PIEZOELECTRIC BEAMS WITH MAGNETIC EFFECT AND NONLINEAR DAMPING: GENERAL STABILITY

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Abstract. A coupled model of piezoelectric beams with magnetic effect and nonlinear damping is considered. Assuming adequate hypotheses on nonlinear damping, uniform decay rates for the solution is established.

Keywords: Piezoelectric beams; energy decay; nonlinear damping.
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1. Introduction

Piezoelectric materials are known to have the valuable property of converting mechanical energy into electromagnetic energy and vice versa (see [6] for details). In this direction, piezoelectric actuators are generally scalable, smaller, economically cheaper and more efficient than traditional actuators and therefore a competitive choice for many tasks, particularly the control of flexible structures. They are used in civil engineering, industrial, automotive, aeronautical and space structures.

Taking into account these important applications in 2013, Morris and Özer [8], used a variational approach to construct a coupled model of piezoelectric beams with magnetic effect given by

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, & \text{in} \quad (0,L)\times(0,\infty), \\ \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & \text{in} \quad (0,L)\times(0,\infty). \end{cases} \tag{1}$$

To the above system they added boundary conditions

$$\begin{cases} v(0,t) = p(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & \forall t \geq 0, \\ p_x(L,t) - \gamma v_x(L,t) = -\frac{V(t)}{h}, & \forall t \geq 0 \end{cases} \tag{2}$$

and initial conditions

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ p(x,0) = p_0(x),\ p_t(x,0) = p_1(x) \quad \text{in} \quad (0,L), \tag{3}$$

where the functions v and p are used to denote the transverse displacement of the beam and the total load of the electric displacement along the transverse direction at each point x respectively. Here ρ, α, γ, β, μ, L, h and V denote mass density per unit volume, elastic stiffness, piezoelectric coefficient, magnetic permeability, impermittivity, length, thickness and the prescribed voltage on electrodes of beam respectively. In addition, the following relationship was considered

$$\alpha = \alpha_1 + \gamma^2\beta. \tag{4}$$

Remark 1.1. *It is important to note that since*

$$p(x,t)=\int_0^x D(\xi,t)d\xi,$$

where $D(x,t)$ represents the electric displacement in the direction z, then $p(0,t)=0$ and still $p(L,t)=\int_0^L D(\xi,t)d\xi$ may not be zero, because the boundary condition $p(L,t)=0$ does not represent the fixation of the beam on both sides. In fact, the fixation is due to the boundary condition $v(0,t)=v(L,t)=0$, where v is the transverse displacement of the beam.

In [9] Morris and Özer, taking $V(t)=p_t(L,t)$ in (2), proved that the magnetic effect, despite being relatively small, has a strong interference in stabilizing and controlling the obtained system. Despite being strongly stable, it is not exponentially stable for almost all system parameters, unlike the classical model, consisting of a single wave equation as studied in [4, 5, 10].

As shown in [9], the model with the magnetic effects proved not to be exactly observable/stabilizable exponentially in the energy space for almost all material parameter choices. In addition, even strong stability is not achievable for many material parameter values.

It is well known that a single piezoelectric beam model without the magnetic effects is known to be exactly observable and exponentially stabilized in the energy space. Thus, taking into account this fact, in [2] Ramos et. al. considered $V(t)=0$ and a damping term v_t and proved that the system is exponentially stable in the energy space.

In this paper, we consider the following piezoelectric beams with magnetic effect and nonlinear damping

$$\begin{cases} \rho v_{tt}-\alpha v_{xx}+\gamma\beta p_{xx}+a\,h\,(v_t)=0, & in \quad (0,L)\times(0,\infty), \\ \\ \mu p_{tt}-\beta p_{xx}+\gamma\beta v_{xx}=0, & in \quad (0,L)\times(0,\infty), \end{cases} \tag{5}$$

where $a>0$. The system (5), we consider the initial conditions given by

$$v(x,0)=v_0(x),\ v_t(x,0)=v_1(x),\ p(x,0)=p_0(x),\ p_t(x,0)=p_1(x) \quad in \quad (0,L), \tag{6}$$

and boundary conditions given by

$$\begin{cases} v(0,t)=\alpha v_x(L,t)-\gamma\beta p_x(L,t)=0, & \forall t\geq 0, \\ p(0,t)=p_x(L,t)-\gamma v_x(L,t)=0, & \forall t\geq 0. \end{cases} \tag{7}$$

On the nonlinear term h, we assume, as in Lasiecka and Tataru ([10]), that it satisfies the following hypotheses:

$(A1)$ $h:\mathbb{R}\longrightarrow\mathbb{R}$ is a non-decreasing C^0-function such that there exist positive constants c_1,c_2,ε, and a strictly increasing function $H\in C^1([0,+\infty))$, with $H(0)=0$, and H is linear or strictly convex C^2-function on $(0,\varepsilon]$ such that

$$\begin{cases} y^2+h^2(y)\leq H^{-1}(y\,h(y))\,;\ \text{for all } |y|\leq\varepsilon, \\ c_1|y|\leq|h(y)|\leq c_2|y|\,;\ \text{for all } |y|\geq\varepsilon. \end{cases}$$

Remark 1.2 *Due to the relation (4), the boundary condition (7) is equivalent to*

$$\begin{cases} v(0,t)=v_x(L,t)=0, & \forall t>0, \\ p(0,t)=p_x(L,t)=0, & \forall t>0. \end{cases} \tag{8}$$

Hypothesis $(A1)$ implies that $y\,h(y)>0$, for all $y\neq 0$.

With motivations coming from various physical applications, our goal in this paper is to prove a general stability result for the system (5)-(6) with boundary conditions (8). This result is new and generalizes the result obtained in [2]. The corresponding Lyapunov functional in this paper is easier than the one constructed with the linear damping. In Section 2, using the energy method we prove the general decay.

2. General stability result

In this section, we state and prove a general stability result for (5) with the initial and boundary conditions given by (6) and (8), respectively.

We state without proof a global existence result. Throughout this paper, C is used to denote a generic positive constant, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the $L^2(0,L)$ norm and inner product. Our objective is to prove the general decay using the energy method. The well-posedness of (5) is stated in the following proposition.

Proposition 2.1 *Assume that* $(A1)$ *is satisfied, then for all initial data* $((v_0,v_1),(p_0,p_1)) \in \left(\left(H^2(0,L)\cap H^1_*(0,L)\right)\times H^1_*(0,L)\right)^2$, *the system (5) has a unique solution* (v,p) *that verifies*

$$(v,p)(t) \in C\left(\mathbb{R}_+;\left(H^2(0,L)\cap H^1_*(0,L)\right)^2\right)\cap C^1\left(\mathbb{R}_+;\left(H^1_*(0,L)\right)^2\right),$$

where

$$H^1_*(0,L) = \left\{w \in H^1(0,L) : w(0) = 0\right\}.$$

Proof. This proposition can be proved using the nonlinear semigroup method as in [3]. □

The first-order energy associated with (5) is given by:

$$E(t) = \frac{1}{2}\left(\rho\|v_t\|^2 + \alpha\|v_x\|^2 + \mu\|p_t\|^2 + \beta\|p_x\|^2\right) - \gamma\beta\,\mathrm{Re}\langle v_x,p_x\rangle. \tag{1}$$

Theorem 2.2 *Let* $((v_0,v_1),(p_0,p_1)) \in \left(\left(H^2(0,L)\cap H^1_*(0,L)\right)\times H^1_*(0,L)\right)^2$ *be given and assume that* $(A1)$ *is satisfied. Then, there exist three positive constants* a_1,a_2,a_3 *and* ϵ_0 *such that*

$$E(t) \le a_1 H_3^{-1}(a_2 t + a_3);\ t \ge 0, \tag{2}$$

where

$$H_3(t) = \int_t^1 \frac{1}{H_2(y)}dy \quad\text{and}\quad H_2(t) = t\,H'(\epsilon_0 t).$$

Here H_3 *is a strictly decreasing and convex function on* $(0,1]$, *with* $\lim_{t\to 0} H_3(t) = +\infty$.

The proof of this Theorem will be established through several lemmas.

Lemma 2.3 Let (v,p) *be the solution of (5),* then the energy functional E, defined by (1) satisfies

$$E'(t) = -a\,\langle h(v_t),v_t\rangle \le 0. \tag{3}$$

Proof. Multiplying the first equation of (5) by v_t , the second by p_t, integrating by parts over $(0,L)$ and using the boundary conditions (8), then summing up, we obtain the result. □

Lemma 2.4 Let (v,p) *be the solution of (5). Then the functional*

$$I_1(t) = \langle v, v_t\rangle + \frac{\gamma\mu}{\rho}\left\langle v_x, \int_x^L p_t(y)\,dy\right\rangle, \tag{4}$$

satisfies, for all $\varepsilon_1 > 0$*, the following estimate*

$$I_1'(t) \leq -\frac{\alpha_1}{2\rho}\|v_x\|^2 + \varepsilon_1\|p_t\|^2 + C\int_0^L \left(v_t^2 + h^2(v_t)\right)dx. \tag{5}$$

Proof. sing $(5)_1$, $(5)_2$, integrating by parts over $(0;L)$ and using the boundary conditions (8), we obtain

$$\begin{aligned} I_1'(t) &= \|v_t\|^2 + \langle v, v_{tt}\rangle + \frac{\gamma\mu}{\rho}\left\langle v_{tx}, \int_x^L p_t(y)\,dy\right\rangle + \frac{\gamma\mu}{\rho}\left\langle v_x, \int_x^L p_{tt}(y)\,dy\right\rangle \\ &= \|v_t\|^2 - \frac{\alpha_1}{\rho}\|v_x\|^2 - \frac{a}{\rho}\langle v, h(v_t)\rangle - \frac{\gamma\mu}{\rho}\langle v_t, p_t\rangle. \end{aligned} \tag{6}$$

By using Young's, Cauchy–Schwarz and Poincaré's inequalities, we have

$$\begin{cases} \begin{cases} \left|\dfrac{a}{\rho}\langle v, h(v_t)\rangle\right| \leq \dfrac{\alpha_1}{2\rho C_p}\|v\|^2 + C\|h(v_t)\|^2 \\ \qquad\qquad\qquad\quad \leq \dfrac{\alpha_1}{2\rho}\|v_x\|^2 + C\|h(v_t)\|^2, \end{cases} \\ \dfrac{\gamma\mu}{\rho}|\langle v_t, p_t\rangle| \leq \varepsilon_1\|p_t\|^2 + C\|v_t\|^2, \end{cases} \tag{7}$$

by (6) and (7) and assume that $(A1)$ is satisfied then we deduce the result. □

Lemma 2.5 Let (v,p) *be the solution of (5). Then the functional*

$$I_2(t) = -\langle v_t, p\rangle - \frac{\gamma\mu}{2\rho}\langle p, p_t\rangle, \tag{8}$$

satisfies the following estimate

$$I_2'(t) \leq -\frac{\gamma\mu}{4\rho}\|p_t\|^2 - \frac{\gamma\beta}{4\rho}\|p_x\|^2 + C\|v_x\|^2 + C\int_0^L \left(v_t^2 + h^2(v_t)\right)dx. \tag{9}$$

Proof. Using $(5)_1$, $(5)_2$,integrating by parts over $(0;L)$ and using the boundary conditions (8), we obtain

$$\begin{aligned} I_2'(t) &= -\langle v_{tt}, p\rangle - \langle v_t, p_t\rangle - \frac{\gamma\mu}{2\rho}\|p_t\|^2 - \frac{\gamma\mu}{2\rho}\langle p, p_{tt}\rangle \\ &= -\frac{\gamma\mu}{2\rho}\|p_t\|^2 - \frac{\gamma\beta}{2\rho}\|p_x\|^2 + \frac{[\alpha+\alpha_1]}{2\rho}\langle v_x, p_x\rangle \\ &\quad + \frac{a}{\rho}\langle h(v_t), p\rangle - \langle v_t, p_t\rangle. \end{aligned} \tag{10}$$

By using Young's, Cauchy–Schwarz and Poincaré's inequalities, we have

$$\begin{cases} \dfrac{[\alpha+\alpha_1]}{2\rho}\left|\langle v_x,p_x\rangle\right| \le \dfrac{\gamma\beta}{8\rho}\left\|p_x\right\|^2 + C\left\|v_x\right\|^2, \\ \left|\langle v_t,p_t\rangle\right| \le \dfrac{\gamma\mu}{4\rho}\left\|p_t\right\|^2 + C\left\|v_t\right\|^2, \\ \\ \begin{cases} \dfrac{a}{\rho}\left|\langle\, h\left(v_t\right),p\rangle\right| \le \dfrac{\gamma\beta}{8\rho C_p}\left\|p\right\|^2 + C\left\|h\left(v_t\right)\right\|^2 \\ \qquad\qquad\qquad\quad \le \dfrac{\gamma\beta}{8\rho}\left\|p_x\right\|^2 + C\left\|h\left(v_t\right)\right\|^2, \end{cases} \end{cases} \tag{11}$$

by (10) and (11) and assume that $(A1)$ is satisfied then we deduce the result. □

To complete the proof of **Theorem 2.2**, we introduce the following Lyapunov functional $\mathcal{L}$

$$\mathcal{L}\left(t\right) = N\,E\left(t\right) + N_1 I_1\left(t\right) + I_2\left(t\right), \tag{12}$$

where N and $N_1$are positive constants to be fixed later.

Lemma 2.6 For N large enough, there exist two positive constants δ_1 and δ_2 such that

$$\delta_1 E\left(t\right) \le \mathcal{L}\left(t\right) \le \delta_2 E\left(t\right). \tag{13}$$

Proof. Let's define the following functional

$$\mathcal{L}_1\left(t\right) = N_1 I_1\left(t\right) + I_2\left(t\right).$$

Using Cauchy–Schwarz,Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} \frac{\gamma\mu}{\rho}N_1\left|\left\langle v_x,\int_x^L p_t\left(y\right)dy\right\rangle\right| &\le \frac{\gamma\mu}{2\rho}N_1\left\|v_x\right\|^2 + \frac{\gamma\mu L^2}{2\rho}N_1\left\|p_t\right\|^2, \\ N_1\left|\langle v,v_t\rangle\right| &\le \frac{N_1 C_p}{2}\left\|v_x\right\|^2 + \frac{N_1}{2}\left\|v_t\right\|^2, \\ \left|\langle v_t,p\rangle\right| &\le \frac{1}{2}\left\|v_t\right\|^2 + \frac{C_p}{2}\left\|p_x\right\|^2, \\ \frac{\gamma\mu}{2\rho}\left|\langle p,p_t\rangle\right| &\le \frac{\gamma\mu}{4\rho}\left\|p_t\right\|^2 + \frac{\gamma\mu C_p}{4\rho}\left\|p_x\right\|^2, \end{aligned}$$

$$\begin{aligned} \left|\mathcal{L}_1\left(t\right)\right| \le\ & \left(\frac{\gamma\mu}{2\rho}N_1 + \frac{N_1 C_p}{2}\right)\left\|v_x\right\|^2 + \frac{\gamma\mu}{4\rho}\left(1+2L^2N_1\right)\left\|p_t\right\|^2 \\ & + \frac{1}{2}\left[\left(1+N_1\right)\right]\left\|v_t\right\|^2 + \frac{C_p}{2}\left(1+\frac{\gamma\mu}{2\rho}\right)\left\|p_x\right\|^2. \end{aligned}$$

Then by (1) and (4), we get

$$\left|\mathcal{L}_1\left(t\right)\right| \le c\,E\left(t\right).$$

Consequently

$$\left|\mathcal{L}\left(t\right) - N\,E\left(t\right)\right| \le c\,E\left(t\right),$$

which implies that

$$\left(N-c\right)E\left(t\right) \le \mathcal{L}\left(t\right) \le \left(N-c\right)E\left(t\right).$$

Choosing N large enough, then we have (13). □

Proof of **theorem 2.2**.
Differentiating (12), exploiting (3), (5) and (9), we obtain the following estimates:

$$\begin{aligned}\mathcal{L}'(t) &\leq -\left(\frac{\alpha_1}{2\rho}N_1 - C\right)\|v_x\|^2 + (N_1C + C)\int_0^L \left(v_t^2 + h^2(v_t)\right)dx - Na\int_0^L (v_t h(v_t))\,dx \\ &\quad -\left(\frac{\gamma\mu}{4\rho} - N_1\varepsilon_1\right)\|p_t\|^2 - \frac{\gamma\beta}{4\rho}\|p_x\|^2.\end{aligned}$$

Now, we will choose carefully the constants, first we choose

$$\begin{cases} \varepsilon_1 = \dfrac{\gamma\mu}{8\rho N_1} \text{ then we obtain } \left(\dfrac{\gamma\mu}{4\rho} - N_1\varepsilon_1\right) = \dfrac{\gamma\beta}{8\rho N_1}, \\ N_1 = \dfrac{2\rho}{\alpha_1}(1 + C) \text{ then we obtain } \left(\dfrac{\alpha_1}{2\rho}N_1 - C\right) = 1, \end{cases}$$

then, for some $\kappa > 0$, we have

$$\mathcal{L}'(t) \leq -\kappa\left(\|v_x\|^2 + \|p_t\|^2 + \|p_x\|^2\right) + C\int_0^L \left(v_t^2 + h^2(v_t)\right)dx - Na\int_0^L v_t h(v_t)\,dx,$$

and by using the energy functional defined by (1), then we obtain, for some $k > 0$

$$\mathcal{L}'(t) \leq -kE(t) + C\int_0^L \left(v_t^2 + h^2(v_t)\right)dx, \qquad \forall t \geq 0. \tag{14}$$

Case I: H **is linear**. In this case, from $(A1)$, we have

$$c_1'|y| \leq |h(y)| \leq c_2'|y|; \qquad \forall y \in \mathbb{R}, \tag{15}$$

then, we deduce

$$\begin{cases} h^2(y) \leq c_2'\, h(y)\, y; & \forall y \in \mathbb{R}, \\ y^2 \leq \dfrac{1}{c_1'} h(y)\, y; & \forall y \in \mathbb{R}, \end{cases} \tag{16}$$

then, by *(3), (14), (15) and (16)*

$$\mathcal{L}'(t) \leq -kE(t) + C\int_0^L h(v_t)\,v_t\,dx = -kE(t) - CE'(t); \qquad \forall t \geq 0,$$

which gives

$$[\mathcal{L}(t) + CE(t)]' \leq -kE(t).$$

By using *(13)*, we have $\mathcal{L}(t) + CE(t) \sim E(t)$, then we easily obtain

$$E(t) \leq C_0\, e^{-C_1 t};\ \forall t \geq 0, \tag{17}$$

Case II: H **is nonlinear** on $[0, \varepsilon]$. Following the same approach as in ([1]), in this case, as in ([10]), we choose $0 \prec \epsilon_1 \leq \varepsilon$ such that

$$h(y)\ y \leq \min\{\varepsilon, H(\varepsilon)\}, \qquad \forall |y| \leq \epsilon_1.$$

we deduce from the hypothesis $(A1)$ that

$$\begin{cases} y^2 + h^2(y) \leq H^{-1}(y\,h(y))\,;\ \text{for all } |y| \leq \epsilon_1, \\ c_1|y| \leq |h(y)| \leq c_2|y|\,;\ \text{for all } |y| \geq \epsilon_1. \end{cases} \tag{18}$$

To estimate the last term of *(14)*, as in ([7]), we consider the following partition of $(0, L)$:

$$\Omega_1 = \{x \in (0, L) : |v_t| \leq \epsilon_1\}\,;\ \Omega_2 = \{x \in (0, L) : |v_t| > \epsilon_1\}\,.$$

Now, we apply Jensen's inequality to the following term

$$I\ (t) = \frac{1}{|\Omega_1|} \int\limits_{\Omega_1} v_t\, h\,(v_t)\ dx, \tag{19}$$

as H^{-1} is concave, we infer that

$$H^{-1}\,(I\ (t)) \geq C \int\limits_{\Omega_1} H^{-1}\,(v_t\, h\,(v_t))\ dx. \tag{20}$$

Using *(3), (18)* and *(20)*, then the right-hand side of *(14)* becomes

$$\begin{aligned} \int_0^L \left(v_t^2 + h^2\,(v_t)\right)\,dx &= \int\limits_{\Omega_1} \left(v_t^2 + h^2\,(v_t)\right)\,dx + \int\limits_{\Omega_2} \left(v_t^2 + h^2\,(v_t)\right)\,dx \\ &\leq \int\limits_{\Omega_1} H^{-1}\,(v_t\, h\,(v_t))\ dx + C \int\limits_{\Omega_2} v_t\, h\,(v_t)\ dx \leq C\, H^{-1}\,(I\ (t)) - C\, E'\,(t)\,. \end{aligned} \tag{21}$$

Consequently, the estimate *(14)* gives

$$\mathcal{F}_0'\,(t) \leq -kE\,(t) + C\, H^{-1}\,(I\ (t))\,;\ \forall t \geq 0, \tag{22}$$

where $\mathcal{F}_0\,(t) = \mathcal{L}\,(t) + CE\,(t)$. On the one hand for $\epsilon_0 \prec \varepsilon$, $\delta_0 > 0$, using *(22)*, $H' > 0$ and $H'' > 0$ over $(0, \varepsilon]$ and $E' \leq 0$, the functional

$$\mathcal{F}_1\,(t) = H'\left(\epsilon_0 \frac{E\,(t)}{E\,(0)}\right) \mathcal{F}_0\,(t) + \delta_0 E\,(t)\,,$$

is equivalent to $E\,(t)$.

$$c_1 \mathcal{F}_1\,(t) \leq E\,(t) \leq c_2 \mathcal{F}_1\,(t)\,;\ c_1, c_2 > 0. \tag{23}$$

On the other hand, using the fact that $\epsilon_0 \dfrac{E'\,(t)}{E\,(0)} H''\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{F}_0\,(t) \leq 0$ and *(22)* we conclude that

$$\begin{aligned} \mathcal{F}_1'\,(t) &= \epsilon_0 \frac{E'\,(t)}{E\,(0)} H''\left(\epsilon_0 \frac{E\,(t)}{E\,(0)}\right) \mathcal{F}_0\,(t) + H'\left(\epsilon_0 \frac{E\,(t)}{E\,(0)}\right) \mathcal{F}_0'\,(t) + \delta_0 E'\,(t) \\ &\leq -kE\,(t)\, H'\left(\epsilon_0 \frac{E\,(t)}{E\,(0)}\right) + C\, H'\left(\epsilon_0 \frac{E\,(t)}{E\,(0)}\right) H^{-1}\,(I\ (t)) + \delta_0 E'\,(t)\,. \end{aligned} \tag{24}$$

In order to estimate the second term in the right-hand side of *(24)*, we introduce the convex conjugate H^* of H defined by

$$H^*\,(y) = y\ (H')^{-1}\,(y) - H\left[(H')^{-1}\,(y)\right] \leq y\ (H')^{-1}\,(y)\,;\ \text{for } y \in (0, H'\,(\varepsilon)]\,, \tag{25}$$

and H^* satisfies the following Young inequality:

$$A\,B \leq H^*\,(A) + H\,(B)\,;\ \text{for } A \in (0, H'\,(\varepsilon)]\,,\ B \in (0, \varepsilon]\,. \tag{26}$$

Now, taking $A = H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)$and $B = H^{-1}\left(I\ (t)\right)$ and using $H' > 0$ over $(0, \varepsilon]$, *(3), (19), (24) and (25)*, we obtain

$$\begin{aligned} \mathcal{F}_1'(t) &\leq -k\,E(t)\,H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + CH^*\left(H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\right) + CH\left(H^{-1}\left(I\ (t)\right)\right) + \delta_0 E'(t) \\ &\leq -k\,E(t)\,H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + C\,\epsilon_0 \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - C\,H\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - C\,E'\ (t) + \delta_0 E'(t) \\ &\leq -k\,E(t)\,H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + C\,\epsilon_0 \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + (\delta_0 - C)\,E'(t). \end{aligned}$$

With a suitable choice of ϵ_0 and δ_0, we deduce from the last inequality that

$$\mathcal{F}_1'(t) \leq -\left(k\,E(0) - C\,\epsilon_0\right) \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \leq -a_1 H_2\left(\frac{E(t)}{E(0)}\right), \tag{27}$$

where $a_1 = \left(k\,E(0) - C\,\epsilon_0\right) > 0$ and $H_2(t) = t\,H'(\epsilon_0 t)$. Since $E(t) \sim \mathcal{F}_1(t)$ by *(23), we set now* $\mathcal{F}_2(t) = \dfrac{c_1 \mathcal{F}_1(t)}{E(0)}$, It is clear that

$$E(t) \sim \mathcal{F}_2(t). \tag{28}$$

We use the fact that $H_2'(t)$, $H_2(t) > 0$ over $(0, 1]$ (this is due to the fact that H is strictly convex on $(0, \varepsilon]$) and we deduce from *(23) and (27)* that

$$\mathcal{F}_2'(t) \leq -a_2 H_2\left(\frac{E(t)}{E(0)}\right) \leq -a_2 H_2\left(\mathcal{F}_2(t)\right);\ \forall t \geq 0, \tag{29}$$

with $a_2 > 0$. Inequality *(29)*implies that $\left[H_3\left(\mathcal{F}_2(t)\right)\right]' \geq a_2$, where

$$H_3(t) = \int_t^1 \frac{1}{H_2(y)} dy.$$

Thus, by integrating over $[0, t]$, using the properties of $H_2$and the fact that H_3 is strictly decreasing on $(0, 1]$ we obtain, for some $a_3 > 0$,

$$\mathcal{F}_2(t) \leq H_3^{-1}\left(a_2 t + a_3\right);\ \forall t \geq 0, \tag{30}$$

by using *(28)* and *(30)* we obtain the result *(2)*. ■

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