Existence of global weak solutions for the high frequency and small displacement oscillation fluid-structure interactions systems*

Lin Shen, Shu Wang†, Yuehong Feng

College of Applied Science, Beijing University of Technology, Ping Le Yuan 100 Beijing 100124, P. R. China;

March 10, 2020

Abstract The purpose of this paper is to study the fluid-structure interaction (FSI) problem which is a simplified model to describe high frequency and small displacement oscillation of elastic structure in fluids. The elastic structure displacement is modeled by a fourth order nonlinear hyperbolic square equations, the motion of fluid is modeled by the time-dependent incompressible Navier-Stokes equations. we prove the existence of at least one weak solutions (global in time) to this problem by compactness method. The result both holds for two-dimensional and three-dimensional cases.

Key words FSI; Global Weak Solutions; Navier-Stokes equations; Coupled PDEs; Compactness method

2000 Mathematics Subject Classification 35Q30, 74F10, 76D03, 76D05.

---

*Project supported by NSFC (No. 11831003, No. 11771031, No. 11531010 and No. 11726625) of China and NSF (No. 2017-ZJ-908) of Qinghai Province.

† Corresponding author: Shu Wang, E-mail:wangshu@bjut.edu.cn

1 Introduction

We investigate in this paper the existence of global weak solutions for fluid-structure interaction (FSI) systems, which takes the following form

$$\begin{cases} u_{tt} + \nabla \cdot \sigma = 0, & x \in \Omega_s \times (0,T), \\ v_t + v \cdot \nabla v - \Delta v + \nabla p = 0, & x \in \Omega_f \times (0,T), \\ \nabla \cdot v = 0, & x \in \Omega_f \times (0,T), \\ v = 0, & x \in \Gamma_{out} \times (0,T), \\ u_t = v(x), & x \in \Gamma \times (0,T), \\ \nabla u \cdot n = 0, & x \in \Gamma \times (0,T), \\ -\sigma n = \frac{1}{2} v \cdot nv - n \cdot \nabla v + pn, & x \in \Gamma \times (0,T), \\ u(x,0) = u_0, u_t(x,0) = u_1, & x \in \Omega_s, \\ v(x,0) = v_0, & x \in \Omega_f. \end{cases} \tag{1.1}$$

where u is the displacement of elastic structure, v is the incompressible fluid velocity, p denotes the pressure of fluid, $\sigma = \nabla(a(x)\Delta u) - \frac{1}{2}|\nabla u|^2 \nabla u$ is the stress tensor. we consider that elastic structure Ω_s and fluid Ω_f is in a fixed connected bounded domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, and $\Omega_s \cap \Omega_f = \emptyset$, Denote the interface of structure and fluid as $\Gamma = \partial\Omega_s \cap \partial\Omega_f$, the interface of fluid and total domain Ω as $\Gamma_{out} = \partial\Omega_f \setminus \Gamma$. n is the outer normal vector of fluids boundary Γ, the d-order matrix $a(x)$ satisfies the uniformly elliptic condition. For the derivation of $(1.1)_1$, see [1].

In this paper, we assume no rigid motion and the structure undergoes only the high frequency, small displacement oscillation. Then, since the motion of FSI is wholly determined by the small elastic displacement, one may assume that the interface is stationary(see [2] for more information). Thus, system (1.1) can simulate the high frequency and small displacement oscillation of an elastic structure immersed in an incompressible fluid.

The high frequency and small displacement oscillation of a structure immersed in an incompressible fluid have been studied extensively, see [2–5] and therein references. In chapter one, section 9 of [3], Lions discusses FSI model in geology, which the structure motion is described by the wave equation $u_{tt} - \Delta u = g$. The existence of global weak solution has been proved and the uniqueness in two-dimensional space has been got in [3]. In [2], the structure motion is described by the following elastodynamics equation with St.Ventant-Kirchhoff elastic material

$$u_{tt} = \mu \nabla \cdot (\nabla u + \nabla u^T) - \lambda \nabla(\nabla \cdot u) + f.$$

The existence and uniqueness of the global weak solution have been derived, and the existence of an L^2 integrable pressure field has been established after the verification of an inf-sup condition in [2]. Other related studies about fixed interface can be found in the references [4–10].

When the structure have a rigid motion or large displacement oscillation, the motion would result in a moving fluid-structure interface. For the existence of solutions for moving FSI problems, see [11–20] and therein references. For a numerical algorithm for solving moving FSI problems, see [21–23] and therein references.

In the paper, we are interested in the high frequency and small displacement oscillation FSI model. The main purpose of this paper is to establish the global existence of the weak solution to FSI systems in two and three-dimension. The FSI problem with moving interface shall be addressed in later work.

2 Main results

We begin by specifying our notation. For any given domain $\Omega_f, \Omega_s \subset R^d, \Omega = \bar{\Omega}_s \cup \bar{\Omega}_f, \Gamma = \bar{\Omega}_s \cap \bar{\Omega}_f$, we define

$$
\begin{aligned}
&\mathbf{L}_s^m = (L^m(\Omega_s))^d, \mathbf{H}_s^m = (H^m(\Omega_s))^d,\\
&\mathbf{L}_f^m = (L^m(\Omega_f))^d, \mathbf{H}_f^m = (H^m(\Omega_f))^d,\\
&\mathbf{L}^m = (L^m(\Omega))^d, \mathbf{H}^m = (H^m(\Omega))^d,\\
&\mathbf{L}_\Gamma^m = (L^m(\Gamma))^d, \mathbf{H}_\Gamma^m = (H^m(\Gamma))^d,\\
&\mathbf{V}_f^1 = \left\{\psi \in \mathbf{H}_f^1 | \psi(x) = 0, x \in \Gamma_{out}, \nabla \cdot \psi = 0, x \in \Omega_f\right\},\\
&\mathbf{V}_f^2 = \left\{\psi \in \mathbf{H}_f^2 | \psi(x) = 0, x \in \Gamma_{out}, \nabla \cdot \psi = 0, x \in \Omega_f\right\},\\
&\mathbf{V}_s^2 = \left\{\varphi \in \mathbf{H}_s^2 | \nabla\varphi \cdot n = 0, x \in \Gamma\right\},\\
&\mathbf{V} = \left\{\phi \in \mathbf{H}^2 | \phi(x) = 0, x \in \Gamma_{out}, \nabla \cdot \phi = 0, x \in \Omega_f, \nabla\varphi \cdot n = 0, x \in \Gamma\right\},\\
&(f, \phi) = \int_\Omega f\phi dx, (u, \varphi)_s = \int_{\Omega_s} u\varphi dx, (v, \psi)_f = \int_{\Omega_f} v\psi dx,
\end{aligned}
$$

Then, we give the weak formulations of (1.1) as follows.

Definition 2.1 (Weak Solution). *We say function*

$$u \in L^\infty(0, T; \mathbf{V}_s^2), v \in L^\infty(0, T; \mathbf{L}_f^2) \cap L^2(0, T; \mathbf{V}_f^1),$$

is a weak solution of (1.1) *provided*

(i) $u_t \in L^\infty(0, T; \mathbf{L}_s^2), \{u_{tt}, v_t\} \in L^2(0, T; \mathbf{V}')$;

(ii)

$$
\begin{aligned}
&(u_{tt}, \varphi)_s + (a(x)\Delta u, \Delta\varphi)_s + \frac{1}{2}\left(|\nabla u|^2 \nabla u, \nabla\varphi\right)_s\\
&\qquad + (v_t, \psi)_f + (v \cdot \nabla v, \psi)_f + (\nabla v, \nabla\psi)_f - \frac{1}{2}\int_\Gamma (v \cdot n) v\psi dx = 0,
\end{aligned}
\tag{2.1}
$$

for each $\varphi \in \mathbf{V}_s^2, \psi \in \mathbf{V}_f^1, \psi = \varphi, x \in \Gamma$ *and a.e.* $0 \le t \le T$.

(iii) $u_t = v, x \in \Gamma \times (0, T), u(x, 0) = u_0, u_t(x, 0) = u_1, x \in \Omega_s \times (0, T), v(x, 0) = v_0, x \in \Omega_f \times (0, T)$.

Remark:(1) Assume $\varphi \in (C_0^\infty(\Omega_s))^d$ and $\psi = 0$, we can deduce $(1.1)_1$ from (2.1); assume $\psi \in (C_0^\infty(\Omega_f))^d$ and $\varphi = 0$, we can deduce $(1.1)_2, (1.1)_3$ from (2.1).

(2)Multiply $(1.1)_1, (1.1)_2$ by φ, ψ, respectively, sum the two equations to find

$$
\begin{aligned}
&(u_{tt}, \varphi)_s + (a(x)\Delta u, \Delta\varphi)_s + \frac{1}{2}\left(|\nabla u|^2 \nabla u, \nabla\varphi\right)_s + (v_t, \psi)_f + (v\cdot\nabla v, \psi)_f + (\nabla v, \nabla\psi)_f \\
&\qquad + \int_\Gamma \left(\nabla\left(a(x)\Delta u\right) - \frac{1}{2}|\nabla u|^2\nabla u\right) n\varphi d\Gamma + \int_\Gamma \left(p - \nabla v\right) n\psi d\Gamma = 0
\end{aligned}
\tag{2.2}
$$

Comparing (2.1) and (2.2), we deduce $(1.1)_6$ and $(1.1)_7$ by using $\psi = \varphi, x \in \Gamma$ and the arbitrariness of ψ, φ .

Now, we state our main Theorems as follows.

Theorem 2.1. *Assume* $a(x) \in \mathbf{L}^\infty(\Omega_s), a(x)\Delta$ *is uniformly elliptic operator , and* $u_0 \in \mathbf{H}^2(\Omega_s), u_1 \in \mathbf{L}^2(\Omega_s), v_0 \in \mathbf{L}^2(\Omega_f)$ *,there exists a solution of* (1.1). *Moreover*

$$
\begin{cases}
u \in L^\infty(0,T;\mathbf{V}_s^2), \\
u_t \in L^\infty(0,T;\mathbf{L}_s^2), \\
v \in L^\infty(0,T;\mathbf{L}_f^2) \cap L^2(0,T;\mathbf{V}_f^1), \\
\{u_{tt}, v_t\} \in L^2(0,T;\mathbf{V}').
\end{cases}
$$

Remark: uniqueness is complex since the coupled FSI systems has strong nonlinear term.

We now briefly outline the proof in the following:

Step 1: employing Galerkin's method to construct solutions of certain finite-dimensional approximations to (1.1);

Step 2: using the energy method to find the uniform estimates of the finite-dimensional approximations solutions;

Step 3: using compactness method to obtain the weak solutions of (1.1), the main difficulty is dealing with nonlinear terms.

In order to prove Theorem 2.1, we list a few basic tools for bounded domains to be used in the subsequent sections. We start with the well-known Gagliardo-Nirenberg interpolation inequality for bounded domains (see, e.g. [24]).

Lemma 2.1. *Let* $\Omega_0 \subset R^d$ *be a bounded domain with smooth boundary. Let* $1 \le p, q, r \le \infty$ *be real numbers and* $j \le m$ *be non-negative integers. If a real number* α *satisfies*

$$
\frac{1}{p} - \frac{j}{d} = \alpha\left(\frac{1}{r} - \frac{m}{d}\right) + (1-\alpha)\frac{1}{q}, \quad \frac{j}{m} \le \alpha \le 1
$$

Then

$$
\left\|\mathrm{D}^j f\right\|_{L^p(\Omega_0)} \le C_1 \left\|\mathrm{D}^m f\right\|_{L^r(\Omega_0)}^\alpha \|f\|_{L^q(\Omega_0)}^{1-\alpha} + C_2\|f\|_{L^s(\Omega_0)}
$$

where $s > 0$, *and the constants* C_1 *and* C_2 *depend upon* Ω_0 *and the indices* p, q, r, m, j, s *only.*

According to the Gagliardo-Nirenberg interpolation inequality and trace theorem, we can deduce the following Lemma without any essential difficulty.

Lemma 2.2. *Let $\Omega_0 \subset R^d$ be a bounded domain with smooth boundary, $\frac{2}{p} + \frac{1}{d} = 1, 0 < \varepsilon < \frac{1}{2}$, assume $\phi \in \mathbf{H}^2(\Omega_0), \varphi \in \mathbf{H}^1(\Omega_0)$, then*

$$\|\varphi\|_{\mathbf{L}^p(\Omega_0)} \le C_1 \|\nabla\varphi\|^{\frac{1}{2}}_{\mathbf{L}^2(\Omega_0)} \|\varphi\|^{\frac{1}{2}}_{\mathbf{L}^2(\Omega_0)} + C_2\|\varphi\|_{\mathbf{L}^2(\Omega_0)} \le c\|\varphi\|_{\mathbf{H}^1(\Omega_0)} \tag{2.3}$$

$$\|\phi\|_{\mathbf{L}^\infty(\Omega_0)} \le c\|\varphi\|_{\mathbf{H}^2(\Omega_0)}, \tag{2.4}$$

$$\|\nabla\phi\|_{\mathbf{L}^d(\Omega_0)} \le c\|\varphi\|_{\mathbf{H}^{\frac{d}{2}}(\Omega_0)} \le c\|\varphi\|_{\mathbf{H}^1(\Omega_0)}, \tag{2.5}$$

$$\|\phi\|_{\mathbf{L}^3(\Omega_0)} \le c_3\|\phi\|_{\mathbf{H}^{\frac{1}{2}}(\Omega_0)} \le c_4\|\phi\|_{\mathbf{H}^{1-\varepsilon}(\Omega_0)}. \tag{2.6}$$

Lemma 2.3. *Let $\Omega_0 \subset R^d$ be a bounded domain with smooth boundary Γ_0, assume $\phi \in \mathbf{H}^2(\Omega_0), \varphi \in \mathbf{H}^1(\Omega_0)$, thus*

$$\|\varphi\|_{\mathbf{L}^3(\Gamma_0)} \le c_1\|\varphi\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma_0)} \le c_2\|\varphi\|_{\mathbf{H}^1(\Omega_0)}, \tag{2.7}$$

$$\|\phi\|_{\mathbf{L}^6(\Gamma_0)} \le c_3\|\phi\|_{\mathbf{H}^1(\Gamma_0)} \le c_4\|\phi\|_{\mathbf{H}^2(\Omega_0)}. \tag{2.8}$$

Now, we give the Aubin-Lions Lemma (see, e.g. [3]).

Lemma 2.4. *Suppose B_0, B, B_1 are Banach Space, if*
(i)$\{u_i\}_{i=1}^\infty$ is bounded in $L^{p_0}(0,T;B_0)$;
(ii)$\{u_{i,t}\}_{i=1}^\infty$ is bounded in $L^{p_1}(0,T;B_1)$;
(iii) $B_0 \circlearrowleft\circlearrowleft B \circlearrowleft B_1$,
then $\{u_i\}_{i=1}^\infty$ admits a strongly converging subsequence in $L^{p_0}(0,T;B)$, provided $p_0 < \infty, p_1 > 1$.

3 The proof of Theorem 2.1

Step 1: Galerkin approximations

We construct our weak solution by first solving a finite dimensional approximation. Assume

$$\{\phi_j\}_{j=1}^\infty \text{ is an orthogonal basis of } \mathbf{V}, \tag{3.1}$$

and

$$\{\phi_j\}_{j=1}^\infty \text{ is an orthonormal basis of } \mathbf{L}^2. \tag{3.2}$$

the function $\varphi_j = \phi_j(x), x \in \Omega_s, \psi_j = \phi_j(x), x \in \Omega_f (j = 1, 2, \cdots)$, $\psi_j = \varphi_j, x \in \Gamma$

Fix a positive integer m. We will look for function $\{u_m, v_m\} : [0,T] \to \mathbf{V}$ of the form

$$u_m(t) = \sum_{j=1}^m a_j^m(t)\varphi_j(x), v_m(t) = \sum_{j=1}^m b_j^m(t)\psi_j(x), \tag{3.3}$$

where we hope to select the vector $a_j^m(t), b_j^m(t) (0 \le t \le T, j = 1, 2, \cdots)$ so that

$$\begin{aligned} &(u_{m,tt}, \varphi_j)_s + \big(a(x)\Delta u_m, \Delta\varphi_j\big)_s + \frac{1}{2}(|\nabla u_m|^2\nabla u_m, \varphi_j)_s \\ &\quad + \int_\Gamma \Big(\nabla\big(a(x)\Delta u_m\big) - \frac{1}{2}|\nabla u_m|^2\nabla u_m\Big) n\varphi_j d\Gamma = 0, \end{aligned} \tag{3.4}$$

$$(v_{m,t},\psi_j)_f+(v_m\cdot\nabla v_m,\psi_j)_f+(\nabla v_m,\nabla\psi_j)_f-\frac{1}{2}\int_\Gamma(v_m\cdot n_f)v_m\psi_j d\Gamma \\ -\int_\Gamma\Big(\nabla\big(a(x)\Delta u_m\big)-\frac{1}{2}|\nabla u_m|^2\nabla u_m\Big)n\psi_j d\Gamma=0, \tag{3.5}$$

and when $m\to\infty$,

$$u_m(0)=u_{0m}=\sum_j^m\alpha_j^m\varphi_j(x)\to u_0,\quad \text{in } \mathbf{H}^2(\Omega_s); \tag{3.6}$$

$$u_{m,t}(0)=u_{1m}=\sum_j^m\beta_j^m\varphi_j(x)\to u_1,\quad \text{in } \mathbf{L}^2(\Omega_s); \tag{3.7}$$

$$v_m(0)=v_{0m}=\sum_j^m\gamma_j^m\psi_j(x)\to v_0,\quad \text{in } \mathbf{L}^2(\Omega_f). \tag{3.8}$$

Lemma 3.1. *Give $a(x),u_0,u_1,v_0$ satisfying the assumptions of Theorem 2.1, for each integer $m=1,2,\cdots$, there exists a unique function u_m,v_m of the form (3.3) satisfying (3.4)-(3.8) for $0\le t\le t_m$.*

Proof. We first note form (3.2) that

$$(u_{m,tt},\varphi_j)_s=a_{j,tt}^m(t),(v_{m,t},\psi_j)_f=b_{j,t}^m(t). \tag{3.9}$$

Let us now define the time-dependent form

$$B_1[u_m,\varphi_j;t]:=\big(a(x)\Delta u_m,\Delta\varphi_j\big)_s+\frac{1}{2}(|\nabla u_m|^2\nabla u_m,\varphi_j)_s, \tag{3.10}$$

$$B_2[u_m,\varphi_j;t]:=\int_\Gamma\Big(\nabla\big(a(x)\Delta u_m\big)-\frac{1}{2}|\nabla u_m|^2\nabla u_m\Big)\cdot n\varphi_j d\Gamma, \tag{3.11}$$

$$B_3[v_m,\psi_j;t]:=(v_m\cdot\nabla v_m,\psi_j)_f+(\nabla v_m,\nabla\psi_j)_f \tag{3.12}$$

$$B_4[v_m,\psi_j;t]:=-\frac{1}{2}\int_\Gamma(v_m\cdot n_f)v_m\psi_j d\Gamma-B_2[u_m,\varphi_j;t] \tag{3.13}$$

Furthermore

$$B_1[u_m,\varphi_j;t]:=\sum_{i=1}^m a_i^m\big(a\Delta\varphi_i,\Delta\varphi_j\big)_s+\frac{1}{2}\sum_{i,k=1}^m(a_i^m)^2a_k^m(|\nabla\varphi_i|^2\nabla\varphi_k,\varphi_j)_s, \tag{3.14}$$

$$B_2[u_m,\varphi_j;t]:=\int_\Gamma\Big(\sum_{i=1}^m a_i^m\nabla\big(a\Delta\varphi_i\big)-\frac{1}{2}\sum_{i,k=1}^m(a_i^m)^2a_k^m|\nabla\varphi_i|^2\nabla\varphi_k\Big)\cdot n\varphi_j d\Gamma, \tag{3.15}$$

$$B_3[v_m,\psi_j;t]:=\sum_{i,k=1}^m b_i^m(t)b_k^m(t)(\psi_i\cdot\nabla\psi_k,\psi_j)_f+\sum_{i=1}^m b_i^m(t)(\nabla\psi_i,\nabla\psi_j)_f, \tag{3.16}$$

$$B_4[v_m,\psi_j;t]:=-\frac{1}{2}\sum_{i,k=1}^m b_i^m(t)b_k^m(t)\int_\Gamma(\psi_i\cdot n_i)\psi_k\psi_j d\Gamma-B_2[u_m,\varphi_j,t]. \tag{3.17}$$

Then (3.4),(3.5) become the following system of ODE

$$a_{j,tt}^m(t)+B_1[u_m,\varphi_j,t]+B_2[u_m,\varphi_j,t]=0, \tag{3.18}$$

$$b_{j,t}^m(t)+B_3[v_m,\psi_j,t]+B_4[v_m,\psi_j,t]=0. \tag{3.19}$$

subject to the initial condition

$$a_j^m(0) = \alpha_j^m,\ a_{j,t}^m(0) = \beta_j^m,\ b_j^m(0) = \gamma_j^m. \tag{3.20}$$

From (3.14)-(3.17), $B_i, i = 1, 2, 3, 4$ are the polynomial function of $a_i^m(t)$ and $b_i^m(t)$. Since the polynomial function is Lipschitz continuous, according to standard existence theory for ODE , there exists a unique absolutely continuous function $a_j^m, b_j^m, j = 1, \cdots, m$ satisfying (3.18)-(3.20) for $0 \le t \le t_m := t(m)$. And then u_m, v_m defined by (3.3) solves (3.4)-(3.8) for $0 \le t \le t_m$.

Step 2: Energy estimates

Lemma 3.2. *Give $a(x), u_0, u_1, v_0$ satisfying the assumptions of Theorem 2.1, there exists a positive constant C, depending only on $\Omega_f, \Omega_s, T, a(x)$, such that*

$$\begin{aligned}\|u_{m,t}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_s))} + \|u_m\|^2_{L^\infty(0,T;\mathbf{V}_s^2)} + \|v_m\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_f))} + \|v_m\|^2_{L^2(0,T;\mathbf{V}_f^1)} \\ \le C\left(\|u_0\|^2_{\mathbf{H}^2(\Omega_s)} + \|u_1\|^2_{\mathbf{L}^2(\Omega_s)} + \|v_0\|^2_{\mathbf{L}^2(\Omega_f)}\right)\end{aligned} \tag{3.21}$$

for $m = 1, 2, \cdots$.

Proof. Multiplying equation (3.4) , (3.5) by $(a_j^m(t))_t, b_j^m(t)$,respectively, summing the resulting two equations, then summing for $j = 1, 2, \cdots, m$, recalling $\varphi(x) = \psi(x), x \in \Gamma$ and (3.3), we find

$$(u_{m,tt}, u_{m,t})_s + B_1[u_m, u_{m,t}; t] + (v_{m,t}, v_m)_f + (\nabla v_m, \nabla v_m)_f = 0. \tag{3.22}$$

for a.e. $0 \le t \le t_m, m = 1, 2, \cdots$, where $(v_m \cdot \nabla v_m, v_m)_f - \frac{1}{2}\int_\Gamma (v_m \cdot n) v_m^2 dx = 0$ is used.

Furthermore

$$\begin{cases} (u_{m,tt}, u_{m,t})_s = \dfrac{1}{2}\dfrac{d}{dt}\|u_{m,t}\|^2_{\mathbf{L}^2(\Omega_s)}, (v_{m,t}, v_m)_f = \dfrac{1}{2}\dfrac{d}{dt}\|v_m\|^2_{\mathbf{L}^2(\Omega_f)}, \\ (\nabla v_m, \nabla v_m)_f = \|\nabla v_m\|^2_{\mathbf{L}^2(\Omega_f)}, (|\nabla u_m|^2 \nabla u_m, \nabla u_{m,t})_{\mathbf{L}^2(\Omega_s)} = \dfrac{1}{4}\dfrac{d}{dt}\|\nabla u_m\|^4_{\mathbf{L}^4(\Omega_s)}. \end{cases} \tag{3.23}$$

Consequently (3.22) yields the inequality

$$\frac{d}{dt}\left(\|u_{m,t}\|^2_{\mathbf{L}^2(\Omega_s)} + \int_{\Omega_s} a(x)|\Delta u_m|^2 dx + \|v_m\|^2_{\mathbf{L}^2(\Omega_s f)} + \frac{1}{4}\frac{d}{dt}\|\nabla u_m\|^4_{\mathbf{L}^4(\Omega_s)}\right) + 2\|\nabla v_m\|^2_{\mathbf{L}^2(\Omega_s)} = 0. \tag{3.24}$$

for a.e. $0 \le t \le t_m, m = 1, 2, \cdots$. Then,we can deduce (3.21), where we used the inequality

$$c\|\Delta u_m\|^2_{\mathbf{L}^2(\Omega_s)} \le \int_{\Omega_s} a(x)|\Delta u_m|^2 dx \tag{3.25}$$

which follows from the uniformly elliptic condition.

Remark: Form the Energy estimates, we obtain C is independent on m. Since $u(t_m) \in \mathbf{H}^2(\Omega_s), u_t(t_m) \in \mathbf{L}^2(\Omega_s), v(t_m) \in \mathbf{L}^2(\Omega_s)$, by using Lemma 3.1, we deduce that there exists a unique absolutely continuous function $a_j^m, b_j^m, j = 1, \cdots, m$ satisfying (3.18)-(3.20) for $t_m \le t \le 2t_m$, and so on, we finally get $t_m \to T, m \to \infty$.

Lemma 3.3. *Give $a(x), u_0, u_1, v_0$ satisfying the assumptions of Theorem 2.1, and assume $\{u_{m,tt}, v_{m,t}\}$ is the solutions of* (3.4)-(3.8)*, thus*

$$\{u_{m,tt}, v_{m,t}\} \in L^2(0,T;\mathbf{V}') \tag{3.26}$$

Proof. For any $w \in \mathbf{V}$, we define η, ϕ_j, Q_m, as follow

$$\eta = \begin{cases} u_t, x \in \Omega_s \\ v, x \in \Omega_f \end{cases}, \phi_j = \begin{cases} \varphi_j, x \in \Omega_s \\ \psi_j, x \in \Omega_f \end{cases} \tag{3.27}$$

$$Q_m : \mathbf{V} \mapsto \mathbf{V}, \tag{3.28}$$

satisfly

$$Q_m w = \sum_{j=1}^{m} (w, \phi_j)\phi_j, a.e. x \in \Omega_s, x \in \Omega_f;\ \|Q_m w\|_{\mathbf{V}} \le C\|w\|_{\mathbf{V}}. \tag{3.29}$$

Multiplying (3.4), (3.5) by $(w,\phi_j)_s, (w,\phi_j)_f$, respectively, summing for $j = 1, 2\cdots, m$, then we observe that

$$\begin{aligned}(\eta_{m,t}, Q_m w) = &- \big(a(x)\Delta u_m, \Delta(Q_m w)\big)_s - \frac{1}{2}(|\nabla u_m|^2 \nabla u_m, Q_m w)_s \\ &- (v_m \cdot \nabla v_m, Q_m w)_f - (\nabla v_m, \nabla(Q_m w))_f + \frac{1}{2}\int_\Gamma (v_m \cdot n) v_m (Q_m w) d\Gamma.\end{aligned} \tag{3.30}$$

Since,

$$\frac{1}{2}(v_m \cdot \nabla v_m, \psi_j)_f = \frac{1}{2}\int_\Gamma (v_m \cdot n) v_m \psi_j d\Gamma - \frac{1}{2}(v_m \cdot \nabla \psi_j, v_m)_f. \tag{3.31}$$

Comparing (3.30) and (3.31), we conclude

$$\begin{aligned}(\eta_{m,t}, Q_m w) = &- \big(a(x)\Delta u_m, \Delta(Q_m w)\big)_s - \frac{1}{2}(|\nabla u_m|^2 \nabla u_m, Q_m w)_s \\ &- \frac{1}{2}(v_m \cdot \nabla v_m, Q_m w)_f + \frac{1}{2}(v_m \cdot \nabla(Q_m w), v_m)_f - (\nabla v_m, \nabla(Q_m w))_f. \\ \le &C\Big(\|a(x)\|_{\mathbf{L}^\infty(\Omega_s)}\|\Delta u_m\|_{\mathbf{L}^2(\Omega_s)}\|\Delta Q_m w\|_{\mathbf{L}^2(\Omega_s)} + \|\nabla u_m\|^3_{\mathbf{L}^6(\Omega_s)}\|Q_m w\|_{\mathbf{L}^2(\Omega_s)} \\ &+ \|v_m\|_{\mathbf{L}^2(\Omega_f)}\|\nabla v_m\|_{\mathbf{L}^2(\Omega_f)}\|Q_m w\|_{\mathbf{L}^\infty(\Omega_f)} + \|v_m\|^2_{\mathbf{L}^p(\Omega_f)}\|\nabla(Q_m w)\|_{\mathbf{L}^d(\Omega_f)} \\ &+ \|\nabla v_m\|_{\mathbf{L}^2(\Omega_f)}\|\nabla(Q_m w)\|_{\mathbf{L}^2(\Omega_f)}\Big) \\ \le &C\Big(\|u_m\|_{\mathbf{V}^2_s} + \|u_m\|^3_{\mathbf{V}^2_s} + \|v_m\|_{\mathbf{H}^1_f} + \|v_m\|^2_{\mathbf{H}^1_f}\Big)\|Q_m w\|_{\mathbf{V}} \\ \le &C\Big(\|u_m\|_{\mathbf{V}^2_s} + \|u_m\|^3_{\mathbf{V}^2_s} + \|v_m\|_{\mathbf{H}^1_f} + \|v_m\|^2_{\mathbf{H}^1_f}\Big)\|w\|_{\mathbf{V}}.\end{aligned} \tag{3.32}$$

where (3.29) and Lemma 2.2 are used.

We write w as $w = Q_m w + (w - Q_m w)$, and then we have

$$(\eta_{m,t}, w) = (\eta_{m,t}, Q_m w) + (\eta_{m,t}, w - Q_m w) = (\eta_{m,t}, Q_m w). \tag{3.33}$$

where $(\phi_j, \phi_k) = 0, j = 1, \cdots, m, k > m$ is used. Comparing (3.32) and (3.33), we have

$$(\eta_{m,t}, w) \le C\Big(\|u_m\|_{\mathbf{V}^2_s} + \|u_m\|^3_{\mathbf{V}^2_s} + \|v_m\|_{\mathbf{H}^1_f} + \|v_m\|^2_{\mathbf{H}^1_f}\Big)\|w\|_{\mathbf{V}}. \tag{3.34}$$

Then we deduce

$$\|\eta_{m,t}\|_{\mathbf{V}'} \leq C\Big(\|u_m\|_{\mathbf{V}_s^2} + \|u_m\|_{\mathbf{V}_s^2}^3 + \|v_m\|_{\mathbf{H}_f^1} + \|v_m\|_{\mathbf{H}_f^1}^2\Big). \tag{3.35}$$

According to (3.21) and (3.35), we deduce

$$\eta_{m,t} \text{ is bounded in } L^2(0,T;\mathbf{V}'). \tag{3.36}$$

Thus,

$$\{u_{m,tt}, v_{m,t}\} \text{ is bounded in } L^2(0,T;(\mathbf{V}_s^2)' \times (\mathbf{V}_f^2)').$$

Step 3: Limiting processes and existence of the weak solutions

1. According to the energy estimates (3.21) and (3.36), we see that

$$u_m \text{ is bounded in } L^\infty(0,T;\mathbf{V}_s^2), \tag{3.37}$$

$$u_{m,t} \text{ is bounded in } L^\infty(0,T;\mathbf{L}_s^2), \tag{3.38}$$

$$v_m \text{ is bounded in } L^\infty(0,T;\mathbf{L}_f^2) \cap L^2(0,T;\mathbf{V}_f^1), \tag{3.39}$$

$$\{u_{m,tt}, v_{m,t}\} \text{ is bounded in } L^2(0,T;\mathbf{V}'). \tag{3.40}$$

Consequently there exists subsequence $\{u_\mu\}_{\mu=1}^\infty \subset \{u_m\}_{m=1}^\infty, \{v_\mu\}_{\mu=1}^\infty \subset \{v_m\}_{m=1}^\infty$ and function

$$u \in L^\infty(0,T;\mathbf{V}_s^2),$$

$$u_t \in L^\infty(0,T;\mathbf{L}_s^2),$$

$$v \in L^\infty(0,T;\mathbf{L}_f^2) \cap L^2(0,T;\mathbf{V}_f^1),$$

$$\{u_{tt}, v_t\} \in L^2(0,T;\mathbf{V}'),$$

such that

$$u_\mu \rightharpoonup u \text{ weakly * in } L^\infty(0,T;\mathbf{V}_s^2), \tag{3.41}$$

$$u_{\mu,t} \rightharpoonup u_t \text{ weakly * in } L^\infty(0,T;\mathbf{L}_s^2), \tag{3.42}$$

$$v_\mu \rightharpoonup v \text{ weakly in } L^\infty(0,T;\mathbf{L}_f^2) \cap L^2(0,T;\mathbf{V}_f^1), \tag{3.43}$$

$$\{u_{\mu,tt}, v_{\mu,t}\} \rightharpoonup \text{ weakly in } L^2(0,T;\mathbf{V}'). \tag{3.44}$$

2. Thanks to Aubin-Lions Lemma, we deduce the following conclusion for $0 < \varepsilon < \frac{1}{2}$,

$$u_\mu \to u \text{ strong in } L^\infty(0,T;\mathbf{H}_s^{2-\varepsilon}), \tag{3.45}$$

$$\{u_{\mu,t}, v_\mu\} \to \{u_t, v\} \text{ strong in } L^2(0,T;\mathbf{H}_s^{-\varepsilon} \times \mathbf{H}_f^{1-\varepsilon}). \tag{3.46}$$

where $\mathbf{H}_s^2 \circlearrowleft\circlearrowleft \mathbf{H}_s^{2-\varepsilon}, \mathbf{L}_s^2 \circlearrowleft\circlearrowleft \mathbf{H}_s^{-\varepsilon}, \mathbf{H}_f^1 \circlearrowleft\circlearrowleft \mathbf{H}_f^{1-\varepsilon}$ are used. By using trace theorem, we have mappings $v \to v|_\Gamma$ is continuous mapping from $\mathbf{H}_f^{1-\varepsilon}$ to $\mathbf{H}_f^{\frac{1}{2}-\varepsilon}$. Then we find

$$v_\mu|_\Gamma \to v|_\Gamma \text{ strong in } L^2(0,T;\mathbf{H}^{\frac{1}{2}-\varepsilon}(\Gamma)) \tag{3.47}$$

and in particular

$$v_\mu|_\Gamma \to v|_\Gamma \text{ strong in } L^2(0,T;\mathbf{L}^2(\Gamma)). \tag{3.48}$$

Next we pass to limits as $m \to \infty$, let us first prove

$$\int_\Gamma (v_\mu \cdot n) v_\mu \psi_j d\Gamma \to \int_\Gamma (v \cdot n) v \psi_j d\Gamma \quad \text{weakly in } L^2(0,T). \tag{3.49}$$

Denote

$$\int_\Gamma (v_\mu \cdot n) v_\mu \psi_j - (v \cdot n) v \psi_j d\Gamma = I_1 + I_2. \tag{3.50}$$

where

$$I_1 = \int_\Gamma \big((v_\mu - v) \cdot n\big) v_\mu \psi_j d\Gamma \tag{3.51}$$

$$I_2 = \int_\Gamma (v \cdot n)(v_\mu - v) \psi_j d\Gamma \tag{3.52}$$

Thanks to Holder inequality , we have

$$I_1 \le c \|v_\mu - v\|_{\mathbf{L}^2(\Gamma)} \|v_\mu\|_{\mathbf{L}^3(\Gamma)} \|\psi_j\|_{\mathbf{L}^6(\Gamma)} \tag{3.53}$$

Utilizing Lemma 2.3 and (3.48), we have

$$I_1 \to 0, \text{ in } L^2(0,T), \text{ as } \mu \to \infty. \tag{3.54}$$

Similarly, we deduce

$$I_2 \to 0, \text{ in } L^2(0,T), \text{ as } \mu \to \infty. \tag{3.55}$$

Combining (3.54) and (3.55), we finally conclude (3.49).

Now we prove when $\mu \to \infty$,

$$(v_\mu \cdot \nabla v_\mu, \psi_j)_f \to (v \cdot \nabla v, \psi_j)_f \quad \text{weakly in } L^2(0,T). \tag{3.56}$$

We note that

$$(v_\mu \cdot \nabla v_\mu, \psi_j)_f - (v \cdot \nabla v, \psi_j)_f = I_3 + I_4. \tag{3.57}$$

where

$$I_3 = \big((v_\mu - v) \cdot \nabla v_\mu, \psi_j\big), \tag{3.58}$$

$$I_4 = \big(v \cdot \nabla (v_\mu - v), \psi_j\big). \tag{3.59}$$

Thanks to Holder inequality , we have

$$I_3 \le c \|v_\mu - v\|_{\mathbf{L}^3(\Omega_f)} \|\nabla v_\mu\|_{\mathbf{L}^2(\Omega_f)} \|\psi_j\|_{\mathbf{L}^6(\Omega_f)} \le c \|v_\mu - v\|_{\mathbf{H}_f^{1-\varepsilon}} \|\nabla v_\mu\|_{\mathbf{L}^2(\Omega_f)} \|\psi_j\|_{\mathbf{L}^6(\Omega_f)} \tag{3.60}$$

Utilizing Lemma 2.2 and (3.39), (3.46), we have

$$I_3 \to 0, \text{ in } L^2(0,T), \text{ as } \mu \to \infty. \tag{3.61}$$

According to (3.43) and $\|v^k\psi_j^i\|_{\mathbf{L}_f^2} \leq c$, we find

$$I_4 \to 0, \text{ in } L^2(0,T), \text{ as } \mu \to \infty. \tag{3.62}$$

where v^k is the k-th component of v, ψ_j^i is the i-th component of ψ_j. Combining (3.61) and (3.62), we finally conclude (3.56).

Now we prove

$$(|\nabla u_\mu|^2 \nabla u_\mu, \varphi_j)_s \to (|\nabla u|^2 \nabla u, \varphi_j)_s \text{ weakly* in } L^\infty(0,T). \tag{3.63}$$

We note that

$$\begin{aligned}(|\nabla u_\mu|^2 \nabla u_\mu - |\nabla u|^2 \nabla u, \varphi_j)_s &\leq 3\big(sup(|\nabla u_\mu|^2, |\nabla u|^2)|\nabla(u_\mu - u)|, |\varphi_j|\big)_s \\ &\leq c(\|\nabla u_\mu\|_{\mathbf{L}^6(\Omega_s)} + \|\nabla u\|_{\mathbf{L}^6(\Omega_s)})\|\nabla(u_\mu - u)\|_s \|\varphi_j\|_{\mathbf{L}^6(\Omega_s)} \\ &\leq c(\|u_\mu\|_{\mathbf{V}_s} + \|u\|_{\mathbf{V}_s})\|\nabla(u_\mu - u)\|_s \|\varphi_j\|_{\mathbf{V}_s} \\ &\leq c\|\nabla(u_\mu - u)\|_s.\end{aligned} \tag{3.64}$$

Thanks to (3.45) , we deduce

$$\nabla u_\mu \to \nabla u \text{ strong in } C(0,T;\mathbf{L}^2(\Omega_s)). \tag{3.65}$$

Combining (3.64) and (3.65), we finally get (3.63).

According to (3.41)-(3.44), we discover

$$\begin{aligned}&(u_{\mu,tt}, \varphi_j)_s \to (u_{tt}, \varphi_j)_s \text{ in } \mathscr{D}'(0,T), \\ &(v_{\mu,t}, \psi_j)_f \to (v_t, \psi_j)_f \text{ in } \mathscr{D}'(0,T), \\ &(a(x)\Delta u_\mu, \Delta\varphi_j)_s \to (a(x)\Delta u, \Delta\varphi_j)_s \ \text{ weakly * in } L^\infty(0,T), \\ &(\nabla v_\mu, \nabla\psi_j)_f \to (\nabla v, \nabla\psi_j)_f \ \text{ weakly in } L^2(0,T),\end{aligned} \tag{3.66}$$

Combining (3.49), (3.56), (3.63), (3.66) , we deduce

$$\begin{aligned}(u_{tt}, \varphi_j)_s + \big(a(x)\Delta u, \Delta\varphi_j\big)_s + \frac{1}{2}(|\nabla u|^2 \nabla u, \varphi_j)_s + (v_t, \psi_j)_f \\ + (v \cdot \nabla v, \psi_j)_f + (\nabla v, \nabla\psi_j)_f - \frac{1}{2}\int_\Gamma (v \cdot n) v \psi_j d\Gamma = 0,\end{aligned}$$

According to (3.1) and (3.2) , we have

$$\begin{aligned}(u_{tt}, \varphi)_s + \big(a(x)\Delta u, \Delta\varphi\big)_s + \frac{1}{2}(|\nabla u|^2 \nabla u, \varphi)_s + (v_t, \psi)_f \\ + (v \cdot \nabla v, \psi)_f + (\nabla v, \nabla\psi)_f - \frac{1}{2}\int_\Gamma (v \cdot n) v \psi d\Gamma = 0,\end{aligned} \tag{3.67}$$

for each $\{\varphi, \psi\} \in \mathbf{V}$. Then we can deduce that equation (3.67) holds for each $\varphi \in \mathbf{V}_s^2, \psi \in \mathbf{V}_f^1$.

3. We must now verify

$$u(0) = u_0, u_t(0) = u_1, v(0) = v_0. \tag{3.68}$$

According to (3.37) - (3.40), we discover

$$u_\mu \in C(0,T;\mathbf{L}^2(\Omega_s)), \{u_{\mu,t}, v_\mu\} \in C(0,T;\mathbf{V}'). \tag{3.69}$$

Then, we have

$$\begin{aligned}(u_\mu(0),\varphi)_s &\to (u(0),\varphi)_s\\ (u_{\mu,t}(0),\varphi)_s &\to (u_t(0),\varphi)_s\\ (v_\mu(0),\psi_j)_f &\to (v(0),\psi_j)_f\end{aligned} \tag{3.70}$$

On the other hand, according to (3.6)-(3.8), we find

$$\begin{aligned}&u_\mu(0) \to u_0, && \text{in } \mathbf{H}^2(\Omega_s);\\ &u_{\mu,t}(0) \to u_1, && \text{in } \mathbf{L}^2(\Omega_s);\\ &v_\mu(0) \to v_0, && \text{in } \mathbf{L}^2(\Omega_f);\end{aligned} \tag{3.71}$$

Combining (3.70) and (3.71), we deduce (3.68).

Combining the three steps above, we have proved the existence of global weak solutions of (1.1).

4 References

References

[1] Yao M H, Chen Y P, Zhang W. Nonlinear vibrations of blade with varying rotating speed[J]. Nonlinear Dynamics, 2012, 68(4):487-504.

[2] Du Q, Gunzburger M D, Hou L S, Lee J. Analysis of a linear fluid-structure interaction problem [J]. Discrete and Continuous Dynamical Systems, 2003, 9(3):633-650.

[3] Lions J L. Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Paris, 1969.

[4] Barbu V, Grujic Z, Irena L, et al. Existence of the energy level weak solutions for a nonlinear fluid-structure interaction Model[J]. Fluids and Waves, 2007, 440:55-82.

[5] Barbu V, Grujic Z, Lasiecka I, et al. Smoothness of weak solutions to a nonlinear fluid-structure interaction model[J]. Indiana University Mathematics Journal, 2008, 57(3):1173-1208.

[6] Hsiao G, Kleinman R, Roach G, Weak solutions of fluid-solid interaction problems. Mathematische Nachrichten, 2000,218:139-163.

[7] Sarrate J, Huerta A, Donea J. Arbitrary Lagrangian-Eulerian formulation for fluidCrigid body interaction[J]. Computer Methods in Applied Mechanics and Engineering, 2001, 190(24-25):3171-3188.

[8] Li X Z, Ya K G, Hong M Z. Analysis of fully coupled flow-induced vibration of structure under small deformation with GMRES[J]. Applied Mathematics and Mechanics, 2010, 31(1):81-90.

[9] Bazilevs Y, Hsu M C, Takizawa K, et al. ALE-VMS and ST-VMS methods for computer modeling of wind-turbine rotor aerodynamics and fluid-structure interaction[J]. Mathematical Models and Methods in Applied Sciences, 2012, 22(supp02):1230002.

[10] Hsu M C, Bazilevs Y. Fluid-structure interaction modeling of wind turbines: simulating the full machine[J]. Computational Mechanics, 2012, 50(6):821-833.

[11] Coutand D, Shkoller S. Motion of an elastic solid inside an incompressible viscous fluid[J]. Archive for Rational Mechanics and Analysis, 2005, 176(1):25-102.

[12] Boulakia M, Guerrero . Regular solutions of a problem coupling a compressible fluid and an elastic structure[J]. Journal de Mathématiques Pures et Appliquées, 2010, 94(4):341-365.

[13] Desjardins B, Esteban M J. Existence of weak solutions for the motion of rigid bodies in a viscous Fluid[J]. Archive for Rational Mechanics and Analysis, 1999, 146(1):59-71.

[14] Desjardins B. On weak solutions for fluid-rigid structure interaction:compressible and incompressible Models[J]. Communications in Partial Differential Equations, 2000, 25(7):263-285.

[15] Boulakia M. Existence of weak solutions for the motion of an elastic structure in an incompressible viscous fluid[J]. Comptes rendus Mathematique, 2003, 336(12):985-990.

[16] Boulakia M, Osses A. Two-dimensional local null controllability of a rigid structure in a NavierCStokes fluid[J]. Academie des Sciences. Comptes Rendus, Mathematique, 2006, 343(2):105-109.

[17] Boulakia M. Existence of weak solutions for the three-dimensional motion of an elastic structure in an incompressible fluid[J]. Journal of Mathematical Fluid Mechanics, 2007, 9(2):262-294.

[18] Boulakia M, Schwindt E, Takéo T. Existence of strong solutions for the motion of an elastic structure in an incompressible viscous fluid[J]. Interfaces and Free Boundaries, 2012, 14(3):273-306.

[19] Boulakia M, Guerrero S. On the interaction problem between a compressible fluid and a Saint-Venant Kirchhoff elastic structure. Advances in Differential Equations, Khayyam Publishing, 2017, 22(1-2).

[20] Grandmont, Cline, Maday Y . Existence for an unsteady fluid-structure interaction problem[J]. ESAIM Mathematical Modelling and Numerical Analysis, 2000, 34(3):609-636.

[21] Yang K, Sun P, Wang L, et al. Modeling and simulations for fluid and rotating structure interactions[J]. Computer Methods in Applied Mechanics and Engineering, 2016, 311:788-814.

[22] Leng W, Zhang C S, Sun P, et al. Numerical simulation of an immersed rotating structure in fluid for hemodynamic applications[J]. Journal of Computational Science, 2019, 30:79-89.

[23] Sun P, Xu J, Zhang L. Full Eulerian finite element method of a phase field model for fluidCstructure interaction problem[J]. Computers and Fluids, 2014, 90:1-8.

[24] Nirenberg L. On elliptic partial differential equations[J]. IL Principio Di Minimo E Sue Applicazioni Alle Equazioni Funzionali, 1959, 13(1):1-48.