}_{j=1}^{N_1}$ and $B=\left\{\boldsymbol{\psi}_j\right\}_{j=1}^{N_2}$ are basis for V_h and $\boldsymbol{W}_h$, respectively, therefore from (12) we have

$$
\begin{aligned}
&\left(u_h|_{t=0},\varphi_j\right)_{\Omega}=\left(U_0,\varphi_j\right)_{\Omega}\quad j=0,1,2,\ldots,N_1, &&\left(\boldsymbol{w_h}|_{t=0},\boldsymbol{\psi}_j\right)_{\Omega}=\left(\nabla U_0,\boldsymbol{\psi}_j\right)_{\Omega}\quad j=0,1,2,\ldots,N_2,\\
&\left(\frac{\partial}{\partial t}u_h\Big|_{t=0},\varphi_j\right)_{\Omega}=\left(U_1,\varphi_j\right)_{\Omega}\quad j=0,1,2,\ldots,N_1, &&\left(v_h|_{t=0},\varphi_j\right)_{\Omega}=\left(V_0,\varphi_j\right)_{\Omega}\quad j=0,1,2,\ldots,N_1,\\
&\left(\boldsymbol{z_h}|_{t=0},\boldsymbol{\psi}_j\right)_{\Omega}=\left(\nabla V_0,\boldsymbol{\psi}_j\right)_{\Omega}\quad j=0,1,2,\ldots,N_2 &&\left(\frac{\partial}{\partial t}v_h\Big|_{t=0},\varphi_j\right)_{\Omega}=\left(V_1,\varphi_j\right)_{\Omega}\quad j=0,1,2,\ldots,N_1,
\end{aligned}
$$

Using the above relations and applying (16)-(19), we find that

$$\left\|e_{1,h}(0)\right\|_{\Omega}^{2} \leq \|U_0 - \Pi_h U_0\|_{\Omega}^{2} \leq c_{29} h^{2l} \|U_0\|_{l,\Omega}^{2}, \tag{73}$$

$$\left\|e_{3,h}(0)\right\|_{\Omega}^{2} \leq \|\nabla U_0 - R_h \nabla U_0\|_{\Omega}^{2} \leq c_{30} h^{2l} \|\nabla U_0\|_{l,\Omega}^{2}, \tag{74}$$

$$\left\|\frac{\partial}{\partial t} e_{1,h}(0)\right\|_{\Omega}^{2} \leq c_{31} h^{2l} \|U_1\|_{l,\Omega}^{2}, \tag{75}$$

$$\left\|e_{2,h}(0)\right\|_{\Omega}^{2} \leq \|V_0 - \Pi_h V_0\|_{\Omega}^{2} \leq c_{32} h^{2l} \|V_0\|_{l,\Omega}^{2}, \tag{76}$$

$$\left\|e_{4,h}(0)\right\|_{\Omega}^{2} \leq \|\nabla V_0 - R_h \nabla V_0\|_{\Omega}^{2} \leq c_{33} h^{2l} \|\nabla V_0\|_{l,\Omega}^{2}, \tag{77}$$

$$\left\|\frac{\partial}{\partial t} e_{2,h}(0)\right\|_{\Omega}^{2} \leq c_{36} h^{2l} \|V_1\|_{l,\Omega}^{2}. \tag{78}$$

Also it is easy to see that

$$\left\|e_{1,h}\right\|_{\Omega}^{2} - \left\|e_{1,h}(0)\right\|_{\Omega}^{2} \leq \left\|\int_0^t \frac{\partial}{\partial t} e_{1,h}\, \mathrm{d}r\right\|_{\Omega}^{2} \leq \int_0^t \left\|\frac{\partial}{\partial t} e_{1,h}\, \mathrm{d}r\right\|_{\Omega}^{2}, \tag{79}$$

$$\left\|e_{2,h}\right\|_{\Omega}^{2} - \left\|e_{2,h}(0)\right\|_{\Omega}^{2} \leq \left\|\int_0^t \frac{\partial}{\partial t} e_{2,h}\, \mathrm{d}r\right\|_{\Omega}^{2} \leq \int_0^t \left\|\frac{\partial}{\partial t} e_{2,h}\, \mathrm{d}r\right\|_{\Omega}^{2}. \tag{80}$$

Now substituting (73) - (78) into (72) and using (79) and (80), we complete the proof.

4. Numerical example

In this section, in order to validate the explained discretization strategy, we introduce the following numerical example. Here, the computational geometry ($\Omega = [0, 1] \times [0, 1]$) including the boundaries is shown in Figure 2. In the numerical example, we use spaces $X_h(K) = \mathcal{P}_1(K)$ and $M_h(K) = \mathcal{P}_1(K) \oplus (x, y)\mathcal{P}_1(K)$ where $\oplus$ indicates the direct sum, $(x, y) \in \Omega$ and for the triangle K

$$\mathcal{P}_1(K) = \{v : v \text{ is a polynoimal of degree at most 1 on } K\}.$$

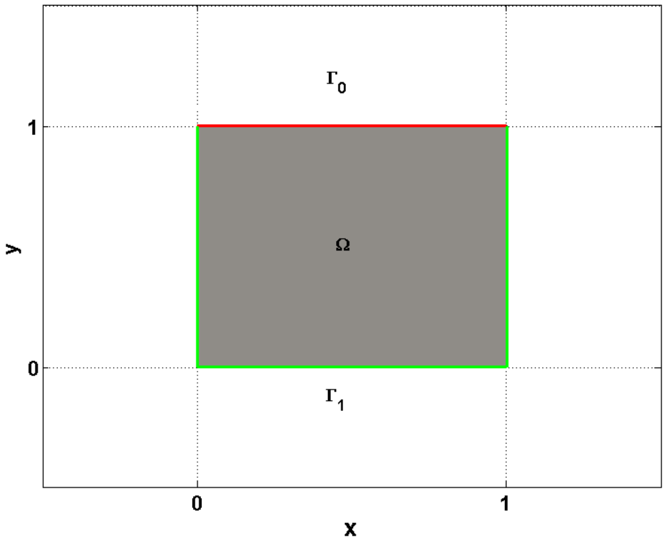

Figure 2: The domain of (81) and its respective boundaries Γ_0 and Γ_1.

We rewrite the boundary value problem (1) as

$$\begin{cases} U_{tt}(x,y,t) - \left(1 + \|\nabla U\|_\Omega^2 + \|\nabla V\|_\Omega^2\right) \Delta U(x,y,t) - \Delta U_t(x,y,t) = f(x,y,t) & (x,y) \in \Omega, \quad t \in [0,1], \\ V_{tt}(x,y,t) - \left(1 + \|\nabla U\|_\Omega^2 + \|\nabla V\|_\Omega^2\right) \Delta V(x,y,t) - \Delta V_t(x,y,t) = f(x,y,t) & (x,y) \in \Omega \quad t \in [0,1]\,, \\ U = V = \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \Sigma_1 = \Gamma_1 \times [0,1]\,, \\ \left(1 + \|\nabla U\|_\Omega^2 + \|\nabla V\|_\Omega^2\right) \frac{\partial U}{\partial \nu} + \frac{\partial U_t}{\partial \nu} + U + U_t + g(t)|U_t|^p U_t = g(t) * |U|^\gamma U & \text{on } \Sigma_0 = \Gamma_0 \times [0,1]\,, \\ \left(1 + \|\nabla U\|_\Omega^2 + \|\nabla V\|_\Omega^2\right) \frac{\partial V}{\partial \nu} + \frac{\partial V_t}{\partial \nu} + V + V_t + g(t)|V_t|^p V_t = g(t) * |V|^\gamma V & \text{on } \Sigma_0 = \Gamma_0 \times [0,1]\,, \\ U(x,y,0) = \left(\sin(y) + y^2(y-1) - y\sin(y)\right) \times \left(\sin(x) + x^2(x-1) - x\sin(x)\right) & (x,y) \in \Omega, \\ U_t(x,y,0) = -\left(\sin(y) + y^2(y-1) - y\sin(y)\right) \times \left(\sin(x) + x^2(x-1) - x\sin(x)\right) & (x,y) \in \Omega, \\ V(x,y,0) = \left(\sin(y) + y^2(y-1) - y\sin(y)\right) \times \left(\sin(x) + x^2(x-1) - x\sin(x)\right) & (x,y) \in \Omega, \\ V_t(x,y,0) = -\left(\sin(y) + y^2(y-1) - y\sin(y)\right) \times \left(\sin(x) + x^2(x-1) - x\sin(x)\right) & (x,y) \in \Omega. \end{cases} \tag{81}$$

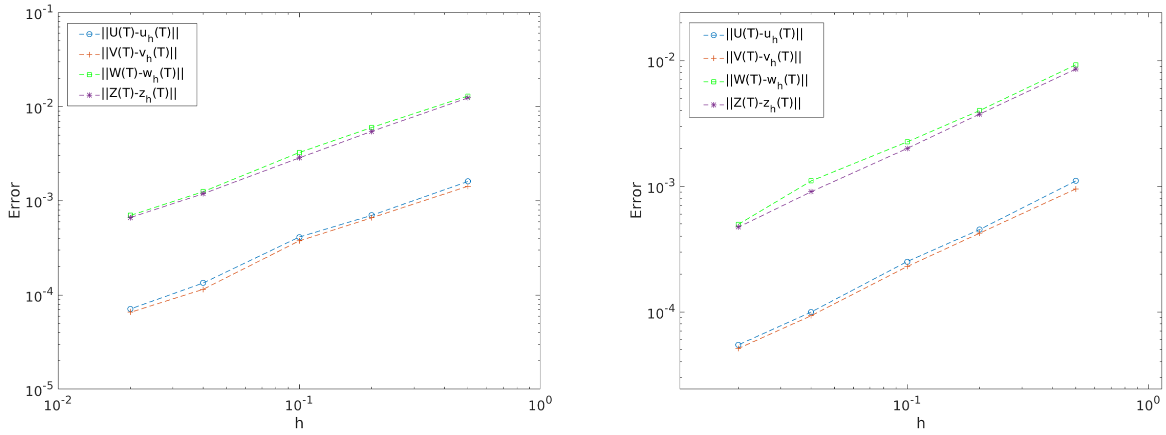

Figure 3: The L^2-error of different solutions of (81) for $\Delta t = 0.005$ (left) and $\Delta t = 0.003$ (right) at $T = 1$

The right hand side of the first two equations is given by

$$
\begin{aligned}
f(x, y, t) = \exp(-t)\Big(&\big(\sin(x) + x^2(x-1) - x\sin(x)\big)(2\cos(y) - 6y + \sin(y) - y\sin(y) + 2) + \\
&\exp(-t)\,\big(\sin(y) + y^2(y-1) - y\sin(y)\big)(2\cos(x) - 6x + \sin(x) - x\sin(x) + 2)\Big)\times \\
&(0.0046\exp(-2t) + 1) + \exp(-t)\Big(\big(\sin(x) + x^2(x-1) - x\sin(x)\big)\times \\
&\big(\sin(y) + y^2(y-1) - y\sin(y)\big)\big(\sin(x) + x^2(x-1) - x\sin(x)\big)\times \\
&(2\cos(y) - 6y + \sin(y) - y\sin(y) + 2) - \big(\sin(y) + y^2(y-1) - y\sin(y)\big)\times \\
&(2\cos(x) - 6x + \sin(x) - x\sin(x) + 2)\Big).
\end{aligned}
$$

The boundary consists of Γ_0 and Γ_1 (see Figure 1) where in the example, zero Neumann and Dirichlet boundary conditions are applied on Γ_1. Moreover, we use $g(t) = 0.1\exp(-t)$, $p = 3.5$, $\gamma = 2.5$. Finally, the exact solution of the equations is

$$U(x, y, t) = V(x, y, t) = \exp(-t)\big(\sin(y) + y^2(y-1) - y\sin(y)\big)\times\big(\sin(x) + x^2(x-1) - x\sin(x)\big).$$

In order to solve the mentioned numerical example, we use the mixed variational formulation given in (11). To that end, we define two auxiliary variables (i.e., $\boldsymbol{W} = \nabla U$ and $\boldsymbol{Z} = \nabla V$) to apply the Raviart-Thomas mixed finite element. In time discretization scheme, we use implicit finite difference (Crank-Nicolson method) to estimate the second-order derivative in the sense that in each time step (Δt), we solved a system of equations. In fact, the method can use a larger range step size than the other explicit (e.g., Euler method) scheme because of its unconditional stability. For the nonlinear terms (on Σ_1), the Newton method is used to obtain the solution. Also, we consider that U_0, U_1, V_0 and V_1 satisfy the condition Eq. 2. The numerical results for two different time steps ($\Delta t = 0.005$ and $\Delta t = 0.003$) are

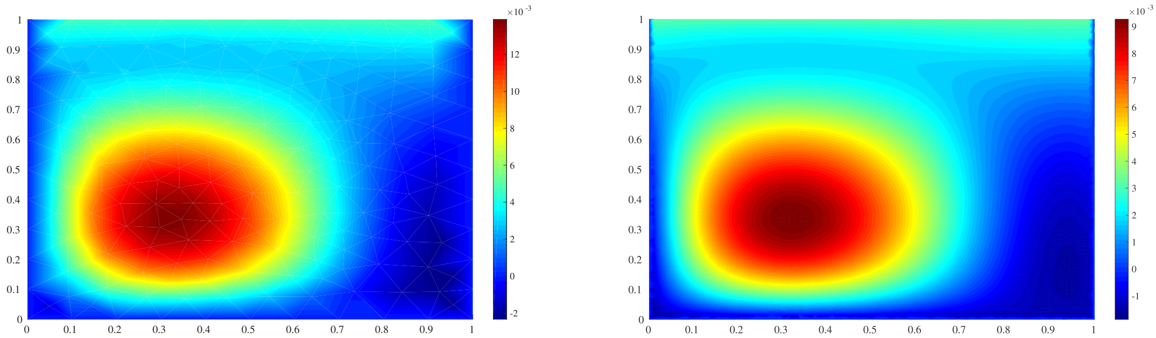

Figure 4: The approximated error of $v_h(T)$ for $h = 0.1$ (left) and $u_h(T)$ for $h = 0.01$ (right) with $\Delta t = 0.005$ at $T = 1$.

given in Figure 3. In both cases, the convergence rates agree very well with Theorem 2 since respectively $l = 0.91$ and $l = 0.89$ are achieved.

As the last step, the discretization error ($U(T) - u_h(T)$ and $V(T) - v_h$) with respect to the exact solutions for two specific mesh sizes are illustrated in Figure 4. Due to the nonlinearity, the estimated solutions show error on Γ_0. The results show that lower mesh sizes decrease the error.

5. Conclusions

In this paper, we considered the coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory term. Due to the structure of the problem, the optimum degree of convergence cannot be found using the classical finite element methods. In order to gain this aim, we presented the Raviart-Thomas mixed finite element method for solving (1). In other words, a theorem about the convergence of semi-discretized Raviart-Thomas mixed finite element method was proved and the optimal degree of convergence was obtained. The theorem has shown that total order of convergence with $O\left(h^{2l}\right)$. Finally, a numerical example is given to validate the error estimation.

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