

Riemann-Hilbert approach and N -soliton solutions for a new four-component nonlinear Schrödinger equation*

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Abstract

A new four-component nonlinear Schrödinger equation is first proposed in this work and studied by Riemann-Hilbert approach. Firstly, we derive a Lax pair associated with a 5×5 matrix spectral problem for the four-component nonlinear Schrödinger equation. Then based on the Lax pair, we analyze the spectral problem and the analytical properties of the Jost functions, from which the Riemann-Hilbert problem of the equation is successfully established. Moreover, we obtain the N -soliton solutions of the equation by solving the Riemann-Hilbert problem without reflection. Finally, we derive two special cases of the solutions to the equation for $N = 1$ and $N = 2$, and the local structure and dynamic behavior of the one-and two-soliton solutions are analyzed graphically.

Key words: A four-component nonlinear Schrödinger equation; Riemann-Hilbert approach; N -soliton solutions.

1 Introduction

The nonlinear Schrödinger equation (NLS) is an important integrable model. It is closely related to many nonlinear problems in theoretical physics such as nonlinear optics and ion acoustic waves of plasmas. Some higher-order coupled NLS equations are proposed, to describe more deep physical effects, including self-deepening, self-frequency shifting, and cubic-quintic nonlinearity. Among the different solutions of these models, soliton solutions play a crucial role in explaining some related complex nonlinear phenomena. With the development of nonlinear science, there are many ways to find solutions for nonlinear integrable models, including inverse scattering transform [1], Darboux transform [2], Hirota bilinear method [3], Lie group method [4], etc. Among them, inverse scattering

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transform method is one of the most effective tools for solving the initial value problem of nonlinear integrable systems to get the soliton solutions. For second-order spectral problems, inverse scattering theory is equivalent to Riemann-Hilbert (RH) approach, but for higher-order spectral problems the development of inverse scattering theory is not perfect, part of the inverse scattering problem needs to be transformed into RH problem. RH approach is developed by Zakharov et al [5], applied to integrable systems [6]-[31] as a more general method than inverse scattering method. This method has been successfully used to study the integrable system with single component. However, to the best of authors' knowledge, there are very few studies on the multi-component problems. The well-known general two-component coupled nonlinear Schrödinger equation of the form [32]

$$\begin{cases} ip_t + p_{xx} + 2(a|p|^2 + c|q|^2 + bpq^* + b^*qp^*)p = 0, \\ iq_t + q_{xx} + 2(a|p|^2 + c|q|^2 + bpq^* + b^*qp^*)q = 0, \end{cases} \quad (1.1)$$

where a and c are real constants, b is a complex constant, and “ $*$ ” denotes complex conjugation. In physics, a and c describe the SPM and XPM effects, and b and b^* describe the four-wave mixing effects.

In this work, we first propose an interesting equation named by a new four-component nonlinear Schrödinger (FCNLS) equation

$$\begin{cases} iq_{1t} + q_{1xx} - 2[a_{11}|q_1|^2 + a_{22}|q_2|^2 + a_{33}|q_3|^2 + a_{44}|q_4|^2 \\ + 2\text{Re}(a_{12}q_1^*q_2 + a_{13}q_1^*q_3 + a_{14}q_1^*q_4 + a_{23}q_2^*q_3 + a_{24}q_2^*q_4 + a_{34}q_3^*q_4)]q_1 = 0, \\ iq_{2t} + q_{2xx} - 2[a_{11}|q_1|^2 + a_{22}|q_2|^2 + a_{33}|q_3|^2 + a_{44}|q_4|^2 \\ + 2\text{Re}(a_{12}q_1^*q_2 + a_{13}q_1^*q_3 + a_{14}q_1^*q_4 + a_{23}q_2^*q_3 + a_{24}q_2^*q_4 + a_{34}q_3^*q_4)]q_2 = 0, \\ iq_{3t} + q_{3xx} - 2[a_{11}|q_1|^2 + a_{22}|q_2|^2 + a_{33}|q_3|^2 + a_{44}|q_4|^2 \\ + 2\text{Re}(a_{12}q_1^*q_2 + a_{13}q_1^*q_3 + a_{14}q_1^*q_4 + a_{23}q_2^*q_3 + a_{24}q_2^*q_4 + a_{34}q_3^*q_4)]q_3 = 0, \\ iq_{4t} + q_{4xx} - 2[a_{11}|q_1|^2 + a_{22}|q_2|^2 + a_{33}|q_3|^2 + a_{44}|q_4|^2 \\ + 2\text{Re}(a_{12}q_1^*q_2 + a_{13}q_1^*q_3 + a_{14}q_1^*q_4 + a_{23}q_2^*q_3 + a_{24}q_2^*q_4 + a_{34}q_3^*q_4)]q_4 = 0, \end{cases} \quad (1.2)$$

where a_{11} , a_{22} , a_{33} and a_{44} are real constants, a_{12} , a_{13} , a_{14} , a_{23} , a_{24} and a_{34} are complex constant, “ $*$ ” denotes complex conjugation, and “Re” denotes the real part. The FCNLS equation includes group velocity dispersion, self-phase modulation, cross-phase modulation and paired tunnel modulation. The RH problem of the FCNLS equation and its multi-soliton solutions have not been studied so far, which is studied in this work. Here are the reasons for taking the FCNLS equation as the model. The Eq. (1.2) can be reduced to the three-component nonlinear Schrödinger equation (1.3) given by

$$\begin{cases} iq_{1t} + q_{1xx} - 2[a|q_1|^2 + c|q_2|^2 + f|q_3|^2 + 2\text{Re}(bq_1^*q_2 + dq_1^*q_3 + eq_2^*q_3)]q_1 = 0, \\ iq_{2t} + q_{2xx} - 2[a|q_1|^2 + c|q_2|^2 + f|q_3|^2 + 2\text{Re}(bq_1^*q_2 + dq_1^*q_3 + eq_2^*q_3)]q_2 = 0, \\ iq_{3t} + q_{3xx} - 2[a|q_1|^2 + c|q_2|^2 + f|q_3|^2 + 2\text{Re}(bq_1^*q_2 + dq_1^*q_3 + eq_2^*q_3)]q_3 = 0, \end{cases} \quad (1.3)$$

where a , c and f are real constants, b , d and e are complex constant, “ $*$ ” denotes complex conjugation, and “Re” denotes the real part. This equation can be reduced to equation (1.1). Eq. (1.3) is studied by extending the Fokas unified approach by Yan in [33]. Eq. (1.3) can be reduced to different three-component NLS equations with the different conditions of the six parameters a , b , c , d , e and f , such as

- The three-component focused NLS equation for $a = c = f = -1$ and $b = d = e = 0$.
- The three-component defocused NLS equation for $a = c = f = 1$ and $b = d = e = 0$.
- The three-component mixed NLS equation for $a = -1, c = f = 1$ and $b = d = e = 0$ or $a = 1, c = f = -1$ and $b = d = e = 0$. If one takes other parameter values, equation (1.3) can be reduced to other three-component NLS equations.

As the same as the three-component NLS equations (1.3), the FCNLS equation (1.2) can also be reduced to different four-component NLS equations with the different conditions of the ten parameters. Therefore, it is of great significance to construct the RH problem of FCNLS equation involving 5×5 matrix spectrum problem for finding the multi-soliton solutions.

The structure of this work is as follows. In the second part, we derive a Lax pair associated with a 5×5 matrix spectral problem for the FCNLS equation (1.2). Then based on the Lax pair with a 5×5 matrix, we analyze the spectral problem and the analytical properties of the Jost functions. In the third part, we establish the RH problem based on the previous conclusions. Next, we give the symmetry of the scattering matrix, and study the temporal and spatial evolution of the scattering data. In the fourth part, by solving the RH problem, we obtain the N -soliton solutions of the FCNLS equation (1.2), and analyze the propagation behaviors of one-soliton solutions and two-soliton solutions. Finally, some conclusions are presented in the last section.

2 Spectral analysis

2.1 The Lax Pair and eigenfunction

We first derive the Lax pair of the FCNLS equation (1.2) via the following theorem.

Theorem 2.1. *The FCNLS equation (1.2) admits the following Lax pair*

$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi, \quad (2.1)$$

where Φ is a column vector function, and matrices U and V are written as

$$U = \begin{pmatrix} -i\lambda & 0 & 0 & 0 & q_1 \\ 0 & -i\lambda & 0 & 0 & q_2 \\ 0 & 0 & -i\lambda & 0 & q_3 \\ 0 & 0 & 0 & -i\lambda & q_4 \\ p_1 & p_2 & p_3 & p_4 & i\lambda \end{pmatrix}, \quad (2.2)$$

$$V = -2i\lambda^2\Lambda + 2\lambda P + V_0, \quad V_0 = -i(P_x + P^2), \quad (2.3)$$

here λ being the spectral parameter and $p_1 = a_{11}q_1^* + a_{21}q_2^* + a_{31}q_3^* + a_{41}q_4^*$, $p_2 = a_{21}q_1^* + a_{22}q_2^* + a_{32}q_3^* + a_{42}q_4^*$, $p_3 = a_{31}q_1^* + a_{32}q_2^* + a_{33}q_3^* + a_{43}q_4^*$, $p_4 = a_{41}q_1^* + a_{42}q_2^* + a_{43}q_3^* + a_{44}q_4^*$, with

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & q_3 \\ 0 & 0 & 0 & 0 & q_4 \\ p_1 & p_2 & p_3 & p_4 & 0 \end{pmatrix}. \quad (2.4)$$

Proof. The compatibility condition of the two equations in Eqs.(2.1)

$$U_t - V_x + [U, V] = 0, \quad (2.5)$$

with $[U, V] = UV - VU$, which is reduced to the FCNLS equation (1.2). \square

Then we obtain that

$$\begin{aligned} \Phi_x + i\lambda\Lambda\Phi &= P\Phi, \\ \Phi_t + 2i\lambda^2\Lambda\Phi &= Q\Phi, \end{aligned} \quad (2.6)$$

where $Q = 2\lambda P + V_0$.

From Eqs.(2.6) when $|x| \rightarrow \infty$, one has

$$\Phi \propto e^{-i\lambda\Lambda x - 2i\lambda^2\Lambda t}. \quad (2.7)$$

Letting

$$\mu = \Phi e^{i\lambda\Lambda x + 2i\lambda^2\Lambda t}, \quad (2.8)$$

then we can get the equivalent Lax pair

$$\begin{aligned} \mu_x + i\lambda[\Lambda, \mu] &= P\mu, \\ \mu_t + 2i\lambda^2[\Lambda, \mu] &= Q\mu, \end{aligned} \quad (2.9)$$

where $[\Lambda, \mu] = \Lambda\mu - \mu\Lambda$ is the commutator. We can get the following full differential

$$d\left(e^{i(\lambda x + 2\lambda^2 t)\bar{\Lambda}}\mu\right) = e^{i(\lambda x + 2\lambda^2 t)\bar{\Lambda}}[(Pdx + Qdt)\mu], \quad (2.10)$$

where $e^{\lambda\bar{\Lambda}}\mu = e^{\lambda\Lambda}\mu e^{-\lambda\Lambda}$.

2.2 Asymptotic analysis

To formulate an RH problem, we seek solutions of the spectral problem with the 5×5 unit matrix as $\lambda \rightarrow \infty$. Let us consider the solution of Eq.(2.10) as follows

$$\mu = \mu^{(0)} + \frac{\mu^{(1)}}{\lambda} + \frac{\mu^{(2)}}{\lambda^2} + o\left(\frac{1}{\lambda^3}\right), \quad \lambda \rightarrow \infty, \quad (2.11)$$

where $\mu^{(0)}$, $\mu^{(1)}$ and $\mu^{(2)}$ are independent of λ . Substituting Eq.(2.11) into Eqs.(2.9), and comparing the same order of frequency for λ , we obtain

$$\begin{aligned} o(1) &: \mu_x^{(0)} + i\lambda[\Lambda, \mu^{(1)}] = P\mu^{(0)}, \\ o(\lambda) &: i\lambda[\Lambda, \mu^{(0)}] = 0, \\ o(\lambda) &: 2i\lambda^2[\Lambda, \mu^{(1)}] = 2\lambda P\mu^{(0)}. \end{aligned} \quad (2.12)$$

Eqs.(2.12) implies $\mu^{(0)}$ is a diagonal matrix, $\mu_x^{(0)} = 0$. This means that $\mu^{(0)}$ is not related to x . Then

$$I = \lim_{\lambda \rightarrow \infty} \lim_{|x| \rightarrow \infty} \mu = \mu^{(0)}. \quad (2.13)$$

Now, two solutions $\mu_{\pm} = \mu_{\pm}(x, \lambda)$ are constructed for Eq.(2.9)

$$\begin{aligned}\mu_+ &= ([\mu_+]_1, [\mu_+]_2, [\mu_+]_3, [\mu_+]_4, [\mu_+]_5), \\ \mu_- &= ([\mu_-]_1, [\mu_-]_2, [\mu_-]_3, [\mu_-]_4, [\mu_-]_5),\end{aligned}\tag{2.14}$$

with the asymptotic conditions

$$\begin{aligned}\mu_+ &\rightarrow I \quad \text{as } x \rightarrow +\infty, \\ \mu_- &\rightarrow I \quad \text{as } x \rightarrow -\infty,\end{aligned}\tag{2.15}$$

here each $[\mu_{\pm}]_l (l = 1, 2, 3, 4, 5)$ denotes the l -th column of the matrices $[\mu_{\pm}]$, respectively. The symbol I is the 5×5 unit matrix, and the two solutions $[\mu_{\pm}]$ are uniquely determined by the Volterra integral equations for $\lambda \in R$

$$\begin{aligned}\mu_+(x, \lambda) &= I - \int_x^{+\infty} e^{-i\lambda\Lambda(x-y)} P(y) \mu_+(y, \lambda) e^{i\lambda\Lambda(x-y)} dy, \\ \mu_-(x, \lambda) &= I + \int_{-\infty}^x e^{-i\lambda\Lambda(x-y)} P(y) \mu_-(y, \lambda) e^{i\lambda\Lambda(x-y)} dy.\end{aligned}\tag{2.16}$$

Then we analysis the Eqs.(2.16),

$$e^{-i\lambda\Lambda(x-y)} P e^{i\lambda\Lambda(x-y)} = \begin{pmatrix} 0 & 0 & 0 & 0 & q_1 e^{-2i\lambda(x-y)} \\ 0 & 0 & 0 & 0 & q_2 e^{-2i\lambda(x-y)} \\ 0 & 0 & 0 & 0 & q_3 e^{-2i\lambda(x-y)} \\ 0 & 0 & 0 & 0 & q_4 e^{-2i\lambda(x-y)} \\ p_1 e^{2i\lambda(x-y)} & p_2 e^{2i\lambda(x-y)} & p_3 e^{2i\lambda(x-y)} & p_4 e^{2i\lambda(x-y)} & 0 \end{pmatrix}.\tag{2.17}$$

To find the analytic area of each column, we just consider $\text{Re}[2i\lambda(x-y)] < 0$ and $\text{Re}[-2i\lambda(x-y)] < 0$. One obtains $[\mu_-]_1, [\mu_-]_2, [\mu_-]_3, [\mu_-]_4$ and $[\mu_+]_5$ are analytic in the upper half-plane C^+ . Similarly, $[\mu_+]_1, [\mu_+]_2, [\mu_+]_3, [\mu_+]_4$ and $[\mu_-]_5$ are analytic in the lower half-plane C^- . Now we investigate the properties of μ_{\pm} . Since $\text{tr}(P) = 0$ and Liouville's formula, we know that the determinants of μ_{\pm} are independent of the variable x . Therefore we obtain from Eq.(2.15) that

$$\det \mu_{\pm} = 1, \quad \lambda \in R.\tag{2.18}$$

Since $\mu_{\pm} E$ are both matrix solutions of the spectral problem Eqs.(2.9), where $E = e^{-i\lambda\Lambda x}$. Therefore, these two solutions are interdependent, and they must be related by a scattering matrix $S(\lambda) = (s_{kj})_{5 \times 5}$

$$\mu_- E = \mu_+ E S(\lambda), \quad \lambda \in R.\tag{2.19}$$

From Eq.(2.18) and Eq.(2.19), we have

$$\det S(\lambda) = 1, \quad \lambda \in R.\tag{2.20}$$

To formulate an RH problem for the FCNLS equations (1.2), we consider the inverse matrices of μ_{\pm} as

$$\mu_{\pm}^{-1} = \begin{pmatrix} [\mu_{\pm}^{-1}]^1 \\ [\mu_{\pm}^{-1}]^2 \\ [\mu_{\pm}^{-1}]^3 \\ [\mu_{\pm}^{-1}]^4 \\ [\mu_{\pm}^{-1}]^5 \end{pmatrix},\tag{2.21}$$

where each $[\mu_{\pm}^{-1}]^l$, ($l = 1, 2, 3, 4, 5$) denotes the l -th row of μ_{\pm}^{-1} , respectively.

Theorem 2.2. *Letting*

$$S(\lambda) = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} \end{pmatrix},$$

$$R(\lambda) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} \\ r_{51} & r_{52} & r_{53} & r_{54} & r_{55} \end{pmatrix},$$

$s_{11}, s_{12}, s_{13}, s_{14}, s_{21}, s_{22}, s_{23}, s_{24}, s_{31}, s_{32}, s_{33}, s_{34}, s_{41}, s_{42}, s_{43}$ and s_{44} are analytic for $\lambda \in C^+$, s_{55} is analytic in C^- . r_{55} is analytic in C^+ , $r_{11}, r_{12}, r_{13}, r_{14}, r_{21}, r_{22}, r_{23}, r_{24}, r_{31}, r_{32}, r_{33}, r_{34}, r_{41}, r_{42}, r_{43}$ and r_{44} are analytic for $\lambda \in C^-$.

Proof. Using Eq.(2.9) it is easy to verify that μ_{\pm}^{-1} satisfy the equation of K

$$K_x = -i\lambda[\Lambda, K] - KP. \quad (2.22)$$

According to (2.19), it's easy to find

$$E^{-1}\mu_-^{-1} = R(\lambda)E^{-1}\mu_+^{-1}. \quad (2.23)$$

From Eq.(2.19), we have $E^{-1}\mu_+^{-1}\mu_-E = S(\lambda)$.

$$S(\lambda) = E^{-1} \begin{pmatrix} [\mu_+^{-1}]^1[\mu_-]_1 & [\mu_+^{-1}]^1[\mu_-]_2 & [\mu_+^{-1}]^1[\mu_-]_3 & [\mu_+^{-1}]^1[\mu_-]_4 & [\mu_+^{-1}]^1[\mu_-]_5 \\ [\mu_+^{-1}]^2[\mu_-]_1 & [\mu_+^{-1}]^2[\mu_-]_2 & [\mu_+^{-1}]^2[\mu_-]_3 & [\mu_+^{-1}]^2[\mu_-]_4 & [\mu_+^{-1}]^2[\mu_-]_5 \\ [\mu_+^{-1}]^3[\mu_-]_1 & [\mu_+^{-1}]^3[\mu_-]_2 & [\mu_+^{-1}]^3[\mu_-]_3 & [\mu_+^{-1}]^3[\mu_-]_4 & [\mu_+^{-1}]^3[\mu_-]_5 \\ [\mu_+^{-1}]^4[\mu_-]_1 & [\mu_+^{-1}]^4[\mu_-]_2 & [\mu_+^{-1}]^4[\mu_-]_3 & [\mu_+^{-1}]^4[\mu_-]_4 & [\mu_+^{-1}]^4[\mu_-]_5 \\ [\mu_+^{-1}]^5[\mu_-]_1 & [\mu_+^{-1}]^5[\mu_-]_2 & [\mu_+^{-1}]^5[\mu_-]_3 & [\mu_+^{-1}]^5[\mu_-]_4 & [\mu_+^{-1}]^5[\mu_-]_5 \end{pmatrix} E,$$

according to the analytic property of μ_+^{-1} and μ_- , we can proof the theorem. The matrix $R(\lambda)$ can be analyzed in the same way. \square

3 Riemann-Hilbert problem

In this part, an RH problem is formulated by using the properties of μ_{\pm} . We construct matrix function $P_1 = P_1(x, \lambda)$ and $P_2 = P_2(x, \lambda)$. The function $P_1 = P_1(x, \lambda)$ is analytic in C^+ , and the function $P_2 = P_2(x, \lambda)$ is analytic in C^- .

Let

$$P_1 = ([\mu_-]_1, [\mu_-]_2, [\mu_-]_3, [\mu_-]_4, [\mu_+]_5), \quad (3.1)$$

$$P_2 = \begin{pmatrix} [\mu_-^{-1}]^1 \\ [\mu_-^{-1}]^2 \\ [\mu_-^{-1}]^3 \\ [\mu_-^{-1}]^4 \\ [\mu_+^{-1}]^5 \end{pmatrix}, \quad (3.2)$$

with

$$\begin{aligned} P_1 &\rightarrow I, \quad \text{as } \lambda \rightarrow +\infty, \\ P_2 &\rightarrow I, \quad \text{as } \lambda \rightarrow -\infty. \end{aligned} \quad (3.3)$$

At present, we restrict P_1 to the left-hand side of the real λ -axis as P_+ , and the restrict P_2 to the right-hand side of the real λ -axis as P_- . On the real line, they are meet

$$P_-(x, \lambda)P_+(x, \lambda) = G(x, \lambda), \quad \lambda \in R, \quad (3.4)$$

with

$$G(x, \lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & r_{15}e^{-2i\lambda x} \\ 0 & 1 & 0 & 0 & r_{25}e^{-2i\lambda x} \\ 0 & 0 & 1 & 0 & r_{35}e^{-2i\lambda x} \\ 0 & 0 & 0 & 1 & r_{45}e^{-2i\lambda x} \\ s_{51}e^{2i\lambda x} & s_{52}e^{2i\lambda x} & s_{53}e^{2i\lambda x} & s_{54}e^{2i\lambda x} & 1 \end{pmatrix}. \quad (3.5)$$

According to Eq.(3.3), we obtain the canonical normalization conditions as follows

$$\begin{aligned} P_1 &\rightarrow I, \quad \text{as } \lambda \rightarrow +\infty, \\ P_2 &\rightarrow I, \quad \text{as } \lambda \rightarrow -\infty. \end{aligned}$$

To solve the RH problem, we consider the following theorem.

Proposition 3.1.

$$\begin{aligned} \det P_1 &= r_{55}, \quad \lambda \in C^+, \\ \det P_2 &= s_{55}, \quad \lambda \in C^-. \end{aligned}$$

Proof. According to Eq.(3.1) and Eq.(3.2), we write P_1 and P_2 in the form of

$$\begin{aligned} P_1 &= \mu_- H_1 + \mu_- H_2 + \mu_- H_3 + \mu_- H_4 + \mu_+ H_5, \\ P_2 &= H_1 \mu_-^{-1} + H_2 \mu_-^{-1} + H_3 \mu_-^{-1} + H_4 \mu_-^{-1} + H_5 \mu_+^{-1}, \end{aligned} \quad (3.6)$$

where $H_1 = \text{dig}(1, 0, 0, 0, 0)$, $H_2 = \text{dig}(0, 1, 0, 0, 0)$, $H_3 = \text{dig}(0, 0, 1, 0, 0)$, $H_4 = \text{dig}(0, 0, 0, 1, 0)$ and $H_5 = \text{dig}(0, 0, 0, 0, 1)$. Hence

$$\begin{aligned} \det P_1(\lambda) &= \det(\mu_- H_1 + \mu_- H_2 + \mu_- H_3 + \mu_- H_4 + \mu_+ H_5) \\ &= \det \mu_- \cdot \det(H_1 + H_2 + H_3 + H_4 + ER(\lambda)E^{-1}H_5) \\ &= 1 \cdot \det \begin{pmatrix} 1 & 0 & 0 & 0 & r_{15}e^{-2i\lambda x} \\ 0 & 1 & 0 & 0 & r_{25}e^{-2i\lambda x} \\ 0 & 0 & 1 & 0 & r_{35}e^{-2i\lambda x} \\ 0 & 0 & 0 & 1 & r_{45}e^{-2i\lambda x} \\ 0 & 0 & 0 & 0 & r_{55} \end{pmatrix} \\ &= r_{55}, \end{aligned} \quad (3.7)$$

apply the same method to $P_2(\lambda)$,

$$\det P_2(\lambda) = s_{55}. \quad (3.8)$$

From Eq.(2.18), we know that $\det \mu_{\pm} = 1$, according to above analysis, we can get $\det P_1 = r_{55}, \lambda \in C^+$, and $\det P_2 = s_{55}, \lambda \in C^-$. \square

As we can see matrix P has the symmetry relation

$$P^\dagger = -BPB^{-1}, \quad (3.9)$$

symbol “ \dagger ” represents the Hermitian of a matrix, and

$$B = \begin{pmatrix} a_{11} & a_{21}^* & a_{31}^* & a_{41}^* & 0 \\ a_{21} & a_{22} & a_{32}^* & a_{42}^* & 0 \\ a_{31} & a_{32} & a_{33} & a_{43}^* & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3.10)$$

According to Eq.(2.9) and Eq.(3.9), μ_{\pm} meet the following relation

$$B^{-1}\mu_{\pm}^\dagger(\lambda^*)B = \mu_{\pm}^{-1}(\lambda), \quad (3.11)$$

the scattering matrix $S(\lambda)$ satisfies the equation

$$B^{-1}S^\dagger(\lambda^*)B = S^{-1}(\lambda) = R(\lambda). \quad (3.12)$$

Eq.(3.12) evidently shows

$$\begin{aligned} r_{55}(\lambda) &= s_{55}^*(\lambda^*), \quad \lambda \in C^+, \\ -s_{51}^*(\lambda) &= a_{11}r_{15} + a_{21}^*r_{25} + a_{31}^*r_{35} + a_{41}^*r_{45}, \quad \lambda \in R, \\ -s_{52}^*(\lambda) &= a_{21}r_{15} + a_{22}r_{25} + a_{32}^*r_{35} + a_{42}^*r_{45}, \quad \lambda \in R, \\ -s_{53}^*(\lambda) &= a_{31}r_{15} + a_{32}r_{25} + a_{33}r_{35} + a_{43}^*r_{45}, \quad \lambda \in R, \\ -s_{54}^*(\lambda) &= a_{41}r_{15} + a_{42}r_{25} + a_{43}r_{35} + a_{44}r_{45}, \quad \lambda \in R. \end{aligned} \quad (3.13)$$

Theorem 3.2.

$$P_1^\dagger(\lambda^*) = BP_2(\lambda)B^{-1}, \quad \lambda \in C^-. \quad (3.14)$$

Proof. According to Eq.(3.6), we have $P_1 = \mu_-H_1 + \mu_-H_2 + \mu_-H_3 + \mu_-H_4 + \mu_+H_5$,

$$\begin{aligned} P_1^\dagger(\lambda^*) &= (\mu_-(\lambda^*)H_1 + \mu_-(\lambda^*)H_2 + \mu_-(\lambda^*)H_3 + \mu_-(\lambda^*)H_4 + \mu_+(\lambda^*)H_5)^\dagger \\ &= H_1\mu_-^\dagger(\lambda^*) + H_2\mu_-^\dagger(\lambda^*) + H_3\mu_-^\dagger(\lambda^*) + H_4\mu_-^\dagger(\lambda^*) + H_5\mu_+^\dagger(\lambda^*) \\ &= BP_2(\lambda)B^{-1}, \quad \lambda \in C^-. \end{aligned} \quad (3.15)$$

So that

$$P_1^\dagger(\lambda^*) = BP_2(\lambda)B^{-1}, \quad \lambda \in C^-.$$

\square

From Eq.(3.7), Eq.(3.8) and Eq.(3.13), we see $\det P_1(\lambda) = (\det P_2(\lambda^*))^*$, if $\det P_1$ have a zero λ , $\det P_2$ have a zero λ^* . So we suppose that $\det P_1$ has N simple zeros $\{\lambda_j\}_1^N$ in C^+ , and $\det P_2$ has N simple zeros $\{\lambda_j^*\}_1^N$ in C^- . These zeros with the nonzero vectors v_j and \hat{v}_j , set up of the full generic discrete data, which satisfy the equations

$$\begin{aligned} P_1(\lambda_j)v_j &= 0, \\ \hat{v}_j P_2(\lambda_j^*) &= 0, \end{aligned} \quad (3.16)$$

where v_j is column vector, and \hat{v}_j is the row vector. From Eq.(3.14), Eqs.(3.16) one obtains that the eigenvectors admit the following relation.

$$\hat{v}_j = v_j^\dagger B, \quad 1 \leq j \leq N. \quad (3.17)$$

Then we analyze the time-spatial revolution with v_j . We take the derivative of the first equation of Eqs.(3.16) with respect to x , apply the same method to t .

$$\begin{aligned} P_{1,x}v_j + P_1v_{j,x} &= 0, \\ P_{1,t}v_j + P_1v_{j,t} &= 0. \end{aligned} \quad (3.18)$$

On the basis of

$$\begin{aligned} P_{1,x} &= (\mu_-H_1 + \mu_-H_2 + \mu_-H_3 + \mu_+H_4)_x \\ &= \mu_{-,x}H_1 + \mu_{-,x}H_2 + \mu_{-,x}H_3 + \mu_{+,x}H_4, \end{aligned}$$

and the Lax pair Eqs.(2.9), we have

$$\begin{aligned} P_{1,x} &= [-i\lambda(\Lambda\mu_- - \mu_- \Lambda) + P\mu_-]H_1 + [-i\lambda(\Lambda\mu_- - \mu_- \Lambda) + P\mu_-]H_2 \\ &\quad + [-i\lambda(\Lambda\mu_- - \mu_- \Lambda) + P\mu_-]H_3 + [-i\lambda(\Lambda\mu_- - \mu_- \Lambda) + P\mu_-]H_4 \\ &\quad + [-i\lambda(\Lambda\mu_+ - \mu_+ \Lambda) + P\mu_+]H_5 \\ &= -i\lambda\Lambda P_1 + i\lambda P_1 \Lambda + PP_1 \\ &= -i\lambda[\Lambda, P_1] + PP_1. \end{aligned} \quad (3.19)$$

Applying the same method to $P_{1,t}$, we get

$$P_{1,t} = -2i\lambda^2[\Lambda, P_1] + QP_1. \quad (3.20)$$

Inserting Eq.(3.19) and Eq.(3.20) into the first equation of Eqs.(3.18) and the second equation of Eqs.(3.18), respectively. Noticing that $P_1v_j = 0$, we have

$$\begin{aligned} i\lambda\Lambda v_j + v_{j,x} &= 0, \\ 2i\lambda^2\Lambda v_j + v_{j,t} &= 0. \end{aligned} \quad (3.21)$$

According to Eqs.(3.21), we have

$$v_j = e^{-i(\lambda_j x + 2\lambda_j^2 t)\Lambda} v_{j,0},$$

where $v_{j,0}$ are complex constant vectors. From Eq.(3.17), we have

$$\hat{v}_j = v_j^\dagger(\lambda_j)B = v_{j,0}^\dagger e^{i(\lambda_j^* x + 2\lambda_j^{*2} t)\Lambda} B.$$

4 Multi-soliton solutions

Now, we are going to expand $P_1(\lambda)$ at large- λ as

$$P_1(\lambda) = I + \frac{P_1^{(1)}}{\lambda} + \frac{P_1^{(2)}}{\lambda^2} + o\left(\frac{1}{\lambda^3}\right), \quad \lambda \rightarrow \infty. \quad (4.1)$$

Inserting Eq.(4.1) into Eq.(2.9)

$$o(1) : i[\Lambda, P_1^{(1)}] = P. \quad (4.2)$$

From Eq.(4.2), we can generate

$$\begin{aligned} q_1(x, t) &= 2i(P_1^{(1)})_{15}, \\ q_2(x, t) &= 2i(P_1^{(1)})_{25}, \\ q_3(x, t) &= 2i(P_1^{(1)})_{35}, \\ q_4(x, t) &= 2i(P_1^{(1)})_{45}, \end{aligned} \quad (4.3)$$

where $(P_1^{(1)})_{ij}$ is the (i, j) -entry of matrix $P_1^{(1)}$.

To obtain soliton solutions, we set $G = I$ in (3.4). The solutions for this special RH problem (3.4) can be given as

$$\begin{aligned} P_1(\lambda) &= I - \sum_{k=1}^N \sum_{j=1}^N \frac{v_k \hat{v}_j (M^{-1})_{kj}}{\lambda - \hat{\lambda}_j}, \\ P_2(\lambda) &= I + \sum_{k=1}^N \sum_{j=1}^N \frac{v_k \hat{v}_j (M^{-1})_{kj}}{\lambda - \lambda_j}, \end{aligned} \quad (4.4)$$

where M is a $N \times N$ matrix with entries

$$M_{kj} = \frac{\hat{v}_k v_j}{\lambda_j - \hat{\lambda}_k}, \quad (4.5)$$

and $(M^{-1})_{kj}$ means the (k, j) -entry of the inverse matrix of M . From expression Eqs.(4.4), one has

$$P_1^{(1)} = - \sum_{k=1}^N \sum_{j=1}^N v_k \hat{v}_j (M^{-1})_{kj}. \quad (4.6)$$

Then setting nonzero vectors $v_{k,0} = (\alpha_k, \beta_k, \tau_k, \zeta_k, \gamma_k)^T$ and $\theta_k = -i(\lambda_k x + 2\lambda_k^2 t)$, we generate

$$v_k = e^{\theta_k \Lambda} v_{k,0} = \begin{pmatrix} e^{\theta_k} & 0 & 0 & 0 & 0 \\ 0 & e^{\theta_k} & 0 & 0 & 0 \\ 0 & 0 & e^{\theta_k} & 0 & 0 \\ 0 & 0 & 0 & e^{\theta_k} & 0 \\ 0 & 0 & 0 & 0 & e^{-\theta_k} \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \\ \tau_k \\ \zeta_k \\ \gamma_k \end{pmatrix} = \begin{pmatrix} \alpha_k e^{\theta_k} \\ \beta_k e^{\theta_k} \\ \tau_k e^{\theta_k} \\ \zeta_k e^{\theta_k} \\ \gamma_k e^{-\theta_k} \end{pmatrix}, \quad (4.7)$$

$$\hat{v}_j = v_j^\dagger(\lambda_j)B$$

$$\begin{aligned}
&= (\alpha_k e^{\theta_k}, \beta_k e^{\theta_k}, \tau_k e^{\theta_k}, \zeta_k e^{\theta_k}, \gamma_k e^{-\theta_k}) \begin{pmatrix} a_{11} & a_{21}^* & a_{31}^* & a_{41}^* & 0 \\ a_{21} & a_{22} & a_{32}^* & a_{42}^* & 0 \\ a_{31} & a_{32} & a_{33} & a_{43}^* & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \\
&= (a_{11}\alpha_j^* e^{\theta_j^*} + a_{21}\beta_j^* e^{\theta_j^*} + a_{31}\tau_j^* e^{\theta_j^*} + a_{41}\zeta_j^* e^{\theta_j^*}, a_{21}^*\alpha_j^* e^{\theta_j^*} + a_{22}\beta_j^* e^{\theta_j^*} + a_{32}\tau_j^* e^{\theta_j^*} \\
&\quad + a_{42}\zeta_j^* e^{\theta_j^*}, a_{31}^*\alpha_j^* e^{\theta_j^*} + a_{32}^*\beta_j^* e^{\theta_j^*} + a_{33}\tau_j^* e^{\theta_j^*} + a_{43}\zeta_j^* e^{\theta_j^*}, a_{41}^*\alpha_j^* e^{\theta_j^*} + a_{42}^*\beta_j^* e^{\theta_j^*} \\
&\quad + a_{43}^*\tau_j^* e^{\theta_j^*} + a_{44}\zeta_j^* e^{\theta_j^*}, -\gamma_j^* e^{-\theta_j^*}).
\end{aligned} \tag{4.8}$$

Obviously

$$v_k \hat{v}_j = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{pmatrix}, \tag{4.9}$$

$$\begin{aligned}
\hat{v}_k v_j &= (a_{11}\alpha_k^* \alpha_j + a_{21}\beta_k^* \alpha_j + a_{31}\tau_k^* \alpha_j + a_{41}\zeta_k^* \alpha_j + a_{21}^*\alpha_k^* \beta_j + a_{22}\beta_k^* \beta_j + a_{32}\tau_k^* \beta_j \\
&\quad + a_{42}\zeta_k^* \beta_j + a_{31}^*\alpha_k^* \tau_j + a_{32}^*\beta_k^* \tau_j + a_{33}\tau_k^* \tau_j + a_{43}\zeta_k^* \tau_j + a_{41}^*\alpha_k^* \zeta_j + a_{42}^*\beta_k^* \zeta_j \\
&\quad + a_{43}^*\tau_k^* \zeta_j + a_{44}\zeta_k^* \zeta_j) e^{\theta_k^* + \theta_j} - \gamma_k^* \gamma_j e^{-\theta_k^* - \theta_j}.
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
b_{15} &= -\alpha_k \gamma_j^* e^{\theta_k - \theta_j^*}, & b_{25} &= -\beta_k \gamma_j^* e^{\theta_k - \theta_j^*} & b_{35} &= -\tau_k \gamma_j^* e^{\theta_k - \theta_j^*}, & b_{45} &= -\zeta_k \gamma_j^* e^{\theta_k - \theta_j^*}, \\
b_{55} &= -\gamma_k \gamma_j^* e^{\theta_k - \theta_j^*}.
\end{aligned}$$

It should be noted that the parameter b_{ij} ($i \leq 5, j \leq 4$) do not work on the construction solutions, so the specific expression is not given for convenience.

As a consequence, general N -soliton solution for the FCNLS equation (1.2) can be derived as follows

$$\begin{aligned}
q_1 &= 2i \sum_{k=1}^N \sum_{j=1}^N \alpha_k \gamma_j^* e^{\theta_k - \theta_j^*} (M^{-1})_{kj}, \\
q_2 &= 2i \sum_{k=1}^N \sum_{j=1}^N \beta_k \gamma_j^* e^{\theta_k - \theta_j^*} (M^{-1})_{kj}, \\
q_3 &= 2i \sum_{k=1}^N \sum_{j=1}^N \tau_k \gamma_j^* e^{\theta_k - \theta_j^*} (M^{-1})_{kj}, \\
q_4 &= 2i \sum_{k=1}^N \sum_{j=1}^N \zeta_k \gamma_j^* e^{\theta_k - \theta_j^*} (M^{-1})_{kj},
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
M_{kj} &= \frac{1}{\lambda_j - \hat{\lambda}_k} [(a_{11}\alpha_k^* \alpha_j + a_{21}\beta_k^* \alpha_j + a_{31}\tau_k^* \alpha_j + a_{41}\zeta_k^* \alpha_j + a_{21}^*\alpha_k^* \beta_j + a_{22}\beta_k^* \beta_j \\
&\quad + a_{32}\tau_k^* \beta_j + a_{42}\zeta_k^* \beta_j + a_{31}^*\alpha_k^* \tau_j + a_{32}^*\beta_k^* \tau_j + a_{33}\tau_k^* \tau_j + a_{43}\zeta_k^* \tau_j + a_{41}^*\alpha_k^* \zeta_j \\
&\quad + a_{42}^*\beta_k^* \zeta_j + a_{43}^*\tau_k^* \zeta_j + a_{44}\zeta_k^* \zeta_j) e^{\theta_k^* + \theta_j} - \gamma_k^* \gamma_j e^{-\theta_k^* - \theta_j}], 1 \leq k, j \leq N.
\end{aligned} \tag{4.12}$$

To make the expression (4.11) simpler, we define the following matrix F , G , H and K .

$$F = \begin{pmatrix} 0 & \alpha_1 e^{\theta_1} & \alpha_2 e^{\theta_2} & \dots & \alpha_N e^{\theta_N} \\ \gamma_1^* e^{-\theta_1^*} & M_{11} & M_{12} & \dots & M_{1N} \\ \gamma_2^* e^{-\theta_2^*} & M_{21} & M_{22} & \dots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_N^* e^{-\theta_N^*} & M_{N1} & M_{N2} & \dots & M_{NN} \end{pmatrix}, \quad (4.13)$$

$$G = \begin{pmatrix} 0 & \beta_1 e^{\theta_1} & \beta_2 e^{\theta_2} & \dots & \beta_N e^{\theta_N} \\ \gamma_1^* e^{-\theta_1^*} & M_{11} & M_{12} & \dots & M_{1N} \\ \gamma_2^* e^{-\theta_2^*} & M_{21} & M_{22} & \dots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_N^* e^{-\theta_N^*} & M_{N1} & M_{N2} & \dots & M_{NN} \end{pmatrix}, \quad (4.14)$$

$$H = \begin{pmatrix} 0 & \tau_1 e^{\theta_1} & \tau_2 e^{\theta_2} & \dots & \tau_N e^{\theta_N} \\ \gamma_1^* e^{-\theta_1^*} & M_{11} & M_{12} & \dots & M_{1N} \\ \gamma_2^* e^{-\theta_2^*} & M_{21} & M_{22} & \dots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_N^* e^{-\theta_N^*} & M_{N1} & M_{N2} & \dots & M_{NN} \end{pmatrix}. \quad (4.15)$$

$$K = \begin{pmatrix} 0 & \zeta_1 e^{\theta_1} & \zeta_2 e^{\theta_2} & \dots & \zeta_N e^{\theta_N} \\ \gamma_1^* e^{-\theta_1^*} & M_{11} & M_{12} & \dots & M_{1N} \\ \gamma_2^* e^{-\theta_2^*} & M_{21} & M_{22} & \dots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_N^* e^{-\theta_N^*} & M_{N1} & M_{N2} & \dots & M_{NN} \end{pmatrix}. \quad (4.16)$$

On the basis of matrix (4.13), (4.14), (4.15) and (4.16), we can get

$$q_1 = -2i \frac{\det F}{\det M}, \quad q_2 = -2i \frac{\det G}{\det M}, \quad q_3 = -2i \frac{\det H}{\det M}, \quad q_4 = -2i \frac{\det K}{\det M}. \quad (4.17)$$

In the remainder of this section, we figure out the soliton solutions in the case of $N = 1$ and $N = 2$. In the case of $N = 1$, we obtain the one-soliton solution

$$\begin{aligned} q_1 &= 2i \frac{\alpha_1 \gamma_1^* e^{\theta_1 - \theta_1^*}}{M_{11}}, \\ q_2 &= 2i \frac{\beta_1 \gamma_1^* e^{\theta_1 - \theta_1^*}}{M_{11}}, \\ q_3 &= 2i \frac{\tau_1 \gamma_1^* e^{\theta_1 - \theta_1^*}}{M_{11}}, \\ q_4 &= 2i \frac{\zeta_1 \gamma_1^* e^{\theta_1 - \theta_1^*}}{M_{11}}, \end{aligned} \quad (4.18)$$

$$\begin{aligned} M_{11} &= \frac{1}{\lambda_1 - \hat{\lambda}_1} [(a_{11}|\alpha_1|^2 + a_{22}|\beta_1|^2 + a_{33}|\tau_1|^2 + a_{44}|\zeta_1|^2 + a_{21}\beta_1^* \alpha_1 + a_{31}\tau_1^* \alpha_1 \\ &\quad + a_{41}\zeta_1^* \alpha_1 + a_{21}^* \alpha_1^* \beta_1 + a_{32}\tau_1^* \beta_1 + a_{42}\zeta_1^* \beta_1 + a_{31}^* \alpha_1^* \tau_1 + a_{32}^* \beta_1^* \tau_1 + a_{43}\zeta_1^* \tau_1 \\ &\quad + a_{41}^* \alpha_1^* \zeta_1 + a_{42}^* \beta_1^* \zeta_1 + a_{43}^* \tau_1^* \zeta_1) e^{\theta_1 + \theta_1^*} - |\gamma_1|^2 e^{-(\theta_1 + \theta_1^*)}], \end{aligned}$$

where $\theta_1 = -i(\lambda_1 x + 2\lambda_1^2 t)$. Furthermore, fixing $\gamma_1 = 1$, $\lambda_1 = n_1 + im_1$ and setting

$$\begin{aligned} & -(a_{11}|\alpha_1|^2 + a_{22}|\beta_1|^2 + a_{33}|\tau_1|^2 + a_{44}|\zeta_1|^2 + a_{21}\beta_1^*\alpha_1 + a_{31}\tau_1^*\alpha_1 + a_{41}\zeta_1^*\alpha_1 \\ & + a_{21}^*\alpha_1^*\beta_1 + a_{32}\tau_1^*\beta_1 + a_{42}\zeta_1^*\beta_1 + a_{31}^*\alpha_1^*\tau_1 + a_{32}^*\beta_1^*\tau_1 + a_{43}\zeta_1^*\tau_1 + a_{41}^*\alpha_1^*\zeta_1 \\ & + a_{42}^*\beta_1^*\zeta_1 + a_{43}^*\tau_1^*\zeta_1) = e^{2\xi_1}, \end{aligned} \quad (4.19)$$

the expression (4.18) can be written as

$$\begin{aligned} q_1 &= 2m_1\alpha_1\gamma_1^*e^{-\xi_1}e^{\theta_1-\theta_1^*}\text{sech}(\theta_1^* + \theta_1 + \xi_1), \\ q_2 &= 2m_1\beta_1\gamma_1^*e^{-\xi_1}e^{\theta_1-\theta_1^*}\text{sech}(\theta_1^* + \theta_1 + \xi_1), \\ q_3 &= 2m_1\tau_1\gamma_1^*e^{-\xi_1}e^{\theta_1-\theta_1^*}\text{sech}(\theta_1^* + \theta_1 + \xi_1), \\ q_4 &= 2m_1\zeta_1\gamma_1^*e^{-\xi_1}e^{\theta_1-\theta_1^*}\text{sech}(\theta_1^* + \theta_1 + \xi_1). \end{aligned} \quad (4.20)$$

According to the notation above, we have

$$\begin{aligned} \theta_1 - \theta_1^* &= -2in_1x - 4in_1^2t, \\ \theta_1 + \theta_1^* &= 2m_1x + 4im_1^2t + 8n_1m_1t. \end{aligned} \quad (4.21)$$

Thus the one-soliton solutions in (4.18) can be further written as

$$\begin{aligned} q_1 &= 2m_1\alpha_1\gamma_1^*e^{-\xi_1}e^{-2in_1x-4in_1^2t}\text{sech}(2m_1x + 4im_1^2t + 8n_1m_1t + \xi_1), \\ q_2 &= 2m_1\beta_1\gamma_1^*e^{-\xi_1}e^{-2in_1x-4in_1^2t}\text{sech}(2m_1x + 4im_1^2t + 8n_1m_1t + \xi_1), \\ q_3 &= 2m_1\tau_1\gamma_1^*e^{-\xi_1}e^{-2in_1x-4in_1^2t}\text{sech}(2m_1x + 4im_1^2t + 8n_1m_1t + \xi_1), \\ q_4 &= 2m_1\zeta_1\gamma_1^*e^{-\xi_1}e^{-2in_1x-4in_1^2t}\text{sech}(2m_1x + 4im_1^2t + 8n_1m_1t + \xi_1). \end{aligned} \quad (4.22)$$

From the Eq.(4.22), we can know that the one-soliton solutions of q_1 , q_2 , q_3 and q_4 can be described by hyperbolic cosecant function. Taking an example, q_1 has the peak amplitude

$$\Upsilon_1 = 2m_1\alpha_1e^{-\xi_1},$$

and the velocity

$$\varpi_1 = 2im_1t + 4n_1t,$$

Similarly, we can know the peak amplitude and velocity of q_2 , q_3 and q_4 , respectively.

$$\Upsilon_2 = 2m_1\beta_1e^{-\xi_1}, \quad \Upsilon_3 = 2m_1\tau_1e^{-\xi_1}, \quad \Upsilon_4 = 2m_1\zeta_1e^{-\xi_1},$$

$$\varpi_2 = 2im_1t + 4n_1t, \quad \varpi_3 = 2im_1t + 4n_1t, \quad \varpi_4 = 2im_1t + 4n_1t.$$

From the expressions of Υ_1 , Υ_2 , Υ_3 and Υ_4 , ϖ_1 , ϖ_2 , ϖ_3 , ϖ_4 we can know that all of them rely on both the real part n_1 and the imaginary part m_1 of the eigenvalue λ_1 . Figure 1, Figure 2, Figure 3 and Figure 4 represent the localized structures and dynamic behaviors of the single-soliton solution. All the analysis of q_1 be the same with q_2 , q_3 and q_4 .

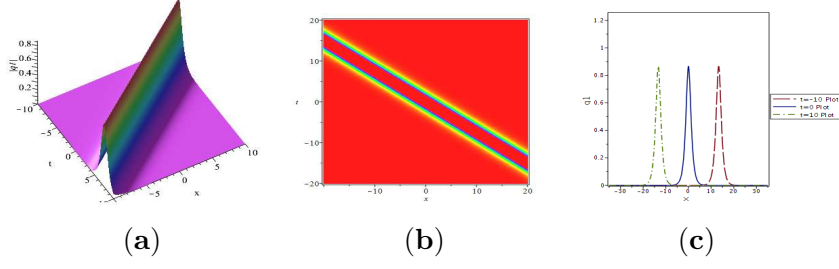


Figure 1. Plots of the single-soliton solution q_1 , with the parameters chosen as $a_{11} = a_{22} = a_{33} = a_{44} = 0$, $a_{21} = a_{31} = a_{41} = a_{32} = a_{42} = a_{43} = -\frac{1}{9}$, $\alpha_1 = \tau_1 = \beta_1 = \zeta_1 = \frac{1}{2} - \frac{\sqrt{2}}{2}i$, $\gamma_1 = 1$, $n_1 = \frac{1}{3}$, $m_1 = \frac{1}{2}$. (a) three dimensional plot at time $t = 0$, (b) density plot, (c) the wave propagation along the x -axis with different time.

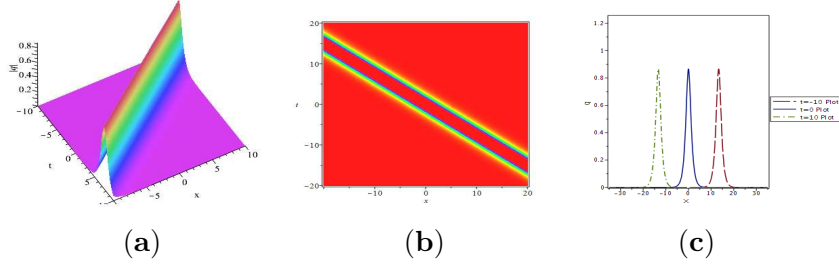


Figure 2. Plots of the single-soliton solution q_2 , with the parameters chosen as $a_{11} = a_{22} = a_{33} = a_{44} = 0$, $a_{21} = a_{31} = a_{41} = a_{32} = a_{42} = a_{43} = -\frac{1}{9}$, $\alpha_1 = \tau_1 = \beta_1 = \zeta_1 = \frac{1}{2} - \frac{\sqrt{2}}{2}i$, $\gamma_1 = 1$, $n_1 = \frac{1}{3}$, $m_1 = \frac{1}{2}$. (a) three dimensional plot at time $t = 0$, (b) density plot, (c) the wave propagation along the x -axis with different time.

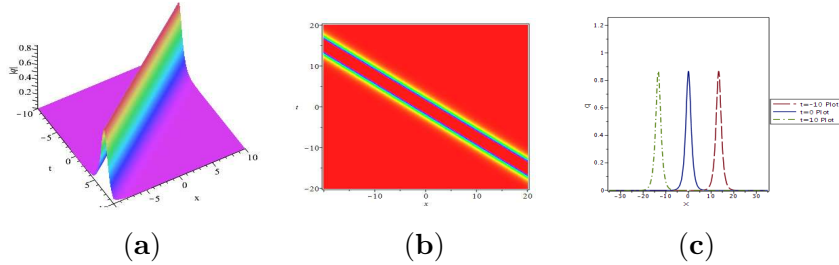


Figure 3. Plots of the single-soliton solution q_3 , with the parameters chosen as $a_{11} = a_{22} = a_{33} = a_{44} = 0$, $a_{21} = a_{31} = a_{41} = a_{32} = a_{42} = a_{43} = -\frac{1}{9}$, $\alpha_1 = \tau_1 = \beta_1 = \zeta_1 = \frac{1}{2} - \frac{\sqrt{2}}{2}i$, $\gamma_1 = 1$, $n_1 = \frac{1}{3}$, $m_1 = \frac{1}{2}$. (a) three dimensional plot at time $t = 0$, (b) density plot, (c) the wave propagation along the x -axis with different time.

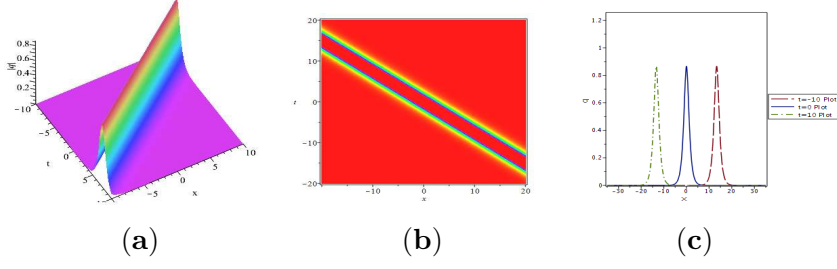


Figure 4. Plots of the single-soliton solution q_3 , with the parameters chosen as $a_{11} = a_{22} = a_{33} = a_{44} = 0$, $a_{21} = a_{31} = a_{41} = a_{32} = a_{42} = a_{43} = -\frac{1}{9}$, $\alpha_1 = \tau_1 = \beta_1 = \zeta_1 = \frac{1}{2} - \frac{\sqrt{2}}{2}i$, $\gamma_1 = 1$, $n_1 = \frac{1}{3}$, $m_1 = \frac{1}{2}$. (a) three dimensional plot at time $t = 0$, (b) density plot, (c) the wave propagation along the x -axis with different time.

In the case of $N = 2$, the two-soliton solutions can be obtained as

$$\begin{aligned}
q_1 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}} (-\alpha_1\gamma_1^*e^{\theta_1-\theta_1^*}M_{22} + \alpha_1\gamma_2^*e^{\theta_1-\theta_2^*}M_{12} \\
&\quad + \alpha_2\gamma_1^*e^{\theta_2-\theta_1^*}M_{21} - \alpha_2\gamma_2^*e^{\theta_2-\theta_2^*}M_{11}), \\
q_2 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}} (-\beta_1\gamma_1^*e^{\theta_1-\theta_1^*}M_{22} + \beta_1\gamma_2^*e^{\theta_1-\theta_2^*}M_{12} \\
&\quad + \beta_2\gamma_1^*e^{\theta_2-\theta_1^*}M_{21} - \beta_2\gamma_2^*e^{\theta_2-\theta_2^*}M_{11}), \\
q_3 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}} (-\tau_1\gamma_1^*e^{\theta_1-\theta_1^*}M_{22} + \tau_1\gamma_2^*e^{\theta_1-\theta_2^*}M_{12} \\
&\quad + \tau_2\gamma_1^*e^{\theta_2-\theta_1^*}M_{21} - \tau_2\gamma_2^*e^{\theta_2-\theta_2^*}M_{11}), \\
q_4 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}} (-\zeta_1\gamma_1^*e^{\theta_1-\theta_1^*}M_{22} + \zeta_1\gamma_2^*e^{\theta_1-\theta_2^*}M_{12} \\
&\quad + \zeta_2\gamma_1^*e^{\theta_2-\theta_1^*}M_{21} - \zeta_2\gamma_2^*e^{\theta_2-\theta_2^*}M_{11}),
\end{aligned} \tag{4.23}$$

where

$$\left\{ \begin{array}{l} M_{11} = \frac{1}{\lambda_1 - \lambda_1^*} [(a_{11}|\alpha_1|^2 + a_{22}|\beta_1|^2 + a_{33}|\tau_1|^2 + a_{44}|\zeta_1|^2 + a_{21}\beta_1^*\alpha_1 + a_{31}\tau_1^*\alpha_1 \\ \quad + a_{41}\zeta_1^*\alpha_1 + a_{21}^*\alpha_1^*\beta_1 + a_{32}\tau_1^*\beta_1 + a_{42}\zeta_1^*\beta_1 + a_{31}^*\alpha_1^*\tau_1 + a_{32}^*\beta_1^*\tau_1 + a_{43}\zeta_1^*\tau_1 \\ \quad + a_{41}^*\alpha_1^*\zeta_1 + a_{42}^*\beta_1^*\zeta_1 + a_{43}^*\tau_1^*\zeta_1)e^{\theta_1^*+\theta_1} - |\gamma_1|^2 e^{-(\theta_1^*+\theta_1)}], \\ M_{12} = \frac{1}{\lambda_2 - \lambda_1^*} [(a_{11}\alpha_1^*\alpha_2 + a_{21}\beta_1^*\alpha_2 + a_{31}\tau_1^*\alpha_2 + a_{41}\zeta_1^*\alpha_2 + a_{21}^*\alpha_1^*\beta_2 + a_{22}\beta_1^*\beta_2 \\ \quad + a_{32}\tau_1^*\beta_2 + a_{42}\zeta_1^*\beta_2 + a_{31}^*\alpha_1^*\tau_2 + a_{32}^*\beta_1^*\tau_2 + a_{33}\tau_1^*\tau_2 + a_{43}\zeta_1^*\tau_2 + a_{41}^*\alpha_1^*\zeta_2 \\ \quad + a_{42}^*\beta_1^*\zeta_2 + a_{43}^*\tau_1^*\zeta_2 + a_{44}\zeta_1^*\zeta_2)e^{\theta_1^*+\theta_2} - \gamma_1^*\gamma_2 e^{-\theta_1^*-\theta_2}], \\ M_{21} = \frac{1}{\lambda_1 - \lambda_2^*} [(a_{11}\alpha_2^*\alpha_1 + a_{21}\beta_2^*\alpha_1 + a_{31}\tau_2^*\alpha_1 + a_{41}\zeta_2^*\alpha_1 + a_{21}^*\alpha_2^*\beta_1 + a_{22}\beta_2^*\beta_1 \\ \quad + a_{32}\tau_2^*\beta_1 + a_{42}\zeta_2^*\beta_1 + a_{31}^*\alpha_2^*\tau_1 + a_{32}^*\beta_2^*\tau_1 + a_{33}\tau_2^*\tau_1 + a_{43}\zeta_2^*\tau_1 + a_{41}^*\alpha_2^*\zeta_1 \\ \quad + a_{42}^*\beta_2^*\zeta_1 + a_{43}^*\tau_2^*\zeta_1 + a_{44}\zeta_2^*\zeta_1)e^{\theta_2^*+\theta_1} - \gamma_2^*\gamma_1 e^{-\theta_2^*-\theta_1}], \\ M_{22} = \frac{1}{\lambda_2 - \lambda_2^*} [(a_{11}|\alpha_2|^2 + a_{22}|\beta_2|^2 + a_{33}|\tau_2|^2 + a_{44}|\zeta_2|^2 + a_{21}\beta_2^*\alpha_2 + a_{31}\tau_2^*\alpha_2 \\ \quad + a_{41}\zeta_2^*\alpha_2 + a_{21}^*\alpha_2^*\beta_2 + a_{32}\tau_2^*\beta_2 + a_{42}\zeta_2^*\beta_2 + a_{31}^*\alpha_2^*\tau_2 + a_{32}^*\beta_2^*\tau_2 + a_{43}\zeta_2^*\tau_2 \\ \quad + a_{41}^*\alpha_2^*\zeta_2 + a_{42}^*\beta_2^*\zeta_2 + a_{43}^*\tau_2^*\zeta_2)e^{\theta_2^*+\theta_2} - |\gamma_2|^2 e^{-(\theta_2^*+\theta_2)}], \end{array} \right.$$

$\theta_1 = -i(\lambda_1 x + 2\lambda_1^2 t)$, $\theta_2 = -i(\lambda_2 x + 2\lambda_2^2 t)$, $\lambda_1 = n_1 + im_1$ and $\lambda_2 = n_2 + im_2$. If we let $\gamma_1 = \gamma_2 = 1$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\tau_1 = \tau_2$, $\zeta_1 = \zeta_2$ and $-(a_{11}|\alpha_1|^2 + a_{22}|\beta_1|^2 + a_{33}|\tau_1|^2 + a_{44}|\zeta_1|^2 + a_{21}\beta_1^*\alpha_1 + a_{31}\tau_1^*\alpha_1 + a_{41}\zeta_1^*\alpha_1 + a_{21}^*\alpha_1^*\beta_1 + a_{32}\tau_1^*\beta_1 + a_{42}\zeta_1^*\beta_1 + a_{31}^*\alpha_1^*\tau_1 + a_{32}^*\beta_1^*\tau_1 + a_{43}\zeta_1^*\tau_1 + a_{41}^*\alpha_1^*\zeta_1 + a_{42}^*\beta_1^*\zeta_1 + a_{43}^*\tau_1^*\zeta_1) = e^{2\xi_1}$, then the two-soliton solutions in (4.23) have the following form

$$\begin{aligned} q_1 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}} (-\alpha_1 e^{\theta_1 - \theta_1^*} M_{22} + \alpha_1 e^{\theta_1 - \theta_2^*} M_{12} \\ &\quad + \alpha_2 e^{\theta_2 - \theta_1^*} M_{21} - \alpha_2 e^{\theta_2 - \theta_2^*} M_{11}), \\ q_2 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}} (-\beta_1 e^{\theta_1 - \theta_1^*} M_{22} + \beta_1 e^{\theta_1 - \theta_2^*} M_{12} \\ &\quad + \beta_2 e^{\theta_2 - \theta_1^*} M_{21} - \beta_2 e^{\theta_2 - \theta_2^*} M_{11}), \\ q_3 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}} (-\tau_1 e^{\theta_1 - \theta_1^*} M_{22} + \tau_1 e^{\theta_1 - \theta_2^*} M_{12} \\ &\quad + \tau_2 e^{\theta_2 - \theta_1^*} M_{21} - \tau_2 e^{\theta_2 - \theta_2^*} M_{11}), \\ q_4 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}} (-\zeta_1 e^{\theta_1 - \theta_1^*} M_{22} + \zeta_1 e^{\theta_1 - \theta_2^*} M_{12} \\ &\quad + \zeta_2 e^{\theta_2 - \theta_1^*} M_{21} - \zeta_2 e^{\theta_2 - \theta_2^*} M_{11}), \end{aligned} \tag{4.24}$$

where

$$\begin{cases} M_{11} = \frac{-e^{\xi_1}}{im_1} \cosh(\theta_1^* + \theta_1 + \xi_1), \\ M_{12} = \frac{-2e^{\xi_1}}{n_2 - n_1 + i(m_1 + m_2)} \cosh(\theta_1^* + \theta_2 + \xi_1), \\ M_{21} = \frac{-2e^{\xi_1}}{n_1 - n_2 + i(m_1 + m_2)} \cosh(\theta_2^* + \theta_1 + \xi_1), \\ M_{22} = \frac{-e^{\xi_1}}{im_2} \cosh(\theta_2^* + \theta_2 + \xi_1). \end{cases} \quad (4.25)$$

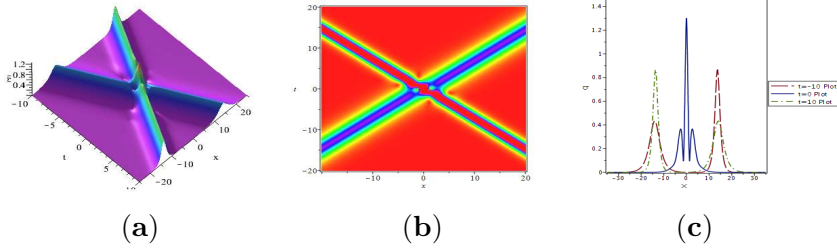


Figure 5. Plots of the double-soliton solution q_1 , with the parameters chosen as $a_{11} = a_{22} = a_{33} = a_{44} = 0$, $a_{21} = a_{31} = a_{41} = a_{32} = a_{42} = a_{43} = -\frac{1}{9}$, $\alpha_1 = \tau_1 = \beta_1 = \zeta_1 = \frac{1}{2} - \frac{\sqrt{2}}{2}i$, $\gamma_1 = \gamma_2 = 1$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\tau_1 = \tau_2$, $\zeta_1 = \zeta_2$, $n_1 = -\frac{1}{3}$, $n_2 = \frac{1}{3}$, $m_1 = 0.25$, $m_2 = 0.5$. (a) three dimensional plot at time $t = 0$, (b) density plot, (c) the wave propagation along the x -axis with different time.

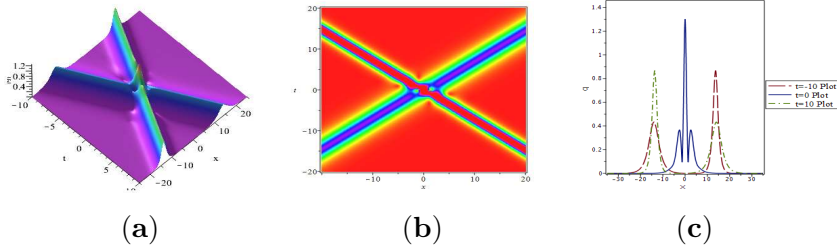


Figure 6. Plots of the double-soliton solution q_1 , with the parameters chosen as $a_{11} = a_{22} = a_{33} = a_{44} = 0$, $a_{21} = a_{31} = a_{41} = a_{32} = a_{42} = a_{43} = -\frac{1}{9}$, $\alpha_1 = \tau_1 = \beta_1 = \zeta_1 = \frac{1}{2} - \frac{\sqrt{2}}{2}i$, $\gamma_1 = \gamma_2 = 1$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\tau_1 = \tau_2$, $\zeta_1 = \zeta_2$, $n_1 = -\frac{1}{3}$, $n_2 = \frac{1}{3}$, $m_1 = 0.25$, $m_2 = 0.5$. (a) three dimensional plot at time $t = 0$, (b) density plot, (c) the wave propagation along the x -axis with different time.

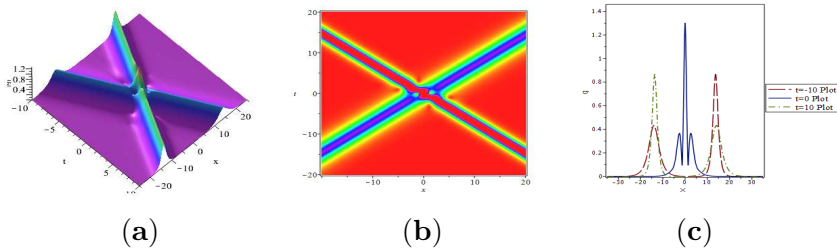


Figure 7. Plots of the double-soliton solution q_1 , with the parameters chosen as $a_{11} = a_{22} = a_{33} = a_{44} = 0$, $a_{21} = a_{31} = a_{41} = a_{32} = a_{42} = a_{43} = -\frac{1}{9}$, $\alpha_1 = \tau_1 = \beta_1 = \zeta_1 = \frac{1}{2} - \frac{\sqrt{2}}{2}i$, $\gamma_1 = \gamma_2 = 1$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\tau_1 = \tau_2$, $\zeta_1 = \zeta_2$, $n_1 = \frac{-1}{3}$, $n_2 = \frac{1}{3}$, $m_1 = 0.25$, $m_2 = 0.5$. (a) three dimensional plot at time $t = 0$, (b) density plot, (c) the wave propagation along the x -axis with different time.

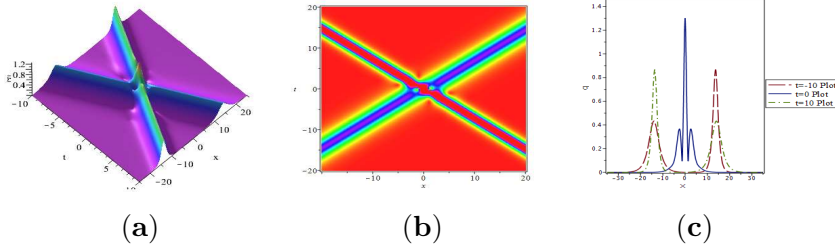


Figure 8. Plots of the double-soliton solution q_1 , with the parameters chosen as $a_{11} = a_{22} = a_{33} = a_{44} = 0$, $a_{21} = a_{31} = a_{41} = a_{32} = a_{42} = a_{43} = -\frac{1}{9}$, $\alpha_1 = \tau_1 = \beta_1 = \zeta_1 = \frac{1}{2} - \frac{\sqrt{2}}{2}i$, $\gamma_1 = \gamma_2 = 1$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\tau_1 = \tau_2$, $\zeta_1 = \zeta_2$, $n_1 = \frac{-1}{3}$, $n_2 = \frac{1}{3}$, $m_1 = 0.25$, $m_2 = 0.5$. (a) three dimensional plot at time $t = 0$, (b) density plot, (c) the wave propagation along the x -axis with different time.

5 Conclusions and discussions

In this work, we have proposed a FCNLS equation (1.2) associated with a 5×5 Lax pair, which was investigated via the RH approach. Based on the Lax pair with a 5×5 matrix, we start with the analyze of the spectral problem and the analytical properties of the Jost functions, from which the RH problem of the equation is established. Then, we obtain the N -soliton solutions of the FCNLS equation (1.2), by solving the RH problem without reflection. Finally, we derive two special cases of the solutions to the equation for $N = 1$ and $N = 2$, and the local structure and dynamic behavior of the one-and two-soliton solutions are analyzed graphically.

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Conflict of interest

This work does not have any conflicts of interest.

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