Dynamics of soliton solutions of the fifth-order nonlinear Schrödinger equation via the Riemann-Hilbert approach *

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Abstract

The theory of inverse scattering is developed to investigate the initial-value problem for the fifth-order nonlinear Schrödinger (foNLS) equation under the zero boundary conditions at infinity. The spectral analysis is performed in the direct scattering process, including the establishment of the analytical, asymptotic and symmetric properties of the scattering matrix and the Jost functions. In the inverse scattering process, a suitable Riemann-Hilbert (RH) problem is successfully established by using the modified eigenfunctions and scattering data, and the relationship between the potential function and the solution of the RH problem is successfully established. In order to further analyze the propagation behavior of the solutions of the foNLS equation, we present some new phenomena of studying the one-, two-, and three- soliton solutions corresponding to simple zeros in scattered data. Finally, we also analyze the one- and two-soliton solutions corresponding to double zeros.

Key words: The fifth-order nonlinear Schrödinger equation; Riemann Hilbert approach; Soliton solutions.

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1 Introduction

Nonlinear integrable partial differential equations (PDEs) can be used to establish connections with many fields, including electromagnetics, plasma, and equations of motion in Euclidean space. By studying these integrable models, one can obtain some important data analysis, including soliton solution, as an accurate solution has the good property of maintaining the speed and shape of the propagation process. Therefore, many scholars

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work hard to find the exact solutions of integrable PDEs by this reason, and have also successfully put forward many effective methods, including the Darboux transformation [1] and the Lie symmetric [2] and inverse scattering transformation (IST) [3], etc. Among these methods, the IST is a very powerful tool, which is based on the Gel'fand-Levitan-Marchenko [4, 5] integral equation at first, and then the Riemann-Hilbert (RH) problem [6] was proposed, which greatly simplified the IST. At present, the RH problem has been widely used in the related problems of integrable systems.

In this work, we mainly investigate the following simplified fifth-order nonlinear Schrödinger equation (foNLS) equation proposed by Radha in [7] via the RH problem with the potential function decays rapidly at infinity

$$iq_t - i\epsilon q_{xxxxx} - 10i\epsilon|q|^2 q_{xxx} - 20i\epsilon q_x q^* q_{xx} - 30i\epsilon|q|^4 q_x - 10i\epsilon(|q_x|^2 q)_x + q_{xx} + 2q|q|^2 - iq_x = 0. \tag{1.1}$$

where $q = q(x,t)$ is a complex variable, ϵ is the real constant, and the star $*$ represents the complex conjugate. In [8], one of our authors Tian and his collaborators obtained the breather solutions and rouge wave solutions of (1.1) by using n-fold Darboux transformation. Conservation laws and solitons of the inhomogeneous foNLS were investigated by Wang in [9]. Darboux transformation and conservation laws for the inhomogeneous foNLS were studied in [10].

In recent years, the RH problem has many applications in the zero boundary condition (ZBC), that is, the potential function decays rapidly at infinity, including generalized Sasa-Satsuma equation [11], Korteweg-de Vries equation, [12, 13] nonlinear Schrödinger equation [14], coupled derivative Schrödinger equation [15], etc. [16]$^-$[21] In addition, the RH problem can also be used to deal with special boundaries. [23, 22] It is worth noting that when many authors construct N-soliton solutions of integrable equations, they often use a transformation to transform an irregular RH problem into a regular RH problem when considering the RH problem with discrete spectral points. Motivated by the idea from, [23]$^-$[25] these authors transformed the irregular RH problem (only simple zeros) into a regular RH problem by subtracting the asymptotic property and the residue generated at the zero point. Therefore, we will combine these two ideas in this work, but we should pay attention to the problem that the potential function does not tend to zero at infinity, that is, the non-zero boundary value condition, which makes our work different from it. Furthermore, we also consider the case that the scattering data has double zeros. As far as we know, these are not reported before.

The frame of the work is arranged as: In section 2, the analytical property, asymptotic property and symmetry of Jost function and scattering matrix are established by spectral analysis. In section 3, according to the modified eigenfunction and the analyticity of scattering data, we establish a suitable RH problem and derive the corresponding residue by two kinds of zeros including simple zeros and double zeros, which is necessary to regularize the original RH problem. In section 4, according to the reconstruction potential function formula given in the third section under the condition of no reflection, we further analyze the propagation behavior of the solutions, including the one-, two-, and three-soliton solutions under simple zeros, and one- and two-soliton solutions under double zeros. Finally some conclusions and discussions are presented in the last section.

2 The direct problem for the foNLS equation

The Lax pair of (1.1) in [7] can be written as

$$\psi_x = U\psi, \qquad \psi_t = V\psi \tag{2.1}$$

where

$$U = -ik\sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\begin{aligned} V =&(-16ik^5\epsilon - 2ik - ik)\sigma_3 + 4k^2\epsilon[Q_x, Q]\sigma_3 - 2ik\epsilon(3Q^4 - Q_xQ - QQ_{xx})\sigma_3 \\ &- iQ^2\sigma_3 + \epsilon\left([Q, Q_{xxx}] + [Q_{xx}, Q_x] + 6Q^2[Q, Q_x]\right)\sigma_3 + 16k^4\epsilon Q + 8ik^3Q_x\sigma_3 \\ &- 4k^2Q_{xx}\epsilon + 8k^2Q^3 - 2ikQ_{xxx}\epsilon + 12ikQ^2Q_x\epsilon + 2kQ + iQ_x + Q \\ &+ \epsilon(Q_{xxxx} - 8Q^2Q_{xx} - 2QQ^*Q_x^* - 4Q_x^2Q - 6Q_xQ_x^*Q^* + 6Q^5). \end{aligned}$$

Considering the initial value

$$q(x, t = 0) = q_0(x) \in \mathbf{S}(R), \tag{2.2}$$

where the $\mathbf{S}(R)$ represents the Schwartz space, one can obtain the Jost functions

$$\psi_\pm \sim e^{-i[kx+(16k^5\epsilon+2k^2+k)t]\sigma_3}, \quad x \to \pm\infty, \tag{2.3}$$

note that $\det\psi_\pm(x, t; k) = 1$ and we take $\theta(x, t; k) = kx + (16k^5\epsilon + 2k^2 + k)t$. Then introducing the modified eigenfunctions

$$u_\pm(x, t; k) = \psi_\pm(x, t; k)e^{i\theta(x,t;k)\sigma_3} \to \mathbb{I}, \quad x \to \pm\infty. \tag{2.4}$$

and substituting (2.4) into the Lax pair (2.1), one has

$$u_{\pm,x}(x, t; k) + ik[\sigma_3, u_\pm(x, t; k)] = Q(x, t)u_\pm(x, t; k), \tag{2.5a}$$

$$u_{\pm,t}(x, t; k) + i(16k^5\epsilon + 2k^2 + k)[\sigma_3, u_\pm(x, t; k)] = \hat{Q}(x, t)u_\pm(x, t; k), \tag{2.5b}$$

where

$$\begin{aligned} \hat{Q}(x, t) =&4k^2\epsilon[Q_x, Q]\sigma_3 - 2ik\epsilon(3Q^4 - Q_xQ - QQ_{xx})\sigma_3 + 8ik^3Q_x \\ &- iQ^2\sigma_3 + \epsilon\left([Q, Q_{xxx}] + [Q_{xx}, Q_x] + 6Q^2[Q, Q_x]\right)\sigma_3 + 16k^4\epsilon Q\sigma_3 \\ &- 4k^2Q_{xx}\epsilon + 8k^2Q^3 - 2ikQ_{xxx}\epsilon + 12ikQ^2Q_x\epsilon + 2kQ + iQ_x + Q \\ &+ \epsilon(Q_{xxxx} - 8Q^2Q_{xx} - 2QQ^*Q_x^* - 4Q_x^2Q - 6Q_xQ_x^*Q^* + 6Q^5). \end{aligned}$$

The ordinary differential equations of the functions $u_\pm(x, t; k)$ can be expressed by the following integral equations

$$\begin{aligned} u_-(x, t; k) = \mathbb{I} + \textstyle\int_{-\infty}^{x} e^{-ik(x-y)\sigma_3}Q(y, t)u_-(y, t; k)e^{ik(x-y)\sigma_3}\, dy, \\ u_+(x, t; k) = \mathbb{I} - \textstyle\int_{x}^{\infty} e^{-ik(x-y)\sigma_3}\hat{Q}(y, t)u_+(y, t; k)e^{ik(x-y)\sigma_3}\, dy. \end{aligned} \tag{2.6}$$

Similar to the analysis in [23], we can establish the analyticity of the modified eigenfunctions as follow:

Proposition 2.1. *The modified eigenfunctions $u_{\pm}(x,t;k)$ admit that: $u_{-,1}(x,t;k)$ and $u_{+,2}(x,t;k)$ are analytic in the region C^+, in addition, $u_{-,2}(x,t;k)$ and $u_{+,1}(x,t;k)$ are analytic in the region C^-, where $u_{\pm,i}(x,t;k)$ $(i=1,2)$ denote the i-th column of $u_{\pm}(x,t;k)$, C^+ denotes the upper half-plane: $C^+ = \{k \in C | \Im(k) > 0\}$, and similarly C^- denotes the lower half-plane: $C^- = \{k \in C | \Im(k) < 0\}$.*

Since the spectral problem is a first order homogeneous ordinary differential equation, it has a unique solution. Therefore, for the two Jost functions solutions $\psi_{\pm}(x,t;k)$ of the spectral problem, they are linearly related, that is, there exits a matrix $S(k)$ (it is independent of variables x and t) which makes the following expression hold

$$\psi_+(x,t;k) = \psi_-(x,t;k)S(k), \quad S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}, \tag{2.7}$$

note that $S(k)$ is generally called the scattering matrix, and its elements are called scattering data. Through direct calculation, these scattering data can be expressed using the Jost functions and the Wronskian determinant $Wr[\bullet,\bullet]$, i.e.,

$$s_{11}(z) = Wr[\psi_{+,1}, \psi_{-,2}], \quad s_{22}(z) = Wr[\psi_{-,1}, \psi_{+,2}], \tag{2.8}$$

and combining with the proposition 2.1 one can get the following corollary

Corollary 2.2. *The scattering data $s_{11}(k)$ and $s_{22}(k)$ are analytic in C^- and C^+, respectively. In addition, the scattering data $s_{12}(k)$ and $s_{21}(k)$ are not analytic but continuous to the real axis.*

Corollary 2.3. *The modified eigenfunction $u_{\pm}(x,t;k)$ and scattering matrix $S(k)$ satisfy the following relations respectively*

$$u^*_{\pm}(k^*) = \sigma u_{\pm}(k)\sigma, \quad S(k) = -\sigma S^*(k^*)\sigma, \tag{2.9}$$

where $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof: Given that the modified eigenfunctions $u_{\pm}(k)$ satisfie (2.5a), i.e.,

$$G_x(x,t) + ik[\sigma_3, G(x,t)] = Q(x,t)G(x,t),$$

it is not difficult to verify that the functions $u^*_{\pm}(k^*)$ and $\sigma u_{\pm}(k)\sigma$ also satisfy the equation. Then Combining with the asymptotic property (2.4), the first relation can be obtained. Combined with (2.7), it is not difficult to verify that the second equation is also true.

Corollary 2.4. *The componentwise satisfy:*

$$u_{\pm,1}(k) = \sigma u^*_{\pm,2}(k^*), \qquad u_{\pm,2}(k) = -\sigma u^*_{\pm,1}(k^*), \tag{2.10a}$$
$$u_{\pm,11}(k) = u^*_{\pm,22}(k^*), \qquad u_{\pm,22}(k) = u^*_{\pm,11}(k^*), \tag{2.10b}$$
$$u_{\pm,12}(k) = -\sigma u^*_{\pm,21}(k^*), \quad u_{\pm,21}(k) = -\sigma u^*_{\pm,12}(k^*), \tag{2.10c}$$
$$s_{11}(k) = s^*_{22}(k^*), \qquad s_{22}(k) = s^*_{11}(k^*). \tag{2.10d}$$

Next, we will analyze the asymptotic property of the modified eigenfunctions and scattering matrix, as well as establish the relationship between the potential function and the modified eigenfunctions. Taking the following expansion about $u_{\pm}(k)$

$$u_{\pm} = u_{\pm}^{(0)} + \frac{u_{\pm}^{(1)}}{k} + \cdots \quad k \to \infty, \tag{2.11}$$

and substituting (2.11) into (2.5a) we can get the following relationships by comparing the coefficient of k^i $(i = 0, 1)$

$$0 = i[\sigma_3, u_{\pm}^{(0)}], \tag{2.12a}$$

$$u_{\pm,x}^{(0)} = i[u_{\pm}^{(1)}, \sigma_3] + Qu_{\pm}^{(0)}. \tag{2.12b}$$

Similarly substituting (2.11) into (2.5b) yields

$$0 = i[16\epsilon u_{\pm}^{(1)}, \sigma_3] + 16\epsilon Q u_{\pm}^{(0)}. \tag{2.13}$$

Summing up equations (2.12a)-(2.13), we obtain that $u_{\pm}^{(0)}$ is a diagonal matrix and is independent of x. Note that (2.4) $u_{\pm} \to \mathbb{I}$ implies

$$u_{\pm}^{(0)} \to \mathbb{I}, \quad x, k \to \infty. \tag{2.14}$$

In addition, from (2.12b), the relationship between potential function and modified eigenfunctions

$$q(x,t) = 2i \lim_{k\to\infty} (k u_{\pm})_{12}, \tag{2.15}$$

where A_{ij} $(i, j = 1, 2)$ denote that the element of the matrix A. Finally the asymptotic behavior of scattering matrix $S(k)$ can be derived by (2.4), (2.7) and (2.14)

$$S(k) \to \mathbb{I}, \quad k \to \infty. \tag{2.16}$$

3 The inverse problem for the foNLS equation

To sum up the above analysis, we have established the analytical and asymptotic properties of the modified eigenfunctions and scattering matrix. Next, we will combine the modified eigenfunction and scattering data to construct a suitable RH problem. The purpose is to reconstruct the expression of the modified eigenfunction and finally restore the potential function $q(x,t)$. Introducing the sectionally meromorphic functions

$$M(k) = \begin{cases} \left(u_{-,1}(k), \dfrac{u_{+,2}(k)}{s_{22}(k)} \right), & k \in C^{+}, \\ \left(\dfrac{u_{+,1}(k)}{s_{11}(k)}, u_{-,2}(k) \right), & k \in C^{-}. \end{cases} \tag{3.1}$$

$\varepsilon, k \in R$, we can establish the RH problem:

Definition 3.1. The RH problem admits the following conditions:

(a) : $M(k)$ is meromorphic function in $C^{\pm}$,
(b) : The jump condition

$$M^{-}(x,t;k) = M^{+}(x,t;k)(\mathbb{I} - G(x,t;k)), \tag{3.2}$$

where $M^{\pm}(k) = \lim_{\varepsilon\to 0_+} M(k \pm i\varepsilon)$, with the matrix

$$G(x,t;z) = \left(\begin{array}{cc} 0 & -\tilde{\rho}(k)e^{-2i\theta(k)} \\ \rho(k)e^{2i\theta(k)} & \rho(k)\tilde{\rho}(k) \end{array} \right),$$

(c) : $M(k) \to \mathbb{I}, \quad k \to \infty$,
where the reflection coefficients $\rho(k) = \frac{s_{21}(k)}{s_{11}(k)}$ and $\tilde{\rho}(k) = \frac{s_{12}(k)}{s_{22}(k)}$.

It is noticed that when the scattering coefficients $s_{11}(k)$ and $s_{22}(k)$ are not equal to zero for any $k \in C^{\pm}$, the RH problem is regular and can be solved directly by Plemelj's formulate. When there are some points that make the scattering coefficients equal to zero, i.e., $s_{11}(k_j) = 0$ $(k_j \in C^-, j = 1, 2, \cdots, N)$, it is an irregular RH problem. This problem can be solved by transforming the irregular RH problem into a regular RH problem (please refer to Ref.[26] for details). However, we will not adopt this idea in our work, but calculate the residue at the singularity, and regularize RH problem by subtracting the asymptotic behavior and the contribution value of the singularity. In this work, we will discuss the residue of the singularity in two cases, including simple zeros and double zeros.

Case(A) : Similar to [23], suppose the scattering coefficient $s_{11}(k)$ is of N simple zero in the region C^-, i.e., $s_{11}(k_n) = 0$ $(k_n \in C^-, n = 1, 2, \cdots, N)$ and $s'_{11}(k_n) \neq 0$. Similarly, from symmetry (2.10d), we know that the scattering coefficient $s_{22}(k)$ has N simple zeros in C^+, i.e., $s_{22}(k_n^*) = 0$ $(k_n \in C^+, n = 1, 2, \cdots, N)$ and $s'_{22}(k_n^*) \neq 0$, where the notation $'$ indicates the derivation of the independent variable k.

Since $k_n \in C^-$ and $k_n^* \in C^+$ are the zero point of $s_{11}(k)$ and $s_{22}(k)$, respectively, we know from expression (2.8) that there are the constants b_n and d_n satisfy that

$$\psi_{+,1}(k_n) = b_n\psi_{-,2}(k_n), \quad \psi_{+,2}(k_n^*) = d_n\psi_{-,1}(k_n^*), \tag{3.3}$$

which imply taht

$$u_{+,1}(k_n) = b_n e^{2i\theta(k_n)}u_{-,2}(k_n), \quad u_{+,2}(k_n^*) = d_n e^{-2i\theta(k_n^*)}u_{-,1}(k_n^*). \tag{3.4}$$

Obviously, we can calculate the residue of meromorphic function $M(k)$ at zero points k_n and k_n^*, namely

$$\begin{aligned} \underset{k=k_n}{Res}\, M_1^-(k) &= \underset{k=k_n}{Res}\, \frac{u_{+,1}(k)}{s_{11}(k)} = \frac{b_n}{s'_{11}(k_n)}e^{2i\theta(k_n)}u_{-,2}(k_n) \triangleq C_n(k_n)e^{2i\theta(k_n)}u_{-,2}(k_n), \\ \underset{k=k_n^*}{Res}\, M_2^+(k) &= \underset{k=k_n^*}{Res}\, \frac{u_{+,2}(k)}{s_{22}(k)} = \frac{d_n}{s'_{22}(k_n^*)}e^{-2i\theta(k_n^*)}u_{-,1}(k_n^*) \triangleq \tilde{C}_n(k_n^*)e^{-2i\theta(k_n^*)}u_{-,2}(k_n^*). \end{aligned} \tag{3.5}$$

It is not difficult to verify the following relationship from the symmetry (2.10a)

$$b_n = -d_n^*, \tag{3.6}$$

which can lead to

$$C_n(k_n) = -\tilde{C}_n^*(k_n^*), \tag{3.7}$$

combining with the symmetry (2.10c).

Case(B) : Similar to [24], suppose the scattering coefficient $s_{11}(k)$ is of N double zero in the region C^-, i.e., $s_{11}(k_n) = s'_{11}(k_n) = 0$ $(k_n \in C^-, n = 1, 2, \cdots, N)$ and $s''_{11}(k_n) \neq 0$. Similarly, from symmetry (2.10d), we know that the scattering coefficient $s_{22}(k)$ has N simple zeros in C^+, i.e., $s_{22}(k_n^*) = s'_{22}(k_n^*) = 0$ $(k_n \in C^+, n = 1, 2, \cdots, N)$ and $s''_{22}(k_n^*) \neq 0$.

Since $k_n \in C^-$ and $k_n^* \in C^+$ are the double zero point of $s_{11}(k)$ and $s_{22}(k)$, respectively, similar to the **Case(A)**, there are the constants f_n and h_n satisfy that

$$u_{+,1}(k_n) = f_n e^{2i\theta(k_n)} u_{-,2}(k_n), \tag{3.8}$$

$$u'_{+,1}(k_n) = e^{2i\theta(k_n)} \left[(h_n + 2i\theta'(k_n) f_n) u_{-,2}(k_n) + f_n u'_{-,2}(k_n)\right]. \tag{3.9}$$

Obviously for the double zeros $k_n^* \in C^+$ of the $s_{22}(k)$, one has

$$u_{+,2}(k_n^*) = \tilde{f}_n e^{-2i\theta(k_n^*)} u_{-,1}(k_n^*), \tag{3.10}$$

$$u'_{+,2}(k_n^*) = e^{2i\theta(k_n^*)} \left[(\tilde{h}_n - 2i\theta'(k_n^*) \tilde{f}_n) u_{-,1}(k_n^*) + \tilde{f}_n u'_{-,1}(k_n^*)\right]. \tag{3.11}$$

Also from the symmetries (2.10a) and (2.10d), one has

$$\tilde{f}_n = -f_n^*, \quad \tilde{h}_n = -h_n^*, \tag{3.12}$$

In what follows, we will establish the residue corresponding to the double zeros. Similar to the double zeros in the Ref.[24], we can get

$$\begin{aligned}
\underset{k=k_n}{Res}\left[M_1^-(k)\right] &= \underset{k=k_n}{Res}\left[\frac{u_{+,1}(k)}{s_{11}(k)}\right] = \frac{2u'_{+,1}(k_n)}{s''_{11}(k_n)} - \frac{2u_{+,1}(k_n) s'''_{11}(k_n)}{3[s''_{11}(k_n)]^2} \\
&= \frac{2f_n}{s''_{11}(k_n)} e^{2i\theta(k_n)} \left[u'_{-,2}(k_n) + u_{-,2}(k_n)\left(\frac{f_n}{h_n} + 2i\theta'(k_n) - \frac{s'''_{11}(k_n)}{3s''_{11}(k_n)}\right)\right], \\
\underset{k=k_n}{P_{-,2}}\left[M_1^-(k)\right] &= \underset{k=k_n}{P_{-,2}}\left[\frac{u_{+,1}(k)}{s_{11}(k)}\right] = \frac{2u_{+,1}(k_n)}{s''_{11}(k_n)} = \frac{2f_n}{s''_{11}(k_n)} e^{2i\theta(k_n)} u_{-,2}(k_n),
\end{aligned}$$

where $\underset{k=k_n}{P_{-,2}}[\bullet]$ represents the coefficient of $\frac{1}{(k-k_n)^2}$ in Laurent series of $\bullet$ at $k = k_n$. For the double zeros $k_n^* \in C^+$, a similar result is obtained

$$\begin{aligned}
\underset{k=k_n^*}{Res}\left[\frac{u_{+,2}(k)}{s_{22}(k)}\right] &= \frac{2\tilde{f}_n}{s''_{22}(k_n^*)} e^{-2i\theta(k_n^*)} \left[u'_{-,1}(k_n^*) + u_{-,1}(k_n^*)\left(\frac{\tilde{f}_n}{\tilde{h}_n} - 2i\theta'(k_n^*) - \frac{s'''_{22}(k_n^*)}{3s''_{22}(k_n^*)}\right)\right], \\
\underset{k=k_n^*}{P_{-,2}}\left[\frac{u_{+,2}(k)}{s_{22}(k)}\right] &= \frac{2u_{+,2}(k_n^*)}{s''_{22}(k_n^*)} = \frac{2\tilde{f}_n}{s''_{22}(k_n^*)} e^{2i\theta(k_n^*)} u_{-,1}(k_n^*),
\end{aligned}$$

for convenience, we take the notation

$$A_n = \frac{2f_n}{s''_{11}(k_n)}, \quad B_n = \frac{f_n}{h_n} - \frac{s'''_{11}(k_n)}{3s''_{11}(k_n)}, \tag{3.13a}$$

$$\tilde{A}_n = \frac{2\tilde{f}_n}{s''_{22}(k_n^*)}, \quad \tilde{B}_n = \frac{\tilde{f}_n}{\tilde{h}_n} - \frac{s'''_{22}(k_n^*)}{3s''_{22}(k_n^*)}. \tag{3.13b}$$

The expression (3.12) implies

$$\tilde{A}_n = -A_n^*, \quad \tilde{B}_n = -B_n^*. \tag{3.14}$$

So far, we have completed the establishment of the residues in the two cases of single zeros and double zeros. The next goal is to solve the Riemann-Hilbert problem (3.2) for these two situations.

For **Case**(**A**), by using (3.5), we get

$$\underset{k=k_n}{Res}\, M^-(k) = \left(C_n(k_n)e^{2i\theta(k_n)}u_{-,2}(k_n), 0\right), \tag{3.15}$$

$$\underset{k=k_n^*}{Res}\, M^+(k) = \left(0, \tilde{C}_n(k_n^*)e^{-2i\theta(k_n^*)}u_{-,1}(k_n^*)\right). \tag{3.16}$$

In order to solve RH problem (3.2), it is regularized it by subtracting the asymptotic behavior and the contribution value of singularity, namely

$$M^+ - \mathbb{I} - \sum_{n=1}^{N}\left(\frac{\underset{k=k_n^*}{Res}\, M^+}{k-k_n^*} + \frac{\underset{k=k_n}{Res}\, M^-}{k-k_n}\right) = M^- - \mathbb{I} - \sum_{n=1}^{N}\left(\frac{\underset{k=k_n^*}{Res}\, M^+}{k-k_n^*} + \frac{\underset{k=k_n}{Res}\, M^-}{k-k_n}\right) - M^-G. \tag{3.17}$$

By using the projection operator defined in [27], we can obtain

$$M(x,t;k) = \mathbb{I} + \sum_{n=1}^{N}\left(\frac{\underset{k=k_n^*}{Res}\, M^+}{k-k_n^*} + \frac{\underset{k=k_n}{Res}\, M^-}{k-k_n}\right) + \frac{1}{2\pi i}\int_R \frac{M^-(\zeta)G(\zeta)}{\zeta-k}\,d\zeta. \tag{3.18}$$

Taking $M = M^-$ and comparing the element of $M^(-)_{12}$ can yield that

$$u_{-,12}(x,t;k) = \sum_{n=1}^{N}\frac{\underset{k=k_n^*}{Res}\, M^+}{k-k_n^*}u_{-,11}(x,t;k_n^*) + \frac{1}{2\pi i}\int_R \frac{(M^-(\zeta)G(\zeta))_{12}}{\zeta-k}\,d\zeta. \tag{3.19}$$

The equation (2.15) means that the reconstruction formula of the solution to (1.1) can be expressed as

$$q(x,t) = 2i\sum_{n=1}^{N}\tilde{C}_n(k_n^*)e^{-2i\theta(k_n^*)}u_{-,11}(x,t;k_n^*) - \frac{1}{\pi}\int_R (M^-(\zeta)G(\zeta))_{12}\,d\zeta. \tag{3.20}$$

Considering the second column of M^- at $k = k_n$ in C^-, one has

$$u_{-,2}(k_n) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{N}\frac{\tilde{C}_n(k_j^*)}{k_n-k_j^*}e^{-2i\theta(k_j^*)}u_{-,1}(k_j^*) + \frac{1}{2\pi i}\int_R \frac{(M^-(\zeta)G(\zeta))_2}{\zeta-k}\,d\zeta. \tag{3.21}$$

Similarly, considering the first column of M^+ at $k = k_n^*$ in C^+, one has

$$u_{-,1}(k_n^*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{N}\frac{C_n(k_j)}{k_n^*-k_j}e^{2i\theta(k_j)}u_{-,2}(k_j) + \frac{1}{2\pi i}\int_R \frac{(M^-(\zeta)G(\zeta))_2}{\zeta-k}\,d\zeta. \tag{3.22}$$

Considering the soliton solution without reflection, i.e., $G=0$, we can obtain

$$\begin{aligned} u_{-,12}(k_n) &= \sum_{j=1}^{N} \frac{\tilde{C}_n(k_j^*)}{k_n - k_j^*} e^{-2i\theta(k_j^*)} u_{-,11}(k_j^*), \\ u_{-,1}(k_n^*) &= 1 + \sum_{j=1}^{N} \frac{C_n(k_j)}{k_n^* - k_j} e^{2i\theta(k_j)} u_{-,12}(k_j). \end{aligned} \tag{3.23}$$

Introducing the notation

$$\varpi_j(k) = \frac{C_j(k_j)}{k - k_j} e^{2i\theta(k_j)}, \quad j = 1, 2, \cdots, N.$$

we can transform the expression (3.23) into

$$u_{-,12}(k_j) = -\sum_{l=1}^{N} \varpi_l^*(k_j^*) u_{-,11}(k_l^*), \tag{3.24a}$$

$$u_{-,11}(k_n^*) = 1 + \sum_{j=1}^{N} \varpi_j(k_n^*) u_{-,12}(k_j). \tag{3.24b}$$

Substituting (3.24a) into (3.24b) yields

$$u_{-,11}(k_n^*) = 1 - \sum_{j=1}^{N=1} \sum_{l=1}^{N=1} \varpi_j(k_n^*) \varpi_l^*(k_j^*) u_{-,11}(k_l^*). \tag{3.25}$$

Defining $\mathbf{X} = (X_1, X_2, \cdots, X_N)^T$ and $\mathbf{I}_{N\times 1} = (1, 1, \cdots, 1)^T$, where $X_n = u_{-,11}(k_n^*)$, and introducing the $N \times N$ matrix $A = (A_{n,l})$

$$A_{n,l} = A = \sum_{j=1}^{N=1} \varpi_j(k_n^*) \varpi_l^*(k_j^*) u_{-,11}(k_l^*), \quad k, l = 1, 2, \cdots, N,$$

then (3.25) can be written in the matrix from, i.e.,

$$(\mathbf{I} + A)\mathbf{X} = \mathbf{I} \triangleq \mathscr{M}\mathbf{X} = \mathbf{I} \tag{3.26}$$

which can be solved by Cramer's law, namely $X_n = \frac{\det \mathscr{M}_n^{rep}}{\det \mathscr{M}}$ for $n = 1, 2, \cdots, N$, where

$$\mathscr{M}_n^{rep} = (\mathscr{M}_1, \cdots, \mathscr{M}_{n-1}, \mathbf{I}, \mathscr{M}_{n+1}, \cdots, \mathscr{M}_N).$$

Substituting the $X_1, \cdots, X_N$ obtained by (3.26) into (3.25) yeilds

$$q(x,t) = -2i \frac{\det \mathscr{M}^{aug}}{\det \mathscr{M}}, \tag{3.27}$$

where the $(N+1)\times(N+1)$ matrix $\mathscr{M}^{aug}$ defined by $\mathscr{M}^{aug} = \begin{pmatrix} 0 & Y \\ \mathbf{I} & \mathscr{M} \end{pmatrix}$, and

$$Y = (\tilde{C}_1(k_1^*) e^{-2i\theta(k_1^*)}, \tilde{C}_2(k_2^*) e^{-2i\theta(k_2^*)}, \cdots, \tilde{C}_N(k_N^*) e^{-2i\theta(k_N^*)})$$

For the double zeros, i.e., **Case**(**B**), using the same technique as **Case**(**A**), we can express the solution of RH problem (3.2) as

$$M(x,t;k)=\mathbb{I}+\sum_{n=1}^{N}\left(\frac{\underset{k=k_n^*}{Res}\,M^+}{k-k_n^*}+\frac{\underset{k=k_n^*}{P_{-,2}}\,M^+}{(k-k_n^*)^2}+\frac{\underset{k=k_n}{P_{-,2}}\,M^-}{(k-k_n)^2}+\frac{\underset{k=k_n}{Res}\,M^-}{k-k_n}\right)+\frac{1}{2\pi i}\int_R\frac{M^-(\zeta)G(\zeta)}{\zeta-k}\,d\zeta. \tag{3.28}$$

Taking $M=M^-$ and comparing the 12 element of (3.28) one can get the potential for the double zeros without the refelection

$$q(x,t)=2i\left(\sum_{n=1}^{N}\tilde{A}_ne^{-2i\theta(k_n^*)}\left[u'_{-11}(k_n^*)+u_{-,11}(k_n^*)(\tilde{B}_n-2i\theta'(k_n^*))\right]\right), \tag{3.29}$$

combining with the (2.15), where $u'_{-11}(k_n^*)$ and $u_{-,11}(k_n^*)$ defined by (3.33).

In the case of no reflection, i.e., $G=0$, considering the second column of the M^- at $k=k_j$ in C^-, we have

$$u_{-,2}(k_j)=\begin{pmatrix}0\\1\end{pmatrix}+\sum_{n=1}^{N}\left(\frac{\tilde{A}_ne^{-2i\theta(k_n^*)}\left[u'_{-1}(k_n^*)+u_{-,1}(k_n^*)(\tilde{B}_n-2i\theta'(k_n^*))\right]}{k_j-k_n^*}+\frac{\tilde{A}_ne^{-2i\theta(k_n^*)}u_{-,1}(k_n^*)}{(k_j-k_n^*)^2}\right), \tag{3.30}$$

then considering the first column of M^+ at $k=k_j^*$ in C^+, one has

$$u_{-,1}(k_j^*)=\begin{pmatrix}1\\0\end{pmatrix}+\sum_{n=1}^{N}\left(\frac{A_ne^{2i\theta(k_n)}\left[u'_{-2}(k_n)+u_{-,2}(k_n)(B_n+2i\theta'(k_n))\right]}{k_j^*-k_n}+\frac{A_ne^{2i\theta(k_n)}u_{-,2}(k_n)}{(k_j^*-k_n)^2}\right). \tag{3.31}$$

Taking these notations

$$\tilde{C}_n(k_l)=\frac{\tilde{A}_ne^{-2i\theta(k_n^*)}}{k_l-k_n^*},\quad \tilde{D}_n=\tilde{B}_n-2i\theta'(k_n^*),$$
$$C_n(k_l^*)=\frac{A_ne^{2i\theta(k_n)}}{k_l^*-k_n},\quad D_n=B_n+2i\theta'(k_n),$$

we can transform (3.30) and (3.31) into the following expressions

$$\begin{cases} u_{-,2}(k_l)=\begin{pmatrix}0\\1\end{pmatrix}+\sum_{n=1}^{N}\tilde{C}_n(k_l)\left[u'_{-,1}(k_n^*)+u_{-,1}(k_n^*)\left(\tilde{D}_n+\frac{1}{k_l-k_n^*}\right)\right],\\ u'_{-,2}(k_l)=-\sum_{n=1}^{N}\frac{\tilde{C}_n(k_l)}{k_l-k_n^*}\left[u'_{-,1}(k_n^*)+u_{-,1}(k_n^*)\left(\tilde{D}_n+\frac{2}{k_l-k_n^*}\right)\right],\\ u_{-,1}(k_l^*)=\begin{pmatrix}1\\0\end{pmatrix}+\sum_{n=1}^{N}C_n(k_l^*)\left[u'_{-,2}(k_n)+u_{-,2}(k_n)\left(D_n+\frac{1}{k_l^*-k_n}\right)\right],\\ u'_{-,1}(k_l^*)=-\sum_{n=1}^{N}\frac{C_n(k_l^*)}{k_l^*-k_n}\left[u'_{-,2}(k_n)+u_{-,2}(k_n)\left(D_n+\frac{2}{k_l^*-k_n}\right)\right], \end{cases} \tag{3.32}$$

further one has

$$
\begin{cases}
u_{-,12}(k_l) = \displaystyle\sum_{n=1}^{N} \tilde{C}_n(k_l) \left[u'_{-,11}(k_n^*) + u_{-,11}(k_n^*) \left(\tilde{D}_n + \frac{1}{k_l - k_n^*} \right) \right], \\
u'_{-,12}(k_l) = -\displaystyle\sum_{n=1}^{N} \frac{\tilde{C}_n(k_{1l})}{k_l - k_n^*} \left[u'_{-,1}(k_n^*) + u_{-,11}(k_n^*) \left(\tilde{D}_n + \frac{2}{k_l - k_n^*} \right) \right], \\
u_{-,11}(k_l^*) = 1 + \displaystyle\sum_{n=1}^{N} C_n(k_l^*) \left[u'_{-,12}(k_n) + u_{-,12}(k_n) \left(D_n + \frac{1}{k_l^* - k_n} \right) \right], \\
u'_{-,11}(k_l^*) = -\displaystyle\sum_{n=1}^{N} \frac{C_n(k_l^*)}{k_l^* - k_n} \left[u'_{-,12}(k_n) + u_{-,12}(k_n) \left(D_n + \frac{2}{k_l^* - k_n} \right) \right].
\end{cases}
\tag{3.33}
$$

4 The solution for the foNLS equation

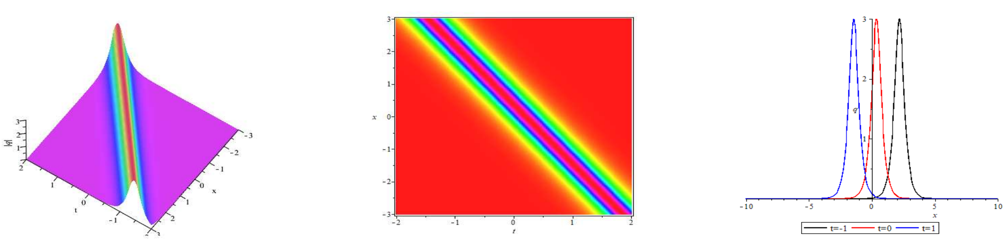
In this section, we will select the appropriate parameters to observe the propagation behavior of the solutions (3.27) and (3.29) according to the potential function reconstruction formula obtained under the two zero points in the previous section.

For **Case**(**A**) : For the case of simple zeros, we mainly discuss three cases, that is, assuming that there are one, two and three simple zeros respectively.

Case (a): Suppose that $N = 1$, then the solution (3.27) can be written as

$$
q(x,t) = -2i \frac{-\tilde{C}_1(k_1^*) e^{-2i\theta(k_1^*)}}{1 - \frac{C_1(k_1) e^{2i\theta(k_1)}}{k_1^* - k_1} \frac{\tilde{C}_1(k_1^*)}{k_1 - k_1^*} e^{-2i\theta(k_1^*)}}, \tag{4.1}
$$

in addition, selecting $k_1 = -1.5i$,$\epsilon = 0.01$ and $C_1(k_1) = 1$ we show the propagation behavior of the corresponding solution of the equation in figure. 1.



(**a**) (**b**) (**c**)

Figure 1. **(a)** The bright soliton solution to the solution (3.27). **(b)** is the density map of the bright soliton solution. **(c)** The propagation view of corresponding solution at different times.

From Figure 1, we observe that the solution maintains its shape and velocity during propagation, and the wave appears in the upper half plane, so we know that this is a bright soliton solution.

Case (b): Suppose that $N=2$, then the solution (3.27) can be written as

$$q(x,t)=-2i\frac{\det\begin{pmatrix}0 & \tilde{C}_1(k_1^*) & \tilde{C}_2(k_2^*)\\ 1 & 1+A_{11} & A_{12}\\ 1 & A_{21} & 1+A_{22}\end{pmatrix}}{\det\begin{pmatrix}1+A_{11} & A_{12}\\ A_{21} & 1+A_{22}\end{pmatrix}},\tag{4.2}$$

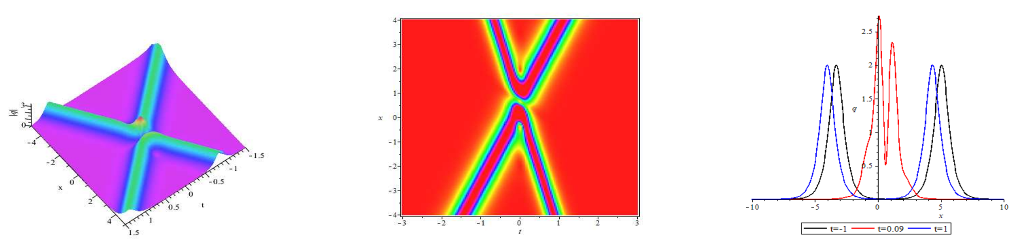
where

$$\begin{aligned}
&A_{11}=\varpi_1(k_1^*)\varpi_1^*(k_1^*)+\varpi_2(k_1^*)\varpi_1^*(k_2^*), && A_{12}=\varpi_1(k_1^*)\varpi_2^*(k_1^*)+\varpi_2(k_1^*)\varpi_2^*(k_2^*)\\
&A_{21}=\varpi_1(k_2^*)\varpi_1^*(k_1^*)+\varpi_2(k_2^*)\varpi_1^*(k_2^*), && A_{22}=\varpi_1(k_2^*)\varpi_2^*(k_1^*)+\varpi_2(k_2^*)\varpi_2^*(k_2^*)\\
&\varpi_1(k_1^*)=\frac{C_1(k_1)}{k_1^*-k_1}e^{2i\theta(k_1)}, && \varpi_1(k_2^*)=\frac{C_1(k_1)}{k_2^*-k_1}e^{2i\theta(k_1)},\\
&\varpi_2(k_1^*)=\frac{C_2(k_2)}{k_1^*-k_2}e^{2i\theta(k_2)}, && \varpi_2(k_2^*)=\frac{C_2(k_2)}{k_2^*-k_2}e^{2i\theta(k_2)},\\
&\varpi_1^*(k_1^*)=-\frac{\tilde{C}_1(k_1^*)}{k_1-k_1^*}e^{-2i\theta(k_1^*)}, && \varpi_1^*(k_2^*)=-\frac{\tilde{C}_1(k_1^*)}{k_2-k_1^*}e^{-2i\theta(k_1^*)},\\
&\varpi_2^*(k_1^*)=-\frac{\tilde{C}_2(k_2^*)}{k_1-k_2^*}e^{-2i\theta(k_2^*)}, && \varpi_2^*(k_2^*)=-\frac{\tilde{C}_2(k_2^*)}{k_2-k_2^*}e^{-2i\theta(k_2^*)}.
\end{aligned}$$

When taking the following parameters $k_1=1-i$, $k_2=-1-i$, $\epsilon=0.01$, $C_1(k_1)=1$ and $C_2(k_2)=i$, the solution (4.2) can be expressed by

$$q(x,t)=16i\frac{\left(ie^{73it-75ix-g}+8ie^{91it-25ix+g}-e^{-0.08(127it+75ix-g)}+8e^{-0.08(109it+25ix+g)}\right)}{e^{2.88t+8x}+16e^{17.44t+4x}-8e^{0.16g(1-i)}-8e^{0.16g(1+i)}+16e^{-14.56t+4x}+64},\tag{4.3}$$

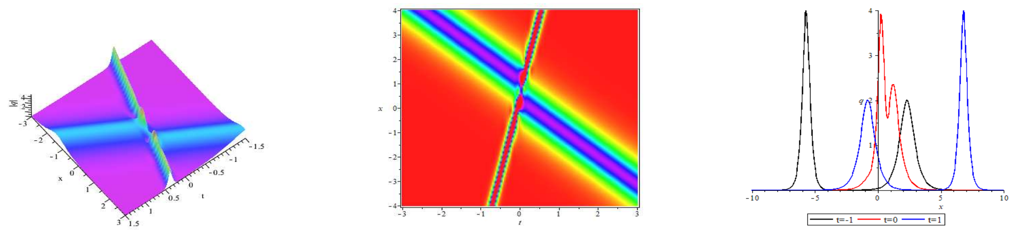
where $g=g(x,t)=9t+25x$, and the propagation behavior is shown in figure 2.



(**a**) (**b**) (**c**)

Figure 2. **(a)** The bright soliton solution to the solution (3.27). **(b)** is the density map of the bright soliton solution. **(c)** The propagation view of corresponding solution at different times.

When taking the following parameters $k_1=-i$, $k_2=-1-2i$, $\epsilon=0.01$, $C_1(k_1)=1$ and $C_2(k_2)=i$, the solution (4.2) shown in the figure 3.



(**a**) (**b**) (**c**)

Figure 3. **(a)** The bright soliton solution to the solution (3.27). **(b)** is the density map of the bright soliton solution. **(c)** The propagation view of corresponding solution at different times.

Both Fig. 2 and Fig. 3 are two soliton solutions. The difference is that the two spectrum parameters in Fig. 2 both contain real parts, while only one spectrum parameter in Fig. 3 has real parts. The difference in solution behavior is that for the first case it is the solution interaction of two bright solitons. At time $t = 0$, the two solitons collide, and the two solitons keep the energy unchanged before and after the collision. For the second case, the Fig. 3 shows the interaction of a soliton solution and a breather-type solution, and the energy before and after the collision also remains unchanged.

Case (c): Suppose that $N = 3$, then the solution (3.27) can be written as

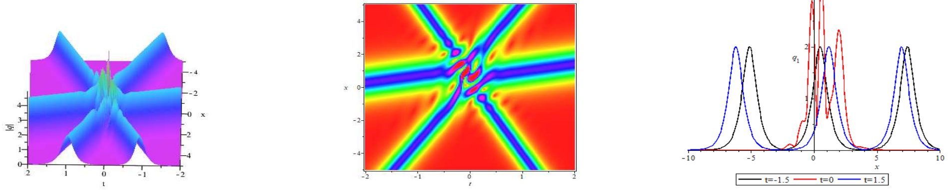
$$q(x,t) = -2i\frac{\det\begin{pmatrix} 0 & \tilde{C}_1(k_1^*) & \tilde{C}_2(k_2^*) & \tilde{C}_3(k_3^*) \\ 1 & 1+A_{11} & A_{12} & A_{13} \\ 1 & A_{21} & 1+A_{22} & A_{23} \\ 1 & A_{31} & A_{32} & 1+A_{33} \end{pmatrix}}{\det\begin{pmatrix} 1+A_{11} & A_{12} & A_{13} \\ A_{21} & 1+A_{22} & A_{23} \\ A_{31} & A_{32} & 1+A_{33} \end{pmatrix}}, \tag{4.4}$$

where

$$A = A_{n,l} = \sum_{j=1}^{3} \varpi_j(k_n^*)\varpi_l^*(k_j^*), \quad n,l = 1,2,3,$$

$$\varpi_j(k_n^*) = \frac{C_j(k_j)}{k_n^* - k_j} e^{2i\theta(k_j)}, \quad \varpi_j^*(k_n^*) = -\frac{\tilde{C}_j(k_j^*)}{k_n - k_j^*} e^{-2i\theta(k_j^*)} \quad n,j = 1,2,3.$$

These parameters $k_1 = -1 - i$, $k_2 = 1 - i$, $k_1 = -2 - i$, $\epsilon = 0.01$, $C_1(k_1) = C_3(k_3) = 1$ and $C_2(k_2) = i$ can be selected to give the propagation behavior of the solution (4.4) shown in figure 4.



(**a**) (**b**) (**c**)

Figure 4. **(a)** The bright soliton solution to the solution (3.27). **(b)** is the density map of the bright soliton solution. **(c)** The propagation view of corresponding solution at different times.

Figure 4 shows the propagation behavior of the three-soliton. When $t = 0$, from the interaction of the three soliton, we observe the energy changes irregularly, and the energy is constant at other times.

For **Case**(**B**) : For the case of double zeros, we mainly discuss two cases, that is, assuming that there are one and two double zeros respectively.

Case (a): At first, when $N = 1$, we give the concrete expression of the solution of the equation (1.1) combining with (3.29) and (3.33). Note that

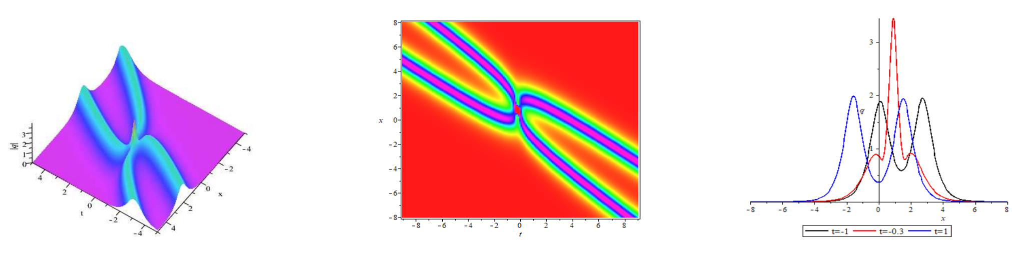
$$\begin{cases} u_{-,12}(k_1) = \tilde{C}_1(k_1)\left[u'_{-,11}(k_1^*) + u_{-,11}(k_1^*)\left(\tilde{D}_1 + \dfrac{1}{k_1 - k_1^*}\right)\right], \\ u'_{-,12}(k_1) = -\dfrac{\tilde{C}_1(k_1)}{k_1 - k_1^*}\left[u'_{-,1}(k_1^*) + u_{-,11}(k_1^*)\left(\tilde{D}_1 + \dfrac{2}{k_1 - k_1^*}\right)\right], \\ u_{-,11}(k_1^*) = 1 + C_1(k_1^*)\left[u'_{-,12}(k_1) + u_{-,12}(k_1)\left(D_1 + \dfrac{1}{k_1^* - k_1}\right)\right], \\ u'_{-,11}(k_1^*) = -\dfrac{C_1(k_1^*)}{k_1^* - k_1}\left[u'_{-,12}(k_1) + u_{-,12}(k_1)\left(D_1 + \dfrac{2}{k_1^* - k_1}\right)\right]. \end{cases}$$

with $\tilde{D}_1 = \tilde{B}_1 - 2i\theta'(k_1^*)$, $\quad \tilde{C}_1(k_1) = \frac{\tilde{A}_1 e^{-2i\theta(k_1^*)}}{k_1 - k_1^*}, D_1 = B_1 + 2i\theta'(k_1)$, $\quad C_1(k_1^*) = \frac{A_1 e^{2i\theta(k_1)}}{k_1^* - k_1}$.

The elements $u'_{-11}(k_n^*)$ and $u_{-11}(k_n^*)$ of the solution (3.29) can be derived by the following equation

$$\begin{pmatrix} 1 & 0 & -C_1(k_1^*)\left(D_1 + \frac{1}{k_1^*-k_1}\right) & -C_1(k_1^*) \\ 0 & 1 & \frac{C_1(k_1^*)}{k_1^*-k_1}\left(D_1 + \frac{2}{k_1^*-k_1}\right) & \frac{C_1(k_1^*)}{k_1^*-k_1} \\ -\tilde{C}_1(k_1^*)\left(\tilde{D}_1 + \frac{1}{k_1^*-k_1}\right) & -\tilde{C}_1(k_1^*) & 1 & 0 \\ \frac{\tilde{C}_1(k_1^*)}{k_1^*-k_1}\left(\tilde{D}_1 + \frac{2}{k_1^*-k_1}\right) & \frac{\tilde{C}_1(k_1^*)}{k_1^*-k_1} & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{-,11}(k_1^*) \\ u'_{-,11}(k_1^*) \\ u_{-,12}(k_1^*) \\ u'_{-,12}(k_1^*) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.5}$$

Based on the solution (3.27) combining with the expression about $u'_{-11}(k_n^*)$ and $u_{-11}(k_n^*)$, we can show the propagation behavior in figure 5.

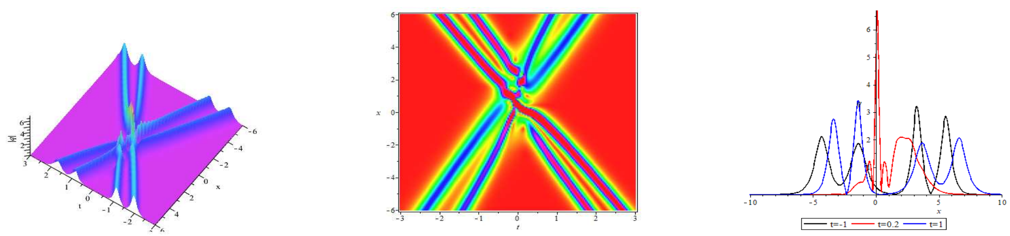


(**a**) (**b**) (**c**)

Figure 5. The parameters $k_1 = -\frac{1}{10} - i$, $\epsilon = 0.01$, $A_1 = -1$, and $B_1 = 2$. **(a)** The bright soliton solution to the solution (3.27). **(b)** is the density map of the bright soliton solution. **(c)** The propagation view of corresponding solution at different times.

It can be observed from figure 5 that the 1-soliton solution under the condition of double zeros is the interaction of two bright soliton solutions. Except for the energy superposition of the two bright soliton solutions near time $t = -0.3$, the two soliton solutions present a symmetrical structure, and the shape and size remain unchanged before and after the collision.

Case (b): When $N = 2$, the expressions of $u_{-,11}(k_1^*)$, $u'_{-,11}(k_1^*)$, $u_{-,11}(k_2^*)$ and $u'_{-,11}(k_2^*)$ are so complex that it is also very complicated to bring the obtained results into the solution (3.29) of the equation (1.1). Therefore, for the sake of simplicity, we omit the specific expression of the solution of the equation. But the figure 6 shows the propagation behavior of the solution at this time with the parameters $k_1 = -1 - i$, $k_2 = 1 - i$, $\epsilon = 0.01$, $A_1 = B_1 = 1$, and $A_2 = B_2 = i$.



(**a**) (**b**) (**c**)

Figure 6. **(a)** The bright soliton solution to the solution (3.27). **(b)** is the density map of the bright soliton solution. **(c)** The propagation view of corresponding solution at different times.

Comparing with Fig. 5 and Fig. 6, we observe that the phenomenon in Fig. 6 is a superposition of the solutions in Fig. 5, that is, two solutions in Fig. 5 interact with each other. However, it is noted that the energy of the solutions in Fig. 5 hardly shifts after the collision, while the phenomena in Fig. 6 show that the solutions transfer after the collision, and the energy lost in the collision is transferred to the other soliton solutions.

5 Conclusions

In this work, we use the RH problem to analyze the foNLS equation with the potential function decays rapidly at infinity. Our motivation is mainly derived from the idea of Biondini and his collaborators who have established a suitable RH problem in the Res.[23]. When discussing the irregular RH problem of scattering data with zero points, they usually discuss the residues generated at simple zeros (including the residues generated by double zeros and the coefficients corresponding to the -2 power of Laurent series in the Ref.[24]).

However, many authors often use a transformation when dealing with integrable equations under zero boundary conditions to transform the irregular RH problem into a regular RH problem [11, 12, 15, 19, 20, 21, 28], so as to avoid discussing the properties of the singularity, and our goal in this work is to combine these two woks, the idea of Biondini's work is used to deal with the integrable equation with zero boundary conditions at infinity, which means that the problem of zero points is not circumvented, and the situation where the scattered data has double zero points is further discussed and the expected results are given.

Of course, this idea can be further promoted to discuss the existence of three zeros or even N zeros in scattering data. Only by calculating the coefficients of each order corresponding to Laurent series, and then subtracting the contribution value of zero points and the asymptotic behavior from the original RH problem, the original RH problem can be regularized. Finally, the connection between the corresponding zero point potential function and the RH problem can be established, which means that the reconstruction of the potential function is completed.

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Conflict of interest

This work does not have any conflicts of interest.

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