Collective periodic motions in a multi-particle model involving processing delay *

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Abstract

How to understand the dynamical collective performances is of particular significance in both theories and applications. In this paper, we are interested in investigating the combined influences of local interaction and processing delay on the asymptotic behaviour in a particle model with local communication weights. As new observations, we show that the desired particle system undergoes both periodic flocking and periodic clustering behaviors when the processing delay crosses a threshold value and the eigenvalue 1 of average matrix is semi-simple. In this case, the connectedness of the particle system may be absent. Also, the number of clusters is discussed by using the subspace analysis. In results, some criterion of flocking and clustering emergence with exponential convergent rate are established by the standard functional differential equations analysis when the processing delay is small. When the processing delay reaches the threshold value, the system undergoes periodic flocking and periodic clustering emergence. It also shows that the processing time lags qualitatively change the emergent performances in a nonlinear way. Finally, we conclude this study with several numerical simulations that intuitively illustrate the validity of the theoretical results and address some discussions for both variable communication weight and distributed processing delay.

Keywords Periodic flock; periodic cluster; multi-particle model; processing delay; semi-simple eigenvalue.

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1 Introduction

As we know, self-organized collective systems arise very naturally in artificial intelligence, physical, biological and social science. Such systems seem to have remarkable capability to regulate the flow of information from distinct and independent components to achieve a prescribed performance. It is of particular interest, in both theories and applications, to understand how self-propelled individuals use only limited environmental information and simple rules to organize into an ordered motion. As the modelling and analysis frame, the

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particle models are widely introduced to verify the practical observations. See [2, 20, 33] for examples, and the references therein. The main motivation in current work is to analyze and explain the dynamical emergent patterns in a delayed particle model, while individual particle interacts locally and the processing time lags are also involved in.

In this paper, we consider a desired N-particle self-organized system with a processing delay, reading as,

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{\lambda}{\bar{N}_i(t)} \sum\limits_{j \in \bar{\mathbb{N}}_i(t)} \chi_r(|\bar{x}_j(t) - \bar{x}_i(t)|)(\bar{v}_j(t) - \bar{v}_i(t)), \quad i = 1, 2, \cdots, N, \end{cases} \tag{1.1}$$

where $x_i \in \mathbb{R}^n$ denotes the position of i-th particle and $v_i \in \mathbb{R}^n$ stands for its velocity. r is a constant denoting the size of neighbourhood, λ is a constant measuring the coupling strength, $N_i(t) = card\{j : |x_j(t) - x_i(t)| < r\}$ is the number of neighbours, $\mathbb{N}_i(t) = \{j : |x_j(t) - x_i(t)| < r\}$ is the set of neighbours of i. $\bar{v}_j(t) = v_j(t - \tau)$, where τ denotes the maximum processing delay from j to i. $\bar{N}_i(t)$, $\bar{x}_i(t)$ and $\bar{x}_j(t)$ are similar. The processing delay is a part of the total system delay, which occurs during node-to-node transmissions. It also includes the decision times about what to deal with and where to send information. The cut-off weight is given as

$$\chi_r(s) = \begin{cases} 1, & \text{if } 0 \le s < r, \\ 0, & \text{if } s \ge r. \end{cases}$$

For more delayed collective models, see [2, 4, 5, 10, 12, 21, 23, 27, 28, 31] for examples.

To achieve the ordered collective performances, there are three-fold facts included into the modeling and qualitative analysis. One is symmetry, which means that the interactions between each pair of particles are same; The second is global interaction for all individuals, and the last is the connectedness of the adjacency structure. Within these consideration, the celebrated Cucker-Smale model [7, 8] proposed in 2007 provided a framework to examine the emergent properties of flocks and explain self-organized behaviours arising from a kind of complex systems. In the successive contributions, the non-symmetric interaction, local interaction weight and the time lag arguments are all incorporated in the more general model settings. For more detailed discussions, we refer the readers to [2, 6, 9, 11, 13, 14, 18, 20, 22, 26, 29, 32, 34] and the references therein. These various literatures imply that the connectedness of the underlying adjacency matrix plays a crucial role in the analysis of synchronization. Naturally, how to remove the troublesome connectedness condition or to find the balance between local interaction and connectedness is a difficult problem in theory.

Aiming to make the dynamical behaviors clear, we firstly focus on the case of delay free. Mathematically, set $\tau = 0$, we have $\bar{v}_j(t) = v_j(t)$, $\bar{x}_j(t) = x_j(t)$ and $\bar{x}_i(t) = x_i(t)$. Thus the system (1.1) becomes

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{\lambda}{N_i(t)} \sum\limits_{j=1}^{N} \chi_r(|x_j(t) - x_i(t)|)(v_j(t) - v_i(t)). \end{cases} \tag{1.2}$$

In this case, Jin [20] posted some criterions to achieve flocks. With our observations, when r is large enough, v_i will be independent of $|x_i - x_j|$, and then system (1.1) degenerates into a first-order delayed system, as an opinion model, which has been investigated by Atay [1] and Cheng eta.[3].

Since the parameter r is sensitive to the dynamics of system (1.2), we consider two special cases: r is sufficiently small and r is infinity. When r is small enough, each particle will almost have no interaction from others at initial time. In this case, the dynamics of most particles evolutes for himself and strongly depends on its initial value. The dynamical behaviours of system (1.2) are complex. When r is large enough, each particle will be effected by all others. And the system (1.2) becomes

$$\dot{x}_i = v_i, \quad \dot{v}_i = \frac{\lambda}{N} \sum_{j=1}^{N} (v_j(t) - v_i(t)) \quad \text{for all } i. \tag{1.3}$$

Let $v_c(t) = \frac{1}{N} \sum\limits_{j=1}^{N} v_j(t)$, then $\dot{v}_c(t) = 0$. Thus $v_c(t) \equiv \frac{1}{N} \sum\limits_{j=1}^{N} v_j(0)$ for all t. From the equation (1.3), we have

$$\dot{v}_i = -\lambda v_i(t) + \lambda v_c \quad \text{for all } i.$$

Thus

$$\lim_{t\to\infty} v_i(t) = v_c = \frac{1}{N} \sum_{j=1}^{N} v_j(0).$$

In this case, the system achieves synchronization for any initial values. The dynamics of system (1.2) becomes simple. A natural question is how to verify the dynamical behaviours of system (1.1) when the processing delay is involved. Our motivation is to give some new viewpoints to the dynamical behaviours of system (1.1).

Now, we give the mathematical definition for periodic flock and periodic cluster.

Definition 1.1 *Suppose $(x_i(t), v_i(t)) \in \mathbb{R}^n \times \mathbb{R}^n (i = 1, 2, \cdots, N)$ is a solution to (1.1). The above system is said to achieve a periodic flock, if there are periodic functions $\phi_{pi}(t)$ with a same period such that*

$$\sup_{t\geq 0, \forall i,j} |x_i(t) - x_j(t)| < +\infty, \qquad \lim_{t\to\infty} (v_i(t) - \phi_{pi}(t)) = v_\infty, i = 1, 2, \cdots, N,$$

where $v_\infty \in \mathbb{R}^n$ is a constant vector. Especially, if all $\phi_{pi}(t) = 0$, the system (1.1) is said to achieve a flock.

Definition 1.2 *Suppose $(x_i(t), v_i(t)) \in \mathbb{R}^n \times \mathbb{R}^n (i = 1, 2, \cdots, N)$ is a solution to (1.1). The above system is said to achieve periodic multi-cluster if there exist periodic functions $\phi_{pi}(t)$ with a same period, some vectors $\varphi_j \in \mathbb{R}^n$ and sets $S_j \subset \{1, 2, \cdots, N\}$ satisfying $S_j \cap S_i = \emptyset$ (empty set), $\varphi_j \neq \varphi_i($ whenever $i \neq j)$ and $\cup_j S_j = \{1, 2, \cdots, N\}$, such that $\lim_{t\to\infty} (v_i(t) - \phi_{pi}(t)) = \varphi_j$ for all $i \in S_j$, $j = 1, 2, \cdots, k$. φ_j $(j = 1, \cdots, k)$ are then called the clustering value. Especially, if all $\phi_{pi}(t) = 0$, the system (1.1) is said to achieve k-cluster.*

This paper is organized as follows: in Section 2, we state more preliminaries for model analysis and give some assumptions. In Section 3, we address our main results on two-fold: one is to establish some criterions of flocking and clustering emergences with exponential convergent rate by the standard functional differential equations analysis when the processing delay is small. The other is to show that the desired system undergoes both periodic flocking and periodic clustering behaviours when the processing delay crosses a critical value and the eigenvalue 1 of average matrix is semi-simple. In Section 4, we give all the proof of main results and show that the time lags will make the emergent property changed from flock to periodic flock, from cluster to periodic cluster in a nonlinear way. Finally, a brief discussion is arranged for two cases: the variable communication weight and distributed processing delay.

2 Preliminaries

Firstly, define the adjacency matrix and average matrix of the system (1.1) by $A(t) = (a_{ij}(t))_{N\times N}$ and $P(t) = (p_{ij}(t))_{N\times N}$, respectively, where

$$a_{ij}(t) = \begin{cases} 1, & j \in \mathbb{N}_i(t), \\ 0, & otherwise, \end{cases} \quad \text{and} \quad p_{ij}(t) = \frac{a_{ij}(t)}{N_i(t)}. \tag{2.1}$$

Recalling the matrix theory, a matrix $S = (s_{ij})_{N\times N}$ is called stochastic if $s_{ij} \geq 0$ and $\sum_{j=1}^{N} s_{ij} = 1$. It is easy to find $\sum_{j=1}^{N} p_{ij} = 1$. Thus $P(t)$ is a stochastic matrix for all t. The matrix S is said to be connected if for arbitrary integers i and j $(1 \leq i, j \leq N)$, there are a sequence of integers $k_1, k_2, \cdots, k_q$ such that $s_{k_{l-1},k_l} > 0$, $l = 1, 2, \cdots, q+1$, where $k_0 = i, k_{q+1} = j$. From the Perron-Frobenius theorem [30], the matrix S is connected if and only if the eigenvalue 1 of S is simple. An eigenvalue is called to be semi-simple if its algebraic multiplicity equals its geometric multiplicity. The matrix $L(t) = I - P(t)$ is called the Laplacian matrix corresponding to the system (1.1).

Also, to quantize the sensitiveness of the average matrix when the distance of two particles is near r, we use the following variables on time t:

$$l_{ij}(t) = |x_j(t) - x_i(t)| \ \text{ and } \ \Gamma(t) = \min\{r - \max_{j\in\mathbb{N}_i(t)} l_{ij}(t), \min_{j\notin\mathbb{N}_i(t)} l_{ij}(t) - r\},$$

$d_M(t) = \max\{l_{ij}(t) : 1 \leq i, j \leq N\}$ and $d_m(t) = \min\{l_{ij}(t) : 1 \leq i, j \leq N\}$ for all $t > 0$. Naturally, we have $\Gamma(t) \geq 0$. If $\Gamma(t) > 0$, we call this case non-critical neighborhood situation. If $\Gamma(t) = 0$, we call it general neighborhood situation.

To specify a solution for the self-organized system (1.1), we need to specify the initial conditions

$$x_i(\theta) = f_i(\theta), v_i(\theta) = g_i(\theta) \quad \text{for } \theta \in [-\tau, 0], i = 1, 2, \cdots, N, \tag{2.2}$$

where f_i and g_i are given continuous vector-value functions.

Noting that the average matrix will change when the distance from another particle is across r. By the continuity of the trajectory of $x_i (1 \leq i \leq N)$, there exists $t_1 > 0$ such that the average matrix $P(t)$ remains unchanged on $[0, t_1)$, and changes at $t = t_1$. Denote t_n be the switching moments at nth time. Then $\{t_n\}$ is called the switching time sequence, which would be finite or infinity. Since the average matrix keeps unchanged at each interval (t_n, t_{n+1}), $n = 0, 1, 2, \cdots$, $(t_0 = 0)$, the matrix $P(t)$ will be a constant matrix on (t_n, t_{n+1}), say $P(t_n)$. Assume the initial average matrix keeps unchanged, saying $P(\theta) = P_0$ for $\theta \in [-\tau, 0]$.

In the sequel, we need the following vector norm. Define

$$\mathbf{V} = (v_1, v_2, \cdots, v_N)^T, \quad v_i \in \mathbb{R}^n, \quad 1 \leq i \leq N,$$

and the Euclidean modulus as

$$\|\mathbf{V}\|_2 = \left(\sum_{i=1}^{N} |v_i|\right)^{\frac{1}{2}}, \text{ where } |v_i| \text{ is the Euclidean norm of } v_i,$$

and the norm of a real matrix $S \in R^{N\times n}$ as

$$\|S\| = \sup_{|\alpha|\neq 0} \frac{|S\alpha|}{|\alpha|}, \quad \alpha \in R^n.$$

If S is a square matrix, then $\|S\|$ is the largest eigenvalue of S. If O is an orthogonal matrix, then $\|O\| = 1$. Using the above definitions and the Cauchy-Schwarz inequality, we see that

$$\|\mathbf{V}\| \leq \|\mathbf{V}\|_2 \leq \sqrt{n}\|\mathbf{V}\|.$$

From Lemma 5.1 in Appendix, the matrix P_0 is a diagonalizable matrix and its all eigenvalues are real. Throughout the paper, we assume the eigenvalue 1 of stochastic matrix P_0 is semi-simple with algebraic multiplicity n_0. And the other different eigenvalues of P_0 are $\mu_i (i = 2, 3, \cdots, m_0)$ with the algebraic multiplicity p_i. All eigenvalues of P_0 satisfy the order

$$1 = \mu_1 > \mu_2 > \cdots > \mu_{m_0}.$$

Naturally, if $n_0 = 1$, then the matrix P_0 is a connected matrix, and whenever $n_0 > 1$, the connectedness of matrix P_0 will be absent. From the matrix theory, we see that there is a non-degenerate matrix T_0 such that $P_0 = T_0 J_0 T_0^{-1}$, where J_0 is a diagonal matrix with the first block being I_{n_0}, say $J_0 = \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & J^* \end{pmatrix}$. Let $N_{max} = \max\{N_1(0), \cdots, N_N(0)\}$ and $N_{min} = \min\{N_1(0), \cdots, N_N(0)\}$, then it follows from Lemma 5.1 in Appendix that

$$\|T_0\|\|T_0^{-1}\| \leq \sqrt{\frac{N_{max}}{N_{min}}} =: D. \tag{2.3}$$

To find the qualitative behavior, consider the equation

$$\dot{\mathbf{w}} = -\lambda\bar{\mathbf{w}}(t) + \lambda J^*\bar{\mathbf{w}}(t), \tag{2.4}$$

and its characteristic equation is

$$h_0(z) = Det\left(zI + \lambda e^{-z\tau}(I - J^*)\right) = 0. \tag{2.5}$$

Lemma 2.1 *([15], Theorem 5.2) If $a_0 = \max\{Rez : h_0(z) = 0\}$, then , for any $c > a_0$, there is a constant $K = K(c)$ such that the fundamental solution $S_v(t)$ of the equation (2.4) satisfies the inequality*

$$\|S_v(t)\| \leq Ke^{ct}.$$

Finally, for the convenience of the reader, we give an overview of the qualitative behavior of solution to the equation $\dot{w} = -\lambda w(t-\tau)$ with $\lambda > 0$, subject to a constant nonzero initial value.

Lemma 2.2 *[15] For the equation $\dot{w} = -\lambda w(t-\tau)$ with positive constants λ and τ, then*

- *If $\lambda\tau \leq \frac{1}{e}$, the solution monotonically converges to zero as $t \to \infty$, hence no oscillations occur.*
- *If $\frac{1}{e} < \lambda\tau < \frac{\pi}{2}$, oscillations appear, however, with asymptotically vanishing amplitude.*
- *If $\lambda\tau = \frac{\pi}{2}$, periodic solutions exist.*
- *If $\lambda\tau > \frac{\pi}{2}$, the amplitude of the oscillations diverges as $t \to \infty$.*

Also, the next lemma is essential in the sequels.

Lemma 2.3 *([24], Lemma3.2) Assume zero is a semi-simple eigenvalue of the Laplacian $L = I - P_0$ with multiplicity n_0. Then there exists a unique family normal zero-one vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{n_0}$ such that $L\boldsymbol{a}_i = 0$ and $\boldsymbol{a}_1 + \boldsymbol{a}_2 + \cdots + \boldsymbol{a}_{n_0} = \boldsymbol{1}$.*

3 Main Results

In this section, we will address our main results. For a brief statement, we achieve the following new observations.

- If $r = +\infty$ and $0 < \lambda\tau < \frac{\pi}{2}$, the system (1.1) achieves a flock, and a periodic flocking performance occurs when $\lambda\tau = \frac{\pi}{2}$. It will diverge when $\lambda\tau > \frac{\pi}{2}$.
- If $r < +\infty$ and the adjacency matrix keeps unchanged.
  - If $0 < \lambda\tau(1-\mu_{m_0}) < \frac{\pi}{2}$, the system (1.1) achieves a flock when $n_0 = 1$, and achieves a flock or multi-cluster when $n_0 > 1$.
  - If $\lambda\tau(1-\mu_{m_0}) = \frac{\pi}{2}$, the system (1.1) achieves a periodic flock when $n_0 = 1$, and achieves a periodic flock or periodic multi-cluster when $n_0 > 1$.
  - If $\lambda\tau(1-\mu_{m_0}) > \frac{\pi}{2}$, the system (1.1) will diverge.
- If $r < +\infty$ and the adjacency matrix doesn't change frequently and sharply.
  - If $0 < \lambda\tau(1-\mu_{m_0}) < \frac{\pi}{2}$, the system (1.1) achieves a flock when $n_0 = 1$, and achieves a flock or multi-cluster when $n_0 > 1$.
  - If $\lambda\tau(1-\mu_{m_0}) = \frac{\pi}{2}$, the system (1.1) achieves a periodic flock when $n_0 = 1$, and achieves a periodic flock or periodic multi-cluster when $n_0 > 1$.

When $\lambda\tau(1-\mu_{m_0}) > \frac{\pi}{2}$, there exists positive real part roots in equation $h_0(z) = 0$. Thus the system (1.1) will diverge when the adjacency matrix keeps unchanged. But when the adjacency matrix changes, it is difficulty to determine whether the system (1.1) is divergent. There are no more discussions for this case in current work.

Let $g_0 = \sup_{\theta\in[-\tau,0]} \|\mathbf{g}(\theta)\|$ and

$$c_1 = \max_{2\le i\le m_0} \sup\{Re(z) : z = -\lambda(1-\mu_i)e^{-z\tau},\ 0 \le \lambda\tau(1-\mu_i) < \frac{\pi}{2}\},$$

then it follows from Lemma 5.2 in Appendix that $c_1 < 0$.

Theorem 3.1 *Let* 1 *be a* n_0*-multiple eigenvalue of the matrix* P_0*. Assume*

$$0 \le \lambda\tau(1-\mu_{m_0}) < \frac{\pi}{2} \quad \textit{and} \quad \sqrt{2n}Dg_0K < |c_1|\Gamma(0)$$

hold for the constant K *in Lemma 2.1. Then there is a constant* $c \in (\frac{\sqrt{2n}Dg_0K}{\Gamma(0)}, -c_1)$ *such that the system is convergent with*

$$\lim_{t\to\infty} \mathbf{V}(t) = T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{V}(0) := \mathbf{V}_\infty,$$

and

$$\|\mathbf{V}(t) - \mathbf{V}_\infty\| \le Dg_0Ke^{-ct}.$$

Especially, when $n_0 = 1$*, the system achieves a flock. When* $n_0 > 1$*, the system achieves a flock or multi-cluster.*

Let

$$\begin{aligned}\mathbf{V}_p(t) &= \cos(\frac{\pi t}{2\tau})T_0\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & I_{p_{m_0}}\end{pmatrix}T_0^{-1}\mathbf{g}(0)\\ &- \sin(\frac{\pi t}{2\tau})T_0\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & I_{p_{m_0}}\end{pmatrix}T_0^{-1}\mathbf{g}(-\tau),\end{aligned}$$

then we have the next result.

Theorem 3.2 *Let 1 be a n_0-multiple eigenvalue of matrix P_0 and $\lambda\tau(1-\mu_{m_0}) = \frac{\pi}{2}$. Assume*

$$\sqrt{2n}D\left(\frac{2\tau}{\pi}\|\mathbf{g}(0)\| + \frac{4\tau}{\pi}\|\mathbf{g}(-\tau)\| + \frac{g_0 K}{|c_1|}\right) < \Gamma(0)$$

hold for the constant K in Lemma 2.1. Then there is a constant $c \in (0, -c_1)$ such that the system converges to a periodic velocity emergence,

$$\lim_{t\to\infty}(\mathbf{V}(t) - \mathbf{V}_p(t)) = \mathbf{V}_\infty.$$

and

$$\|\mathbf{V}(t) - \mathbf{V}_p(t) - \mathbf{V}_\infty\| \le Dg_0Ke^{-ct}.$$

Especially, when $n_0 = 1$, the system achieves a periodic flock. When $n_0 > 1$, the system achieves a periodic flock or periodic multi-cluster.

If $r = +\infty$, the adjacency matrix keeps unchanged all time and $P_0 = \frac{1}{N}\mathbf{1}_{N\times N}$. In this case, we find that the eigenvalue 1 is simple and $\mu_2 = 0$ is a $(N-1)$-multiple eigenvalue. From Theorem 3.1 and Theorem 3.2, we have the following corollary.

Corollary 3.1 *If $r = +\infty$ and $0 \le \lambda\tau < \frac{\pi}{2}$, the system (1.1) achieves a flock with*

$$\lim_{t\to\infty}\mathbf{V}(t) = T_0\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}T_0^{-1}\mathbf{V}(0).$$

If $r = +\infty$ and $\lambda\tau = \frac{\pi}{2}$, a periodic flocking motion occurs with

$$\lim_{t\to\infty}[\mathbf{V}(t) - \mathbf{V}_{p0}(t)] = T_0\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}T_0^{-1}\mathbf{V}(0),$$

where

$$\mathbf{V}_{p0}(t) = \cos(\lambda t)T_0\begin{pmatrix}0 & \mathbf{0}\\ \mathbf{0} & I_{N-1}\end{pmatrix}T_0^{-1}\mathbf{g}(0) - \sin(\lambda t)T_0\begin{pmatrix}0 & \mathbf{0}\\ \mathbf{0} & I_{N-1}\end{pmatrix}T_0^{-1}\mathbf{g}(-\tau).$$

To understand well the dynamics of system (1.1) in the general case, we assume that the adjacency matrix doesn't change frequently and sharply, and consider the following assumptions.

(A_1) -There exist positive constants δ, γ and a sequence $t_n^* \in (t_n, t_{n+1})$ such that

$$t_{n+1} - t_n \ge \delta, \quad t_{n+1} - t_n^* \ge \tau \quad \text{and} \quad \Gamma(t_n^*) \ge \gamma, \quad \forall n.$$

(A_2) - Assume that the amplitude $\|P(t) - P_0\|$ is bounded uniformly on t. Set $\eta = \sup_{t\ge 0}\|P(t) - P_0\|$.

Theorem 3.3 *Let* 1 *be a* n_0*-multiple eigenvalue of matrix* P_0*. Assume* (A_1)*-*(A_2) *,*

$$0 \le \lambda\tau(1-\mu_{m_0}) < \frac{\pi}{2} \text{ and } \lambda\eta DK < |c_1|$$

hold for the constant K *in Lemma 2.1. Then there are constants* Q *and* $c \in (\lambda\eta DK, -c_1)$ *such that the system is convergent with*

$$\lim_{t\to\infty} \mathbf{V}(t) = \mathbf{V}_\infty + \lambda\mathbf{W}_\infty,$$

and

$$\|\mathbf{V}(t) - \mathbf{V}_\infty - \lambda\mathbf{W}_\infty\| \le Qe^{-(c-\lambda\eta DK)t},$$

where $\mathbf{W}_\infty$ *and* Q *will be formulated in (4.22) and (4.23). Especially, when* $n_0 = 1$*, the system achieves a flock. When* $n_0 > 1$*, the system achieves a flock or multi-cluster.*

Theorem 3.4 *Let* 1 *be a* n_0*-multiple eigenvalue of matrix* P_0*. Assume* (A_1)*-*(A_2) *,*

$$\lambda\tau(1-\mu_{m_0}) = \frac{\pi}{2} \text{ and } \lambda\eta DK < |c_1|$$

hold for the constant K *in Lemma 2.1. Then there ia a constant* $c \in (\lambda\eta DK, -c_1)$ *such that the system converges to a periodic velocity emergence,*

$$\lim_{t\to\infty} (\mathbf{V}(t) - \mathbf{V}_p(t)) = \mathbf{V}_\infty + \lambda W_{p\infty},$$

where $W_{p\infty}$ *will be formulated in (4.25). Especially, when* $n_0 = 1$*, the system achieves a periodic flock. When* $n_0 > 1$*, the system achieves a periodic flock or periodic multi-cluster.*

Remark 3.1 *We remark that the final value* $T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{V}(0)$ *is independent on the choice of* T_0*. Indeed, Take* $T_0 = (\boldsymbol{c}_1, \cdots, \boldsymbol{c}_{n_0}, *)$ *and* $T_0^{-1} = \left(\boldsymbol{r}_1^T, \cdots, \boldsymbol{r}_{n_0}^T, *\right)^T$*, then* $\mathbf{r}_i \cdot \mathbf{c}_i = 1$ *for* $i = 1, 2, \cdots, n_0$*. If we select* $k_i\mathbf{c}_i$ $(k_i \ne 0)$ *as the ith column of* T_0*, then the ith row of* T_0^{-1} *would be* $\frac{1}{k_i}\mathbf{r}_i$*. Then,*

$$\begin{aligned} T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{V}(0) &= T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \frac{1}{k_1}\boldsymbol{r}_1\mathbf{V}(0) \\ \frac{1}{k_2}\boldsymbol{r}_2\mathbf{V}(0) \\ \vdots \\ * \end{pmatrix} \\ &= (\frac{1}{k_1}\boldsymbol{r}_1\mathbf{V}(0)) \otimes (k_1\mathbf{c}_1) + \cdots + (\frac{1}{k_{n_0}}\boldsymbol{r}_{n_0}\mathbf{V}(0)) \otimes (k_{n_0}\mathbf{c}_{n_0}) \\ &= (\boldsymbol{r}_1\mathbf{V}(0)) \otimes \mathbf{c}_1 + \cdots + (\boldsymbol{r}_{n_0}\mathbf{V}(0)) \otimes \mathbf{c}_{n_0}, \end{aligned}$$

where $\otimes$ *denotes the Kronecker product. Thus, the final value is independent on the choice of* T_0*.*

Remark 3.2 *If* $\tau = 0$*, then* $c_1 = -\lambda(1-\mu_2)$*, where* μ_2 *is the second maximum eigenvalue of matrix* P_0*. In this case, Theorem 2.1 in [20] is a special case of Theorem 3.1.*

Remark 3.3 *In the sense of average, it would be better to understand the periodic flocking and periodic clustering phenomenon in Theorem 3.2. Indeed, each component of* $\mathbf{V}_p(t)$ *is* 4τ*- periodic. Thus*

$$\lim_{t\to\infty} \frac{1}{4\tau}\int_t^{t+4\tau} \mathbf{V}(s)ds = \lim_{t\to\infty} \frac{1}{4\tau}\int_t^{t+4\tau} (\mathbf{V}(s) - \mathbf{V}_p(s))ds = \mathbf{V}_\infty.$$

Although the final amplitude of each v_i *is different, its average system will achieve a flock or multi-cluster.*

4 Proof of the Main Results

4.1 Proof of Theorem 3.1

Let $\mathbf{X} = (x_1, x_2, \cdots, x_N)^T$, $\mathbf{V} = (v_1, v_2, \cdots, v_N)^T$ and $\bar{\mathbf{V}}(t) = \mathbf{V}(t-\tau)$. Thus the system (1.1) can be rewritten with the vector form on $[0, t_1)$, reading as,

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{V}(t), \\ \dot{\mathbf{V}}(t) = -\lambda \bar{\mathbf{V}}(t) + \lambda P_0 \bar{\mathbf{V}}(t), \\ \mathbf{X}(t) = \mathbf{f}(t), \mathbf{V}(t) = \mathbf{g}(t), t \in [-\tau, 0]. \end{cases} \tag{4.1}$$

Recalling $P_0 = T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & J^* \end{pmatrix} T_0^{-1}$ and let $\mathbf{U}(t) = T_0^{-1}\mathbf{V}(t) = (u_1(t), u_2(t), \cdots, u_{n_0}(t), \mathbf{u}^*(t))^T$, then the second equation of (4.1) yields

$$\dot{\mathbf{U}} = -\lambda \bar{\mathbf{U}}(t) + \lambda \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & J^* \end{pmatrix} \bar{\mathbf{U}}(t).$$

That is, $\dot{u}_i(t) = 0$ for $i = 1, 2, \cdots, n_0$, and

$$\dot{\mathbf{u}}^* = -\lambda \bar{\mathbf{u}}^*(t) + \lambda J^* \bar{\mathbf{u}}^*(t). \tag{4.2}$$

Thus the characteristic equation $h_0(z) = 0$ becomes

$$h(z) = \prod_{i=2}^{m_0} \left(z + \lambda(1-\mu_i)e^{-z\tau} \right)^{p_i} = 0, \tag{4.3}$$

where p_i is the algebraic multiplicity of μ_i, m_0 is the number of the different eigenvalues of P_0.

Let $\mathbf{S}^*(t)$ be a fundamental solution operator of the equation (4.2). Then the solution $\mathbf{V}(t)$ of the second equation in (4.1) becomes

$$\mathbf{V}(t+\theta) = T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^*(t) \end{pmatrix} T_0^{-1}\mathbf{g}(\theta), \quad \text{for} \quad t \in [0, t_1), \theta \in [-\tau, 0]. \tag{4.4}$$

Let

$$\mathbf{V}_a(\theta) = T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{g}(\theta). \tag{4.5}$$

By using the equalities (4.4) and (4.5), we have

$$\|\mathbf{V}(t+\theta) - \mathbf{V}_a(\theta)\| = \|T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^*(t) \end{pmatrix} T_0^{-1}\mathbf{g}(\theta)\|. \tag{4.6}$$

Following Lemma 5.2 in Appendix, we see that all roots of the characteristic equation (4.3) have negative real parts when $\lambda\tau(1-\mu_{m_0}) < \frac{\pi}{2}$. And from Lemma 2.1, there are constants $K > 0$ and $c \in (\frac{\sqrt{2n}Dg_0K}{\Gamma(0)}, -c_1)$ such that

$$\|\mathbf{S}^*(t)\| \leq Ke^{-ct}.$$

Thus

$$\begin{aligned}\|\mathbf{V}(t+\theta)-\mathbf{V}_a(\theta)\| &= \|T_0\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}^*(t)\end{pmatrix}T_0^{-1}\mathbf{g}(\theta)\|\\ &\le Dg_0Ke^{-ct}.\end{aligned}$$

This implies that

$$\sup_{\theta\in[-\tau,0]}\|\mathbf{V}(t+\theta)-\mathbf{V}_a(\theta)\| \;\le\; Dg_0Ke^{-ct}, \quad \text{for } t\in[0,t_1) \tag{4.7}$$

Next, we claim that $t_1=\infty$. If $t_1<\infty$, then the average matrix will change at $t=t_1$. Thus there exists (i_0,j_0) such that

$$\bar{l}_{i_0,j_0}(t_1)=|\bar{x}_{i_0}(t_1)-\bar{x}_{j_0}(t_1)|=r.$$

Recalling the first equation of (1.1), we have $\dot{x}_{i_0}(t)=v_{i_0}(t)$ and

$$x_{i_0}(t)-x_{j_0}(t)=x_{i_0}(0)-x_{j_0}(0)+\int_0^t(v_{i_0}(s)-v_{j_0}(s))ds.$$

If $n_0=1$, then the first column vector of T_0 would be selected as $(1,1,\cdots,1)^T$. From (4.7), we get

$$\begin{aligned}|v_{i_0}(s)-v_{j_0}(s)| &\le \sqrt{2}\sup_{\theta\in[-\tau,0]}\|\mathbf{V}(s+\theta)-\mathbf{V}_a(\theta)\|_2\\ &\le \sqrt{2n}\sup_{\theta\in[-\tau,0]}\|\mathbf{V}(s+\theta)-\mathbf{V}_a(\theta)\|\\ &\le \sqrt{2n}Dg_0Ke^{-cs}.\end{aligned}$$

When $\theta\in[-\tau,0]$ and $j_0\notin\mathbb{N}_{i_0}(0)$, we have

$$\begin{aligned}l_{i_0j_0}(t_1+\theta) &= |x_{i_0}(t_1+\theta)-x_{j_0}(t_1+\theta)|\ge l_{i_0j_0}(0)-\sqrt{2n}Dg_0K\int_0^\infty e^{-cs}ds\\ &= l_{i_0j_0}(0)-\frac{\sqrt{2n}Dg_0K}{c}>l_{i_0j_0}(0)-\Gamma(0)\ge r.\end{aligned}$$

This implies that

$$\bar{l}_{i_0j_0}(t_1)=l_{i_0j_0}(t_1-\tau)>r. \tag{4.8}$$

Also, when $\theta\in[-\tau,0]$ and $j_0\in\mathbb{N}_{i_0}(0)$, we have

$$\begin{aligned}l_{i_0j_0}(t_1+\theta) &= |x_{i_0}(t_1+\theta)-x_{j_0}(t_1+\theta)|\le l_{i_0j_0}(0)+\sqrt{2n}Dg_0K\int_0^\infty e^{-cs}ds\\ &= l_{i_0j_0}(0)+\frac{\sqrt{2n}Dg_0K}{c}<l_{i_0j_0}(0)+\Gamma(0)\le r.\end{aligned}$$

This implies that

$$\bar{l}_{i_0j_0}(t_1)=l_{i_0j_0}(t_1-\tau)<r. \tag{4.9}$$

Obviously, the inequalities (4.8) and (4.9) contradict that there exits (i_0, j_0) such that $\bar{l}_{i_0 j_0}(t_1) = r$. Thus $t_1 = \infty$ and $P(t) \equiv P_0$ for all time.

If $n_0 > 1$, without loss of generality (if necessary, we exchange the rows of matrix P_0 and relabel the subscript of v_i), we assume P_0 is a block diagonal matrix, say, $P_0 = diag(Q_1, Q_2, \cdots, Q_{n_0})$ (See Lemma 5.3 in Appendix). In this case, we consider the subsystem given as

$$\dot{\mathbf{V}}_i = -\lambda(I - Q_i)\mathbf{V}_i(t-\tau) \quad \text{for } t \in [0, t_1),$$

where Q_i is also stochastic matrix with a simple eigenvalue 1 for $i = 1, 2, \cdots, n_0$.

Let

$$\mu_{2i} = \max\{Rez : \det((z+\lambda)I - e^{-z\tau}Q_i) = 0, z \neq 1\},$$

$$N_{max}^{Q_i} = \max\{N_k(0) : \text{ the kth row of } P_0 \text{ partially locates in } Q_i\}$$

and

$$N_{min}^{Q_i} = \min\{N_k(0) : \text{ the kth row of } P_0 \text{ partially locates in } Q_i\},$$

then we have

$$\mu_{2i} \le c_1 \text{ and } \sqrt{\frac{N_{max}^{Q_i}}{N_{min}^{Q_i}}} \le \sqrt{\frac{N_{max}}{N_{min}}} = D.$$

Thus

$$\sqrt{2n}\sqrt{\frac{N_{max}^{Q_i}}{N_{min}^{Q_i}}}g_{i0}K \le \sqrt{2n}\sqrt{\frac{N_{max}}{N_{min}}}g_0 K < |c_1|\Gamma(0) \le |\mu_{2i}|\Gamma(0),$$

where g_{i0} is the supremum of partial components of initial value $\mathbf{g}(\theta)$. Since the eigenvalue 1 of Q_i is simple, following the similar arguments in the case of $n_0 = 1$, we conclude $t_1 = \infty$.

Thus, by using (4.7) and the fact $t_1 = \infty$, we have $\lim_{t\to\infty} \mathbf{V}(t+\theta) = \mathbf{V}_a(\theta)$. On the other hand, noting that $\dot{u}_i(t) = 0$ for $i = 1, 2, \cdots, n_0$ and $\lim_{t\to\infty} \mathbf{u}^*(t) = \mathbf{0}$, we conclude that

$$\lim_{t\to\infty} \mathbf{V}(t) = T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{V}(0) := \mathbf{V}_\infty.$$

Thus, from(4.7) again, we have

$$\mathbf{V}_a(\theta) = T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{V}(0) = \mathbf{V}_\infty$$

and

$$\|\mathbf{V}(t) - \mathbf{V}_\infty\| \le \sup_{\theta\in[-\tau,0]} \|\mathbf{V}(t+\theta) - \mathbf{V}_\infty\| \;\; \le \;\; Dg_0Ke^{-ct}.$$

When $n_0 = 1$, all the components of $\mathbf{V}_\infty$ are same. Thus, we have

$$\begin{aligned} \|\mathbf{X}(t) - t\mathbf{V}_\infty\| &= \|\mathbf{X}(0) + \int_0^t (\mathbf{V}(s) - \mathbf{V}_\infty)ds\| \\ &\le \|\mathbf{f}(0)\| + \int_0^t \|\mathbf{V}(s) - \mathbf{V}_\infty\|ds \\ &\le \|\mathbf{f}(0)\| + Dg_0K\int_0^\infty e^{-cs}ds = \|\mathbf{f}(0)\| + \frac{Dg_0K}{c}. \end{aligned}$$

Thus

$$\begin{aligned}\sup_{t\geq 0}|x_i(t)-x_j(t)| &\leq \sqrt{2}\|\mathbf{X}(t)-t\mathbf{V}_\infty\|_2 \leq \sqrt{2n}\|\mathbf{X}(t)-t\mathbf{V}_\infty\| \\ &\leq \sqrt{2n}\|\mathbf{f}(0)\| + \sqrt{2n}\frac{Dg_0K}{c} < \sqrt{2n}\|\mathbf{f}(0)\| + \Gamma(0).\end{aligned}$$

Thus the system (1.1) achieves a flock as $n_0 = 1$.

When $n_0 > 1$, recalling each of first n_0 columns of T_0 is the eigenvector of matrix $I - P_0$ being eigenvalue 0, from Lemma 2.3 and $L = I - P_0$, we can select zero-one vectors $\mathbf{a}_1, \cdots, \mathbf{a}_{n_0}$ just as the first n_0 columns of T_0, say $T_0 = (\mathbf{a}_1, \cdots, \mathbf{a}_{n_0}, *)$. Let s_i be the *ith* component of $T_0^{-1}\mathbf{V}(0)$, then

$$\begin{aligned}T_0\begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}T_0^{-1}\mathbf{V}(0) &= T_0\begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_N\end{pmatrix} \\ &= s_1\otimes\mathbf{a}_1 + s_2\otimes\mathbf{a}_2 + \cdots + s_{n_0}\otimes\mathbf{a}_{n_0}.\end{aligned}$$

Since $\mathbf{a}_i$ is zero-one vector for $i = 1, 2, \cdots, n_0$, so the number of different each other in vectors set $\{s_1, s_2, \cdots, s_{n_0}\}$ will determine the number of clusters of system (1.1). Indeed, assume that the vectors $\varphi_1, \varphi_2, \cdots, \varphi_k$ are different from each other in the set $\{s_1, s_2, \cdots, s_{n_0}\}$, then

$$T_0\begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}T_0^{-1}\mathbf{V}(0) = \varphi_1\otimes\tilde{\mathbf{a}}_1 + \cdots + \varphi_k\otimes\tilde{\mathbf{a}}_k,$$

where $\tilde{\mathbf{a}}_1, \cdots, \tilde{\mathbf{a}}_k$ are also zero-one vectors. Let

$$S_j = \{i: \text{ the } ith \text{ component of } \tilde{\mathbf{a}}_j \text{ equals } 1\},$$

then $\cup_j S_j = \{1, 2, \cdots, N\}$ and $\lim_{t\to\infty} v_i(t) = \varphi_j$ for all $i \in S_j$. Following Definition 1.1 and Definition 1.2, if $k = 1$, the system (1.1) achieves a flock. And if $k > 1$ the system (1.1) achieves multi-cluster. This completes the proof. □

4.2 Proof of Theorem 3.2

Firstly, the characteristic equation of the equation

$$\dot{u}(t) = -\lambda(1-\mu_{m_0})u(t-\tau) \tag{4.10}$$

becomes $z = -\lambda(1-\mu_{m_0})e^{-\tau z}$. For $\lambda\tau(1-\mu_{m_0}) = \frac{\pi}{2}$, the above equation has pure imaginary roots $\pm\frac{\pi}{2\tau}i$. Thus the solution of equation (4.10) is given as

$$u(t) = \cos(\frac{\pi t}{2\tau})u(0) - \sin(\frac{\pi t}{2\tau})u(-\tau), \quad t\in(0, t_1).$$

Let

$$\begin{aligned}\mathbf{V}_p(t) &= \cos(\frac{\pi t}{2\tau})T_0\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_{m_0}}\end{pmatrix}T_0^{-1}\mathbf{g}(0) \\ &- \sin(\frac{\pi t}{2\tau})T_0\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_{m_0}}\end{pmatrix}T_0^{-1}\mathbf{g}(-\tau),\end{aligned}$$

and rewrite the diagonal matrix J as

$$J = \begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_p^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mu_{m_0} I_{p_{m_0}} \end{pmatrix}.$$

Similarly, let $\mathbf{S}_p^*(t)$ be a fundamental solution operator of the equation

$$\dot{\mathbf{u}}^* = -\lambda \bar{\mathbf{u}}^*(t) + \lambda J_p^* \bar{\mathbf{u}}^*(t). \tag{4.11}$$

Then the solution $\mathbf{V}(t)$ in (4.1) becomes

$$\mathbf{V}(t+\theta) = \mathbf{V}_\infty + \mathbf{V}_p(t) + T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1} \mathbf{g}(\theta), \tag{4.12}$$

for $t \in [0, t_1), \theta \in [-\tau, 0]$.

To find the asymptotic behaviours , we consider the characteristic equation corresponding to (4.11), reading as

$$Det\left(zI + \lambda e^{-z\tau}(I - J_p^*)\right) = 0.$$

By direct computation, the above equation becomes

$$h_1(z) = \prod_{i=2}^{m_0-1} \left(z + \lambda(1-\mu_i)e^{-z\tau}\right)^{p_i} = 0. \tag{4.13}$$

Since all roots of $h_1(z) = 0$ are also the roots of $h_0(z) = 0$, following Lemma 5.2 in Appendix, we see that all roots of the characteristic equation (4.13) have negative real parts when $\lambda\tau(1-\mu_{m_0}) = \frac{\pi}{2}$.

Since

$$\sqrt{2n} D \left(\frac{2\tau}{\pi} \|\mathbf{g}(0)\| + \frac{4\tau}{\pi} \|\mathbf{g}(-\tau)\| + \frac{g_0 K}{|c_1|} \right) < \Gamma(0),$$

following Lemma 2.1, there are constants $K > 0$ and $c \in (0, -c_1)$, such that

$$\|\mathbf{S}_p^*(t)\| \leq K e^{-ct}$$

and

$$\sqrt{2n} D \left(\frac{2\tau}{\pi} \|\mathbf{g}(0)\| + \frac{4\tau}{\pi} \|\mathbf{g}(-\tau)\| + \frac{g_0 K}{c} \right) < \Gamma(0). \tag{4.14}$$

Thus

$$\begin{aligned} \|\mathbf{V}(t+\theta) - \mathbf{V}_\infty - \mathbf{V}_p(t)\| &= \|T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1} \mathbf{g}(\theta)\| \\ &\leq D g_0 K e^{-ct}. \end{aligned}$$

This implies that

$$\sup_{\theta\in[-\tau,0]} \|\mathbf{V}(t+\theta) - \mathbf{V}_\infty - \mathbf{V}_p(t)\| \leq D g_0 K e^{-ct}, \quad \text{for } t \in [0, t_1) \tag{4.15}$$

Next, we claim that $t_1 = \infty$. If $t_1 < \infty$, then the matrix $P(t)$ will change at $t = t_1$. Thus there exists (i_0, j_0) such that

$$\bar{l}_{i_0,j_0}(t_1) = |\bar{x}_{i_0}(t_1) - \bar{x}_{j_0}(t_1)| = r.$$

Recalling the first equation of (1.1), we have $\dot{x}_{i_0}(t) = v_{i_0}(t)$ and

$$x_{i_0}(t) - x_{j_0}(t) = x_{i_0}(0) - x_{j_0}(0) + \int_0^t (v_{i_0}(s) - v_{j_0}(s))ds.$$

Also, it is easy to see that

$$\| \int_0^t \mathbf{V}_p(s)ds\| \leq \frac{2\tau}{\pi} D\|\mathbf{g}(0)\| + \frac{4\tau}{\pi} D\|\mathbf{g}(-\tau)\|.$$

If $n_0 = 1$, from (4.19), we get

$$\begin{aligned}
& | \int_0^t (v_{i_0}(s) - v_{j_0}(s))ds| \\
\leq \ & | \int_0^t ((v_{i_0}(s) - v_{i\infty} - v_{ip}(s)) - (v_{j_0}(s) - v_{j\infty} - v_{jp}(s)))ds| \\
+ \ & | \int_0^t (v_{ip}(s) - v_{jp}(s))ds| \\
\leq \ & \sqrt{2} \left(\sup_{\theta \in [-\tau, 0]} \int_0^t \|\mathbf{V}(s+\theta) - \mathbf{V}_\infty - \mathbf{V}_p(s)\|_2 ds + \| \int_0^t \mathbf{V}_p(s)ds\|_2 \right) \\
\leq \ & \sqrt{2n} \left(\sup_{\theta \in [-\tau, 0]} \int_0^t \|\mathbf{V}(s+\theta) - \mathbf{V}_\infty - \mathbf{V}_p(s)\| ds + \| \int_0^t \mathbf{V}_p(s)ds\| \right) \\
\leq \ & \sqrt{2n} \left(\frac{2\tau}{\pi} D\|\mathbf{g}(0)\| + \frac{4\tau}{\pi} D\|\mathbf{g}(-\tau)\| + \frac{Dg_0K}{c} \right),
\end{aligned}$$

where $v_{j\infty}$ and $v_{jp}(s)$ are the jth component of $\mathbf{V}_\infty$ and $\mathbf{V}_p(t)$, respectively. When $\theta \in [-\tau, 0]$ and $j_0 \notin \mathbb{N}_{i_0}(0)$, from (4.14), we have

$$\begin{aligned}
l_{i_0 j_0}(t_1 + \theta) &= |x_{i_0}(t_1 + \theta) - x_{j_0}(t_1 + \theta)| \\
&\geq l_{i_0 j_0}(0) - | \int_0^t (v_{i_0}(s) - v_{j_0}(s))ds| \\
&= l_{i_0 j_0}(0) - \sqrt{2n} \left(\frac{2\tau}{\pi} D\|\mathbf{g}(0)\| + \frac{4\tau}{\pi} D\|\mathbf{g}(-\tau)\| + \frac{Dg_0K}{c} \right) \\
&> l_{i_0 j_0}(0) - \Gamma(0) \geq r.
\end{aligned}$$

This implies that

$$\bar{l}_{i_0 j_0}(t_1) = l_{i_0 j_0}(t_1 - \tau) > r. \tag{4.16}$$

Similarly, if $j_0 \in \mathbb{N}_{i_0}(0)$, we have

$$\begin{aligned}
l_{i_0 j_0}(t_1 + \theta) &= |x_{i_0}(t_1 + \theta) - x_{j_0}(t_1 + \theta)| \\
&\leq l_{i_0 j_0}(0) + | \int_0^t (v_{i_0}(s) - v_{j_0}(s))ds| \\
&= l_{i_0 j_0}(0) + \sqrt{2n} D \left(\frac{2\tau}{\pi} \|\mathbf{g}(0)\| + \frac{4\tau}{\pi} \|\mathbf{g}(-\tau)\| + \frac{g_0K}{c} \right) \\
&< l_{i_0 j_0}(0) + \Gamma(0) \leq r.
\end{aligned}$$

This implies that

$$\bar{l}_{i_0j_0}(t_1) = l_{i_0j_0}(t_1 - \tau) < r. \tag{4.17}$$

Obviously, the inequalities (4.16) and (4.17) contradict that there exits (i_0, j_0) such that $\bar{l}_{i_0j_0}(t_1) = r$. Thus $t_1 = \infty$ and $P(t) \equiv P_0$ for all time.

If $n_0 > 1$, without loss of generality (if necessary, we exchange the rows of matrix P_0 and relabel the subscript of v_i), we assume P_0 is a block diagonal matrix, say, $P_0 = diag(Q_1, Q_2, \cdots, Q_{n_0})$. In this case, we consider the subsystem given as

$$\dot{\mathbf{V}}_i = -\lambda(I - Q_i)\mathbf{V}_i(t - \tau) \quad \text{for } t \in [0, t_1),$$

where Q_i is also stochastic matrix with a simple eigenvalue 1 for $i = 1, 2, \cdots, n_0$.

If μ_{m_0} isn't an eigenvalue of Q_i, then all eigenvalues μ_{Q_i} of Q_i satisfying $\lambda\tau(1-\mu_{Q_i}) < \frac{\pi}{2}$. From Theorem 3.1 and $\sqrt{2n}Dg_0K < c\Gamma(0)$, we see that $t_1 = \infty$.

If μ_{m_0} is an eigenvalue of Q_i, then the corresponding inequality

$$\sqrt{2n}D\left(\frac{2\tau}{\pi}\|\mathbf{g}(0)\| + \frac{4\tau}{\pi}\|\mathbf{g}(-\tau)\| + \frac{g_0K}{c}\right) < \Gamma(0)$$

holds for the subsystem (substitute for all parameters). Thus we can transform it to the case of $n_0 = 1$ and conclude that $t_1 = \infty$.

Using the inequality (4.15), we see that

$$\|\mathbf{V}(t) - \mathbf{V}_\infty - \mathbf{V}_p(t)\| \le Dg_0Ke^{-ct} \text{ for all } t > 0.$$

Thus

$$\lim_{t\to\infty}[\mathbf{V}(t) - \mathbf{V}_p(t)] = \mathbf{V}_\infty.$$

Also, when $n_0 = 1$, all the components of $\mathbf{V}_\infty$ are same, say v_∞. Furthermore, we have

$$\begin{aligned}
\|\mathbf{X}(t) - t\mathbf{V}_\infty\| &= \|\mathbf{X}(0) + \int_0^t (\mathbf{V}(s) - \mathbf{V}_\infty - \mathbf{V}_p(s))ds + \int_0^t (\mathbf{V}_p(s)ds\| \\
&\le \|\mathbf{f}(0)\| + \int_0^t \|\mathbf{V}(s) - \mathbf{V}_\infty - \mathbf{V}_p(s))\|ds + \|\int_0^t (\mathbf{V}_p(s)ds\| \\
&\le \|\mathbf{f}(0)\| + Dg_0\int_0^\infty Ke^{-cs}ds + \|\int_0^t (\mathbf{V}_p(s)ds\| \\
&\le \|\mathbf{f}(0)\| + \frac{Dg_0K}{c} + \frac{2\tau}{\pi}D\|\mathbf{g}(0)\| + \frac{4\tau}{\pi}D\|\mathbf{g}(-\tau)\|.
\end{aligned}$$

Thus

$$\sup_{t\ge0}|x_i(t) - x_j(t)| \;\le\; \sqrt{2n}\left(\|\mathbf{f}(0)\| + \frac{Dg_0K}{c} + \frac{2\tau}{\pi}D\|\mathbf{g}(0)\| + \frac{4\tau}{\pi}D\|\mathbf{g}(-\tau)\|\right).$$

Furthermore, all the components of $\mathbf{V}_p(t)$ are periodic functions with a same period 4τ and

$$\lim_{t\to\infty}(v_i(t) - v_{ip}(t)) = v_\infty.$$

Thus it follows from Definition 1.1 that the system (1.1) achieves a periodic flock when $n_0 = 1$.

When $n_0 > 1$, with the similar arguments in the proof of Theorem 3.1, we can select a series of vectors $\{\varphi_1, \varphi_2, \cdots, \varphi_k\}$ (different from each other) such that

$$T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1} \mathbf{V}(0) \quad = \quad \varphi_1 \otimes \tilde{\mathbf{a}}_1 + \cdots + \varphi_k \otimes \tilde{\mathbf{a}}_k,$$

where $\tilde{\mathbf{a}}_1, \cdots, \tilde{\mathbf{a}}_k$ are zero-one vectors. Let

$$S_j = \{i : \text{ the } ith \text{ component of } \tilde{\mathbf{a}}_j \text{ equals } 1\},$$

then $\cup_j S_j = \{1, 2, \cdots, N\}$ and $\lim_{t\to\infty}(v_i(t) - v_{ip}(t)) = \varphi_j$ for all $i \in S_j$. Following Definition 1.1 and Definition 1.2, if $k = 1$, the system (1.1) achieves a periodic flock. And if $k > 1$, the system (1.1) achieves periodic k-cluster. This completes the proof. □

4.3 Proof of Theorem 3.3

In the general situation case, the matrix will change at sometimes. To find the collective behaviors, we rewrite the system (1.1) on $[0, t_1)$ as follow

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{V}(t), \\ \dot{\mathbf{V}} = -\lambda(I - P_0)\bar{\mathbf{V}}(t) + \lambda(\bar{P}(t) - P_0)\bar{\mathbf{V}}(t), \\ \mathbf{X}(\theta) = \mathbf{f}(\theta), \mathbf{V}(\theta) = \mathbf{g}(\theta), \theta \in [-\tau, 0], \end{cases} \tag{4.18}$$

where $\bar{P}(t) = P(t - \tau)$, $\quad \bar{\mathbf{V}}(t) = \mathbf{V}(t - \tau)$.

Also, let $\mathbf{S}^*(t)$ be a fundamental solution operator of the equation (4.2), then

$$T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^*(t) \end{pmatrix} T_0^{-1} \mathbf{g}(\theta)$$

is the fundamental solution of the homogeneous system (4.1), by using the variation-of-constant formula, the general solution of (4.18) is given as

$$\begin{aligned} \mathbf{V}(t + \theta) \quad &= \quad T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^*(t) \end{pmatrix} T_0^{-1} \mathbf{g}(\theta) \\ &+ \quad \lambda \int_0^t T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^*(t - s) \end{pmatrix} T_0^{-1} (\bar{P}(s) - P_0)\bar{\mathbf{V}}(s) ds. \end{aligned}$$

Taking

$$\begin{aligned} \hat{\mathbf{V}}(t + \theta) \quad &= \quad T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1} \mathbf{g}(\theta) \\ &+ \quad \lambda \int_0^t T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1} (\bar{P}(s) - P_0)\bar{\mathbf{V}}(s) ds, \end{aligned}$$

and using the fact

$$T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^*(t - s) \end{pmatrix} T_0^{-1} (\bar{P}(s) - P_0)\hat{\mathbf{V}}(s) = \mathbf{0},$$

we have

$$
\begin{aligned}
\mathbf{V}(t+\theta)-\hat{\mathbf{V}}(t+\theta) &= T_0\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}^*(t)\end{pmatrix}T_0^{-1}\mathbf{g}(\theta)\\
&+ \lambda\int_0^t \mathbf{T}_0\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}^*(t-s)\end{pmatrix}T_0^{-1}(\bar{P}(s)-P_0)\bar{\mathbf{V}}(s)ds\\
&= T_0\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}^*(t)\end{pmatrix}T_0^{-1}\mathbf{g}(\theta)\\
&+ \lambda\int_0^t T_0\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}^*(t-s)\end{pmatrix}T_0^{-1}(\bar{P}(s)-P_0)(\bar{\mathbf{V}}(s)-\hat{\mathbf{V}}(s))ds.
\end{aligned}
$$

And from Lemma 2.1, there are constants $K>0$ and $c\in(\lambda\eta DK,-c_1)$ such that

$$\|\mathbf{S}^*(t)\|\le Ke^{-ct}.$$

Thus, using the Assumption (A_2), we have

$$
\begin{aligned}
&\sup_{\theta\in[-\tau,0]}\|\mathbf{V}(t+\theta)-\hat{\mathbf{V}}(t+\theta)\|\\
\le\ & DKg_0e^{-ct}+\lambda DK\int_0^t e^{-c(t-s)}\|P(s)-P_0\|\sup_{\theta\in[-\tau,0]}\|\mathbf{V}(s+\theta)-\hat{\mathbf{V}}(s+\theta)\|ds\\
\le\ & DKg_0e^{-ct}+\lambda\eta DK\int_0^t e^{-c(t-s)}\sup_{\theta\in[-\tau,0]}\|\mathbf{V}(s+\theta)-\hat{\mathbf{V}}(s+\theta)\|ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&e^{ct}\sup_{\theta\in[-\tau,0]}\|\mathbf{V}(t+\theta)-\hat{\mathbf{V}}(t+\theta)\|\\
\le\ & DKg_0+\lambda\eta DK\int_0^t e^{cs}\sup_{\theta\in[-\tau,0]}\|\mathbf{V}(s+\theta)-\hat{\mathbf{V}}(s+\theta)\|ds.
\end{aligned}
$$

By solving the above Gronwall inequality, we get

$$\sup_{\theta\in[-\tau,0]}\|\mathbf{V}(t+\theta)-\hat{\mathbf{V}}(t+\theta)\| \le KDg_0e^{-(c-\lambda\eta DK)t}. \tag{4.19}$$

Next, we claim that $t_{k_1+1}=\infty$. Indeed, if $t_{k_1+1}<\infty$ then there exists (i_0,j_0) such that $\bar{l}_{i_0j_0}(t_{k_1+1})=r$. Following Assumption (A_1), there are positive constants δ, γ and $t^*_{k_1}\in(t_{k_1},t_{k_1+1})$ such that

$$t_{k_1+1}-t_{k_1}\ge\delta,\quad k_1+1-t^*_{k_1}\ge\tau\quad\text{and}\quad\Gamma(t^*_{k_1})\ge\gamma.$$

Recalling the inequality $\lambda\eta DK<c$, we find that there is a positive integral number k_1 such that

$$\sqrt{2n}KDg_0e^{-(c-\lambda\eta DK)k_1\delta}<(c-\lambda\eta DK)\gamma.$$

If $n_0 = 1$, for all $\theta \in [-\tau, 0]$ and $j_0 \notin \mathbb{N}_{i_0}(t_{k_1^*})$, we have

$$\begin{array}{rcl} l_{i_0 j_0}(t_{k_1+1}+\theta) & = & |x_{i_0}(t_{k_1+1}+\theta) - x_{j_0}(t_{k_1+1}+\theta)| \\ & \geq & l_{i_0 j_0}(t^*_{k_1}) - \sqrt{2n}KDg_0 \displaystyle\int_{t^*_{k_1}}^{\infty} e^{-(c-\lambda\eta DK)s} ds \\ & \geq & l_{i_0 j_0}(t^*_{k_1}) - \sqrt{2n}KDg_0 \displaystyle\int_{k_1\delta}^{\infty} e^{-(c-\lambda\eta DK)s} ds \\ & = & l_{i_0 j_0}(t^*_{k_1}) - \dfrac{\sqrt{2n}KDg_0}{c-\lambda\eta DK} e^{-(c-\lambda\eta DK)k_1\delta} \\ & > & l_{i_0 j_0}(t^*_{k_1}) - \gamma \geq l_{i_0 j_0}(t^*_{k_1}) - \Gamma(t^*_{k_1}) \\ & \geq & r, \end{array}$$

Thus

$$\bar{l}_{i_0 j_0}(t_{k_1+1}) = l_{i_0 j_0}(t_{k_1+1} - \tau) > r. \tag{4.20}$$

Similarly, if $j_0 \in \mathbb{N}_{i_0}(t_{k_1^*})$, we have

$$\begin{array}{rcl} l_{i_0 j_0}(t_{k_1+1}+\theta) & = & |x_{i_0}(t_{k_1+1}+\theta) + x_{j_0}(t_{k_1+1}+\theta)| \\ & \leq & l_{i_0 j_0}(t^*_{k_1}) + \sqrt{2n}KDg_0 \displaystyle\int_{t^*_{k_1}}^{\infty} e^{-(c-\lambda\eta DK)s} ds \\ & \leq & l_{i_0 j_0}(t^*_{k_1}) + \sqrt{2n}KDg_0 \displaystyle\int_{k_1\delta}^{\infty} e^{-(c-\lambda\eta DK)s} ds \\ & = & l_{i_0 j_0}(t^*_{k_1}) + \dfrac{\sqrt{2n}KDg_0}{c-\lambda\eta DK} e^{-(c-\lambda\eta DK)k_1\delta} \\ & < & l_{i_0 j_0}(t^*_{k_1}) + \gamma \leq l_{i_0 j_0}(t^*_{k_1}) + \Gamma(t^*_{k_1}) \\ & \leq & r, \end{array}$$

This implies that

$$\bar{l}_{i_0 j_0}(t_{k_1+1}) = l_{i_0 j_0}(t_{k_1+1} - \tau) < r. \tag{4.21}$$

Obviously, the inequalities (4.20) and (4.21) contradict that there exits (i_0, j_0) such that $\bar{l}_{i_0 j_0}(t_{k_1+1}) = r$. Thus $t_{k_1+1} = \infty$ and $P(t) \equiv P_{k_1}$ for all time $t > t_{k_1}$.

If $n_0 > 1$, with the similar arguments in the proof of Theorem 3.1, we conclude that $t_{k_1+1} = \infty$ too.

According to Theorem 3.1, we have

$$\lim_{t\to\infty} T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^*(t) \end{pmatrix} T_0^{-1} \mathbf{g}(\theta) = \mathbf{V}_\infty$$

uniformly for $\theta \in [-\tau, 0]$. Also, using the fact $\begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} (\bar{P}(s) - P_0)\hat{\mathbf{V}}(s) = \mathbf{0}$, we see that

$$\begin{aligned}
& \| \int_0^t T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1} (\bar{P}(s) - P_0)\bar{\mathbf{V}}(s) ds \| \\
= & \| \int_0^t T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1} (\bar{P}(s) - P_0)(\bar{\mathbf{V}}(s) - \hat{\mathbf{V}}(s)) ds \| \\
\leq & \ \eta K D^2 g_0 \int_0^t e^{-(c - \lambda\eta DK)s} ds \\
< & \ \frac{\eta K D^2 g_0}{c - \lambda\eta DK} < \infty.
\end{aligned}$$

So the limit

$$\lim_{t\to\infty} \int_0^t T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1} (\bar{P}(s) - P_0)\bar{\mathbf{V}}(s) ds \tag{4.22}$$

exists, say $\mathbf{W}_\infty$. Thus

$$\lim_{t\to\infty} \hat{\mathbf{V}}(t) = \mathbf{V}_\infty + \lambda \mathbf{W}_\infty.$$

And

$$\begin{aligned}
& \|\hat{\mathbf{V}}(t) - \mathbf{V}_\infty - \lambda \mathbf{W}_\infty\| \\
= & \| \int_0^t T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1} (\bar{P}(s) - P_0)(\bar{\mathbf{V}}(s) - \hat{\mathbf{V}}(s)) ds \| \\
\leq & \ \frac{\eta K D^2 g_0}{c - \lambda\eta DK} e^{-(c - \eta D\lambda K)t}.
\end{aligned}$$

Combining the above inequality and (4.19), we have

$$\lim_{t\to\infty} \mathbf{V}(t) = \mathbf{V}_\infty + \lambda \mathbf{W}_\infty,$$

and

$$\|\mathbf{V}(t) - \mathbf{V}_\infty - \lambda \mathbf{W}_\infty\| \leq Q e^{-(c - \eta D\lambda K)t},$$

where

$$Q = KDg_0 + \frac{\eta K D^2 g_0}{c - \lambda\eta DK}. \tag{4.23}$$

With the similar arguments in the proof of Theorem 3.1 and discussions, we conclude that the system achieves a flock when $n_0 = 1$, and achieves a flock or multi-cluster when $n_0 > 1$. This completes the proof. $\square$

4.4 Proof of Theorem 3.4

Also, by using $\lambda\tau(1-\mu_{m_0}) = \frac{\pi}{2}$, we see that the solution of the equation

$$\dot{u}(t) = -\lambda(1-\mu_{m_0})u(t-\tau)$$

is

$$u(t) = \cos(\frac{\pi t}{2\tau})u(0) - \sin(\frac{\pi t}{2\tau})u(-\tau), \quad t \in (0, t_1).$$

Define the solution operator as

$$S_0(t)\phi(\theta) = \cos(\frac{\pi t}{2\tau})\phi(0) - \sin(\frac{\pi t}{2\tau})\phi(-\tau).$$

then the periodic function $\mathbf{V}_p(t)$ can be rewritten as

$$\begin{aligned} \mathbf{V}_p(t) &= T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S_0(t)I_{p_{m_0}} \end{pmatrix} T_0^{-1}\mathbf{g}(\theta) \\ &= \cos(\frac{\pi t}{2\tau})T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_{m_0}} \end{pmatrix} T_0^{-1}\mathbf{g}(0) \\ &- \sin(\frac{\pi t}{2\tau})T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_{m_0}} \end{pmatrix} T_0^{-1}\mathbf{g}(-\tau). \end{aligned}$$

Using the similar denotations in Subsection 4.2, and let $\mathbf{S}_p^*(t)$ be a fundamental solution operator of the equation

$$\dot{\mathbf{u}}^* = -\lambda\bar{\mathbf{u}}^*(t) + \lambda J_p^*\bar{\mathbf{u}}^*(t).$$

Thus

$$T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t)I_{p_{m_0}} \end{pmatrix} T_0^{-1}\mathbf{g}(\theta)$$

is a fundamental solution of the homogeneous system (4.1). By using the variation-of-constant formula, we see that the general solution of (4.18) is given as

$$\begin{aligned} \mathbf{V}(t+\theta) &= T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t)I_{p_{m_0}} \end{pmatrix} T_0^{-1}\mathbf{g}(\theta) \\ &+ \lambda \int_0^t T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t-s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t-s)I_{p_{m_0}} \end{pmatrix} T_0^{-1}(\bar{P}(s) - P_0)\bar{\mathbf{V}}(s)ds. \end{aligned}$$

Also, taking

$$\begin{aligned} \tilde{\mathbf{V}}(t+\theta) &= T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t)I_{p_{m_0}} \end{pmatrix} T_0^{-1}\mathbf{g}(\theta) \\ &+ \lambda \int_0^t T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t-s)I_{p_{m_0}} \end{pmatrix} T_0^{-1}(\bar{P}(s) - P_0)\bar{\mathbf{V}}(s)ds, \end{aligned}$$

and using the fact

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t-s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}(\bar{P}(s) - P_0)\tilde{\mathbf{V}}(s) = \mathbf{0},$$

we have

$$\begin{aligned} \mathbf{V}(t+\theta) - \tilde{\mathbf{V}}(t+\theta) &= T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{g}(\theta) \\ &+ \lambda \int_0^t \mathbf{T}_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t-s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}(\bar{P}(s) - P_0)\bar{\mathbf{V}}(s)ds \\ &= T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{g}(\theta) \\ &+ \lambda \int_0^t T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t-s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}(\bar{P}(s) - P_0)(\bar{\mathbf{V}}(s) - \tilde{\mathbf{V}}(s))ds. \end{aligned}$$

And from Lemma 2.1, there are constants $K > 0$ and $c \in (\lambda\eta DK, -c_1)$ such that

$$\|\mathbf{S}_p^*(t)\| \le Ke^{-ct}.$$

Thus

$$\begin{aligned} & \sup_{\theta\in[-\tau,0]} \|\mathbf{V}(t+\theta) - \tilde{\mathbf{V}}(t+\theta)\| \\ \le\ & DKg_0e^{-ct} + \lambda DK \int_0^t e^{-c(t-s)}\|\bar{P}(s) - P_0\| \sup_{\theta\in[-\tau,0]} \|\mathbf{V}(s+\theta) - \tilde{\mathbf{V}}(s+\theta)\|ds \\ \le\ & DKg_0e^{-ct} + \lambda\eta DK \int_0^t e^{-c(t-s)} \sup_{\theta\in[-\tau,0]} \|\mathbf{V}(s+\theta) - \tilde{\mathbf{V}}(s+\theta)\|ds. \end{aligned}$$

Therefore

$$\begin{aligned} & e^{ct} \sup_{\theta\in[-\tau,0]} \|\mathbf{V}(t+\theta) - \tilde{\mathbf{V}}(t+\theta)\| \\ \le\ & DKg_0 + \lambda\eta DK \int_0^t e^{cs} \sup_{\theta\in[-\tau,0]} \|\mathbf{V}(s+\theta) - \tilde{\mathbf{V}}(s+\theta)\|ds. \end{aligned}$$

By solving the above Gronwall inequality, we get

$$\sup_{\theta\in[-\tau,0]} \|\mathbf{V}(t+\theta) - \tilde{\mathbf{V}}(t+\theta)\| \ \le\ KDg_0e^{-(c-\lambda\eta DK)t}. \tag{4.24}$$

This implies that

$$\|\mathbf{V}(t) - \tilde{\mathbf{V}}(t)\| \ \le\ KDg_0e^{-(c-\lambda\eta DK)t}.$$

Recalling the inequality $\lambda\eta DK < c$, we find that there is a positive integral number k_2 such that

$$\sqrt{2n}KDg_0e^{-(c-\lambda\eta DK)k_2\delta} < (c-\lambda\eta DK)\gamma,$$

where δ and γ given in Assumption (A_1). With the similar arguments in the proof of Theorem 3.3, we conclude that $t_{k_2+1} = \infty$.

From Theorem 3.2, we have

$$\lim_{t\to\infty}\left[T_0\begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t)I_{p_{m_0}} \end{pmatrix} T_0^{-1}\mathbf{g}(\theta) - \mathbf{V}_p(t)\right] = \mathbf{V}_\infty$$

uniformly for $\theta \in [-\tau, 0]$. Also, for the solution operator $\|S_0(t-s)\|$ is bounded, say M_0, we see that

$$\begin{aligned}
& \|\int_0^t T_0\begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t-s)I_{p_{m_0}} \end{pmatrix} T_0^{-1}(\bar{P}(s)-P_0)\bar{\mathbf{V}}(s)ds\| \\
= & \|\int_0^t T_0\begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t-s)I_{p_{m_0}} \end{pmatrix} T_0^{-1}(\bar{P}(s)-P_0)(\bar{\mathbf{V}}(s)-\tilde{\mathbf{V}}(s))ds\| \\
\le & \eta KM_0\frac{N_{max}}{N_{min}}g_0\int_0^t e^{-(c-\lambda\eta DK)s}ds < \frac{\eta KM_0\frac{N_{max}}{N_{min}}g_0}{c-\lambda\eta DK} < \infty.
\end{aligned}$$

So the limit

$$\lim_{t\to\infty}\int_0^t T_0\begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t-s)I_{p_{m_0}} \end{pmatrix} T_0^{-1}(\bar{P}(s)-P_0)\bar{\mathbf{V}}(s)ds \tag{4.25}$$

exists, say $\mathbf{W}_{p\infty}$. Thus

$$\lim_{t\to\infty}(\tilde{\mathbf{V}}(t)-\mathbf{V}_p(t)) = \mathbf{V}_\infty + \lambda\mathbf{W}_{p\infty}.$$

Combining the above inequality and (4.24), we have

$$\lim_{t\to\infty}(\mathbf{V}(t)-\mathbf{V}_p(t)) = \mathbf{V}_\infty + \lambda\mathbf{W}_{p\infty}.$$

With the similar arguments in the proof of Theorem 3.2 and discussion, we confirm that the system achieves periodic flock when $n_0 = 1$, and achieves a periodic flock or periodic multi-cluster when $n_0 > 1$. This completes the proof. □

5 Numerical Simulations and Final Remarks

For simplicity, we consider a 7-particle system for simulating the dynamics of (1.1), and the initial values of position and velocity produced randomly, reading as in Table 1. Let $g_i(\theta) = x_i(0)$ and $f_i(\theta) = v_i(0)$ for $\theta \in [-0.5, 0]$.

Case 1: $\lambda = 2.6, \tau = 0.5$ and $r = 5.5$. In this case, the values of parameters as follows.

$$P_0 = \frac{1}{7}\mathbf{1}_{7\times 7}, \mu_1 = 1, \mu_i = 0, (i = 2, \cdots, 7).$$

Tab 1: Initial values of velocity and position

$N.O.$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
$x_i(0)$	1.9203	3.4998	1.7065	4.8135	2.6083	1.2034	3.8448
$v_i(0)$	2.6665	0.3048	0.1959	0.7029	2.7993	0.1894	0.7927

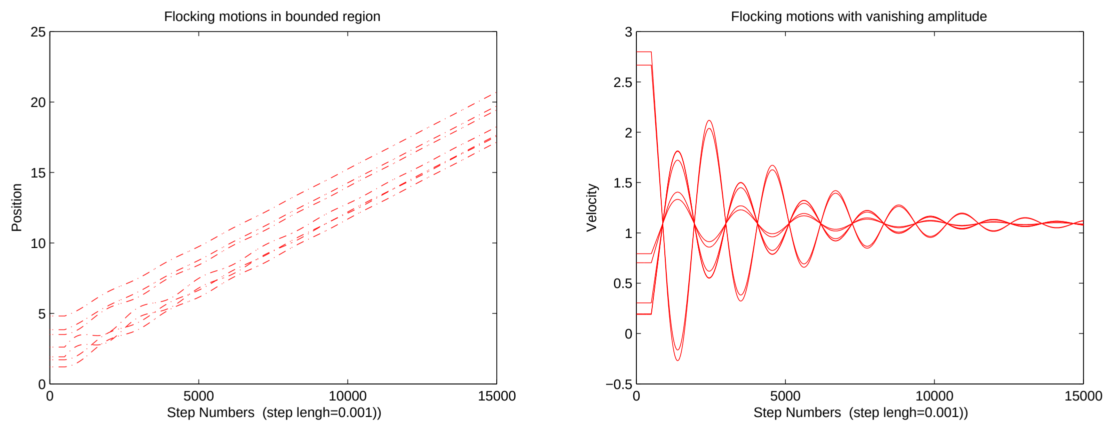

Fig. 1: $\lambda = 2.6, \tau = 0.5, r = 5.5$. The system achieves a flocking motion with a final velocity $v_\infty = 1.0931$ and bounded relative position .

The initial average matrix keeps unchanged, and then the system achieves a flocking motion (See Figure 1). The final velocity value is the average of initial data, saying $\frac{\sum_{i=1}^{7} v_i(0)}{7} = 1.0931$.

Case 2: $\lambda = 2.6, \tau = 0.5, r = 0.65$. In this case, the size of neighbourhood is small. It makes the system undergoing a multi-cluster flocking motion (See Figure 2).

Case 3: $\lambda = \pi, \tau = 0.5, r = 5.5$. In this case, $\lambda\tau = \frac{\pi}{2}$. The system achieves a periodic flock with the period $\frac{2\pi}{\lambda} = 2$ (See Figure 3).

5.1 Final Remarks

About the particle model (1.1), we studied the effects of discrete processing delay on the qualitative dynamics. In reality, there would be two general cases to be involved: variable communication weight and distributed processing delay.

For the variable communication weight case, weight coefficients are given as

$$\chi_r(s) = \begin{cases} a_{ij}, & \text{if } 0 \le s < r, \\ 0, & \text{if } s \ge r, \end{cases}$$

where a_{ij} is selected from the candidate symmetric connected matrix $A = (a_{ij})_{N\times N}$ with $a_{ij} \ge 0$. It would make the designed system achieving a perfect convergent performance when r is large enough. Usually, the balance of the system will be broken for the in-degrees are different. Mathematically, the in-degree of ith node is given as $d_i = \sum_{j\in \mathbf{N}_i(t)} a_{i,j}$. To reestablish the balance, let $C = 1 + \max\{d_i : i = 1, 2, \cdots, N\}$ and reset the coefficients to define the adjacency matrix and average matrix of the sytem by $\tilde{A}(t) = (\tilde{a}_{ij}(t))_{N\times N}$ and

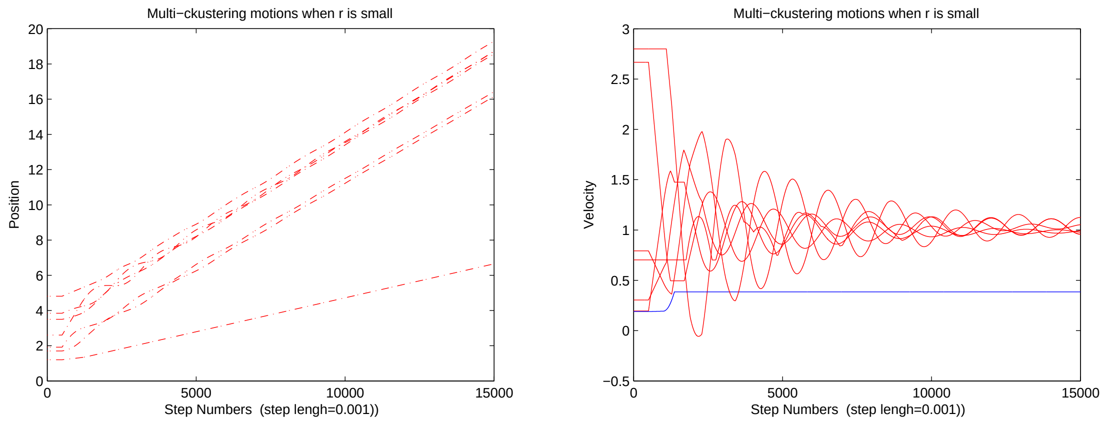

Fig. 2: $\lambda = 2.6; \tau = 0.5, r = 0.65$. The system achieves a two-cluster flocking motion, and the single particle 6 is one cluster, the others concentrate to another cluster.

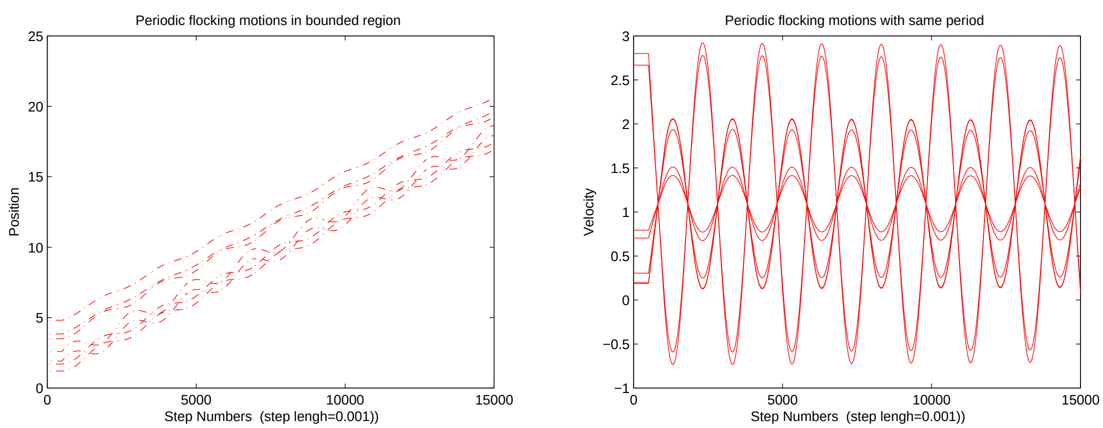

Fig. 3: $\lambda = \pi, \tau = 0.5, r = 5.5$. The system achieves a periodic flock with the period $\frac{2\pi}{\lambda} = 2$.

$P(t) = (p_{ij}(t))_{N\times N}$, respectively, where

$$\tilde{a}_{ij}(t) = \begin{cases} \frac{a_{ij}}{N_i(t)}, & j \in \mathbb{N}_i(t), j \neq i, \\ C - \sum_{j\neq i} \frac{a_{ij}}{N_i(t)}, & i = j, \\ 0, & otherwise, \end{cases} \qquad \text{and} \quad p_{ij}(t) = \frac{\tilde{a}_{ij}(t)}{C}.$$

Then $\sum_{j\in \mathbf{N}_i(t)} \tilde{a}_{i,j} = C$ for all i and $P(t)$ is a stochastic matrix. In this case, let $\tilde{\lambda} = \lambda C$, the vector form of system (1.1) becomes

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{V}(t), \\ \dot{\mathbf{V}}(t) = -\tilde{\lambda}\bar{\mathbf{V}}(t) + \tilde{\lambda} P_0 \bar{\mathbf{V}}(t), \quad t \in [0, t_1). \\ \mathbf{X}(t) = \mathbf{f}(t), \mathbf{V}(t) = \mathbf{g}(t), t \in [-\tau, 0]. \end{cases} \tag{5.1}$$

Thus the similar discussions will yield the corresponding results in Section 3.

For the distributed processing delay case, the effects of time lags work on the whole interval $[-\tau, 0]$, and not at a point $t = -\tau$, where τ denotes the maximum processing delay. To quantize this comprehensive interaction, a distributed function would be involved. Thus, the term $\bar{v}_j(t)$ in system (1.1) is formulated as

$$\bar{v}_i(t) = \int_{-\tau}^{0} \varphi(s) v_i(t+s) ds,$$

where φ is a (positive) normalized distributed function so that $\int_{-\tau}^{0} \varphi(s)ds = 1$. Also, $\bar{v}_j(t)$, $\bar{x}_i(t)$ and $\bar{x}_j(t)$ are similar defined. In this case, to discuss the dynamical behaviors, an additional preliminary is to investigate the roots distribution of the corresponding characteristic equation. And then the corresponding results in Section 3 would be established similarly. There are no more discussions to the details in current work.

Appendix

Lemma 5.1 *The matrix $P(t)$ defined by (2.1) is a diagonalizable matrix and all the eigenvalues are real.*

Proof. Here, we only show that P_0 is a diagonalizable matrix and all the eigenvalues are real. From the matrix theory [30], we see that there is a non-degenerate matrix T_0 such that $P_0 = T_0 J_0 T_0^{-1}$, where J_0 is a Jordan matrix with the first block being 1, say $J_0 = \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & J^* \end{pmatrix}$. Inspired by [20], we reset $p_{ij} = \frac{a_{ij}}{N_i} = \frac{1}{\sqrt{N_i}} \frac{a_{ij}}{\sqrt{N_i N_j}} \sqrt{N_j}$ and $M_0 = (\frac{a_{ij}}{\sqrt{N_i N_j}})_{N\times N}$, then

$$P_0 = diag\{\frac{1}{\sqrt{N_1}}, \frac{1}{\sqrt{N_2}}, \cdots, \frac{1}{\sqrt{N_N}}\} M_0 diag\{\sqrt{N_1}, \sqrt{N_2}, \cdots, \sqrt{N_N}\}.$$

Then M_0 is symmetric matrix and P_0 is similar to M_0. Also, there are an orthogonal matrix O and a diagonal matrix $\tilde{J}_0$ such that $M_0 = O\tilde{J}_0 O^{-1}$. Thus

$$\begin{aligned} P_0 &= T_0 J_0 T_0^{-1} \\ &= diag\{\frac{1}{\sqrt{N_1(0)}}, \cdots, \frac{1}{\sqrt{N_N(0)}}\} M_0 diag\{\sqrt{N_1(0)}, \cdots, \sqrt{N_N(0)}\} \\ &= diag\{\frac{1}{\sqrt{N_1(0)}}, \cdots, \frac{1}{\sqrt{N_N(0)}}\} O\tilde{J}_0 O^{-1} diag\{\sqrt{N_1(0)}, \cdots, \sqrt{N_N(0)}\}. \end{aligned}$$

Since all the diagonal elements of $\tilde{J}_0$ and J_0 are same, Therefore, $\tilde{J}_0 = J_0$ and P_0 is a diagonalizable matrix and all eigenvalues of matrix P_0 are real. The proof of Lemma 5.1 is complete.

Lemma 5.2 *Let $\lambda > 0$. If $0 \leq \lambda\tau(1-\mu_{m_0}) < \frac{\pi}{2}$ and μ_i $(i = 2, \cdots, m_0)$ are eigenvalues of P_0, then all other roots of the equation $z = -\lambda(1-\mu_i)e^{-z\tau}$ have negative real parts and*

$$c_1 := \max_{2\leq i\leq m_0} \sup\{Re(z) : z = -\lambda(1-\mu_i)e^{-z\tau}\} < 0.$$

Proof Since

$$\lambda\tau(1-\mu_i) \leq \lambda\tau(1-\mu_{m_0}) < \frac{\pi}{2}$$

holds for $i = 2, \cdots, m_0$, from Lemma 2.2, we conclude that all other roots of the equation $z = -\lambda(1-\mu_i)e^{-z\tau}$ have negative real parts.

Noting that the set $\{Re(z) : z = -\lambda(1-\mu_i)e^{-z\tau}\}$ is up-bounded, and from above arguments, we see that the supremum

$$c_1 := \sup\{Re(z) : z = -\lambda(1-\mu_i)e^{-z\tau}\} \leq 0.$$

Assume that $c_1 = 0$, then there is a sequence $\{z_n\}$ $(z_n = x_n + iy_n)$ with

$$\lim_{n\to\infty} x_n = 0 \text{ and } \ z_n = -\lambda(1-\mu_i)e^{-z_n\tau}. \tag{5.2}$$

Thus

$$x_n = -\lambda(1-\mu_i)e^{-x_n\tau}\cos(y_n\tau) \ \text{ and } \ y_n = \lambda(1-\mu_i)e^{-x_n\tau}\sin(y_n\tau).$$

For $\lim_{n\to\infty} x_n = 0$, we see that $\lim_{n\to\infty}\cos(y_n\tau) = 0$. Therefore $\lim_{n\to\infty}\sin(y_n\tau) = 1$, and then $\lim_{n\to\infty} y_n = \lambda(1-\mu_i) < \frac{\pi}{2\tau}$. Thus $\lim_{n\to\infty}\sin(y_n\tau) = \sin(\lambda(1-\mu_i)\tau) < 1$. It contradicts with $\lim_{n\to\infty}\sin(y_n\tau) = 1$. Thus $c_1 < 0$. This completes the proof.

□

Lemma 5.3 *If* 1 *is a semisimple eigenvalue of the stochastic matrix P_0 with multiplicity n_0, then P_0 is a block diagonal matrix (if necessary, we exchange the rows of matrix P_0 and renumber the subscript).*

Proof. (The more details refer to [24]) Without loss of generality, we assume the zero-one vectors $\mathbf{a}_1, \cdots, \mathbf{a}_{n_0}$ are of the following forms (if necessary, we exchange the rows of matrix P_0 and relabel the subscript of v_i):

$$\begin{aligned}
\mathbf{a}_1 &= (\overbrace{1,\cdots,1}^{r_1}, 0, \cdots, 0)^T;\\
\mathbf{a}_2 &= (\overbrace{0,\cdots,0}^{r_1}, \overbrace{1,\cdots,1}^{r_2}, 0, \cdots, 0)^T;\\
&\cdots,\cdots\\
\mathbf{a}_{n_0} &= (0,\cdots,0, \overbrace{1,\cdots,1}^{r_{n_0}})^T.
\end{aligned}$$

Also, we see that r_i is the minimum number of components 1 in $\mathbf{a}_i$ and $r_1+r_2+\cdots+r_{n_0} = N$. In this case, noting that $P_0\mathbf{a}_i = \mathbf{a}_i (i = 1, \cdots, n_0)$ and $p_{ij} \geq 0$, we see that P_0 is a block diagonal matrix, that is,

$$P_0 = diag(D_1, D_2, \cdots, D_{n_0}).$$

This complete the proof.

References

[1] F. M. Atay, *The consensus problem in networks with transmission delays.* Phil. Trans. Roy. Soc. A, 371 (2013), 20120460.

[2] N. Bellomo, P. Degond, E. Tadmor (Eds.), *Active Particles Vol. I: Advances in Theory, Models, and Applications, in: Modelling and Simulation in Science and Technology*, Birkhäuser Basel, 2017.

[3] Y. Cheng, X. Wang and B. Shi, *Consensus of a two-agent opinion dynamical system with processing delay,* International Journal of Biomathematics, 11(6)(2018), 1850081.

[4] Y. Choi, J. Haskovec, *Cucker-Smale model with normalized communication weights and time delay*, Kinet. Relat. Models, **10** (2017), 1011-1033.

[5] Y. Choi, Z. Li, *Emergent behavior of Cucker-Smale flocking particles with heterogeneous time delays*, Appl. Math. Lett., **86** (2018), 49-56.

[6] P. Cristina R. V. Irene, *Flocking estimates for the Cucker-Smale model with time lag and hierarchical leadership*, J. Math. Anal. Appl., 464(2)(2018) 1313-1332.

[7] F. Cucker, S. Smale, *Emergent behavior in flocks*, IEEE Trans. Automat. Control, **52** (2007), 852-862.

[8] F. Cucker, S. Smale, *On the mathematics of emergence*, Jpn. J. Math., **2** (2007), 197-227.

[9] F. Cucker, J. Dong, *A general collision-avoiding flocking framework*, IEEE Trans. Automat. Control, **56** (2011), 1124-1129.

[10] J. Dong, S. -Y. Ha, D. Kim, J. Kim, *Time-delay effect on the flocking in an ensemble of thermomechanical Cucker-Smale particles*, J. Differential Equations, https://doi.org/10.1016/j.jde.2018.08.034

[11] J. Dong, S. -Y. Ha,, J. Jung, D. Kim, *On the stochastic flocking of the Cucker-Smale flock with randomly switching topologies*, arXiv:1911.07390v1(2019).

[12] R. Erban, J. Haskovec, Y. Sun, *On Cucker-Smale model with noise and delay,* SIAM J. Appl. Math., **76** (2016), 1535-1557.

[13] S.-Y. Ha, J. Liu, *A simple proof of Cucker-Smale flocking dynamics and mean field limit*, Commun. Math. Sci., **7** (2009), 297-325.

[14] S.-Y. Ha, K. Lee, D. Levy, *Emergence of time-asymptotic flocking in a stochastic Cucker-Smale system*, Commun. Math. Sci., **7** (2009), 453-469.

[15] J.K. Hale, and S.M.V. Lunel, *Introduction to Functional Differential Equations.* Springer-Verlag, Berlin,1993.

[16] J. Haskovec, I. Markou, *Delay Cucker-Smale model with and without noise revisited*, arXiv:1810.01084v2.

[17] R.A. Horn, C.R. Johnson. *Matrix analysis.* Cambridge, UK: Cambridge University Press, 1985.

[18] P.-E. Jabin and S.Motsch, *Clustering and asymptotic behavior in opinion formation.* J. Differ. Equ., 257(11)(2014), 4165-4187.

[19] A. Jadbabaie and J. Lin, *Coordination of groups of mobile autonomous agents using nearest neighbour rules.* IEEE Trans. Automat. Control, 48(6)(2003), 988-1001.

[20] C. Jin , *Flocking of the Motsch-Tadmor model with a cut-Off interaction function* , J. Stat. Phys., 171(2018), 345-360.

[21] L. Li, W. Wang, L. Huang , J. Wu, *Some weak flocking models and its application to target tracking*, J. Math. Anal. Appl., 480 (2019),123440.

[22] Z. Li, X. Xue, *Cucker-Smale flocking under rooted leadership with fixed and switching topologies*, SIAM J. Appl. Math., **70** (2010), 3156-3174.

[23] Y. Liu, J. Wu, *Flocking and asymptotic velocity of the Cucker-Smale model with processing delay*, J. Math. Anal. Appl., 415 (2014), 53-61.

[24] Y. Liu, J. Wu, *Local phase synchronization and clustering for the delayed phase-coupled oscillators with plastic coupling*, J. Math. Anal. Appl., 444 (2016), 947-956.

[25] Y. Liu, J. Wu, *Opinion consensus with delay when the zero eigenvalue of the connection matrix is semi-simple.* J. Dyn. Diff. Equns., 29(4)(2016),1-13.

[26] S. Motsch, E.Tadmor, *A new model for self-organized dynamics and its flocking behavior.* J. Stat. Phys., 144(5)(2011), 923-947

[27] C. Pignotti, E. Trelat, *Convergence to consensus of the general finite-dimensional Cucker-Smale model with time-varying delays*, Commun. Math. Sci., 16(8)(2017), DOI: 10.4310/CMS.2018.v16.n8.a1.

[28] C. Pignotti, I. Vallejo, *Flocking estimates for the Cucker-Smale model with time lag and hierarchical leadership*, J. Math. Anal. Appl., **464** (2018), 1313-1332.

[29] L. Ru, X. Xue, *Multi-cluster flocking behavior of the hierarchical CuckerCSmale model*, J. Franklin Inst., **354** (2017), 2371-2392.

[30] D. Serre, *Matrices.* Graduate Texts in Mathematics 216. Springer (2010)

[31] L. Shi, Y. Cheng, J. Huang, J. Shao, *CuckerCSmale flocking under rooted leadership and time-varying heterogeneous delays*, Appl. Math. Lett., 98 (2019) 453-460.

[32] J. Shen, *Cucker-Smale flocking under hierarchical leadership*, SIAM J. Appl. Math., **68** (2007), 694-719.

[33] T. Vicsek, A. Zafeiris, *Collective motion*, Physics Reports, 2012.

[34] X. Wang, L. Wang, J. Wu, *Impacts of time delay on flocking dynamics of a two-agent flock model*, Commun. Nonlinear Sci. Numer. Simul., **70** (2019), 80-88.