Integrability of systems of ordinary differential equations via Lie point symmetries

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Abstract

The existing literature discusses different strategies to solve a scalar ordinary differential equation using Lie point symmetries. We focus on three of these strategies in order to frame methods for finding solutions of non-linear systems of ordinary differential equations. These include Lie's integration theorem, method of successive reduction of order and the method of using the invariants of the admitted symmetry generators. Illustrative examples and those taken from mechanics are presented to highlight the use of these methods.

Keywords: Nonlinear systems, Lie point symmetries, Integrability, Reduction of order

1 Introduction

Invariance of ordinary differential equations (ODEs) under one-parameter Lie groups of transformations has proved to be successful in reduction of order, linearization, classification into equivalent classes and finding new solutions of the ODEs [1–16]. These successes lead to different integration techniques in order to find complete solution and reduction of the ODEs. Applications of these techniques for non-linear systems of ODEs have been the interest of many researchers during the past few years. For dynamical systems (systems with Lagrangian), Nöether symmetries provide first integrals by using Nöether theorem [1, 16–18]. A large class of dynamical systems appears as geodesic equations for which metric of the spacetime serves as the Lagrangian of the system and the isometries (Killing vectors) serve as Nöether symmetries [1, 19–23]. For non-linear systems of ODEs with no Lagrangian, neither Nöether symmetries nor Killing vectors exist and many techniques were established for using Lie point symmetries in order to integrate, linearize or reduce the order of such systems [24, 26–31].

Lie's integration theorem states that if the r-parameter transitive Lie group of generators in G_r admitted by the nth-order ODE, with $r = n$, is solvable, then the solution of the ODE can be found via quadratures. For a second-order ODE admitting any two-dimensional Lie group, the solution

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is guaranteed by using either canonical forms of the symmetry generators or differential invariants. That is true due to the fact that any two-dimensional Lie group is solvable. For a higher-order ODE, reduction of order depends on the dimension of the solvable Lie group admitted by the ODE. There are various ways to apply Lie's integration theorem. Eisenhart proved that any linear homogeneous first-order PDE in n variables can be integrated via quadratures if it admits a solvable Lie group of dimension $n-1$, provided that the generators of the group are transitive in the space of variables of the PDE [7]. Stephani used Lie's integration theorem to develop an algorithm to integrate an nth-order ODE via line integrals. If the dimension of the solvable Lie group $r < n$, then the order of the ODE can be reduced to $(n-r)$, provided that there exists a transitive n-dimensional Lie group containing the solvable subgroup [1]. Bluman built an iterative algorithm for using a solvable Lie group in reduction of order of an nth-order ODE using differential invariants of the admitted group [25]. Recently, Wafo Soh and Mahomed [26] used Eisenhart's theorem to reduce a system of ODEs to quadratures, benefiting from the equivalence between system of ODEs and the linear homogeneous first-order PDE to be given in eq. (9).

Here we consider three approaches in which two are based on the structure of the admitted Lie group of symmetries including the Lie's integration theorem and the method of successive reduction of order. The first approach is based on Lie's integration theorem applied on an ODE admitting solvable transitive Lie algebra. This method depends on finding the normal forms of the generators in the space of the solutions from which first integrals of the given ODE are obtained. This is as a reformulation of the algorithm in [26]. The advantage of Stephani's algorithm is that it is based on the steps which are almost of algebraic nature except the last step in each iteration where line integrals are found and their existence is guaranteed by Lemma (2.1). Even if the dimension of the admitted solvable Lie algebra is less than kn, we can still apply the algorithm provided that the solvable subalgebra is a subset of a kn-dimensional transitive Lie algebra admitted by the system. This may reduce the system to some integrable form. In a previous work [26], it was pointed out that the method of successive reduction of order fails to be applicable for system of ODEs. In our second integration technique we transform a system into the canonical coordinates associated to the admitted generators in the space of the original variables to successively reduce the order of the ODE provided the symmetry generators satisfy certain structure constants, showing the applicability of the method of successive reduction of order for system of ODEs. We include illustrative examples to demonstrate the applications of these two methods.

The third strategy discussed in [1] is based on using differential invariants of the admitted symmetry generators. In case the admitted Lie group do not satisfy the conditions of either of the previous two strategies, one can find functionally independent differential invariants of the group and can express the ODE in terms of these invariants, which reduces the order of the ODE. Sometimes differential invariants may provide first integrals of the ODE if they are invariant solutions. The method of invariant solutions given by Bluman for a scalar ODE [9, 10] in conjunction with the method of differential invariants were used to find the invariant solutions for systems of ODEs. In the case of non-existence of invariant solutions, differential invariants approach discussed in [1] is applied. Examples were provided to emphasize these results in [32]. For the sake of completeness, Example (4.7) is provided to elaborate this method.

In what follows we consider the system of k nth-order ODEs

$$x_i^{(n)} = \omega_i(t, \mathbf{x}, \dot{\mathbf{x}}, ..., \mathbf{x}^{(n-1)}), \quad i = 1, ..., k, \quad k \geq 1. \tag{1}$$

where $\mathbf{x}^T = [x_1(t) \ \ x_2(t) \ ... \ x_k(t)]$ and $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$, admitting an r-parameter symmetry group with infinitesimals (using Einstein summation convention)

$$X_a = \xi_a \frac{\partial}{\partial t} + \eta_a^i \frac{\partial}{\partial x_i}, \quad i = 1, ..., k, \ a = 1, ..., r. \tag{2}$$

The nth-order prolongation of X_a is given by

$$X_a^{(n)} = \xi_a \frac{\partial}{\partial t} + \eta_a^i \frac{\partial}{\partial x_i} + \left(\eta_a^i\right)^{(1)} \frac{\partial}{\partial \dot{x}_i} + ... + \left(\eta^i{}_a\right)^{(n)} \frac{\partial}{\partial x_i^{(n)}}, \quad i = 1, ..., k, \ a = 1, ..., r,$$

where

$$\left(\eta_a^i\right)^{(m)} = \frac{d}{dt}\left(\eta_a^i\right)^{(m-1)} - x_i^{(m-1)}\frac{d\xi_a}{dt}, \quad m = 1, \ldots, n.$$

Let q be the rank of the matrix of the coefficients of X_a

$$M = \left[\xi_a \ \ \eta_a^i\right], \quad a = 1, ..., r, \ i = 1, ..., k. \tag{3}$$

Then the homogeneous system of partial differential equations

$$X_a\phi(\mathbf{x}) = 0, \quad a = 1, .., r, \tag{4}$$

satisfying $q = r$ and

$$[X_a, X_b] = \zeta_{ab}^d(\mathbf{x})X_d, \ a, b, d = 1, .., r, \tag{5}$$

is called a *complete system* with $(k-r)$ solutions $\phi(\mathbf{x})$ [4, 7].

2 Lie's integration theorem

For r-parameter group of transformations G_r acting on a k-dimensional space, an orbit is defined as the set of points invariant under the action of the group, i.e, the solution $\phi(\mathbf{x})$ = constant of the system (4). The G_r is said to be *transitive* when there is no solution ϕ for the above system, and the space is considered as the unique orbit of the group. This means that the whole space is invariant under the group G_r. In the case where there exists a solution for the latter system, the space is divided into orbits more than one and the group is said to be *intransitive*. Accordingly G_r is transitive if and only if $r \geq k$ and $q = k$ [1, 7].

Lemma 2.1. *Suppose G_r is a transitive Lie group acting on an r-dimensional space. A solution $\phi(\mathbf{x})$, for the system*

$$X_1\phi(\mathbf{x}) = 1, \tag{6a}$$

$$X_d\phi(\mathbf{x}) = 0, \quad d = 2, ..., r, \tag{6b}$$

exists if and only if

$$[X_a, X_b] = \lambda_{ab}^d(\mathbf{x})X_d, \quad a, b = 1, .., r, \ d = 2, .., r, \tag{7}$$

and is given by

$$\phi = \int \frac{\det \begin{bmatrix} dx_1 & \cdots & dx_r \\ & X_2 & \\ & \vdots & \\ & X_r & \end{bmatrix}}{\det(M)}, \tag{8}$$

where X_i denotes the coefficients of the respective generators.

Proof. The G_r is acts transitively on an r-dimensional space, thus G_r is of full rank consisting of r linearly independent operators X_a. Assume condition (7) holds. Then from (5), the system (6b) is a complete system of $r-1$ generators in r variables which means that there exists exactly $r-(r-1) = 1$ solution $\phi(\mathbf{x})$. Since all r generators are linearly independent, $X_1\phi \neq 0$, (if $X_1\phi = 0$, then the generators will not be linearly independent) and under an appropriate change of variables we have

$$X_1\phi = 1.$$

Conversely, assume that the solution $\phi(\mathbf{x})$ of system (6) exists. From the properties of the Lie algebra,

$$[X_a, X_b] = C_{ab}^1 X_1 + C_{ab}^d X_d, a, b = 1, ..., r, d = 2, ..., r.$$

Using the definition of the commutator and applying both sides of the above equation on ϕ yields

$$X_a X_b \phi - X_b X_a \phi = C_{ab}^1 X_1 \phi + C_{ab}^d X_d \phi,$$

which according to Eqs. (6) gives

$$C_{ab}^1 = 0.$$

This means that $X_1 \notin G_r'$, the derived subalgebra of G_r. In order to find $\phi(\mathbf{x})$ we need first to solve the non-homogeneous linear system

$$M\Phi = b,$$

where M is as defined in (3), $\Phi = \begin{bmatrix} \phi_{x_1} & \cdots & \phi_{x_r} \end{bmatrix}^T$, $\phi_{x_i} = \dfrac{\partial \phi}{\partial x_i}$, and $b = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times r}^T$. Due to the transitivity of G_r, we have

$$\phi_{x_i} = \frac{(-1)^{i+1} \det \begin{bmatrix} X_2 \\ \vdots \\ X_r \end{bmatrix}}{\det(M)}.$$

Thus

$$\begin{aligned} \phi = \int \sum_{i=1}^{r} \phi_{x_i} dx_i &= \int \frac{\sum_{i=1}^{r} \left((-1)^{i+1} \det \begin{bmatrix} X_2 \\ \vdots \\ X_r \end{bmatrix} dx_i \right)}{\det(M)} \\ &= \int \frac{\det \begin{bmatrix} dx_1 & \cdots & dx_r \\ & X_2 & \\ & \vdots & \\ & X_r & \end{bmatrix}}{\det(M)}. \end{aligned}$$

□

Now we prove Lie's integration theorem for systems of ODEs.

Theorem 2.2 (**Lie's integration theorem for system of ODEs**). *If a system of k nth-order ODEs admits a solvable Lie group G_{kn} of kn Lie point symmetries that acts transitively in the space of first integrals, then the solution can be given in terms of kn line integrals.*

Proof. Consider the system (1) with $r = kn$ in (2). Let G_{kn} be solvable and transitive in the space of first integrals ϕ_a such that

$$A\phi_a = \left(\frac{\partial}{\partial t} + \dot{x}_i \frac{\partial}{\partial x_i} + ... + \omega_i(t, x_i, \dot{x}_i, ..., x_i^{(n-1)}) \frac{\partial}{\partial x_i^{(n-1)}} \right) \phi_a = 0, \quad i = 1, .., k, \tag{9}$$

where A is the first-order linear partial differential operator associated with the system (1). Due to transitivity of the group in the space of first integrals, we have

$$\Delta = \begin{vmatrix} \mathbf{X}^{(n-1)} \\ A \end{vmatrix} \neq 0, \tag{10}$$

where $\mathbf{X} = [X_1 \ \ ... \ \ X_{kn}]^T$ and $\mathbf{X}^{(n-1)}$ are the $(n-1)$th order prolonged form of $\mathbf{X}$. Since the group G_{kn} is solvable, we can make a chain of derived algebras as follows.

$$\langle X_1, ..., X_{kn}\rangle \supset \langle X_2, ..., X_{kn}\rangle \supset ... \supset \langle X_{kn}\rangle . \tag{11}$$

From the definition of symmetry we have [1]

$$[X_a^{(n-1)}, A] = \lambda_a(t, x_i, \dot{x}_i, ..., x_i^{(n-1)})A, \ \ a = 1, ..., kn, \tag{12}$$

where

$$\lambda_a = -A\xi_a. \tag{13}$$

The sequence of derived algebras (11) with Eq. (12) ensures that the condition (7) of Lemma (2.1) is satisfied. Thus the solution of the system

$$\begin{aligned} &X_1^{(n-1)}\phi_1 = 1,\\ &X_a^{(n-1)}\phi_1 = 0, \ a = 2, .., kn\\ &A\phi_1 = 0 \end{aligned}$$

is guaranteed. Once ϕ_1 is found, using formula (8), it is used as a variable instead of (say) $x_k^{(n-1)}$ and the new set of variables are $(t, x_1, ..., x_k, \dot{x_1}, ..., \dot{x_k}, ..., x_1^{n-1}, ..., x_{k-1}^{n-1}, \phi_1)$. Since

$$\begin{aligned} &A\phi_1 = 0,\\ &X_a^{(n-1)}\phi_1 = 0, \ \ a = 2, .., kn, \end{aligned} \tag{14}$$

then in the space of the new variables, the above generators are independent of the term containing $\dfrac{\partial}{\partial \phi_1}$, and these are exactly the generators admitted by the partial differential equation

$$\hat{A}\phi_a = \left(\frac{\partial}{\partial t} + \dot{x}_i \frac{\partial}{\partial x_i} + ... + x_i^{(n-1)} \frac{\partial}{\partial x_i^{(n-2)}} + \omega_1 \frac{\partial}{\partial x_1^{(n-1)}} + ... + \omega_{k-1} \frac{\partial}{\partial x_{k-1}^{(n-1)}}\right)\phi_a = 0.$$

Due to the chain (11), we can repeat the above discussion for $G_{kn}^{(1)}$ with $\hat{A}$. Repeating the procedure kn times will lead us to find all kn first integrals of the system (1).

$\square$

The extended Lie's integration theorem is implemented by first finding all Lie point symmetries X_N admitted by the system of ODEs, and checking if there exists a transitive solvable Lie subalgebra L_{kn} of dimension kn. If so, to make a chain of derived subalgebras as in (11), and write the generators in the $(n-1)^{th}$ order prolonged form $X_a^{(n-1)}$. In each block of the chain (11), find Δ_a using formula (10), and ϕ_a using Eq. (8) which in the space of first integrals reads

$$\phi_a = \int \frac{\begin{vmatrix} dt \ \ \mathbf{dx} & \mathbf{d\dot{x}} & \cdots \ \ \mathbf{dx^{(n-1)}} \\ & X_{a+1}^{(n-1)} & \\ & \vdots & \\ & X_{nk}^{(n-1)} & \\ & A_a & \end{vmatrix}}{\Delta_a}. \tag{15}$$

We replace one of the space variables with the new first integral ϕ_a and rewrite the generators in the space of the new variables in each iteration. For Abelian groups it doesn't matter by which generator we start in order to find first integrals ϕ_a, and we can use the same Δ without the need for the change of variables in each step. Example (4.5) ascertains this consideration. In general, the algorithm of Lie's integration theorem is illustrated in examples (4.1), (4.5) and (4.6) in section (4).

3 Successive reduction of order

For a symmetry generator

$$X_1 = \xi_1(t, \mathbf{x})\frac{\partial}{\partial t} + \eta_{1i}(t, \mathbf{x})\frac{\partial}{\partial x_i}$$

admitted by system (1), there always exists a change of variables [1–7, 9]

$$r = \phi_\circ(t, \mathbf{x}), \quad u_i = \phi_i(t, \mathbf{x}), \quad i = 1, ..., k \tag{16}$$

such that X_1 can be written in the canonical form

$$\hat{X}_1 = \frac{\partial}{\partial u_1}. \tag{17}$$

Transforming the system of ODEs to the canonical coordinates $(r, \mathbf{u})$, where $\mathbf{u}^T = [u_1(r) \;\; u_2(r) \; ... \; u_k(r)]$, means that the the new system will be independent of the variable u_1, and it is reduced in order by one with respect to the variable u_1 in the space of the canonical variables (16) to give

$$u_i^{(n)} = \hat{\omega}_i(r, u_2, ..., u_k, \dot{\mathbf{u}}, ..., \mathbf{u}^{(n-1)}), \quad i = 1, ..., k. \tag{18}$$

To accomplish more reductions of order we need more symmetries which are admitted by the reduced system. Not all Lie algebras can be used to repeat the process of reduction. The following theorem demonstrates the necessary and sufficient condition.

Theorem 3.1. *Assume that system* (1) *is transformed to system* (18) *in the coordinates of the canonical variables associated with the symmetry generator* X_1*. Then system* (18) *admits the symmetry generator* $\hat{X}_2$ *if and only if*

$$[X_1, X_2] = C_1 X_1, \quad C_1 \text{is a constant}, \tag{19}$$

where $\hat{X}_2$ *is the symmetry generator* X_2 *in the space of variables* (16).

Proof. Suppose that system (1) admits the symmetry generators X_1 and X_2. Transformations (16) maps the system to the new coordinates $(r, \mathbf{u})$ to give the system (18) with operators (17) and

$$\begin{aligned} \hat{A} &= \frac{\partial}{\partial r} + \dot{u}_i\frac{\partial}{\partial u_i} + \ddot{u}_i\frac{\partial}{\partial \dot{u}_i} + ... + \hat{\omega}_i(r, u_2, ..., u_k, \dot{\mathbf{u}}, ..., \mathbf{u}^{(n-1)})\frac{\partial}{\partial u_i^{(n-1)}}, \\ \hat{X}_2^{(n-1)} &= \hat{\xi}_2\frac{\partial}{\partial r} + \hat{\eta}_{2i}\frac{\partial}{\partial u_i} + \hat{\eta}'_{2i}\frac{\partial}{\partial \dot{u}_i} + ... + \hat{\eta}_{2i}^{(n-1)}\frac{\partial}{\partial u_i^{(n-1)}}, \quad i = 1, ..., k. \end{aligned} \tag{20}$$

Since a commutator is invariant under a coordinate transformation [1,4], then due to symmetry conditions (12)-(13), we have

$$[\hat{X}_2^{(n-1)}, \hat{A}] = \hat{\lambda}_2\hat{A}, \quad \hat{\lambda}_2 = -\hat{A}\hat{\xi}_2. \tag{21}$$

Now system (18) is independent of the variable u_1 and so its solution; thus its admitted operators do not contain terms with $\dfrac{\partial}{\partial u_1}$. Hence operators (20) can be written as

$$\begin{aligned} \bar{A} &= \frac{\partial}{\partial r} + \dot{u}_j\frac{\partial}{\partial u_j} + \ddot{u}_i\frac{\partial}{\partial \dot{u}_i} + ... + \hat{\omega}_i(r, u_2, ..., u_k, \dot{\mathbf{u}}, ..., \mathbf{u}^{(n-1)})\frac{\partial}{\partial u_i^{(n-1)}}, \\ \bar{X}_2^{(n-1)} &= \hat{\xi}_2\frac{\partial}{\partial r} + \hat{\eta}_{2j}\frac{\partial}{\partial u_j} + \hat{\eta}'_{2i}\frac{\partial}{\partial \dot{u}_i} + ... + \hat{\eta}_{2i}^{(n-1)}\frac{\partial}{\partial u_i^{(n-1)}}, \quad i = 1, ..., k, \quad j = 2, ..., k. \end{aligned}$$

Accordingly, solutions of system (18) are the constant functions ψ_a such that $\bar{A}\psi_a = 0$. Therefore $\bar{X}_2$ is its symmetry if and only if it satisfies the symmetry condition

$$\left[\bar{X}_2^{(n-1)}, \bar{A}\right] = \bar{\lambda}_2(r, \mathbf{u})\bar{A}. \tag{22}$$

From the definitions of the above operators,

$$\begin{aligned}
&\bar{A} = \hat{A} - \dot{u}_1\hat{X}_1,\\
&\bar{X}_2^{(n-1)} = \hat{X}_2^{(n-1)} - \hat{\eta}_{21}\hat{X}_1.
\end{aligned}$$

Substituting these equations into the LHS of the Eq. (22) and using Eq.(21) yields

$$\left[\hat{X}_2^{(n-1)} - \hat{\eta}_{21}\hat{X}_1, \hat{A} - \dot{u}_1\hat{X}_1\right] = \hat{\lambda}_2\hat{A} - \hat{\eta}_{21}(\hat{\lambda}_1\hat{A}) + \left(\hat{A}\hat{\eta}_{21} - \hat{X}_2^{(n-1)}(\dot{u}_1) - \dot{u}_1\hat{X}_1(\hat{\eta}_{21})\right)\hat{X}_1 - \dot{u}_1[\hat{X}_2^{(n-1)}, \hat{X}_1].$$

From eqs. (13) and (20), we obtain

$$\begin{aligned}
\left[\bar{X}_2^{(n-1)}, \bar{A}\right] &= \hat{\lambda}_2\hat{A} - \hat{\eta}_{21}(-\hat{A}\hat{\xi}_1)\hat{A} + \left(\frac{\partial\hat{\eta}_{21}}{\partial r} + \dot{u}_1\frac{\partial\hat{\eta}_{21}}{\partial u_1} + \dot{u}_j\frac{\partial\hat{\eta}_{21}}{\partial u_j} - \hat{\eta}_{21}' - \dot{u}_1\frac{\partial\hat{\eta}_{21}}{\partial u_1}\right)\hat{X}_1 + \dot{u}_1\left[\hat{X}_1, \hat{X}_2^{(n-1)}\right]\\
&= \hat{\lambda}_2\hat{A} + \left(\frac{\partial\hat{\eta}_{21}}{\partial r} + \dot{u}_j\frac{\partial\hat{\eta}_{21}}{\partial u_j} - \hat{\eta}_{21}'\right)\hat{X}_1 + \dot{u}_1\left[\hat{X}_1, \hat{X}_2^{(n-1)}\right].
\end{aligned}$$

Expanding $\hat{\eta}_1'$, regrouping terms and again using relation (13), the above equation reduces to

$$\begin{aligned}
\left[\bar{X}_2^{(n-1)}, \bar{A}\right] =&\hat{\lambda}_2\hat{A} - \dot{u}_1\left(\frac{\partial\hat{\eta}_{21}}{\partial u_1} + \hat{A}\hat{\xi}_2\right)\hat{X}_1 + \dot{u}_1\left(c_1\hat{X}_1 + c_2\hat{X}_2^{(n-1)}\right)\\
=&\hat{\lambda}_2\hat{A} - \dot{u}_1\left(\frac{\partial\hat{\eta}_{21}}{\partial u_1} - \hat{\lambda}_2 + c_1\right)\hat{X}_1 + c_2\dot{u}_1\hat{X}_2^{(n-1)}\\
=&\hat{\lambda}_2\left(\hat{A} - \dot{u}_1\hat{X}_1\right) - \dot{u}_1\left(\frac{\partial\hat{\eta}_{21}}{\partial u_1} + c_1\right)\hat{X}_1 + c_2\dot{u}_1\hat{X}_2^{(n-1)}
\end{aligned}$$

This statement matches to (22) if and only if

$$c_2 = 0, \quad c_1 = -\frac{\partial\hat{\eta}_{21}}{\partial u_1}, \text{ and } \bar{\lambda}_2 = \hat{\lambda}_2. \tag{23}$$

Conditions (23) demonstrate that the commutator of X_1 and X_2 must be a multiple of only X_1 so that X_2 is admitted by the system (18) which is in the canonical space associated with X_1, i.e,

$$[X_1, X_2] = c_1X_1.$$

□

If we want to completely integrate the system using the Lie algebra admitted by the original given system, we must have enough symmetry generators, at least kn generators, satisfying the chain:

$$\begin{aligned}
&[X_1, X_a] = C_aX_1, \quad a := 2, ..., kn\\
&[X_2, X_b] = C_bX_2, \quad b := 3, ..., kn\\
&\vdots\\
&[X_{kn-1}, X_{kn}] = CX_{kn-1}.
\end{aligned} \tag{24}$$

Transforming the system to the canonical variables corresponding to X_1 reduces the order of the system by one with respect to a selected dependent variable. After writing the rest of the generators in the new coordinates (of canonical variables), we repeat the process using X_2. Repeating this process kn times, the system can be completely integrated. Examples (4.2), (4.3) and (4.6) in the next section demonstrate this algorithm. The advantage of this method is that the system can be integrable without the admitted Lie algebra being solvable, and even if we don't have enough symmetries, we can reduce the order of the system to some form which provides further simplification. This is clear in example (4.3) in which after the second iteration the system can be integrated, whereas in example (4.6), after the second transformation, one equation of the system gets transformed to Bernoulli's equation.

4 Applications

We consider mainly examples taken from the literature.

Example 4.1. We start with the system

$$\ddot{x} = \dot{x} + \frac{\dot{y}}{y}e^{-t}, \;\; \ddot{y} = \frac{\dot{y}^2}{y} + \dot{y} + y \tag{25}$$

admitting the Lie point symmetry generators (in extended form):

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x}, \;\; X_2 = e^t\frac{\partial}{\partial x} + e^t\frac{\partial}{\partial \dot{x}}, \;\; X_3 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \dot{y}\frac{\partial}{\partial \dot{y}}, \\ &X_4 = \left(\ln y + e^t\right)\frac{\partial}{\partial x} + ye^t\frac{\partial}{\partial y} + \left(\frac{\dot{y}}{y} + e^t\right)\frac{\partial}{\partial \dot{x}} + (\dot{y} + y)\, e^t\frac{\partial}{\partial \dot{y}}. \end{aligned}$$

This system has been linearized using four-dimensional Lie group which is intransitive in the space of variables in [29]. Later was solved using the approach of invariant solutions in [32]. Here we intend to apply Theorem (2.2) to find its solution in order to establish the applicability of our algorithm.

$L_4 = \langle X_1, X_2, X_3, X_4\rangle$ is a solvable and a transitive Lie algebra. We have

$$L_4 \supset \langle X_1, X_2, X_3\rangle \supset \langle X_2, X_3\rangle \supset \langle X_3\rangle,$$

and the associated linear partial differential operator to system (25) appears as

$$A_1 = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \left(\dot{x} + \frac{\dot{y}}{y}e^{-t}\right)\frac{\partial}{\partial \dot{x}} + \left(\frac{\dot{y}^2}{y} + \dot{y} + y\right)\frac{\partial}{\partial \dot{y}},$$

whereas Eq. (10) results in

$$\Delta_1 = -y^2e^{2t}.$$

Thus, formula (15) with respect to X_4 gives

$$\begin{aligned} \phi_1 &= \int \frac{\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & e^t & 0 & e^t & 0 \\ 0 & 1 & y & 0 & \dot{y} \\ dt & dx & dy & d\dot{x} & d\dot{y} \\ 1 & \dot{x} & \dot{y} & \left(\dot{x} + \frac{\dot{y}}{y}e^{-t}\right) & \left(\frac{\dot{y}^2}{y} + \dot{y} + y\right) \end{vmatrix}}{\Delta_1} \\ &= \int \left\{ \left(\frac{-\dot{y}}{ye^t} - \frac{1}{e^t}\right) dt - \frac{\dot{y}}{y^2e^t}\, dy + \frac{1}{ye^t}\, d\dot{y} \right\} \\ &= \frac{1}{e^t}\left(\frac{\dot{y}}{y} + 1\right). \end{aligned} \tag{26}$$

Now choosing the variable ϕ_1 instead of (say) $\dot{y}$ and writing the generators in terms of the new variables t, x, y, $\dot{x}$ and ϕ_1 yields

$$\begin{aligned} &\hat{X}_1 = \frac{\partial}{\partial x}, \;\; \hat{X}_2 = e^t\frac{\partial}{\partial x} + e^t\frac{\partial}{\partial \dot{x}}, \;\; \hat{X}_3 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\ &A_2 = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + y(\phi_1 e^t - 1)\frac{\partial}{\partial y} + \left(\dot{x} + \phi_1 - e^{-t}\right)\frac{\partial}{\partial \dot{x}}. \end{aligned} \tag{27}$$

where the operator A_2 replaces the operator A_1 in the space of the new variables. Again using the above procedure with the operators (27) we have

$$\Delta_2 = ye^t,$$

and with respect to $\hat{X}_1$, one gets

$$\phi_2 = (\phi_1 - 1)t + e^{-t} + \phi_1 e^t + x - \ln y - \dot{x}. \tag{28}$$

This gives the new set of variables as t, x, y, ϕ_1 and ϕ_2 and correspondingly the operators reduce to

$$\begin{aligned} &\hat{\hat{X}}_2 = e^t\frac{\partial}{\partial x}, \;\; \hat{\hat{X}}_3 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\ &A_3 = \frac{\partial}{\partial t} + \left((\phi_1 - 1)t + e^{-t} + \phi_1 e^t + x - \ln y - \phi_2\right)\frac{\partial}{\partial x} + y(\phi_1 e^t - 1)\frac{\partial}{\partial y}. \end{aligned}$$

This gives rise to

$$\Delta_3 = ye^t,$$

and with respect to $\hat{\hat{X}}_2$, the first integral is

$$\phi_3 = \frac{1}{2}e^{-2t} - (\phi_2 - \phi_1 - \phi_1 t + \ln y - x + t)e^{-t}. \tag{29}$$

In the last iteration we write the generators in the newest variables t, y, ϕ_1, ϕ_2, and ϕ_3 as

$$\bar{X}_3 = y\frac{\partial}{\partial y}, \;\; A_4 = \frac{\partial}{\partial t} + y(\phi_1 e^t - 1)\frac{\partial}{\partial y},$$

and consequently

$$\Delta_4 = y.$$

whereas

$$\phi_4 = \phi_1 e^t - t - \ln y. \tag{30}$$

Therefore from the first integrals (26)-(30) of system (25), we solve for the dependent variables to deduce

$$\begin{aligned} &x = (\phi_1 + \phi_3)e^t - \frac{1}{2}e^{-t} - \phi_1 t - \phi_1 + \phi_2 - \phi_4, \\ &y = e^{\phi_1 e^t - t - \phi_4},. \end{aligned}$$

Example 4.2. The system

$$\ddot{x} = \frac{1}{\dot{x}^2 - 2\dot{y}}, \;\; \ddot{y} = \frac{1 + \dot{x}}{\dot{x}^2 - 2\dot{y}} \tag{31}$$

admitting the following symmetry ooperators (in extended form)

$$X_1 = \frac{\partial}{\partial t}, \;\; X_2 = \frac{\partial}{\partial x}, \;\; X_3 = \frac{\partial}{\partial y}, \;\; X_4 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial \dot{x}} + \dot{x}\frac{\partial}{\partial \dot{y}},$$

that has been solved in [26] using the solvable Lie algebra L_4. It is already independent of x and y, so it is of first order with respect to the variables $\dot{x}$ and $\dot{y}$.

We intend to demonstrate the use of the method of successive reduction of order on this system by transforming it to canonical space associated with the two symmetry generators X_1 and X_4. This is due to these two generators commuting and thus the condition of Theorem (3.1) is satisfied. Starting with X_4, as a consequence of X_2 and X_3, it can be written as

$$X_4 = \frac{\partial}{\partial \dot{x}} + \dot{x}\frac{\partial}{\partial \dot{y}}.$$

Thus the transformation into the canonical space (r, u, v) associated with the above generator is

$$r = t, \quad u = \dot{y} - \frac{\dot{x}^2}{2}, \quad v = \dot{x} \tag{32}$$

by which $\hat{X}_4 = \dfrac{\partial}{\partial v}$. Accordingly, the system in the canonical space is

$$\dot{u} = \frac{-1}{2u}, \quad \dot{v} = \frac{-1}{2u}.$$

The above system is of order one with respect to the variable v and can be integrated easily. However, we use the generator X_1 to find the full solution to show the efficiency of the method. Now X_1 is already in canonical form and therefore the transformation that leads to our goal, which is the canonical space of variables $(\varepsilon, \phi, \theta)$, is

$$\varepsilon = u, \quad \phi = r, \quad \theta = v, \tag{33}$$

by which the new system is

$$\dot{\phi} = -2\varepsilon, \quad \dot{\theta} = 2,$$

whose solution is

$$\phi = -\varepsilon^2 + c_1, \quad \theta = 2\varepsilon + c_2.$$

Substituting back to the coordinates (r, u, v), using (33), we find

$$u(t) = \pm\sqrt{c_1 - t}, \quad v(t) = \pm 2\sqrt{c_1 - t} + c_2.$$

Then substituting back to the variables (t, x, y), using (32), gives the reductions

$$\dot{x} = \pm 2\sqrt{c_1 - t} + c_2, \quad \dot{y} = \pm\sqrt{c_1 - t} + \frac{1}{2}(\pm\sqrt{c_1 - t} + c_2)^2,$$

whose solution is

$$\begin{aligned}
x(t) &= \mp\frac{2}{3}\,(c_1 - t)^{\frac{3}{2}} + c_2 t + c_3, \\
y(t) &= \pm\sqrt{c_1 - t}\,(c_2 t + t - c_2 - c_1 c_2) - \frac{1}{4}t^2 + \frac{1}{2}(c_1 + c_2^2)t + c_4.
\end{aligned}$$

Example 4.3. The Newtonian system

$$\ddot{x} = \frac{x}{(x^2 + y^2)^2}, \quad \ddot{y} = \frac{y}{(x^2 + y^2)^2}, \tag{34}$$

admits the Lie point symmetry generators in extended form given by

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \qquad X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \dot{x}\frac{\partial}{\partial \dot{x}} - \dot{y}\frac{\partial}{\partial \dot{y}}, \\
X_3 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \dot{y}\frac{\partial}{\partial \dot{x}} - \dot{x}\frac{\partial}{\partial \dot{y}}.
\end{aligned}$$

Despite the above generators forming a solvable Lie algebra, they are not enough to apply Lie's integration theorem. The Lie algebra

$$[X_3, X_1] = [X_3, X_2] = 0, \quad [X_1, X_2] = 2X_1, \tag{35}$$

satisfies the condition of Theorem (3.1) and suggests the use of these operators in the order X_3, X_1 and X_2, respectively to apply successive reduction of order. In the first iteration we transform the system (34) into the canonical coordinates (r, u, v) associated with X_3 for which the canonical form is $\hat{X}_3 = \dfrac{\partial}{\partial v}$. This is done by the transformation

$$r = t, \quad u = \sqrt{x^2 + y^2}, \quad v = \tan^{-1}\left(\frac{x}{y}\right), \tag{36}$$

which leads to the new system

$$\ddot{u} = \frac{u^4 \dot{v}^2 + 1}{u^3}, \quad \ddot{v} = \frac{-2\dot{v}\dot{u}}{u}, \tag{37}$$

admitting the symmetry generators

$$\hat{X}_1 = \frac{\partial}{\partial r}, \qquad \hat{X}_2 = 2r\frac{\partial}{\partial r} + u\frac{\partial}{\partial u} - \dot{u}\frac{\partial}{\partial \dot{u}} - 2\dot{v}\frac{\partial}{\partial \dot{v}}.$$

According to (35), in the second iteration we use $\hat{X}_1$ and transform the system into the coordinates of canonical variables (s, w, z) associated with $\hat{X}_1$. It is already in canonical form with respect to the independent variable r. Hence to achieve our goal we use the transformation

$$s = u, \quad w = r, \quad z = \dot{v}, \tag{38}$$

mapping system (37) to

$$\ddot{w} = -\dot{w}^3\left(sz^2 + \frac{1}{s^3}\right), \quad \dot{z} = \frac{-2z}{s}, \tag{39}$$

which is first order with respect to $\dot{w}$ and z. In the space of the new variables the symmetry generator $\hat{X}_2$ transforms to

$$\hat{\hat{X}}_2 = s\frac{\partial}{\partial s} + \dot{w}\frac{\partial}{\partial \dot{w}} - 2z\frac{\partial}{\partial z} - 3\dot{z}\frac{\partial}{\partial \dot{z}}.$$

At this stage, although the system is integrable, we continue using the symmetries to transform it to an autonomous system. The last iteration uses the above symmetry generator which in the canonical coordinates $(\varepsilon, \theta, \phi)$ is of the form

$$\bar{X}_2 = \frac{\partial}{\partial \varepsilon},$$

via the transformation

$$\varepsilon = \ln s, \quad \theta = zs^2, \quad \phi = \frac{\dot{w}}{s}. \tag{40}$$

The associated canonical form of system (39) is the autonomous system

$$\dot{\theta} = 0, \quad \dot{\phi} = -\phi^3(\theta^2 + 1) - \phi,$$

whose solution is

$$\theta(\varepsilon) = c, \quad \phi(\varepsilon) = \frac{\pm 1}{\sqrt{e^{2\varepsilon} - 1 - c_1 + c_2}}.$$

Now we substitute back to the original variables. From Eq. (40) we have

$$z(s) = \frac{c_1}{s^2}, \quad \dot{w} = \frac{\pm s}{\sqrt{c_2 e^{2\varepsilon} - 1 - c_1}},$$

which gives

$$w(s)=\frac{\pm 1}{c_2}\sqrt{e^{2\varepsilon}-1-c_1}+c_3.$$

Then utilising (38) yields

$$u(r)=\frac{\pm 1}{\sqrt{c_2}}\sqrt{c_2^2(r-c_3)^2+1+c_1},\quad \dot{v}=\frac{c_1}{s^2}=\frac{c_1c_2}{c_2^2(r-c_3)^2+1+c_1}. \tag{41}$$

The above gives

$$v(r)=\frac{c_1}{\sqrt{(1+c_1^2)}}\tan^{-1}\left(\frac{c_2(r-c_3)}{\sqrt{1+c_1^2}}\right). \tag{42}$$

Finally we use the transformation (36) to arrive at

$$x(t)=\frac{u(t)\tan\left(v(t)\right)}{\sqrt{1+\tan^2\left(v(t)\right)}},\quad y(t)=\frac{u(t)}{\sqrt{1+\tan^2\left(v(t)\right)}},$$

where u and v are defined in (41) and (42).

Example 4.4. The Hamiltonian system of equations under the influence of the potential [33, 34]

$$V=\frac{1}{2}(x^2+y^2)+\frac{2}{(x+y)^2}$$

is

$$\ddot{x}+x-\frac{2}{(x+y)^3}=0, \tag{43a}$$

$$\ddot{y}+y-\frac{2}{(x+y)^3}=0, \tag{43b}$$

admits a six-parameter group of Lie point symmetries. Among others, two generators are

$$X_1=\frac{\partial}{\partial t},\quad X_2=-(y-x)\frac{\partial}{\partial x}+(y-x)\frac{\partial}{\partial y}, \tag{44}$$

with commutator

$$[X_1,X_2]=0.$$

So, there is no harm with which we start the reduction process. Starting with X_2, we transform the system into its corresponding canonical coordinates (s,u,v), where

$$s=t,\quad u=\ln\sqrt{x-y},\quad v=x+y. \tag{45}$$

Thus the reduced system is

$$\ddot{u}+2\dot{u}^2+\frac{1}{2}=0,\quad \ddot{v}+v-4v^{-3}=0.$$

Using X_1 which is already in normal form, the above system is transformed into

$$\dot{w}=\frac{-2}{1+4r^2},\quad \dot{z}=z^3(r-\frac{4}{r^3})$$

whose solution can be easily found to be

$$w=-\tan^{-1}2r-c_1,\quad z=\pm\frac{r}{\sqrt{c_3r^2-r^4-4}}.$$

Substituting back to the coordinates (s, u, v), we have the system

$$u(s) = \ln\sqrt{c_2\cos(c_1 - s)}, \quad v(s) = \pm\sqrt{\frac{\sqrt{c_3 - 16}}{2}\sin(\pm 2s + c_4) + \frac{c_3}{2}},$$

which in the space of the original variables is

$$x(t) = \frac{1}{2}\left(\pm\sqrt{\frac{\sqrt{c_3 - 16}}{2}\sin(\pm 2t + c_4) + \frac{c_3}{2}} - c_2\cos(c_1 - t)\right),$$
$$y(t) = \frac{1}{2}\left(\pm\sqrt{\frac{\sqrt{c_3 - 16}}{2}\sin(\pm 2t + c_4) + \frac{c_3}{2}} + c_2\cos(c_1 - t)\right).$$

Example 4.5. The system of equations of motion of the time-dependent n-dimensional oscillator written as

$$\ddot{x}_i + \Omega^2(t)x_i = 0, \quad i = 1, ..., n, \tag{46}$$

admits $n^2 + 4n + 3$ Lie point symmetries from which the Nöether symmetries are singled out to find first integrals of the system by using Nöether theorem [17]. Here we use Lie's integration theorem to solve this system. We choose the Abelian transitive subgroup consisting of the $2n$ symmetry generators

$$X_{1i} = \rho\cos\theta\frac{\partial}{\partial x_i}, \tag{47a}$$
$$X_{2i} = \rho\sin\theta\frac{\partial}{\partial x_i}, \tag{47b}$$

where $\rho(t)$ and $\theta(t)$ are functions satisfying

$$\ddot{\rho} + \Omega^2\rho = \rho^{-3}, \quad \dot{\theta} = \rho^{-2}.$$

This fulfills the conditions of Lie's integration theorem. The linear partial differential operator associated with system (46) is

$$A = \frac{\partial}{\partial t} + \dot{x}_i\frac{\partial}{\partial x_i} - \Omega^2(t)x_i\frac{\partial}{\partial x_i},$$

where the repetition of the indices implies summation of the n terms and the fist order prolongation of the operators (47) are

$$X'_{1i} = \rho\cos\theta\frac{\partial}{\partial x_i} + (\dot{\rho}\cos\theta - \rho\sin\theta\dot{\theta})\frac{\partial}{\partial\dot{x}_i},$$
$$X'_{2i} = \rho\sin\theta\frac{\partial}{\partial x_i} + (\dot{\rho}\sin\theta + \rho\cos\theta\dot{\theta})\frac{\partial}{\partial\dot{x}_i}.$$

Since the group is Abelian, it doesn't matter with which generator we start and therefore we can use the same Δ to find all the first integrals from the determinant of the $(2n+1)\times(2n+1)$ matrix

$$\Delta = \begin{vmatrix} X'_{11} \\ \vdots \\ X'_{1n} \\ X'_{2n} \\ \vdots \\ X'_{2n} \\ A \end{vmatrix} = \begin{vmatrix} 0 & \rho\cos\theta & \mathbf{0} & -\dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta & \mathbf{0} \\ \vdots & \ddots & & \ddots & \\ 0 & \mathbf{0} & \rho\cos\theta & \mathbf{0} & -\dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta \\ 0 & \rho\sin\theta & \mathbf{0} & -\dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta & \mathbf{0} \\ \vdots & \ddots & & \ddots & \\ 0 & \mathbf{0} & \rho\sin\theta & \mathbf{0} & -\dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta \\ 1 & \dot{x}_1\cdots & \cdots\dot{x}_n & -\Omega^2 x_1\cdots & \cdots - \Omega^2 x_n \end{vmatrix}$$
$$= \rho^{2n}\dot{\theta}^n,$$

where $\mathbf{0} = [0 \,...\, 0]$. First integral ϕ_{1i} with respect to X_{1i} is found from the formula

$$\phi_{1i} = \int \frac{\begin{vmatrix} dt \;\; dx_1 \;\; ... \;\; dx_n \;\; d\dot{x}_1 \;\; ... \;\; d\dot{x}_n \\ \mathbf{X'_{1k}} \\ \mathbf{X'_{2j}} \\ A \end{vmatrix}}{\Delta}, \quad i, j, k = 1, ..., n, \; k \neq i,$$

which gives

$$\phi_{1i} = \frac{x_i}{\rho}\cos\theta + (x_i\dot{\rho} - \rho\dot{x}_i)\sin\theta.$$

The same formula is applied to find the first integrals with respect to the operators X_{2i} to give

$$\phi_{2i} = \frac{x_i}{\rho}\sin\theta - (x_i\dot{\rho} - \rho\dot{x}_i)\cos\theta.$$

Using the above first integrals we can find the full set of solutions as

$$x_i = \rho\left(\phi_{1i}\cos\theta + \phi_{2i}\sin\theta\right),$$

which exactly matches the solution deduced in [17, 18].

Example 4.6. The geodesic system of Bertotti-Robinson's electromagnetic universe is given by [20]

$$\ddot{t} + 2\coth r\dot{t}\dot{r} = 0, \quad \ddot{r} + \sinh r\cosh r\dot{t}^2 = 0, \tag{48a}$$
$$\ddot{\phi} + 2\cot z\dot{\phi}\dot{z} = 0, \quad \ddot{z} - \cos z\sin z\dot{\phi}^2 = 0. \tag{48b}$$

This system admits the Lie algebra L_8 consisting of Lie point symmetry operators

$$X_1 = s\frac{\partial}{\partial s}, \;\; X_2 = \frac{\partial}{\partial s}, \;\; X_3 = \frac{\partial}{\partial t}, \;\; X_4 = \frac{\partial}{\partial \phi},$$
$$X_5 = \cot z\cos\phi\frac{\partial}{\partial \phi} + \sin\phi\frac{\partial}{\partial z}, \; X_6 = \cot z\sin\phi\frac{\partial}{\partial \phi} - \cos\phi\frac{\partial}{\partial z},$$
$$X_7 = e^t\coth r\frac{\partial}{\partial t} - e^t\frac{\partial}{\partial r}, \; X_8 = e^{-t}\coth r\frac{\partial}{\partial t} + e^{-t}\frac{\partial}{\partial r},$$

which do not constitute a solvable algebra. It is notable that system (48) is composed of two independent sets of ODEs each consisting of two equations. The first set is composed of Eqs. (48a) admitting the radical subalgebra of L_8

$$\mathrm{rad}(L_8) = \langle X_1, X_2, X_3, X_7, X_8\rangle, \tag{49}$$

and the second one is the set of Eqs. (48b) admitting the non-solvable subalgebra

$$\langle X_1, X_2, X_4, X_5, X_6\rangle. \tag{50}$$

The radical subalgebra (49) provides the necessary condition, in addition to its transitivity, in order to apply Lie's integration theorem for solving the first set of ODEs in $(t(s), r(s))$ space, with associated operator A

$$A = \frac{\partial}{\partial s} + \dot{t}\frac{\partial}{\partial t} + \dot{r}\frac{\partial}{\partial r} - 2\coth r\dot{t}\dot{r}\frac{\partial}{\partial \dot{t}} - \sinh r\cosh r\dot{t}^2\frac{\partial}{\partial \dot{r}}.$$

Hence for the solution in the space of $(t(s), r(s))$, we use Lie's integration theorem, while for the solution in the space of $(\phi(s), z(s))$ we can use the method of successive reduction if the Lie algebra (50) satisfies the chain (24) or a part thereof.

We consider the chain of the derived subalgebras

$$\langle X_1, X_2, X_3, X_7\rangle \supset \langle X_2, X_3, X_7,\rangle \supset \langle X_2, X_7,\rangle \supset \langle X_2\rangle, \tag{51}$$

and accordingly the first integrals are, with respect to the track $X_1 \longrightarrow X_3 \longrightarrow X_7 \longrightarrow X_2$. In the first iteration,

$$\Delta_1 = \begin{vmatrix} X_1' \\ X_2' \\ X_3' \\ X_7' \\ A \end{vmatrix} = \frac{e^t}{\sinh^2 r}\left(\dot{t}\sinh r\cosh r + \dot{r}\right)\left(\dot{t}^2\sinh^2 r - \dot{r}^2\right),$$

with

$$\begin{aligned}\phi_1 &= \int \frac{\dot{t}^2\sinh r\cosh r}{\dot{t}^2\sinh^2 r - \dot{r}^2}dr - \frac{\dot{t}\sinh^2 r}{\dot{t}^2\sinh^2 r - \dot{r}^2}d\dot{t} - \frac{\dot{r}}{\dot{t}^2\sinh^2 r - \dot{r}^2}d\dot{r} \\ &= \ln\left(\frac{1}{\sqrt{\dot{t}^2\sinh^2 r - \dot{r}^2}}\right).\end{aligned}$$

Let ϕ_1 be instead of $\dot{r}$, where

$$\dot{r} = \pm\sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}}.$$

We consider the positive root of $\dot{r}$. Accordingly, in the space of the new set of variables $s, t, r, \dot{t}, \phi_1$, the new operators read

$$\begin{aligned}A_2 &= \frac{\partial}{\partial s} + \dot{t}\frac{\partial}{\partial t} + \sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}}\frac{\partial}{\partial r} - 2\dot{t}\sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}}\coth r\frac{\partial}{\partial \dot{t}}, \\ \hat{X}_2 &= \frac{\partial}{\partial s}, \quad \hat{X}_3 = \frac{\partial}{\partial t}, \\ \hat{X}_7 &= e^t\coth r\frac{\partial}{\partial t} - e^t\frac{\partial}{\partial r} + e^t\left(\coth r\dot{t} - \frac{\dot{t}^2\sinh^2 r - e^{-2\phi_1}}{\sinh^2 r}\right)\frac{\partial}{\partial \dot{t}}.\end{aligned}$$

Thus

$$\Delta_2 = \frac{e^t}{\sinh^2 r}\sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}}\left(\dot{t}\sinh r\cosh r + \sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}}\right).$$

and ϕ_2, with respect to X_3,

$$\phi_2 = t + \ln\left(\sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}} + \dot{t}\sinh r\cosh r\right).$$

So

$$t = \phi_2 - \ln\left(\sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}} + \dot{t}\sinh r\cosh r\right).$$

The third iteration is in the space of the variables $s, r, \dot{t}$, where

$$\begin{aligned}A_3 =& \frac{\partial}{\partial s} + \sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}}\frac{\partial}{\partial r} - 2\dot{t}\sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}}\coth r\frac{\partial}{\partial \dot{t}}, \\ \hat{\hat{X}}_2 =& \frac{\partial}{\partial s}, \\ \hat{\hat{X}}_7 =& -e^{\phi_2 - \ln\left(\sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}} + \dot{t}\sinh r\cosh r\right)}\frac{\partial}{\partial r} \\ &+ \left(\coth r\dot{t} - \frac{\dot{t}^2\sinh^2 r - e^{-2\phi_1}}{\sinh^2 r}\right)e^{\phi_2 - \ln\left(\sqrt{\dot{t}^2\sinh^2 r - e^{-2\phi_1}} + \dot{t}\sinh r\cosh r\right)}\frac{\partial}{\partial \dot{t}},\end{aligned}$$

and the third first integral is, with respect to $\hat{\hat{X}}_7$ according to (51). Hence

$$\Delta_3 = \frac{e^{\phi_2}}{\sinh^2 r}\sqrt{\dot{t}^2 \sinh^2 r - e^{-2\phi_1}},$$

with

$$\phi_3 = \int -e^{-\phi_2}\left(2\dot{t}\sinh r\cosh r dr + \sinh^2 r d\dot{t}\right) = -e^{-\phi_2}\dot{t}\sinh^2 r,$$

from which

$$\dot{t} = -\frac{\phi_3 e^{\phi_2}}{\sinh^2 r}. \tag{52}$$

In the fourth and last iteration, we are in the space of the variables s, r, where

$$A_4 = \frac{\partial}{\partial s} + \sqrt{\frac{\phi_3 e^{\phi_2}}{\sinh^2 r} - e^{-2\phi_1}}\,\frac{\partial}{\partial r}, \quad \tilde{X}_2 = \frac{\partial}{\partial s},$$

and

$$\Delta_4 = \frac{\sqrt{\phi_3 e^{\phi_2} - e^{-2\phi_1}\sinh^2 r}}{\sinh r}.$$

Therefore, the fourth first integral is

$$\begin{aligned}\phi_4 &= \int \frac{\begin{vmatrix} ds & dr \\ 1 & \dfrac{\sqrt{\phi_3 e^{\phi_2} - e^{-2\phi_1}\sinh^2 r}}{\sinh r}\end{vmatrix}}{\Delta_4} \\ &= s - e^{\phi_1}\sin^{-1}\left(\frac{e^{-\phi_1}\cosh r}{\sqrt{\phi_3 e^{\phi_2} + e^{-2\phi_1}}}\right),\end{aligned}$$

Let $\alpha = \phi_3 e^{\phi_2}$, $\beta = e^{-\phi_2}$ and $\phi_4 = 0$. Hence

$$\cosh r = \frac{\sqrt{\alpha^2+\beta^2}}{\beta}\sin(\beta s), \tag{53}$$

and therefore

$$r(s) = \cosh^{-1}\left(\frac{\sqrt{\alpha^2+\beta^2}}{\beta}\sin(\beta s)\right).$$

From (53) we have

$$\sinh r = \frac{1}{\beta}\sqrt{(\alpha^2+\beta^2)\sin^2(\beta s) - \beta^2}.$$

Therefore we can rewrite Eq. (52) as

$$\dot{t} = -\frac{\alpha}{\sinh^2 r} = -\frac{\alpha\beta^2}{(\alpha^2+\beta^2)\sin^2(\beta s) - \beta^2},$$

whose integral is

$$t(s) = \ln\left(\sqrt{\frac{\alpha\tan(\beta s)+\beta}{\alpha\tan(\beta s)-\beta}}\right).$$

For the solution in the space of $(\phi(s), z(s))$, we utilise the non-solvable subalgebra (50) to solve the second set of two ODEs (48b) using the method of successive reduction according to the chain

X_2 then X_1 since $[X_2, X_1] = X_2$. Due to X_4, the system is of first order with respect to $\dot{\phi}$. We want to reduce it to be of first order with respect to $\dot{z}$ too. Thus we use the generator $X_2 = \dfrac{\partial}{\partial s}$ with the transformation

$$t = z, \;\; x = \phi, \;\; y = s. \tag{54}$$

Hence

$$\dot{x} = \frac{\dot{\phi}}{\dot{z}}, \;\; \dot{y} = \frac{1}{\dot{z}},$$

and therefore

$$\ddot{x} = \frac{1}{\dot{z}^3}(\dot{z}\ddot{\phi} - \dot{\phi}\ddot{z}), \;\; \ddot{y} = -\frac{\ddot{z}}{\dot{z}^3}.$$

Substituting for $\ddot{\phi}$ and $\ddot{z}$ from the original system gives the new system

$$\begin{aligned}
&\ddot{x} + 2\cot t\dot{x} + \cos t \sin t\dot{x}^3 = 0,\\
&\ddot{y} + \cos t \sin t\dot{x}^2\dot{y} = 0,
\end{aligned}$$

which is of first order w.r.t.$\dot{x}$ and $\dot{y}$.
In the space of the variables (t, x, y) the symmetry operators are

$$\hat{X}_1 = y\frac{\partial}{\partial y} + \dot{y}\frac{\partial}{\partial \dot{y}}, \;\; \hat{X}_2 = \frac{\partial}{\partial y}, \;\; \hat{X}_4 = \frac{\partial}{\partial x}.$$

In the second reduction we use the operator $\hat{X}_1$. Since the system is independent of y, this latter operator can be written as

$$\hat{X}_1 = \dot{y}\frac{\partial}{\partial \dot{y}},$$

which has the canonical form

$$\hat{\hat{X}}_1 = \frac{\partial}{\partial u},$$

via the transformation

$$u = \ln \dot{y}, \;\; v = \dot{x}, \;\; t = t, \tag{55}$$

where

$$\hat{X}_1 u = 1, \;\; \hat{X}_1 v = \hat{X}_1 t = 0.$$

Thus

$$\dot{u} = \frac{\ddot{y}}{\dot{y}}, \;\; \dot{v} = \ddot{x},$$

and the new reduced system is

$$\dot{v} + 2\cot t v + \cos t \sin t\, v^3 = 0, \tag{56a}$$

$$\dot{u} + \cos t \sin t\, v^2 = 0. \tag{56b}$$

The equation (56a) is Bernoulli's equation which can be solved by the transformation

$$\omega = v^{-2} \;\;\Rightarrow \dot{\omega} = -2v^{-3}\dot{v}. \tag{57}$$

Substituting this into (56a) gives

$$\dot{\omega} = \frac{4\cot t}{v^2} + 2\cos t \sin t,$$

or

$$\dot{\omega} - 4\cot t\omega = 2\cos t \sin t,$$

which is a first-order differential equation which can be solved using the integrating factor

$$\mu = \frac{1}{\sin^4 t}.$$

Thus

$$\omega(t) = \sin^2 t(c_1 \sin^2 t - 1).$$

Inserting back into (57) gives

$$v = \frac{1}{\sqrt{\omega}} = \pm\frac{1}{\sin t\sqrt{c_1 \sin^2 t - 1}}$$

and this result into Eq. (56b) yields

$$\dot{u} = -\frac{\cos t}{\sin t(c_1 \sin^2 t - 1)}$$

whose solution is

$$u(t) = \ln\left(\frac{\sin t}{\sqrt{c_1 \sin^2 t - 1}}\right) + c.$$

From the transformation (55) we find the variables $\dot{x}$ and $\dot{y}$ as

$$\dot{x} = v = \pm\frac{1}{\sin t\sqrt{c_1 \sin^2 t - 1}},$$

$$\dot{y} = e^u = \frac{c_2 \sin t}{\sqrt{c_1 \sin^2 t - 1}},$$

resulting in

$$x = \mp\frac{1}{2}\tan^{-1}\left(\frac{1 - c_1 \sin^2 t + \cos^2 t}{2\cos t\sqrt{c_1 \sin^2 t - 1}}\right) + c_3,$$

$$y = -\frac{c_2}{2\sqrt{c_1}}\tan^{-1}\left(\frac{1 - c_1 + 2\cos^2 t}{2\sqrt{c_1}\cos t\sqrt{c_1 \sin^2 t - 1}}\right) + c_4.$$

Reverting back to the space (s, ϕ, z) via the transformation (54) gives rise to

$$\phi = \mp\frac{1}{2}\tan^{-1}\left(\frac{1 - c_1 \sin^2 z + \cos^2 z}{2\cos z\sqrt{c_1 \sin^2 z - 1}}\right) + c_3,$$

which is an implicit relation between the dependent variables $\phi(s)$ and $z(s)$, where the latter can be found from the implicit relation

$$\tan\left(-\frac{2\sqrt{c_1}}{c_2}(s - c_4)\right) = \frac{1 - c_1 + 2\cos^2 z}{2\sqrt{c_1}\cos z\sqrt{c_1 \sin^2 z - 1}}.$$

Example 4.7. The motion of a unit mass particle in three-dimensional space governed by the potential $V(x, y, z) = xyz$ is described by the Hamiltonian system of equations in Newtonian form [34]

$$\ddot{x} = -yz, \quad \ddot{y} = -xz, \quad \ddot{z} = -xy. \tag{58}$$

This system admits the two-dimensional Lie group of symmetries with generators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z},$$

which is not enough to apply Lie's integration theorem or the method of successive reduction of order. We apply the method of invariants to find a solution for the system (58). Eq. (9) of Theorem (2) in [32] gives

$$Y = \frac{\partial}{\partial t} - \frac{2x}{t}\frac{\partial}{\partial x} - \frac{2y}{t}\frac{\partial}{\partial y} - \frac{2z}{t}\frac{\partial}{\partial z},$$

$$\psi_1 = -\frac{2x}{t}, \quad \psi_2 = -\frac{2y}{t}, \quad \psi_3 = -\frac{2z}{t},$$

with respect to the generator X_2. Consequently

$$Y\psi_1 = \frac{6x}{t^2},\ \ Y\psi_2 = \frac{6y}{t^2},\ \ Y\psi_3 = \frac{6z}{t^2},$$

and

$$Q_1 = \frac{6x}{t^2} + yz,\ \ Q_2 = \frac{6y}{t^2} + xz,\ \ Q_3 = \frac{6z}{t^2} + xy.$$

Now $Q_i = 0$ give the non-trivial solutions

$$\begin{aligned} x(t) &= -\frac{6}{t^2},\ \ y(t) = z(t) = \frac{6}{t^2},\\ y(t) &= -\frac{6}{t^2},\ \ x(t) = z(t) = \frac{6}{t^2},\\ z(t) &= -\frac{6}{t^2},\ \ x(t) = y(t) = \frac{6}{t^2},\end{aligned}$$

and

$$x(t) = y(t) = z(t) = -\frac{6}{t^2}.$$

5 Concluding remarks

Lie's integration theorem for a scalar ODE was extended to a system of ODEs given in the form of theorem (2.2). We noticed that the approach of this theorem is more practical as compared to the previous algorithm [26] for solving a system of ODEs due to its algebraic nature. It may happen that the Lie algebra admitted by a system of ODEs is of high dimension but the condition of transitivity is not satisfied, which prevents the implementation of Lie's integration theorem. In this case the method of successive reduction of order of a system may be an alternative choice as opposed to the remarks made in [26]. Here we have proved theorem (3.1) in which this method is shown to be applicable for a system of ODEs admitting a Lie algebra satisfying the chain (24) in order to use the symmetries admitted by the original system instead of finding new admitted symmetries for the reduced system in each iteration. If a single symmetry exists, there is always a possible reduction with respect to one variable, and if the condition (19) is satisfied, another reduction is attainable. The number of reductions depends on how many commutator brackets one can find in the order of the series (24). We observed that this chain guarantees that the Lie algebra is solvable. However, but the converse may not be true. The additional advantage of this method is that even if the Lie algebra satisfies a part of the chain (24), one can still reduce the system to a lower order. In both of the above mentioned methods, one has to find a suitable chain of derived subalgebras (except in the case of an Abelian Lie algebra) in order to utilise the symmetry generators in a correct order.

If none of the conditions of the above two methods are satisfied by the admitted Lie algebra, one may try to apply the method of invariant solutions. The extended Bluman's theorem [32] for a system of ODEs provides necessary conditions for the system to admit invariant solutions without performing integration. Here we have provided the Example (4.7) to demonstrate the use of this theorem for the sake of completeness. In case one fails to find an invariant solution, the method of differential invariants may provide first integrals, or may be used to reduce the order of the system.

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