

# MULTIPLE SOLUTIONS FOR POLYHARMONIC EQUATIONS WITH POTENTIAL VANISHING AT INFINITY

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ABSTRACT. We are concerned with the following polyharmonic equation:

$$\Delta_p^L u + V(x)|u|^{p-2}u = K(x)f(x, u) \quad \text{and} \quad u > 0 \quad \text{in } \mathbb{R}^N,$$

where  $1 < p < \infty$ ,  $N > Lp$ ,  $L = 1, 2, \dots$  and the potential functions  $V, K : \mathbb{R}^N \rightarrow (0, \infty)$  are continuous. We study the existence and multiplicity of nontrivial positive weak solutions for the problem above via mountain pass theorem and fountain theorem.

## 1. INTRODUCTION

In this paper, we investigate the existence and multiplicity of solutions of the problems involving the  $p$ -polyharmonic operators. To study physical phenomena, Kratochvíl and Nečās introduced the  $p$ -polyharmonic operator in [5, 15]; see also [1, 12, 13, 19] and the references therein.

We are concerned with the following polyharmonic equation:

$$(P) \quad \begin{cases} \Delta_p^L u + V(x)|u|^{p-2}u = K(x)f(x, u) \\ u > 0 \end{cases} \quad \text{in } \mathbb{R}^N.$$

where  $p > 1$ ,  $L = 1, 2, \dots$ , the potential  $V, K : \mathbb{R}^N \rightarrow (0, \infty)$  are continuous functions that vanish at infinity,  $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ , and the  $p$ -polyharmonic operator following [5, 9] is defined by

$$\Delta_p^L u = \begin{cases} -\operatorname{div}(\Delta^{j-1}(|D\Delta^{j-1}u|^{p-2}D\Delta^{j-1}u)) & \text{if } L = 2j - 1, \\ \Delta^j(|\Delta^j u|^{p-2}\Delta^j u) & \text{if } L = 2j, \end{cases} \quad j = 1, 2, \dots,$$

which becomes the usual  $p$ -Laplacian for  $L = 1$ , the  $p$ -biharmonic operator for  $L = 2$  and the polyharmonic operator for  $p = 2$ . For  $L = 1, 2, \dots$ , we introduce the main  $L$ -order differential operator

$$\mathcal{D}_L u = \begin{cases} \mathcal{D}\Delta^{j-1}u & \text{if } L = 2j - 1, \\ \Delta^j u & \text{if } L = 2j, \end{cases} \quad j = 1, 2, \dots.$$

Note that  $\mathcal{D}_L$  is an  $N$ -vectorial operator when  $L$  is odd, while it is a scalar operator when  $L$  is even.

Let us introduce the Sobolev critical exponent  $p_L^*$  and the number  $p_L$  defined respectively by

$$p_L^* = \begin{cases} \frac{Np}{N-Lp} & \text{if } N > Lp, \\ \infty & \text{if } N \leq Lp, \end{cases} \quad p_L = \frac{p_L^*}{p} = \begin{cases} \frac{N}{N-Lp} & \text{if } N > Lp, \\ \infty & \text{if } N \leq Lp. \end{cases}$$

We are going to explore problem (P) with zero mass potential, that is when  $V(x) \rightarrow 0$ , as  $|x| \rightarrow \infty$ . This class was studied by several researchers e.g., in [3, 6, 10, 11, 23] and reference therein, where the main feature is to impose restrictions on  $V, K$ . The involved potentials are allowed for vanishing behavior at infinity to get some compact embedding into a weighted  $L^p$  space. Recently Alves and Souto in [3], in addition to improving all the former restrictions on the potentials, handled subcritical nonlinearities  $f$  which do not satisfy the so-called Ambrosetti–Rabinowitz condition [4], which is commonly called (AR)-condition

$$0 < \zeta F(t) \leq \theta f(t)t \quad \text{for } t > 0 \text{ and some } \theta \in (0, \frac{1}{2}),$$

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where  $F(t) = \int_0^t f(s) ds$ . For the results concerned with fourth-order biharmonic equations involving vanish potential, readers are referred to [6, 10, 23] and references therein.

The purpose of this paper is to study the existence of weak solutions for the problem (P) without (AR)-condition as observing the potentials and nonlinear term. This result extends that in [6, 10, 23] using the arguments in [11] for a generalized polyharmonic elliptic equations. To do this, we think the continuous or compact embedding by restrictions on  $V, K$  to obtain some embedding into a weighted  $L^p$  space.

Second, we show the infinitely many weak solutions to the our problem (P) via fountain theorem. Roughly speaking, we prove the existence of solutions to use the concentration–compactness principle based on the ideas established by Brézis and Nirenberg [8] (see also e.g. [17, 18]) because of dealing the critical growth condition. To the best of our knowledge, there were no such existence results for our problem in this situation.

## 2. PRELIMINARIES

In this section, we see the notation and some results to prove our theorems. First of all, we collect a series of results and notations which will be used throughout the paper. By  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$  we denote a multi-index, with length  $|\alpha| = \sum_{i=1}^N \alpha_i \leq L$  and corresponding partial differentiation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}.$$

The space  $W^{L,p}(\mathbb{R}^N)$  denotes the standard Sobolev space with the standard norm,

$$\|u\|_{W^{L,p}(\mathbb{R}^N)} = \left( \sum_{|\alpha| \leq L} \|D^\alpha u\|_p^p \right)^{1/p}, \quad 1 < p < \infty,$$

where  $\|\cdot\|_p$  denotes the standard  $L^p$ -norm. Note that  $W^{L,p}(\mathbb{R}^N) = (W^{L,p}(\mathbb{R}^N), \|\cdot\|)$  is a separable, uniformly convex, reflexive, real Banach space for all  $L = 1, 2, \dots$  and  $1 < p < \infty$ .

Define the linear subspace

$$X := \left\{ u \in W^{L,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\mathcal{D}_L u|^p dx + \int_{\mathbb{R}^N} V(x) |u|^p dx < \infty \right\}.$$

Then  $X$  is a reflexive separable Banach space with the norm

$$\|u\|_X = \left( \int_{\mathbb{R}^N} |\mathcal{D}_L u|^p dx + \int_{\mathbb{R}^N} V(x) |u|^p dx \right)^{\frac{1}{p}},$$

which is equivalent to the norm  $\|\cdot\|_{W^{L,p}(\mathbb{R}^N)}$  given by

$$\|u\|_{W^{L,p}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |\mathcal{D}_L u|^p dx + \int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}.$$

As in [3], we say  $(V, K) \in \mathcal{K}$  if the following conditions hold.

- (I) (sign of  $V$  and  $K$ )  $V, K$  are continuous,  $V(x), K(x) > 0$  for all  $x \in \mathbb{R}^N$  with  $K \in L^\infty(\mathbb{R}^N)$ .
- (II) (decay of  $K$ ) If  $\{A_n\} \subset \mathbb{R}^N$  is a sequence of Borel sets such that  $|A_n| \leq R$  for all  $n$  and some  $R > 0$ , we have that

$$(K1) \quad \lim_{r \rightarrow \infty} \int_{A_n \cap B_r^c(0)} K(x) dx = 0 \quad \text{uniformly in } n \in \mathbb{N}.$$

- (III) (interrelation between  $V$  and  $K$ ) One of the following conditions occurs:

$$(K2) \quad \frac{K(x)}{V(x)} \in L^\infty(\mathbb{R}^N)$$

or there is a  $p_0 \in (p, p_L^*)$  such that

$$(K3) \quad \frac{K(x)}{|V(x)|^{(p_L^* - p_0)/(p_L^* - p)}} \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Related to the function  $f$ , we assume the following conditions.

(F1) (behavior at zero)  $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

$$\limsup_{|s| \rightarrow 0} \frac{f(x, s)}{|s|^{p-2} s} = 0 \quad \text{if (K2) holds}$$

or

$$\limsup_{|s| \rightarrow 0} \frac{f(x, s)}{|s|^{p_0-2} s} < \infty \quad \text{if (K3) holds.}$$

(F2) (quasi-critical growth)

$$\limsup_{|s| \rightarrow \infty} \frac{f(x, s)}{|s|^{p_L^*-2} s} = 0 \quad \text{if (K2) holds}$$

or

$$\limsup_{|s| \rightarrow \infty} \frac{F(x, s)}{|s|^{p_0}} < \infty \quad \text{if (K3) holds.}$$

(F3) (super-quadraticity)  $\frac{f(x, s)}{s}$  is non-decreasing in  $\mathbb{R}^N \times \mathbb{R}$  and

$$\limsup_{|s| \rightarrow \infty} \frac{F(x, s)}{|s|^p} = \infty \quad \text{uniformly for all } x \in \mathbb{R}^N.$$

(F4)  $f(x, -t) = -f(x, t)$  holds for all  $t \in \mathbb{R}$ .

We recall the well-known embedding results in [16, Lemma 2.1]; see also [7].

**Lemma 2.1.** *The following statements hold:*

- (i) *There is a continuous embedding  $W^{L,p}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$  for any  $s \in [p, p_L^*]$ .*
- (ii) *If  $V$  satisfies the assumption (I), then there is a compact embedding  $X \hookrightarrow L^s(\mathbb{R}^N)$  for any  $s \in [p, p_L^*]$ .*

Denote by  $L_K^q(\mathbb{R}^N)$  the weighted Lebesgue space

$$L_K^q(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^N} K(x) |u|^q dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{L_K^q(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} K(x) |u|^q dx \right)^{\frac{1}{q}}.$$

Throughout this paper, let  $X$  be the completion of  $C_0^\infty(\mathbb{R}^N, \mathbb{R})$ , and  $X^*$  be a dual space of  $X$ . Furthermore,  $\langle \cdot, \cdot \rangle$  denotes the pairing of  $X$  and its dual  $X^*$ . All generic constants will be denoted by  $C$ , which may vary from line to line. In what follows, we assume that  $1 < p < \infty$ ,  $N > Lp$ ,  $L = 1, 2, \dots$ .

First, we prove the following embedding lemmas for any  $q \in [p, p_L^*]$  based on [10, Lemmas 2.1– 2.3].

**Lemma 2.2.** *Assume that  $(V, K) \in \mathcal{K}$ . Then there is a continuous embedding  $X \hookrightarrow L_K^q(\mathbb{R}^N)$  for any  $q \in [p, p_L^*]$  if (K2) holds. Moreover,  $X$  can be continuously embedded in  $L_K^{p_0}(\mathbb{R}^N)$  if (K3) holds.*

*Proof.* The proof is trivial if  $q = p$  or  $p_L^*$ . Now we prove that the embedding holds for  $q \in (p, p_L^*)$  if (K2) holds. For a fixed  $q \in (p, p_L^*)$ , set  $\gamma := (p_L^* - q)/(p_L^* - p)$ . Then  $q = p\gamma + (1 - \gamma)p_L^*$ . Thus we have

$$\begin{aligned}
\int_{\mathbb{R}^N} K(x) |u|^q dx &= \int_{\mathbb{R}^N} K(x) |u|^{p\gamma} |u|^{(1-\gamma)p_L^*} dx \\
&\leq \left( \int_{\mathbb{R}^N} |K(x)|^{1/\gamma} |u|^p dx \right)^\gamma \left( \int_{\mathbb{R}^N} |u|^{p_L^*} dx \right)^{1-\gamma} \\
&\leq \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\gamma} \right) \left( \int_{\mathbb{R}^N} V(x) |u|^p dx \right)^\gamma \left( \int_{\mathbb{R}^N} |u|^{p_L^*} dx \right)^{1-\gamma} \\
&\leq C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\gamma} \right) \left( \int_{\mathbb{R}^N} V(x) |u|^p dx \right)^\gamma \left( \int_{\mathbb{R}^N} |\mathcal{D}_L u|^p dx \right)^{\frac{(1-\gamma)p_L^*}{p}} \\
&\leq C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\gamma} \right) \left( \int_{\mathbb{R}^N} |\mathcal{D}_L u|^p + V(x) |u|^p dx \right)^{\gamma + \frac{(1-\gamma)p_L^*}{p}} \\
&= C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\gamma} \right) \left( \int_{\mathbb{R}^N} |\mathcal{D}_L u|^p + V(x) |u|^p dx \right)^{\frac{q}{p}} \\
&= C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|^\gamma}{|V(x)|^\gamma} |K(x)|^{1-\gamma} \right) \|u\|_X^q.
\end{aligned}$$

Since  $K \in L^\infty(\mathbb{R}^N)$  and  $K/V \in L^\infty(\mathbb{R}^N)$ , we have

$$\|u\|_{L_K^q(\mathbb{R}^N)} \leq C \|u\|_X \quad \text{for } q \in (p, p_L^*).$$

Next, we suppose that (K3) holds. For this, we set  $\gamma_0 := (p_L^* - p_0)/(p_L^* - p)$ . Then we have  $p_0 = p\gamma_0 + (1 - \gamma_0)p_L^*$  and hence using the similar argument as above, we get

$$\int_{\mathbb{R}^N} K(x) |u|^{p_0} dx = \int_{\mathbb{R}^N} K(x) |u|^{p\gamma_0} |u|^{(1-\gamma_0)p_L^*} dx \leq C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|^\gamma}{|V(x)|^\gamma} |K(x)|^{1-\gamma} \right) \|u\|_X^q.$$

By  $K/|V|^{(p_L^* - p_0)/(p_L^* - p)} \in L^\infty(\mathbb{R}^N)$ , we conclude that

$$\|u\|_{L_K^{p_0}(\mathbb{R}^N)} \leq C \|u\|_X.$$

This completes the proof.  $\square$

**Lemma 2.3.** *Assume that  $(V, K) \in \mathcal{K}$ . Then there is a compact embedding  $X \hookrightarrow L_K^q(\mathbb{R}^N)$  for any  $q \in (p, p_L^*)$  if (K2) holds. Moreover,  $X$  can be compactly embedded in  $L_K^{p_0}(\mathbb{R}^N)$  if (K3) holds.*

*Proof.* Let  $q$  be a fixed number in  $(p, p_L^*)$  and  $\varepsilon$  be an arbitrary positive number. Since  $p < q < p_L^*$ , for any small  $\varepsilon > 0$ , there are  $0 < s_0 < s_1$  such that  $|s|^q < \varepsilon |s|^p$  if  $|s| \leq s_0$  and  $|s|^q < \varepsilon |s|^{p_L^*}$  if  $|s| \geq s_1$ . This implies

$$K(x) |s|^q \leq \varepsilon C (V(x) |s|^p + |s|^{p_L^*}) + CK(x) \chi_{[s_0, s_1]}(|s|) |s|^{p_L^*} \quad \text{for all } s \in \mathbb{R}$$

for some constant  $C > 0$ . Hence we have for all  $u \in X$

$$(2.1) \quad \int_{B_r^c(0)} K(x) |u|^q dx \leq \varepsilon C Q(u) + C \int_{A \cap B_r^c(0)} K(x) |u|^{p_L^*} dx,$$

where

$$Q(u) = \int_{\mathbb{R}^N} (V(x) |u|^p + |u|^{p_L^*}) dx$$

and  $A = \{x \in \mathbb{R}^N : s_0 \leq |u(x)| \leq s_1\}$ .

If  $\{v_n\}$  is a sequence such that  $v_n \rightharpoonup v$  in  $X$ , there is  $M_1 > 0$  such that for all  $n \in \mathbb{N}$

$$\int_{\mathbb{R}^N} (|\mathcal{D}_L v_n|^p + V(x) |v_n|^p) dx \leq M_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{p_L^*} dx \leq M_1,$$

which gives that  $\{Q(v_n)\}$  is bounded. On the other hand, setting

$$A_n = \{x \in \mathbb{R}^N : s_0 \leq |v_n(x)| \leq s_1\},$$

the last inequality implies that

$$s_0^{p_L^*} |A_n| \leq \int_{A_n} |v_n|^{p_L^*} dx \leq M_1 \quad \text{for all } n \in \mathbb{N},$$

which gives that  $\sup_{n \in \mathbb{N}} |A_n| < \infty$ . Therefore, from (K1), there is  $r > 0$  such that

$$(2.2) \quad \int_{A_n \cap B_r^c(0)} K(x) dx < \frac{\varepsilon}{s_1^{p_L^*}} \quad \text{for all } n \in \mathbb{N}.$$

From (2.1) and (2.2) we deduce that

$$(2.3) \quad \int_{B_r^c(0)} K(x) |v_n|^q dx \leq 2\varepsilon C M_1 + C s_1^{p_L^*} \int_{A_n \cap B_r^c(0)} K(x) dx < (2C M_1 + C) \varepsilon$$

for all  $n \in \mathbb{N}$ . Since  $q \in (p, p_L^*)$  and  $K$  is a continuous function, it follows from Sobolev embedding on the bounded domain that

$$(2.4) \quad \lim_{n \rightarrow \infty} \int_{B_r(0)} K(x) |v_n|^q dx = \int_{B_r(0)} K(x) |v|^q dx.$$

Combining (2.3) and (2.4),

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |v_n|^q dx = \int_{\mathbb{R}^N} K(x) |v|^q dx,$$

which yields

$$v_n \rightarrow v \quad \text{in } L_K^q(\mathbb{R}^N) \text{ for all } q \in (p, p_L^*).$$

Next we suppose that (K3) holds. We will prove the compact embedding by a similar argument in [10, Lemma 2.2]. It is important to observe that for each  $x \in \mathbb{R}^N$  fixed, the function

$$g(s) = V(x) s^{p-p_0} + s^{p_L^*-p_0} \quad \text{for all } s > 0$$

has  $C_{p_0} V^{(p_L^*-p_0)/(p_L^*-p)}(x)$  as its minimum value, where

$$C_{p_0} = \left( \frac{p_L^* - p}{p_L^* - p_0} \right) \left( \frac{p_0 - p}{p_L^* - p_0} \right)^{(p-p_0)/(p_L^*-p)}.$$

Hence,

$$C_{p_0} V^{(p_L^*-p_0)/(p_L^*-p)}(x) \leq V(x) s^{p-p_0} + s^{p_L^*-p_0} \quad \text{for all } x \in \mathbb{R}^N \text{ and } s > 0.$$

It follows from the assumption (K3) that for given  $\varepsilon \in (0, C_{p_0})$ , there is  $r > 0$  large enough such that

$$K(x) |s|^{p_0} \leq \varepsilon C (V(x) |s|^p + |s|^{p_L^*}) \quad \text{for all } s \in \mathbb{R} \text{ and } |x| \geq r,$$

which leads to

$$\int_{B_r^c(0)} K(x) |u|^{p_0} dx \leq \varepsilon C \int_{B_r^c(0)} (V(x) |u|^p + |u|^{p_L^*}) dx.$$

If  $\{v_n\}$  is a sequence such that  $v_n \rightharpoonup v$  in  $X$ , there is  $M_1 > 0$  such that for all  $n \in \mathbb{N}$

$$\int_{\mathbb{R}^N} V(x) |v_n|^p dx \leq M_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{p_L^*} dx \leq M_1$$

and so

$$(2.5) \quad \int_{B_r^c(0)} K(x) |v_n|^{p_0} dx \leq 2\varepsilon C M_1 \quad \text{for all } n \in \mathbb{N}.$$

Since  $p_0 \in (p, p_L^*)$  and  $K$  is a continuous function, it follows from Sobolev embeddings on the bounded domain that

$$(2.6) \quad \lim_{n \rightarrow \infty} \int_{B_r^c(0)} K(x) |v_n|^{p_0} dx = \int_{B_r^c(0)} K(x) |v|^{p_0} dx.$$

From (2.5) and (2.6),

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |v_n|^{p_0} dx = \int_{\mathbb{R}^N} K(x) |v|^{p_0} dx,$$

which implies that

$$v_n \rightarrow v \quad \text{in } L_K^{p_0}(\mathbb{R}^N).$$

This completes the proof.  $\square$

**Lemma 2.4.** *Assume that  $(V, K) \in \mathcal{K}$  and (F1)-(F2) hold. Let  $\{v_n\}$  be a sequence such that  $v_n \rightharpoonup v$  in  $X$ . Then*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) F(x, v_n) dx = \int_{\mathbb{R}^N} K(x) F(x, v) dx$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) f(x, v_n) v_n dx = \int_{\mathbb{R}^N} K(x) f(x, v) v dx.$$

*Proof.* Assume that (K2) holds. It follows from (F1)-(F2) that for a fixed  $q \in (p, p_L^*)$  and given  $\varepsilon > 0$ , there exists  $C > 0$  such that

$$(2.7) \quad K(x) F(x, t) \leq \varepsilon C (V(x) |t|^p + |t|^{p_L^*}) + C K(x) |t|^q \quad \text{for all } t \in \mathbb{R}.$$

By Lemma 2.3, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |v_n|^q dx = \int_{\mathbb{R}^N} K(x) |v|^q dx$$

and there exists  $r > 0$  such that

$$(2.8) \quad \lim_{n \rightarrow \infty} \int_{B_r^c(0)} K(x) |v_n|^q dx < \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Since  $\{v_n\}$  is bounded, there is  $M_2 > 0$  such that for all  $n \in \mathbb{N}$

$$\int_{\mathbb{R}^N} V(x) |v_n|^p dx \leq M_2 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{p_L^*} dx \leq M_2.$$

By (2.7) and (2.8) we conclude that

$$\left| \lim_{n \rightarrow \infty} \int_{B_r^c(0)} K(x) F(x, v_n) dx \right| < (2CM_2 + C)\varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Next, we assume that (K3) holds. Repeating the same arguments as in the proof of Lemma 2.3, for given  $\varepsilon > 0$  small enough, there is  $r > 0$  large enough such that

$$K(x) \leq \varepsilon (V(x) |s|^{p-p_0} + |s|^{p_L^*-p_0}) \quad \text{for all } s \in \mathbb{R} \text{ and } |x| > r.$$

From (F1) and (F2), for the given  $\varepsilon > 0$ , we have

$$F(x, s) \leq C |s|^{p_0} + \varepsilon |s|^{p_L^*} \quad \text{for all } s \in I,$$

where  $I = \{s \in \mathbb{R} : |s| < s_0 \text{ or } |s| > s_1\}$ . Since  $K \in L^\infty(\mathbb{R}^N)$ , for all  $s \in I$  and  $|x| > r$ , we have

$$\begin{aligned} K(x) F(x, s) &\leq C K(x) |s|^{p_0} + \varepsilon K(x) |s|^{p_L^*} \\ &\leq C \varepsilon (V(x) |s|^{p-p_0} + |s|^{p_L^*-p_0}) |s|^{p_0} + \varepsilon \|K(x)\|_{L^\infty(\mathbb{R}^N)} |s|^{p_L^*} \\ &\leq C \varepsilon (V(x) |s|^{p-p_0} + |s|^{p_L^*}). \end{aligned}$$

Therefore, for any  $u \in X$ , we have the following estimate

$$\int_{B_r^c(0)} K(x) F(x, u) dx \leq \varepsilon C Q(u) + C \int_{A \cap B_r^c(0)} K(x) dx,$$

where

$$Q(u) = \int_{\mathbb{R}^N} (V(x) |u|^p + |u|^{p_L^*}) dx$$

and  $A = \{x \in \mathbb{R}^N : s_0 \leq |u(x)| \leq s_1\}$ . Since  $\{v_n\}$  is bounded in  $X$ , there is  $M_3 > 0$  such that

$$\int_{\mathbb{R}^N} V(x) |v_n|^p dx \leq M_3 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{p_L^*} dx \leq M_3.$$

Thus,

$$\int_{B_r^c(0)} K(x)F(x, v_n) dx \leq 2CM_3 + C \int_{A_n \cap B_r^c(0)} K(x) dx$$

where  $A_n = \{x \in \mathbb{R}^N : s_0 \leq |v_n(x)| \leq s_1\}$ . Repeating the same arguments used in the proof of Lemma 2.3, it follows that

$$\int_{A_n \cap B_r^c(0)} K(x) dx \rightarrow 0 \text{ as } r \rightarrow +\infty$$

and so, for  $n$  large enough,

$$\left| \int_{B_r^c(0)} K(x)F(x, v_n) dx \right| \leq C(2M_3 + 1)\varepsilon.$$

From (2.8), we need to show that

$$\lim_{n \rightarrow \infty} \int_{B_r(0)} K(x)F(x, v_n) dx = \int_{B_r(0)} K(x)F(x, v) dx.$$

However, this limit follows by using Compactness Lemma 2 of Strauss [21]. The proof is completed.  $\square$

**Definition 2.5.** We say that  $u \in X$  is a weak solution of the problem (P) if

$$\int_{\mathbb{R}^N} |\mathcal{D}_L u|^{p-2} \mathcal{D}_L u \mathcal{D}_L v dx + \int_{\mathbb{R}^N} V(x)|u|^{p-2} uv dx = \int_{\mathbb{R}^N} K(x)f(x, u)v dx$$

for all  $v \in X$ .

Let us define the functional  $\Phi : X \rightarrow \mathbb{R}$  by

$$\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\mathcal{D}_L u|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u|^p dx.$$

Under the assumption (I), it is obvious that the functional  $\Phi$  is well defined on  $X$ ,  $\Phi \in C^1(X, \mathbb{R})$  and its Fréchet derivative is given by

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{R}^N} |\mathcal{D}_L u|^{p-2} \mathcal{D}_L u \mathcal{D}_L v dx + \int_{\mathbb{R}^N} V(x)|u|^{p-2} uv dx$$

for any  $u, v \in X$ .

Now, we give some properties of the operator  $\Phi$ . According to similar arguments in [20, Lemma 3.1], the following lemma is easily checked, and thus we omit the proof.

**Lemma 2.6.** Assume that the assumption  $(V, K) \in \mathcal{K}$  holds.

- (1)  $\Phi$  is a continuous, bounded and strictly monotone operator.
- (2)  $\Phi'$  is of type  $(S_+)$ , i.e., if  $u_n \rightharpoonup u$  in  $X$  and

$$\limsup_{n \rightarrow \infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \leq 0,$$

then  $u_n \rightarrow u$  in  $X$  as  $n \rightarrow \infty$ .

- (3)  $\Phi$  is a homomorphism.

Next we define the functional  $\Psi : X \rightarrow \mathbb{R}$  by

$$\Psi(u) = \frac{1}{p} \int_{\mathbb{R}^N} K(x)F(x, u) dx.$$

Then it is easy to check that  $\Psi \in C^1(X, \mathbb{R})$  and its Fréchet derivative is

$$\langle \Psi'(u), v \rangle = \int_{\mathbb{R}^N} K(x)f(x, u)v dx$$

for any  $u, v \in X$ . Also we define the functional  $\mathcal{I} : X \rightarrow \mathbb{R}$  by

$$\mathcal{I}(u) = \Phi(u) - \Psi(u).$$

Then it follows that the functional  $\mathcal{I} \in C^1(X, \mathbb{R})$  and its Fréchet derivative is

$$\langle \mathcal{I}'(u), v \rangle = \int_{\mathbb{R}^N} |\mathcal{D}_L u|^{p-2} \mathcal{D}_L u \mathcal{D}_L v \, dx + \int_{\mathbb{R}^N} V(x) |u|^{p-2} uv \, dx - \int_{\mathbb{R}^N} K(x) f(x, u) v \, dx$$

for any  $u, v \in X$ .

### 3. EXISTENCE OF WEAK SOLUTIONS

In this section, we shall give the proof of the existence of nontrivial positive weak solutions for problem (P), by applying the mountain pass theorem and the fountain theorem. With the aid of Lemmas 2.4 and 2.6, we prove that the energy functional  $\mathcal{I}$  satisfies the Cerami condition. This plays a key role in obtaining the existence of a nontrivial weak solution for the given problem.

**3.1. Existence of weak solutions : Approach to the mountain pass theorem.** The following result is to show that the energy functional  $\mathcal{I}$  satisfies the mountain pass geometry (see e.g. [4], [11, Lemma 2.2]).

**Lemma 3.1.** *Assume that  $(V, K) \in \mathcal{K}$ , (F1) and (F2) hold. Then the functional  $\mathcal{I}$  satisfies the following two properties:*

- (1) *there exist two constants  $\alpha, \rho > 0$  such that  $\mathcal{I}(u) \geq \alpha$  for all  $\|u\| = \rho$ ;*
- (2) *there exists  $e \in X \setminus \{0\}$  with  $\|u\| > \rho$  such that  $\mathcal{I}(e) \leq 0$ .*

*Proof.* It is obvious that (2) holds. Concerning (1), note that  $X$  is continuously embedded into  $L_K^q(\mathbb{R}^N)$  for any  $q \in [p, p_L^*]$  by Lemma 2.2. Then we can write the inequality, for a fixed small  $\varepsilon_0$ ,

$$K(x)F(x, u) \leq \varepsilon_0 V(x) |u|^p + C |u|^{p_L^*} + CK(x) |u|^q \quad \text{for } x \in \mathbb{R}^N$$

and thus the mountain pass geometry can be proved.  $\square$

Therefore, there exists a Cerami sequence ( $(C)$ -sequence for short),  $\{u_n\} \subset X$  such that

$$\mathcal{I}(u_n) \rightarrow c \quad \text{and} \quad \|\mathcal{I}'(u_n)\|_{X^*} (1 + \|u_n\|_X) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where  $c$  is given by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \mathcal{I}(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0 \text{ and } \mathcal{I}(\gamma(1)) \leq 0\}.$$

Next, we consider the boundedness of the Cerami sequence.

**Lemma 3.2.** *Assume that  $(V, K) \in \mathcal{K}$  and (F1)–(F3) hold. If  $\{u_n\}$  in  $X$  is a  $(C)_c$ -sequence of  $\mathcal{I}$ , then  $\{u_n\}$  is bounded in  $X$ .*

*Proof.* The proof is followed the argument in [11]. For  $c \in \mathbb{R}$ , let  $\{u_n\}$  be a  $(C)_c$ -sequence in  $X$ , that is,

$$\mathcal{I}(u_n) \rightarrow c \quad \text{and} \quad \|\mathcal{I}'(u_n)\|_{X^*} (1 + \|u_n\|_X) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This says that

$$c = \mathcal{I}(u_n) + o_n(1) \quad \text{and} \quad \langle \mathcal{I}'(u_n), u_n \rangle = o_n(1),$$

where  $o_n(1) \rightarrow 0$  as  $n \rightarrow \infty$ .

Since  $\mathcal{I}(tu_n)$  is continuous in  $t \in [0, 1]$ , for each  $n \in \mathbb{N}$  there is  $t_n \in [0, 1]$  such that

$$\mathcal{I}(t_n u_n) := \max_{t \in [0, 1]} \mathcal{I}(tu_n).$$

We claim that  $\mathcal{I}(t_n u_n)$  is bounded from above. Without loss of generality, we may assume that  $t_n \in (0, 1)$  for all  $n \in \mathbb{N}$ . Then we have

$$\begin{aligned} p\mathcal{I}(t_n u_n) - o_n(1) &= p\mathcal{I}(t_n u_n) - \langle \mathcal{I}'(t_n u_n), t_n u_n \rangle \\ &= - \int_{\mathbb{R}^N} K(x) F(x, t_n u_n) \, dx + \int_{\mathbb{R}^N} K(x) f(x, t_n u_n) t_n u_n \, dx \\ &= \int_{\mathbb{R}^N} K(x) \mathcal{H}(x, t_n u_n) \, dx \end{aligned}$$



$$\leq \int_{\mathbb{R}^N} K(x) \mathcal{H}(x, u_n) dx,$$

where  $\mathcal{H}(x, s) := sf(x, s) - F(x, s)$  is non-decreasing due to (F3). Since  $t_n \in (0, 1)$ ,

$$p\mathcal{I}(t_n u_n) \leq \int_{\mathbb{R}^N} K(x) \mathcal{H}(x, u_n) dx + o_n(1) = p\mathcal{I}(u_n) - \langle \mathcal{I}'(u_n), u_n \rangle + o_n(1) = p\mathcal{I}(u_n),$$

which proves the claim. Now we prove that  $\{u_n\} \subset X$  is bounded. Assume the contrary that the sequence  $\{u_n\}$  is unbounded in  $X$ . Then we may suppose that  $\|u_n\|_X > 1$  and  $\|u_n\|_X \rightarrow \infty$  as  $n \rightarrow \infty$ . Define a sequence  $\{\ell_n\}$  by  $\ell_n = u_n / \|u_n\|_X$ . Then it is obvious that  $\{\ell_n\} \subset X$ . Hence, up to a subsequence, still denoted by  $\{\ell_n\}$ , we obtain  $\ell_n \rightharpoonup \ell$  in  $X$  as  $n \rightarrow \infty$ .

Now, we claim that  $\ell_n(x) = 0$  for almost all  $x \in \mathbb{R}^N$ . Since  $\mathcal{I}(u_n) \rightarrow c$ , we have

$$o_n(1) + \frac{1}{p} = \int_{\mathbb{R}^N} \frac{K(x)F(x, u_n)}{\|u\|_X^p} dx = \int_{\mathbb{R}^N} \frac{K(x)F(x, u_n)}{|u_n|^p} |\ell_n|^p dx.$$

By (F3), given  $\tau > 0$  there exists  $\xi_\tau > 0$  such that  $F(x, s) \geq \tau|s|^p$  for all  $|s| \geq \xi_\tau$ . Hence,

$$\begin{aligned} o_n(1) + \frac{1}{p} &\geq \int_{\{x \in \mathbb{R}^N : |u_n| \geq \xi_\tau\}} \frac{K(x)F(x, u_n)}{|u_n|^p} |\ell_n|^p dx \\ &\geq \tau \int_{\mathbb{R}^N} K(x) |\ell_n|^p \chi_{\{|u_n| \geq \frac{\xi_\tau}{\|u_n\|}\}} dx. \end{aligned}$$

Since  $|\ell_n|^p \chi_{\{|u_n| \geq \frac{\xi_\tau}{\|u_n\|}\}} \rightarrow |\ell|^p$  as  $n \rightarrow \infty$  for any  $\tau > 0$ , the Fatou lemma implies that

$$\frac{1}{p} \geq \tau \int_{\mathbb{R}^N} K(x) |\ell|^p dx.$$

Since  $K > 0$  and  $\tau$  is arbitrary, we deduce  $\ell(x) = 0$  for almost all  $x \in \mathbb{R}^N$  and the claim follows.

Now, let  $B > 0$ , it is obvious that  $B\|u_n\|^{-1} \in [0, 1]$  for sufficiently large  $n$ . Thus, we have

$$(3.1) \quad \mathcal{I}(t_n u_n) \geq \mathcal{I}(B\ell_n) = \frac{B^2}{p} - \int_{\mathbb{R}^N} K(x)F(x, B\ell_n) dx,$$

because  $t_n$  is a maximum point. By Lemma 2.4, it follows

$$(3.2) \quad \int_{\mathbb{R}^N} K(x)F(x, B\ell_n) dx \rightarrow \int_{\mathbb{R}^N} K(x)F(x, B\ell) dx = 0$$

and  $\mathcal{I}(t_n u_n) + o_n(1) \geq B^2/p$ . Therefore, we deduce  $\sup\{\mathcal{I}(t_n u_n) : n \in \mathbb{N}\} \geq B^2/p$ , which is a contradiction if we take

$$B := p\sqrt{\sup\{\mathcal{I}(t_n u_n) : n \in \mathbb{N}\}} \in (0, \infty).$$

This completes the proof.  $\square$

Using Lemma 3.1, we prove the existence of a nontrivial weak solution for our problem under the assumptions.

**Theorem 3.3.** *Let  $1 < p < \infty$ ,  $N > Lp$ ,  $L = 1, 2, \dots$ . Assume that  $(V, K) \in \mathcal{K}$  and (F1)–(F3) hold. Then the problem (P) admits a nontrivial positive weak solution.*

*Proof.* Note that  $\mathcal{I}(0) = 0$ . In view of Lemma 3.1, the geometric conditions in the mountain pass theorem are fulfilled. And also,  $\mathcal{I}$  satisfies the  $(C)_c$ -condition by Lemma 3.2. Hence, the problem (P) has a nontrivial weak solution. This completes the proof.  $\square$

**3.2. Existence of a sequence of weak solutions : Approach to the fountain theorem.** In this subsection, we will establish the principle of concentration compactness in  $X$ .

In this subsection, we assume that

(IV) Given  $\varepsilon > 0$ , there exist  $r := r_\varepsilon > 0$  and  $C > 0$  (is independent of  $\varepsilon$ ) such that

$$\int_{|x| \leq r} K(x) |u|^{2^*_L} dx \leq C\varepsilon,$$

and  $\lim_{x \rightarrow \infty} K(x) = 0$  (in short  $K(\infty) = 0$ ).

Before the proof the multiplicity of weak solutions to our equations, we introduce the notion and the concentration–compactness lemma in [14]. In particular, the arguments in [14, Theorem 2.2] is almost held for the problem (P), we omit the proof.

**Definition 3.4.** Let  $\Omega$  be an open subset of  $\mathbb{R}^N$  and define

$$K(\Omega) = \{\eta \in C(\Omega) : \text{supp } \eta \text{ is a compact subset of } \Omega\},$$

and

$$BC(\Omega) = \{\eta \in C(\Omega) : |\eta|_\infty = \sup_{x \in \Omega} |\eta(x)| < \infty\}.$$

The space  $C_0(\Omega)$  is the closure of  $K(\Omega)$  in  $BC(\Omega)$  with respect to the uniform norm  $|\cdot|_\infty$ . A finite measure on  $\Omega$  is a continuous linear functional on  $C_0(\Omega)$ . The norm of the finite measure  $\mu$  is defined by

$$\|\mu\| = \sup_{\eta \in C_0(\Omega), |\eta|_\infty = 1} |(\mu, \eta)|,$$

where  $(\mu, \eta) = \int_\Omega \eta d\mu$ . We denote by  $M(\Omega)$  the space of finite non-negative Borel measures on  $\Omega$ . A sequence  $\mu_n \rightarrow \mu$  weakly  $-*$  in  $M(\Omega)$  is defined by  $(\mu_n, \eta) \rightarrow (\mu, \eta)$  for any  $\mu \in C_0(\Omega)$ .

**Lemma 3.5.** Let  $\{u_n\} \subset X$  with  $\|u_n\| \leq 1$  such that

$$\begin{aligned} u_n &\rightarrow u \quad \text{weakly in } X, \\ |\nabla u_n|^2 + V(x)|u_n|^2 &\rightarrow \mu \quad \text{weakly } -* \text{ in } M(\mathbb{R}^N) \end{aligned}$$

and

$$|u_n|^2 \rightarrow \nu \quad \text{weakly } -* \text{ in } M(\mathbb{R}^N)$$

as  $n \rightarrow \infty$ . Denote  $C^* = \sup\{\int_{\mathbb{R}^N} |u|^{2^*} dx : \|u\| \leq 1 \text{ for } u \in X\}$ . Then the limit measures are of the form

$$\mu = |\nabla u|^2 + V(x)|u|^2 + \sum_{j \in J} \mu_j \delta_{x_j} + \tilde{\mu} \quad \text{and} \quad \mu(\mathbb{R}^N) \leq 1$$

and

$$\nu = |u|^{2^*} + \sum_{j \in J} \nu_j \delta_{x_j} \quad \text{and} \quad \nu(\mathbb{R}^N) \leq C^*,$$

where  $J$  is a countable set,  $\{\mu_j\}, \{\nu_j\} \subseteq [0, \infty)$ ,  $\{x_j\} \subset \mathbb{R}^N$  and  $\mu \in M(\mathbb{R}^N)$  is a non-atomic non-negative measure. The atoms and the regular part satisfy the generalized Sobolev inequality

$$\nu(\mathbb{R}^N) \leq 2_L^* C^* \mu(\mathbb{R}^N)$$

and

$$\nu_j \leq C^* \mu^{2^*/2}.$$

Applying the fountain theorem in [22, Theorem 3.6] with the oddity on  $f$ , we investigate infinitely many positive weak solutions for the problem (P). For this, let  $W$  be a reflexive and separable Banach space. Then there are  $\{e_n\} \subseteq W$  and  $\{f_n^*\} \subseteq W^*$  such that

$$W = \overline{\text{span}\{e_n : n = 1, 2, \dots\}}, \quad W^* = \overline{\text{span}\{f_n^* : n = 1, 2, \dots\}},$$

and

$$\langle f_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote  $W_n = \text{span}\{e_n\}$ ,  $Y_k = \bigoplus_{n=1}^k W_n$ , and  $Z_k = \overline{\bigoplus_{n=k}^\infty W_n}$ .

**Theorem 3.6.** Let  $N > 2L$ ,  $L = 1, 2, \dots$ . Assume that  $(V, K) \in \mathcal{K}$ , and (F1)–(F4) hold. Then the problem (P) possesses an unbounded sequence of nontrivial positive weak solutions  $\{u_n\}$  in  $X$  such that  $\mathcal{I}(u_n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

*Proof.* For the time being, we prove the Theorem 3.6 in case (K2). It is obvious that  $\mathcal{I}$  is an even functional and satisfies the  $(C)_c$ -condition. It suffices to show that there exist  $\rho_k > \delta_k > 0$  such that

$$(1) \quad b_k := \inf\{\mathcal{I}(u) : u \in Z_k, \|u\|_X = \delta_k\} \rightarrow \infty \quad \text{as } k \rightarrow \infty;$$

(2)  $\alpha_k := \max\{\mathcal{I}(u) : u \in Y_k, \|u\|_X = \rho_k\} \leq 0$  for  $k$  large enough.

Denote

$$\alpha_k := \sup_{u \in Z_k, \|u\|_X = 1} \int_{\mathbb{R}^N} K(x) |u|^{2_L^*} dx.$$

Then we claim that  $\alpha_k \rightarrow 0$  as  $k \rightarrow \infty$ . It is obvious that  $0 \leq \alpha_{k+1} \leq \alpha_k$ , then  $\alpha_k \rightarrow \alpha \geq 0$  as  $k \rightarrow \infty$ . There exists  $u_k \in Z_k$  with  $\|u_k\|_X = 1$  such that

$$0 \leq \alpha_k - \int_{\mathbb{R}^N} K(x) |u_k|^{2_L^*} dx < \frac{1}{k},$$

for each  $k = 1, 2, \dots$ .

By the boundedness of the sequence  $\{u_k\}$  in  $X$ , we can find an element  $u \in X$  such that  $u_k \rightharpoonup u$  in  $X$  as  $k \rightarrow \infty$  and

$$\langle f_j^*, u \rangle = \lim_{k \rightarrow \infty} \langle f_j^*, u_k \rangle = 0$$

for  $j = 1, 2, \dots$ . Thus we deduce  $u = 0$ .

Now, we show  $\alpha = 0$  based on a argument in [14, Lemma 3.5, pp.1683–1684]. By Theorem 3.5, there exist finite measure  $\nu$  and sequences  $\{x_j\} \subset \mathbb{R}^N$  such that  $|u_k| \rightarrow \nu = \sum_{j \in J} \nu_j \delta_{x_j}$  weakly  $*$  in  $M(\mathbb{R}^N)$ , where  $J$  is a countable set. Similar to the discussion in [14, Lemma 3.4], we obtain  $\nu_j = \nu(\{x_j\}) = 0$  for any  $j \in J$  and  $x_j \neq 0$ .

For any  $0 < r < R$ , take  $\eta \in C_0^\infty(B_{2R}(0))$  such that  $0 \leq \eta \leq 1$ ;  $\eta \equiv 1$  in  $B_R(0) \setminus B_r(0)$ ,  $\eta \equiv 0$  in  $B_{r/2}(0)$ . Then

$$\int_{\mathbb{R}^N} |u_k|^{2_L^*} \eta dx \rightarrow \int_{\mathbb{R}^N} \eta d\nu = \int_{r/2 \leq |x| \leq 2R} \eta d\nu = 0$$

as  $k \rightarrow \infty$ . Note that if  $\int_{r \leq |x| \leq R} |u_k|^{2_L^*} dx \leq \int_{\mathbb{R}^N} |u_k|^{2_L^*} \eta dx$ , then

$$\lim_{k \rightarrow \infty} \int_{r \leq |x| \leq R} |u_k|^{2_L^*} dx = 0.$$

Since  $\lim_{x \rightarrow \infty} K(x) = 0$  for any  $\varepsilon > 0$ , there exists positive number  $R_\varepsilon$  such that

$$(3.3) \quad K(x) < \varepsilon, \quad |x| \geq R_\varepsilon.$$

Thus, for any  $k \in \mathbb{N}$ , we have by the assumption (IV), for  $R_\varepsilon > r_\varepsilon$

$$\int_{|x| \leq r_\varepsilon} K(x) |u_k|^{2_L^*} dx \leq C\varepsilon,$$

and by (3.3),

$$\int_{|x| \geq R_\varepsilon} K(x) |u_k|^{2_L^*} dx \leq \varepsilon \int_{|x| \geq R} |u_k|^{2_L^*} dx \leq C\varepsilon, \quad R > R_\varepsilon.$$

Thus we obtain

$$\lim_{k \rightarrow \infty} \int_{|x| \leq r_\varepsilon} K(x) |u_k|^{2_L^*} dx = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \int_{|x| \geq R_\varepsilon} K(x) |u_k|^{2_L^*} dx = 0,$$

which implies that  $\alpha_k \rightarrow \alpha = 0$  as  $k \rightarrow \infty$ .

For any  $u \in Z_k$ , we may suppose that  $\|u\|_X > 1$ . According to the assumption (F2), we obtain that

$$\begin{aligned} \mathcal{I}(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |\mathcal{D}_L u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |u|^2 dx - \int_{\mathbb{R}^N} K(x) F(x, u) dx \\ &\geq \frac{1}{2} \|u\|_X^2 - \int_{\mathbb{R}^N} K(x) |F(x, u)| dx \\ &\geq \frac{1}{2} \|u\|_X^2 - \int_{\mathbb{R}^N} K(x) \frac{|u|^{2_L^*}}{2_L^*} dx \\ &\geq \frac{1}{2} \|u\|_X^2 - \frac{\alpha_k}{2_L^*} \|u\|_X^{2_L^*} \end{aligned}$$

where we use the definition of  $\alpha_k$ . If we take

$$\delta_k = (\alpha_k)^{1/(2-2_L^*)},$$

then  $\delta_k \rightarrow \infty$  as  $k \rightarrow \infty$  because  $2 < 2_L^*$  and  $\alpha_k \rightarrow 0$  as  $k \rightarrow \infty$ . Hence, if  $u \in Z_k$  and  $\|u\|_X = \delta_k$ , then we conclude that

$$\mathcal{I}(u) \geq \left( \frac{1}{2} - \frac{1}{2_L^*} \right) \delta_k^2 \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

This implies that the condition (1) holds.

The proof of the condition (2) proceeds analogously as in the proof of Theorem 1.3 in [2]. For the reader's convenience, we give the proof. Assume that the condition (2) is not true. Then for some  $k$  there exists a sequence  $\{u_n\}$  in  $Y_k$  such that

$$(3.4) \quad \|u_n\|_X \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \mathcal{I}(u_n) \geq 0.$$

Set  $w_n = u_n / \|u_n\|_X$ . Note that  $\|w_n\|_X = 1$ . Since  $\dim Y_k < \infty$ , there exists  $w \in Y_k \setminus \{0\}$  such that up to a subsequence,

$$\|w_n - w\|_X \rightarrow 0 \quad \text{and} \quad w_n(x) \rightarrow w(x)$$

for almost all  $x \in \mathbb{R}^N$  as  $n \rightarrow \infty$ . If  $w(x) \neq 0$ , then  $|u_n(x)| \rightarrow \infty$  for all  $x \in \mathbb{R}^N$  as  $n \rightarrow \infty$ . Hence we obtain by the assumption (F3) that

$$\lim_{n \rightarrow \infty} \frac{F(x, u_n)}{\|u_n\|_X^2} = \lim_{n \rightarrow \infty} \frac{F(x, u_n)}{|u_n|^2} |w_n|^2 = \infty$$

for all  $x \in \Omega_1 := \{x \in \mathbb{R}^N : w(x) \neq 0\}$ . So then, it follows from the assumption (F3) that for all  $x \in \Omega_1$ , we have

$$(3.5) \quad \lim_{n \rightarrow \infty} \frac{K(x)F(x, u_n)}{\|u_n\|_X^2} = \lim_{n \rightarrow \infty} \frac{K(x)F(x, u_n)}{|u_n|^2} |w_n|^2 = \infty.$$

The assumption (F3) implies that there exists  $t_0 > 1$  such that  $F(x, t) > |t|^2$  for all  $|t| > t_0$ . From the assumptions (F1) and (F2), there exists  $\mathcal{C} > 0$  such that  $|F(x, t)| \leq \mathcal{C}$  for all  $t \in [-t_0, t_0]$ . Therefore we can choose a real number  $\mathcal{C}_0$  such that  $F(x, t) \geq \mathcal{C}_0$  for all  $t \in \mathbb{R}$ , and thus

$$\frac{F(x, u_n) - \mathcal{C}_0}{\|u_n\|_X^2} \geq 0,$$

for all  $n \in \mathbb{N}$ . Since  $K > 0$  and  $K \in L^\infty(\mathbb{R}^N)$ , there exists a real number  $\mathcal{C}_1$  such that

$$\frac{K(x)F(x, u_n) - \mathcal{C}_1}{\|u_n\|_X^2} \geq 0.$$

Thus, we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{K(x)F(x, u_n)}{\|u_n\|_X^2} dx &\geq \liminf_{n \rightarrow \infty} \int_{\Omega_1} \frac{K(x)F(x, u_n)}{\|u_n\|_X^2} dx - \liminf_{n \rightarrow \infty} \int_{\Omega_1} \frac{\mathcal{C}_1}{\|u_n\|_X^2} dx \\ &\geq \liminf_{n \rightarrow \infty} \int_{\Omega_1} \frac{K(x)F(x, u_n) - \mathcal{C}_1}{\frac{1}{2}\|u_n\|_X^2} dx \\ &\geq \int_{\Omega_1} \liminf_{n \rightarrow \infty} \frac{K(x)F(x, u_n) - \mathcal{C}_1}{\|u_n\|_X^2} dx \\ &\geq \int_{\Omega_1} \liminf_{n \rightarrow \infty} \frac{K(x)F(x, u_n)}{\|u_n\|_X^2} dx - \int_{\Omega_1} \limsup_{n \rightarrow \infty} \frac{\mathcal{C}_1}{\|u_n\|_X^2} dx \\ &= \infty. \end{aligned}$$

A similar argument as (3.5) proves that

$$\int_{\Omega_1} \frac{K(x)F(x, u_n)}{\|u_n\|_X^2} dx \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Therefore, we conclude that

$$\begin{aligned}\mathcal{I}(u_n) &= \frac{1}{2}\|u_n\|_X^2 - \int_{\mathbb{R}^N} K(x)F(x, u_n) dx \\ &\leq \|u_n\|_X^2 \left( \frac{1}{2} - \int_{\Omega_1} \frac{K(x)F(x, u_n)}{\|u_n\|_X^2} dx \right) \rightarrow -\infty \quad \text{as } n \rightarrow \infty,\end{aligned}$$

which contradicts (3.4). Moreover, in case (K3), our analysis holds by Lemma 2.1. This completes the proof.  $\square$

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