Received ; | Revised ; | Accepted
DOI: xxx/xxxx

ARTICLE

Panic behavior induces multiple endemic states and backward bifurcation

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Abstract

This paper analyzes a SIS model for infectious diseases with two classes of individuals with different susceptibilities. It focuses in a transition function between both classes of susceptible individuals depending on the density of the infected population. A classification of all the possible bifurcation diagrams that the model can present is done. Specifically, some conditions for the simultaneous existence of backward bifurcation and multiple endemic states are shown.

KEYWORDS:
Awareness, panic behavior, multiple endemic states, backward bifurcation

1 | INTRODUCTION

Human behavior plays an important role in the spread of infectious diseases, and understanding the influence of behavior on the spread of diseases can be key to improving control efforts.[1] During the initial phase of the epidemics, most people and public mass media are in general unaware of the disease, but as the awareness of it disseminates, people respond and eventually change their behavior to reduce their susceptibility. People aware of the danger of the epidemic spread adopt practices to try to minimize their exposure to contagion, a fact that may deeply influence the epidemic pattern.[2,3]

Awareness can have very complex and sometimes unexpected effects on the dynamics of the disease spread. It can have a clearly positive influence, where disease propagation is minimized or fully stopped by various disease control measures. On the other hand, the spread of information about a disease can also result in anxiety and panic, which can lead to undesired consequences. In light of this complexity of behavioral changes in the population in the presence of awareness, it is important to understand how the concurrent spread of the disease and awareness affects disease dynamics.[4,5]

In recent years, theoretical epidemiology has called attention to the existence of multiple endemic equilibria and the necessary conditions for their existence. Some of the cases in the literature that show the existence of multiple endemic states are related to the backward bifurcation phenomena.[6,7,8,9,10,11] The existence of a backward bifurcation has important consequences in the strategies and control policies designed to eradicate or control an infectious disease because the policies of public health when this phenomenon appears change from the classically adopted ones.[12]

In this work we show an epidemiological model which considers behavioral changes of the population caused by the perception they have about the disease. The susceptible population is divided into two classes: aware susceptible and unaware susceptible. There are two transition functions between these classes, depending on the density of infected individuals in the population: the first one from the unaware to the aware class and the other one in the reverse direction.

The objective of this work is to investigate the effect of different preventive behavioral changes in the dynamics of the infectious diseases. These behavioral changes cause the appearance or disappearance of endemic equilibrium in the system described by the model. The different bifurcation diagrams associated to the system that can appear by these changes will be classified, and we will establish the conditions for the presence of backward bifurcation and multiple endemic states.

The paper is organized as follows. In Section 2 we formulate the mathematical model and present its basic properties. We derive the basic reproduction number R_0 in section 3, and we write it as a function of the number of infected individuals, in order to find the endemic states. In section 4 we explore how the parameters of the transition functions between susceptible classes generate bifurcation diagrams with some different topological characteristics, and we make a classification of them. A final discussion section concludes the paper in Section 5.

2 | THE MODEL

We consider a SIS-type epidemic model for two social groups with different susceptibilities. We divide the population, which is assumed to be constant, into susceptibles unaware of the disease, whose proportion is denoted by S_1, susceptibles aware of the disease, whose proportion is denoted by S_2, and infected individuals, whose proportion is denoted by I [13].

As the awareness disseminates, people respond to it and eventually will change their behavior to alter their susceptibility. Usually, aware susceptible individuals contract the disease at a lower rate than unaware individuals. The disease is transmitted from infected to susceptible individuals following a mass action functional form. Also, infected individuals recover through appropriate treatment. We assume that the awareness rate is proportional to the number of infected individuals whereas the depletion of the aware class is inversely proportional to the number of infected individuals. [14,15] With these assumptions, the model has the form:

$$\begin{aligned} S_1' &= -\beta_1 S_1 I - c_1(I)S_1 + c_2(I)S_2 + \gamma I \\ S_2' &= -\beta_2 S_2 I + c_1(I)S_1 - c_2(I)S_2 \\ I' &= (\beta_1 S_1 + \beta_2 S_2)I - \gamma I \end{aligned} \tag{1}$$

where β_1 and β_2 represent the disease transmission rates of each susceptible population, with $\beta_1 > \beta_2 \geq 0$; $\gamma \geq 0$ is the recovery rate; the transition rate from the unaware class to the aware class is expressed by the non decreasing function $c_1(I)$ and the reverse rate is expressed by the not increasing function $c_2(I)$, that is

$$c_1(I) \geq 0,\ c_2(I) \geq 0,\ c_1'(I) \geq 0,\ c_2'(I) \leq 0 \quad \forall I \in [0,1] \tag{2}$$

3 | EQUILIBRIA ANALYSIS

There is a disease free state $(S_1^*, S_2^*, 0)$ with

$$S_1^* = \frac{c_2(0)}{c_1(0)+c_2(0)},\ S_2^* = \frac{c_1(0)}{c_1(0)+c_2(0)} \tag{3}$$

where $c_1(0) + c_2(0) > 0$.

Let $R_1 = \beta_1/\gamma$ and $R_2 = \beta_2/\gamma$ be the basic reproduction numbers for a population consisting only of S_1 or only of S_2 individuals, respectively. Then the basic reproduction number R_0 of the uninfected population is

$$R_0 = \frac{\beta_1}{\gamma}S_1^* + \frac{\beta_2}{\gamma}S_2^* = \frac{R_1 c_2(0) + R_2 c_1(0)}{c_1(0)+c_2(0)} \tag{4}$$

To determine the endemic states, we will find a bifurcation equation for the prevalence I depending on the parameters. This equation assumes the form $F(I) = 0$ [16], where the function $F(I)$ is given by

$$F(I) = c_1(I)[1 - R_2(1-I)] + [\gamma R_2 I + c_2(I)][1 - R_1(1-I)] \tag{5}$$

It follows that $F(0) = 0$ if and only if $R_0 = 1$.

In the case $R_1 < 1$ then $F(I) > 0$ for all $I \in [0, 1]$. In the case $R_2 > 1$ then $F(0) = 0$ only if $c_1(0) = c_2(0) = 0$; but $c_1(0) + c_2(0) \neq 0$ and then from equation (4) R_0 is not well defined. Thus the interesting case occurs if $R_2 < 1 < R_1$.

The equation (5) can be solved for R_1, that is, R_1 can be expressed as a function of I, where $0 \leq I < 1$. This function takes the form

$$R_1(I) = \frac{1 + \dfrac{c_1(I)[1 - R_2(1 - I)]}{\gamma R_2 I + c_2(I)}}{1 - I} \tag{6}$$

This expression of R_1 is introduced into equation (4) to obtain the basic reproduction number as a function of the endemic state

$$R_0(I) = \frac{\left[1 + \dfrac{c_1(I)[1 - R_2(1 - I)]}{\gamma R_2 I + c_2(I)}\right] \dfrac{c_2(0)}{1 - I} + c_1(0) R_2}{c_1(0) + c_2(0)} \tag{7}$$

The function $R_0(I)$ is continuous in $[0, 1)$. It satisfies $R_0(0) = 1$ and $\lim_{I \to 1^-} R_0(I) \to \infty$.

4 | AWARENESS TRANSITION RATE

For model (1) let $c_2(I) = c_2$ be constant and let $c_1(I)$ be the function

$$c_1(I) = \frac{r}{1 + e^{-s(I - I_m)}} \tag{8}$$

where $r \geq 0$ represents the maximum transition rate from the unaware class to the aware class, $s \geq 0$ represents the reaction speed to new infected individuals and $I_m \in [0, 1]$ represents the infection level such that the change in the behavior is the most drastic.

When the disease reaches the infection level I_m, the susceptible individuals change rapidly their behavior from unaware to aware caused by panic to the disease, in order to reduce their susceptibility. Depending on the moment and the speed of this reaction, it could exist different scenarios of the disease. These scenarios are represented by the bifurcation diagrams related to the model (1) and the graphic of the function (7).

4.1 | Backward bifurcation

Finding the direction of the derivative in $I = 0$ of the function $R_0(I)$ defined in (7), we can determine the conditions for the existence of backward bifurcation in the model (1). If $R_0'(0) < 0$ then there is backward bifurcation, that is

$$c_2^2 + \frac{r}{1 + e^{sI_m}} \left[c_2 - \gamma R_2(1 - R_2)\right] + \frac{rse^{sI_m}}{\left(1 + e^{sI_m}\right)^2}(1 - R_2)c_2 < 0 \tag{9}$$

When $R_0'(0) = 0$ it is possible to obtain a function of the parameter I_m depending on the parameter s of the transition rate defined in (8). The graph of this function divides the positive quadrant of the plane $s - I_m$ in two regions: the region where backward bifurcation exists in the model (1) and the region where backward bifurcation does not exist. This function takes the form:

$$I_m(s) = \frac{\log\left[\dfrac{r[(\gamma R_2(1-R_2)-c_2)-s(1-R_2)c_2]-2c_2^2+\sqrt{r^2[(\gamma R_2(1-R_2)-c_2)-s(1-R_2)c_2]^2+4c_2^3 rs(1-R_2)}}{2c_2^2}\right]}{s} \tag{10}$$

The bifurcation diagrams associated to the points (s, I_m) located to the left of the graph of the function $I_m(s)$ present backward bifurcation. The domain of this function is $(0, s^*]$, where

$$s^* = \frac{2r(\gamma R_2(1 - R_2) - c_2) - 4c_2^2}{rc_2(1 - R_2)} \tag{11}$$

is the maximum value of the parameter s where backward bifurcation can exist in the model (1).

Since the function $I_m(s)$ is decreasing when $s^* > 0$, it follows that if the most accelerated behavioral change I_m took place when the infection level is low then it would be necessary a high reaction speed s to eliminate backward bifurcation. On the other hand, when $s^* < 0$ then $R_0'(0) > 0$ for every pair (s, I_m) on the positive quadrant of the plane $s - I_m$ and thus there is not backward bifurcation. Hence, we have the following Theorem:

Theorem 1. There is a region on the positive quadrant of the plane $s - I_m$ where the bifurcation diagram associated to every pair (s, I_m) on this region presents backward bifurcation if and only if $r > \dfrac{2c_2^2}{\gamma R_2(1 - R_2) - c_2}$.

4.2 | Classification of bifurcation diagrams

Analyzing the zeros of the function $R_0'(I)$, it is possible to divide the positive quadrant of the plane $s - I_m$ into regions according to the topological characteristics of the bifurcation diagrams associated to every pair (s, I_m). Changing the value of the parameters s and I_m while fixing the other parameters of the model (1), there are seven different classes of bifurcation diagrams:

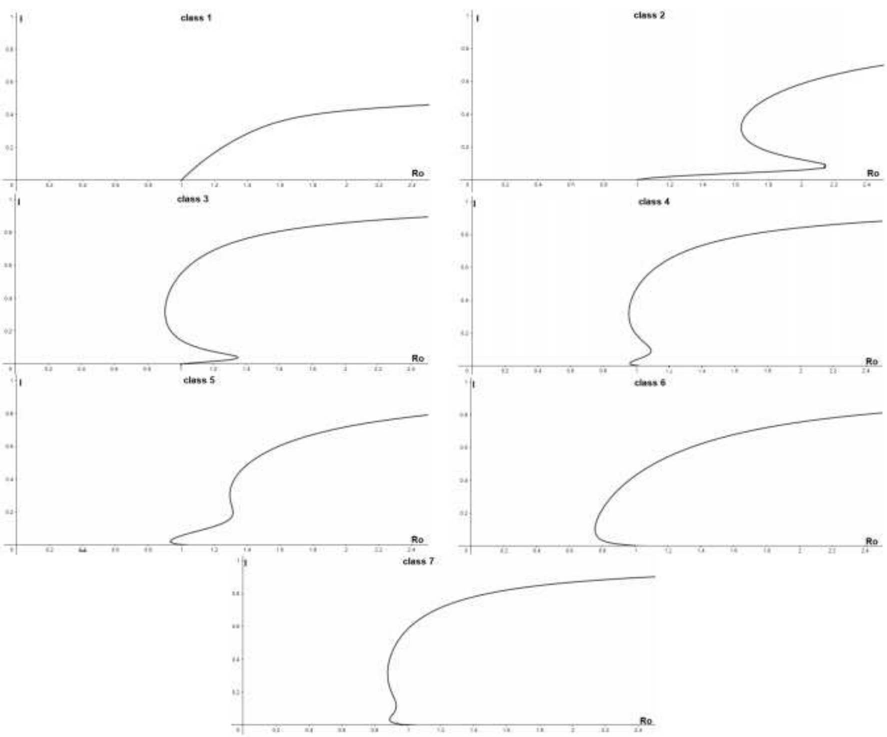

FIGURE 1 Classes of bifurcation diagrams

1. Zero roots of $R_0'(I)$
2. Two roots of $R_0'(I)$ such that $R_0(I_1) > 1$ and $R_0(I_2) > 1$
3. Two roots of $R_0'(I)$ such that $R_0(I_1) > 1$ and $R_0(I_2) < 1$
4. Three roots of $R_0'(I)$ such that $R_0(I_1) < 1$, $R_0(I_2) > 1$ and $R_0(I_3) < 1$
5. Three roots of $R_0'(I)$ such that $R_0(I_1) < 1$, $R_0(I_2) > 1$ and $R_0(I_3) > 1$
6. One root of $R_0'(I)$ such that $R_0(I_1) < 0$
7. Three roots of $R_0'(I)$ such that $R_0(I_1) < 1$, $R_0(I_2) < 1$ and $R_0(I_3) < 1$

Figure 1 shows an example of each one of these classes and Figure 2 shows the regions on the positive quadrant of the plane $s - I_m$ where each one of them can be located.

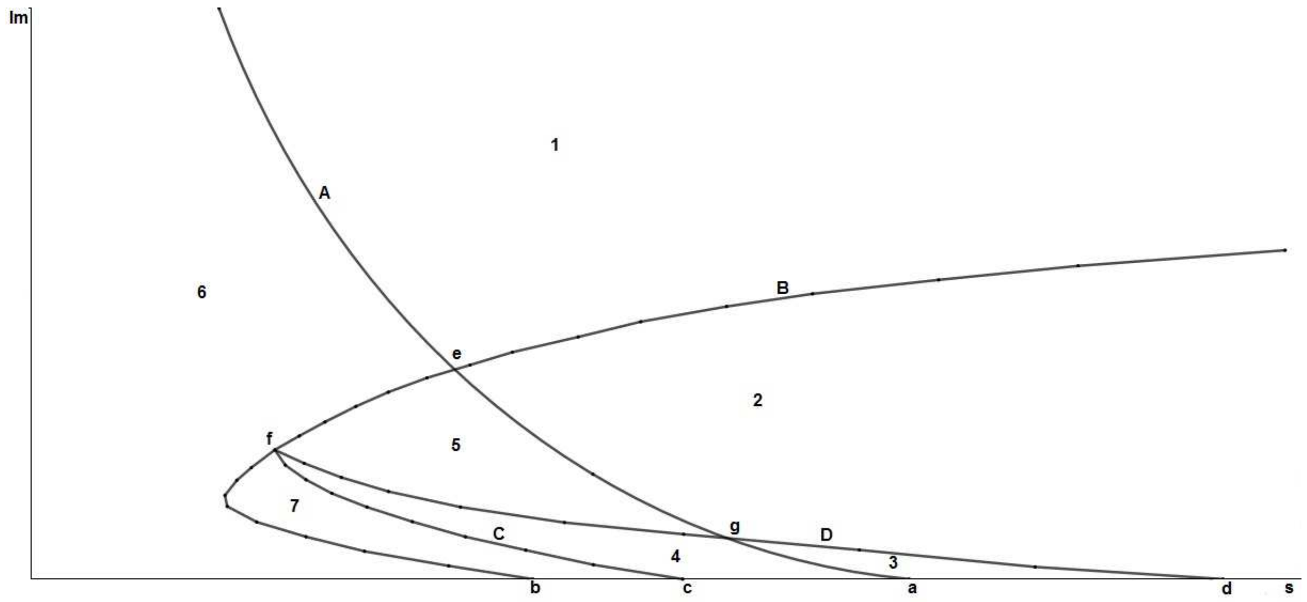

FIGURE 2 Map of classes of bifurcation diagrams on the plane $s - I_m$

From Figure 2 it can be seen that there are four different curves which separate each region according to the characteristics of their bifurcation diagrams:

- The curve A divide the plane $s - I_m$ in two regions: the region formed by the classes 1, 2 and 3 where the model (1) presents forward bifurcation, and the region formed by the classes 4, 5, 6 and 7 where the model (1) presents backward bifurcation.
- The curve B divide the plane $s - I_m$ in two regions: the region formed by the classes 1 and 6 where the model (1) does not present multiple endemic states, and the region formed by the classes 2, 3, 4, 5 and 7 where the model (1) presents multiple endemic states.
- The curve C divide the region where the model (1) presents multiple endemic states in two regions: the region formed by the class 7 where all the roots of the function $R_0'(I)$ satisfy $R_0(I) < 1$, and the region formed by the classes 2, 3, 4 and 5 where at least one root of the function $R_0'(I)$ satisfies $R_0(I) > 1$.
- The curve D divide the region where the model (1) presents multiple endemic states in two regions: the region formed by the classes 3, 4 and 7 where the highest root of the function $R_0'(I)$ satisfies $R_0(I) < 1$, and the region formed by the classes 2 and 5 where the highest root of the function $R_0'(I)$ satisfies $R_0(I) > 1$.

From Figure 2 also it is possible to identify all the natural transitions between different classes of bifurcation diagrams. Figure 3 shows the graph of these transitions.

Further, there are seven points (s, I_m) representing the limit cases of transition between different classes of bifurcation diagrams:

- a is the intersection between the curves $I_m = 0$ and A.
- b is the intersection between the curves $I_m = 0$ and B.

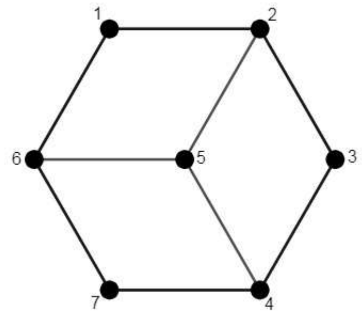

FIGURE 3 Graph of transitions between classes of bifurcation diagrams

- c is the intersection between the curves $I_m = 0$ and $\boldsymbol{C}$.
- d is the intersection between the curves $I_m = 0$ and $\boldsymbol{D}$.
- e is the intersection between the curves $\boldsymbol{A}$ and $\boldsymbol{B}$.
- f is the intersection between the curves $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$.
- g is the intersection between the curves $\boldsymbol{A}$ and $\boldsymbol{D}$.

4.2.1 | Conditions for the existence of bifurcation diagrams

To determine the existence of each class of bifurcation diagram, we will find the existence conditions of the curves $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$.

The curve $\boldsymbol{A}$ is given by the function $I_m(s)$ defined in (10). Further, $a = (s^*, 0)$ where s^* is defined by (11).

The curve $\boldsymbol{B}$ is determined by all the points (s, I_m) whose bifurcation diagrams have an unique value $I \in (0, 1)$ that satisfies $R_0^{'}(I) = 0$ and $R_0^{''}(I) = 0$. We can see that $R_0^{'}(I) = 0$ is equivalent to:

$$e^{s(I_m - I)} = \frac{-(2f(I) + g(I) + h(I)) + \sqrt{[g(I) + h(I)]^2 + 4f(I)h(I)}}{2f(I)} \tag{12}$$

where

$$\begin{aligned} f(I) &= (\gamma R_2 I + c_2)^2 \\ g(I) &= r[c_2 + \gamma[(1 - R_2(1 - I))^2 - (1 - R_2)^2 - R_2(1 - R_2)] \\ h(I) &= rs[\gamma R_2 I + c_2][1 - R_2(1 - I)][1 - I] \end{aligned} \tag{13}$$

We will analyze the values $I \in (0, 1)$ where the equation (12) holds to determine the existence of the curve $\boldsymbol{B}$, and therefore the existence of multiple endemic states in the model (1).

Since $e^{s(I_m - I)} > 0$ for all $I \in (0, 1)$ then the right side of equation (12) must be positive. From (13) when $f(I) + g(I) \geq 0$ the right side of equation (12) is not positive, and since $f(I) + g(I)$ is an increasing function in $I \in [0, 1]$ that satisfies $f(1) + g(1) > 0$, to find a value $I \in (0, 1)$ that holds the equation (12) it is necessary that $f(0) + g(0) < 0$. This is equivalent to $r > \dfrac{c_2^2}{\gamma R_2(1 - R_2) - c_2}$.

Hence, suppose $f(0) + g(0) < 0$ and let $I^* \in (0, 1)$ satisfy $f(I^*) + g(I^*) = 0$. If $I_m < I^*$ then for s sufficiently high it holds:

- If $I \in (I_m, I^*)$ then

$$e^{s(I_m-I)} < \frac{-(2f(I)+g(I)+h(I)) + \sqrt{[g(I)+h(I)]^2 + 4f(I)h(I)}}{2f(I)} \tag{14}$$

- If $I \in (I^*, 1)$ then

$$e^{s(I_m-I)} > 0 > \frac{-(2f(I)+g(I)+h(I)) + \sqrt{[g(I)+h(I)]^2 + 4f(I)h(I)}}{2f(I)} \tag{15}$$

Further, if $s > s^*$ then

$$e^{sI_m} \geq 1 > \frac{-(2f(0)+g(0)+h(0)) + \sqrt{[g(0)+h(0)]^2 + 4f(0)h(0)}}{2f(0)} \tag{16}$$

From (14) to (16) it follows that when $I_m < I^*$ there are two values $I_1, I_2 \in (0, I^*)$ where the equation (12) holds. When $I_1 = I_2$ then $R_0'(I_1) = 0$ and $R_0''(I_1) = 0$ are satisfied and therefore every point (s, I_m) where both conditions hold is on the curve B.

On the other hand, when $I_m > I^*$ and $s > s^*$ it holds

- If $I \in (0, I^*)$ then

$$e^{s(I_m-I)} \geq 1 > \frac{-(2f(I)+g(I)+h(I)) + \sqrt{[g(I)+h(I)]^2 + 4f(I)h(I)}}{2f(I)} \tag{17}$$

- If $I \in (I^*, 1)$ then

$$e^{s(I_m-I)} > 0 > \frac{-(2f(I)+g(I)+h(I)) + \sqrt{[g(I)+h(I)]^2 + 4f(I)h(I)}}{2f(I)} \tag{18}$$

From (17) and (18) the equation (12) does not hold for all $I \in (0, 1)$ and therefore the curve B does not exist when $I_m > I^*$.

We summarize these results in the following theorem.

Theorem 2. There is a region on the positive quadrant of the plane $s - I_m$ where the bifurcation diagram associated to every pair (s, I_m) on this region presents multiple endemic states if and only if $r > \dfrac{c_2^2}{\gamma R_2(1-R_2) - c_2}$. Further, this region is located below $I_m = I^*$.

It can be shown that when the reaction speed s is low, the model (1) does not present multiple endemic states. If $s = 0$ then $c_1(I) = r/2$ and hence

$$R_0'(I) = \frac{c_2\left[(\gamma R_2 I + c_2)^2 + \frac{r}{2}[c_2 + \gamma[(1 - R_2(1-I))^2 - (1-R_2)^2 - R_2(1-R_2)]]\right]}{\left[\frac{r}{2} + c_2\right][\gamma R_2 I + c_2]^2[1-I]^2} \tag{19}$$

The numerator of equation (19) is a polynomial of second grade in the proportion I. It assumes the form $\overline{A}I^2 + \overline{B}I + \overline{C}$ where $\overline{A} > 0$ and $\overline{B} > 0$. Thus:

- In the case $\overline{C} > 0$ the function $R_0'(I)$ does not have positive roots.
- In the case $\overline{C} < 0$ the function $R_0'(I)$ has an unique positive root.

Since the equation (19) has at most one root, it follows that when $s = 0$ the model (1) does not have multiple endemic states. By continuity, there are not multiple endemic states for small values of s. Therefore the curve B never intersects the axis I_m. Hence we have the following result:

Corollary 1. If $c_1(I) = c_1$ and $c_2(I) = c_2$ then the model (1) does not have multiple endemic states.

From Theorem 2 and Corollary 1, we have the following result:

Corollary 2. For every $I_m < I^*$ there is a $s_{I_m} > 0$ such that the bifurcation diagram associated to (s, I_m) presents multiple endemic states if and only if $s > s_{I_m}$.

The curve B is formed with the points (s_{I_m}, I_m). Now, we suppose there are multiple endemic states in the model (1). Using the transition rate (8) it can be shown that there are at most three roots of $R_0'(I) = 0$. We will use these roots to analyse the characteristics of the curves C and D.

The curve C must be located in the region on the plane $s - I_m$ where the bifurcation diagrams associated to the model (1) present backward bifurcation. If this curve were not in that region, there would exist bifurcation diagrams with forward bifurcation and multiple endemic states such that they do not have roots of $R_0'(I)$ with $R_0(I) \geq 1$. But this is impossible because the first root of $R_0'(I)$ satisfies $R_0(I) > 1$ by the forward bifurcation.

Therefore, since there are at most three roots of the function $R_0'(I)$, it follows that the bifurcation diagrams associated to the points on the curve C have three roots I_1, I_2 and I_3 of $R_0'(I)$ such that $R_0(I_1) < 1$, $R_0(I_2) = 1$ and $R_0(I_3) \leq 1$.

The highest root of $R_0'(I)$ satisfies $R_0(I) = 1$ in the bifurcation diagrams associated to the points on the curve D. Let $(\bar{s}, \bar{I}_m)$ be in the intersection of the curves C and D. The roots of $R_0'(I)$ must satisfy $R_0(I_1) < 1$, $R_0(I_2) = 1$ and $R_0(I_3) = 1$, but this can only be possible if $I_2 = I_3$. Thus $R_0'(I_2) = 0$ and $R_0''(I_2) = 0$. Hence $(\bar{s}, \bar{I}_m)$ is located on the curve B, and therefore the point f exists and it is defined by $f = (\bar{s}, \bar{I}_m)$.

In order to find the conditions for the existence of these curves C and D, we will show the existence of the points c and d located on the later curves respectively. To do this, we will use the number of roots of $R_0(I) = 1$. This equation is equivalent to

$$\frac{1}{1 + e^{s(I_m - I)}} = \frac{\gamma R_2 I + c_2}{1 - R_2(1 - I)} \left[\frac{(1 - R_2)(1 - I)}{c_2(1 + e^{sI_m})} - \frac{I}{r} \right] \tag{20}$$

When $I_m = 0$, from equation (20) it is possible to obtain a function of the parameter s depending on I. Using this function, one can determine the number of nonzero roots of $R_0(I) = 1$ for every $s > 0$, in order to find the location and number of endemic states of the model (1). This function is given by

$$s(I) = -\frac{\log(f(I) - 1)}{I} \tag{21}$$

where

$$f(I) = \frac{R_2 I + (1 - R_2)}{(\gamma R_2 I + c_2) \left[\frac{1 - R_2}{2c_2} - I \left(\frac{1}{r} + \frac{1 - R_2}{2c_2} \right) \right]} \tag{22}$$

The function $f(I)$ is continuous in $I \in \left[0, I_\infty\right)$ where $I_\infty = r(1 - R_2)/[r(1 - R_2) + 2c_2]$. It satisfies $f(0) = 2$ and $\lim_{I \to I_\infty^-} f(I) = \infty$. Hence $\log(f(0) - 1) = 0$.

It is easy to see that $f'(0) < 0$ if and only if $s^* > 0$, where s^* is defined by (11). Thus, if $s^* > 0$ then there is an $I_0 \in (0, I_\infty)$ such that $f(I_0) = 2$, $f(I) < 2$ for every $I \in (0, I_0)$ and $f(I) > 2$ for every $I \in (I_0, I_\infty)$.

From equation (21) it follows that $\lim_{I \to 0^+} s(I) = s^*$ and $s(I_0) = 0$. If $f(I) > 1$ for every $I \in [0, I_0]$ then $s(I)$ is a continuous function in $I \in (0, I_0)$. Further $\lim_{I \to 0^+} s'(I) < 0$.

It can be shown that the function $s'(I)$ has at most two roots in $I \in (0, I_0)$. Suppose the function $s'(I)$ has two roots I_c and I_d. We define the points $c = (s(I_c), 0)$ and $d = (s(I_d), 0)$ on the plane $s - I_m$ by the values $s(I_c)$ and $s(I_d)$ where the function $s(I)$ reaches its local minimum and its local maximum respectively.

We can determine the relative position of the point c respect to the points a, b and d. Since $\lim_{I \to 0^+} s'(I) < 0$ then $s(I_c) < s^*$. Further $I_c \leq I_d$ and thus $s(I_c) \leq s(I_d)$. Therefore the point c is located to the left of the points a and d on the axis s. In the case $I_c < I_d$, it follows that $R_0'(I_c) = 0$ but $R_0''(I_c) < 0$, and thus the point c is not located on the curve B. Therefore the point c is located to the right of the point b on the axis s.

The point d can be located to the left or to the right of the point a on the axis s. From the graphic of the function $s(I)$, we determine the existence of different the regions on the plane $s - I_m$ by the relative position of the point d to the point a. In the case $s(I_d) < s^*$ it follows that

- If $0 < s < s(I_c)$ then there is only a root of $R_0(I) = 1$ (classes 6 and 7).
- If $s(I_c) < s < s(I_d)$ then there are three roots of $R_0(I) = 1$ (class 4).
- If $s(I_d) < s < s^*$ then there is only a root of $R_0(I) = 1$ (class 5).
- If $s^* < s$ then there are not roots of $R_0(I) = 1$ (class 2).

Thus there are not values of s where $R_0(I) = 1$ has two roots. Hence it does not exist the region 3 and therefore the point g does not exist. In the case $s(I_d) > s^*$ it follows that

- If $0 < s < s(I_c)$ then there is only a root of $R_0(I) = 1$ (classes 6 and 7).
- If $s(I_c) < s < s^*$ then there are three roots of $R_0(I) = 1$ (class 4).
- If $s^* < s < s(I_d)$ then there are two roots of $R_0(I) = 1$ (class 3).
- If $s(I_d) < s$ then there are not roots of $R_0(I) = 1$ (class 2).

Thus the region 3 exists and then the point g exists. Since the points f and g exist then the region 5 exists, but it does not intersect the axis s. In this case, all the regions with backward bifurcation and multiple endemic states presented previously exist.

Finally, we analyse the cases when the function $s'(I)$ does not have two different roots.

In the case $I_c = I_d$ there is a double root of $s'(I)$, and then $R_0'(I_c) = 0$ and $R_0''(I_c) = 0$. Thus c and d are located on the curve B. Since they are located in the intersection of the curves B, C and D, therefore the point f coincides with the later points.

If $s'(I)$ does not have roots, then the points c and d do not exist and thus the points f and g do not exist too. Hence, if $s < s^*$ then there is an only root of $R_0(I) = 1$, and if $s > s^*$ then there are not roots of $R_0(I) = 1$. Therefore the regions 3, 4 and 7 do not exist.

If there is an $I \in (0, I_0)$ such that $f(I) < 1$ then there is not superior bound to the function (21). Thus, the point d does not exist for a finite value of s. Hence, the curve D does not intersect the axis s. Therefore, for every $s > s^*$ exists an $I_m > 0$ such that (s, I_m) is on the region 3.

4.2.2 | Simultaneous existence of bifurcation diagrams

From the results presented in the last subsection, it can be obtained the possible divisions of the positive quadrant of the plane $s - I_m$ into regions of bifurcation diagrams classes, depending on the value of the parameter r of the function (8).

If $r < c_2^2/[\gamma R_2(1 - R_2) - c_2]$ then from Theorem 2 there are not classes with multiple endemic states. Also it is satisfied $r < 2c_2^2/[\gamma R_2(1 - R_2) - c_2]$ and then from Theorem 1 there are not classes with backward bifurcation. Therefore, on the plane $s - I_m$ there is only the region 1.

If $c_2^2/[\gamma R_2(1 - R_2) - c_2] < r < 2c_2^2/[\gamma R_2(1 - R_2) - c_2]$ then from Theorems 1 and 2 there are classes with multiple endemic states, but there are not classes with backward bifurcation. Since the curve D intersects the curve B at the same time with the curve C, and since there are not classes with backward bifurcation, then the curve C does not exist and hence the curve D does not exist too. Therefore, on the plane $s - I_m$ there are only the regions 1 and 2.

If $r > 2c_2^2/[\gamma R_2(1 - R_2) - c_2]$ then from Theorems 1 and 2 there are classes with backward bifurcation and classes with multiple endemic states. From Corollary 1 since the curve B does not intersect the axis I_m, then the curves A and B do not intersect until r reaches a value r_e where this situation occurs, and then the point e appears.

If $r < r_e$ then the curves A and B do not intersect. Therefore on the plane $s - I_m$ there are only the regions 1, 2 and 6. If $r > r_e$ then the curves A and B intersect, thus the point e exists. By increasing the value of r, then the different classes located in the region with multiple endemic states start to appear.

Since the point c is located to the left of the point a on the axis s, then c is not located on the curve A. It follows that there is a value r_f such that if $r_e < r < r_f$ then the point f does not exist and hence the points c, d and g do not exist too. Therefore on the plane $s - I_m$ there are only the regions 1, 2, 5 and 6.

If $r > r_f$ then there exists the point f, and thus the points c and d exist too. Hence, the regions 4 and 7 appear. Further, if the point d is located to the right of the point a on the axis s then the point g appears and therefore the region 3 appears. Therefore, on the plane $s - I_m$ there are the regions from 1 to 7.

5 | CONCLUSIONS

Awareness among the human population can change the pattern of disease spread. If the information about the presence of infectious diseases is disseminated in the population, people adapt their behavior as a result of their awareness to the disease.[17]

In order to determine the necessary control policies that could eradicate a disease or decrease the infection level, it is important to identify the disease scenario by its associated bifurcation diagram. In the model presented in this work, we decided to focus on the transition rate from the unaware to the aware class defined in (8) to do a classification of all the possible classes of bifurcation diagrams that can appear in this model, selecting the reaction speed s and the moment of change in behavior I_m as the principal parameters. Changes in these parameters are decisive for the transformation of the bifurcation diagrams.

Anticipating the moment of panic reaction in the population and accelerating the speed of this reaction could be a good policy to decrease the infection level. But this policy should be applied carefully, since if the reaction speed is not fast enough, it can generate backward bifurcation in the system. For this reason, it is important to select an adequate control policy that could generate better disease scenarios and eliminate the worst ones.

ACKNOWLEDGMENTS

David Benítez acknowledges the support by CONACyT, Mexico, for the Graduate Fellowship Grant 298170.

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