Nagy's perturbation of a non-selfadjoint operator and application to a Gribov operator in Bargmann space

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Abstract

The purpose of the present paper is to formulate some new supplements to perturbation theory of linear operators [15] by considering a non-analytic perturbation involving more than one perturbation parameter. An application to a Gribov operator in Bargmann space illustrates the mathematical problem involved in this paper.

Key words. Perturbation theory, eigenvalue, eigenvector, Gribov operator, Bargmann space.

1 Introduction

Perturbation theory of linear operators has been pioneered by L. Rayleigh [16] and E. Schrödinger ([18] and [19]) and is the object of many researches until now [12]. In [16], L. Rayleigh has given a formula for computing the natural frequencies and modes of a vibrating system deviating slightly from a simpler system which admits a complete determination of the frequencies and modes. Mathematically speaking, the method was equivalent to an approximation solution of the eigenvalue problem for a linear operator slightly different from a simpler operator for which the problem is completely solved. E. Schrödinger ([18] - [19]) has developed a similar method, with more generality and systematization, for the eigenvalue problems that appear in quantum mechanics. Later, T. Kato [13] and F. Rellich [17] have been mainly concerned with the regular perturbation of self-adjoint operators on a Hilbert space, while some attempts have also been made towards the treatment of non-regular cases which are not less important in applications. Another generalization of the theory has been given by B. Sz. Nagy [15]. By his elegant and powerful method of contour integration, he has been able to transfer most of the theorems for self-adjoint operators to a wider class of closed linear operators in Banach space. More precisely, let $A(\varepsilon)$ be a perturbed operator on a Banach space X, depending on a complex parameter ε as a convergent power series

$$A(\varepsilon) := A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots + \varepsilon^k A_k + \cdots , \tag{1.1}$$

where $\varepsilon \in \mathbb{C}$ and $A_0, A_1, A_2, A_3 ...$ are linear operators on X, having the same domain $\mathcal{D}$ and satisfying the relative boundedness condition

$$\|A_k\varphi\| \leq q^{k-1}(a\|\varphi\| + b\|A_0\varphi\|) \tag{1.2}$$

for all $\varphi \in \mathcal{D}$ and for all $k \geq 1$, with $a, b, q > 0$. Among basic results already obtained by B. Sz. Nagy [15], we focus our attention on the following:
(i) The series

$$A_0\varphi + \varepsilon A_1\varphi + \varepsilon^2 A_2\varphi + \cdots + \varepsilon^k A_k\varphi + \cdots$$

converges for all $\varphi \in \mathcal{D}$ and for all $|\varepsilon| < q^{-1}$. Setting $A(\varepsilon)\varphi$ its limit, we have $A(\varepsilon)$ is a linear operator with domain $\mathcal{D}$.
(ii) If A_0 is closed, then $A(\varepsilon)$ is also closed, for $|\varepsilon| < (q+b)^{-1}$.
(iii) Suppose that the unperturbed operator A_0 has an isolated eigenvalue λ_n with multiplicity one. Then $A(\varepsilon)$ has a unique eigenvalue $\lambda_n(\varepsilon)$ in the neighborhood of λ_n for sufficiently small $|\varepsilon|$ and this eigenvalue can be expanded into a convergent series

$$\lambda_n(\varepsilon) = \lambda_n + \varepsilon\lambda_{n,1} + \varepsilon^2\lambda_{n,2} + \cdots .$$

Moreover, the eigenvector $\varphi_n(\varepsilon)$ of $A(\varepsilon)$ corresponding to the eigenvalue $\lambda_n(\varepsilon)$ depends analytically on ε near 0 :

$$\varphi_n(\varepsilon) = \varphi_n + \varepsilon\varphi_{n,1} + \varepsilon^2\varphi_{n,2} + \cdots .$$

Here φ_n is an eigenvector of A_0 corresponding to the eigenvalue λ_n.

Later, the basis of eigenvectors property was confirmed for the operator $A(\varepsilon)$ in order to describe the radiation of a vibrating structure in a light fluid and to study strong interactions in the context of Reggeon theory (see [3], [6], [7], [8] and [11]). After that [5], it has been proved that the family of exponentials associated to the eigenvalues of the operator $A(\varepsilon)$ forms a Riesz basis. This result was of importance for application to a non-selfadjoint problem deduced from perturbation method for sound radiation.

We now ask what conclusions can be drawn if we consider a more general type of non-analytic perturbation involving more than one perturbation parameter?
Indeed, we investigate under this question and we give new variants to B. Sz. Nagy's results [15]. To this end, we consider the operator

$$T(\xi) := T_0 + \xi_1 T_1 + \xi_2 T_2 + \cdots + \xi_k T_k + \cdots \tag{1.3}$$

where $\xi = (\xi_k)_{k\geq 1}$ is a sequence of complex numbers such that $\tau(\xi) = \sum_{k=1}^{\infty} |\xi_k| < \infty$, T_0 is a closed linear operator on a Banach space X with domain $\mathcal{D}(T_0)$ and $(T_k)_{k\geq 1}$ are linear operators on X, having the same domain $\mathcal{D}$ such that $\mathcal{D}(T_0) \subset \mathcal{D}$ and

$$\|T_k\varphi\| \leq a\|\varphi\| + b\|T_0\varphi\|, \quad \text{for all} \quad \varphi \in \mathcal{D}(T_0), \tag{1.4}$$

where $a,\, b > 0$. We emphasize on the fact that under those considerations, we give an essential improvement to the results developed by B. Sz. Nagy [15] since we deal with a non-analytic perturbation involving more than one perturbation parameter and covering cases where the results developed in [15] can not be applied.

Based on the analysis started in [15], we study the behavior of spectral properties of $T(\xi)$. More precisely, we prove that:

(i) The series

$$T_0\varphi + \xi_1 T_1\varphi + \xi_2 T_2\varphi + \cdots + \xi_k T_k\varphi + \cdots$$

converges for all $\varphi \in \mathcal{D}(T_0)$. If $T(\xi)\varphi$ denotes its limit, then $T(\xi)$ is a linear operator with domain $\mathcal{D}(T_0)$. Moreover, $T(\xi)$ is closed if $\tau(\xi) < \frac{1}{b}$.

(ii) Let λ_n be the eigenvalue number n of T_0. If λ_n is isolated with multiplicity one, then for $\tau(\xi)$ enough small, $T(\xi)$ has a unique simple eigenvalue $\lambda_n(\xi)$ in a small neighborhood of λ_n. Setting $\varphi_n(\xi)$ an eigenvector of $T(\xi)$ associated to $\lambda_n(\xi)$, then one can develop $\lambda_n(\xi)$ and $\varphi_n(\xi)$ into series:

$$\lambda_n(\xi) := \lambda_n + \lambda_{n,1}(\xi) + \lambda_{n,2}(\xi) + \cdots + \lambda_{n,i}(\xi) + \cdots \tag{1.5}$$

and

$$\varphi_n(\xi) := \varphi_n + \varphi_{n,1}(\xi) + \varphi_{n,2}(\xi) + \cdots + \varphi_{n,i}(\xi) + \cdots\ . \tag{1.6}$$

Notice here that if in particular we take $\xi_k = \varepsilon^k q^{k-1}$ and $T_k = \frac{1}{q^{k-1}} A_k$ $(k \geq 1)$ in Eqs (1.3) and (1.4), we recognize the B. Sz. Nagy's perturbation model of linear operators (see [14]-[15]). Moreover, we regain the spectral study of the operator $A(\varepsilon)$ (see Eqs (1.1) and (1.2)) investigated in [15]. The main novelty in this paper is that we give the exact expressions of the coefficients $(\lambda_{n,i})_{i\geq 1}$ and $(\varphi_{n,i})_{i\geq 1}$ in Eqs (1.5) and (1.6) (see Theorems 3.2 and 3.3).
The key tool here was the following equalities,

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} s_k\right)^n = \sum_{n=1}^{\infty} a_n \sum_{l=n}^{\infty} \sum_{i_1+i_2+\ldots+i_n=l} s_{i_1} s_{i_2}\cdots s_{i_n}$$

and

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} A_k\right)^n = \sum_{n=1}^{\infty} a_n \sum_{l=n}^{\infty} \sum_{i_1+i_2+\ldots+i_n=l} A_{i_1} A_{i_2}\cdots A_{i_n},$$

where $(a_k)_{k\geq 0}$ is a complex bounded sequence and $(s_k)_{k\geq 0}$ (resp., $(A_k)_{k\geq 0}$) is a complex sequence (resp., is a sequence of bounded linear operators in a Banach space) satisfying some convergence conditions (see Section 2).

To illustrate the applicability of the abstract results described above, we consider a Gribov operator in Bargmann space (see [1], [2], [9] and [10]) originated from Reggeon theory. This theory was introduced by V. N. Gribov [9] in 1967 to study strong interactions, i.e., the interaction between protons and neutrons among other less stable particles. It is governed by a non-selfadjoint Gribov operator constructed as a polynomial in the standard annihilation operator A and the standard creation operator A^*, defined in the Bargmann space:

$$E = \left\{ \varphi : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire such that } \int_{\mathbb{C}} e^{-|z|^2} |\varphi(z)|^2 dz d\bar{z} < \infty \text{ and } \varphi(0) = 0 \right\}.$$

More precisely, we deal with the operator

$$H_{\lambda",\lambda',\mu,\lambda} := \lambda"(A^*A)^3 + \lambda' A^{*2}A^2 + \mu A^*A + i\lambda A^*(A + A^*)A, \tag{1.7}$$

where $\lambda", \lambda', \mu$ and λ are real numbers. The Bargmann space E is a Hilbert space for the scalar product $\langle ., . \rangle$ defined by

$$\begin{cases} \langle ., . \rangle : E \times E \longrightarrow \mathbb{C} \\ \qquad (\varphi, \psi) \longrightarrow \langle \varphi, \psi \rangle = \displaystyle\int_{\mathbb{C}} e^{-|z|^2} \varphi(z) \bar{\psi}(z) \ dz d\bar{z}, \end{cases}$$

and the associated norm is denoted by $\| \, . \, \|$.

The annihilation operator A and the creation operator A^*, are defined by:

$$\begin{cases} A : \mathcal{D}(A) \subset E \longrightarrow E \\ \qquad \varphi \longrightarrow A\varphi(z) = \dfrac{d\varphi}{dz}(z) \\ \mathcal{D}(A) = \{\varphi \in E \text{ such that } A\varphi \in E\} \end{cases}$$

and

$$\begin{cases} A^* : \mathcal{D}(A^*) \subset E \longrightarrow E \\ \qquad \varphi \longrightarrow A^*\varphi(z) = z\varphi(z) \\ \mathcal{D}(A^*) = \{\varphi \in E \text{ such that } A^*\varphi \in E\}. \end{cases}$$

So, the expression of $H_{\lambda'',\lambda',\mu,\lambda}$ becomes

$$H_{\lambda'',\lambda',\mu,\lambda} := \lambda'' z^3 \ \frac{d^3\varphi}{dz^3}(z) + \big((3\lambda'' + \lambda')z^2 + i\lambda z\big) \ \frac{d^2\varphi}{dz^2}(z) + \big(i\lambda z^2 + (\lambda'' + \mu)z\big) \ \frac{d\varphi}{dz}(z).$$

Regarding the aforementioned theoretical part, we give a characterization of the spectrum of $H_{\lambda'',\lambda',\mu,\lambda}$.

The paper is organized as follow: in Section 2 we develop some preliminary results for future use. Section 3 constitutes the main results of the paper. An illustrative application to a Gribov operator in Bargmann space is the topic of Section 4.

2 Preliminaries

The objective of this section is to establish the equalities

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} s_k \right)^n = \sum_{n=1}^{\infty} a_n \sum_{l=n}^{\infty} \sum_{i_1+i_2+\ldots+i_n=l} s_{i_1} s_{i_2} \cdots s_{i_n} \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} A_k \right)^n = \sum_{n=1}^{\infty} a_n \sum_{l=n}^{\infty} \sum_{i_1+i_2+\ldots+i_n=l} A_{i_1} A_{i_2} \cdots A_{i_n},$$

where $(a_k)_{k\geq 1}$ is a complex bounded sequence and $(s_k)_{k\geq 1}$ (resp., $(A_k)_{k\geq 1}$) is a complex sequence (resp., is a sequence of bounded linear operators in a Banach space) satisfying some convergence conditions.

To attain this goal, we shall first introduce the following technical results for future use.

Lemma 2.1 *Let $s = (s_k)_{k\geq 1}$ be a sequence in $\mathbb{C}$ and $s^\star = (s_k^\star)_{k\geq 0}$ be the transformed sequence given by*

$$s_0^\star = 1, \quad s_k^\star = \sum_{\nu=0}^{k-1} s_\nu^\star \; s_{k-\nu}, \;\; \text{for every} \;\; k \geq 1. \tag{2.2}$$

Then,

$$s_k^\star = \sum_{l=1}^{k} \sum_{i_1+i_2+\ldots+i_l=k} s_{i_1} s_{i_2} s_{i_3} \ldots s_{i_l}, \;\; \text{for every} \;\; k \geq 1. \tag{2.3}$$

(Here $1 \leq i_l \leq k$, for every $l = 1, 2, \cdots, k$ and $k \geq 1$) ♢

Proof. Let us proceed by induction on $k \geq 1$. For $k = 1$, the recurrence property is valid, since $s_1^\star = s_0^\star s_1 = s_1 = \sum_{l=1}^{1} \sum_{i_1=1} s_{i_1}$. Suppose that the recurrence property is valid up to the order k ($k \geq 1$) and let's show that it remains valid to the order $k+1$.
From the recurrence hypothesis and the fact that $s_0^\star = 1$, we get

$$\begin{aligned}
s_{k+1}^\star &= \sum_{\nu=0}^{k} s_\nu^\star s_{k+1-\nu} \\
&= s_0^\star s_{k+1} + \sum_{\nu=1}^{k} s_\nu^\star s_{k+1-\nu} \\
&= s_{k+1} + \sum_{\nu=1}^{k} \left(\sum_{l=1}^{\nu} \sum_{i_1+i_2+\ldots+i_l=\nu} s_{i_1} s_{i_2} s_{i_3} \ldots s_{i_l} \right) s_{k+1-\nu} \\
&= s_{k+1} + \sum_{l=1}^{k} \left(\sum_{\nu=l}^{k} \sum_{i_1+i_2+\ldots+i_l=\nu} s_{i_1} s_{i_2} \ldots s_{i_l} \right) s_{k+1-\nu} \\
&= s_{k+1} + \sum_{l=2}^{k+1} \left(\sum_{\nu=l-1}^{k} \sum_{i_1+i_2+\ldots+i_{l-1}=\nu} s_{i_1} s_{i_2} \ldots s_{i_{l-1}} \right) s_{k+1-\nu} \\
&= s_{k+1} + \sum_{l=2}^{k+1} \sum_{i_1+i_2+\ldots+i_l=k+1} s_{i_1} s_{i_2} \ldots s_{i_l} \\
&= \sum_{l=1}^{k+1} \sum_{i_1+i_2+\ldots+i_l=k+1} s_{i_1} s_{i_2} \ldots s_{i_l}.
\end{aligned}$$

This finishes the proof of the lemma. Q.E.D

Remarks 2.1

1. *For any fixed* λ *in* $\mathbb{C}$, *let* $s_k = \lambda$, *for all* $k \geq 1$. *By an easy induction, we can show that* $s_k^\star = \lambda(1+\lambda)^{k-1}$, *for all* $k \geq 1$.

2. *Due to* (2.2), *for any sequences* $s = (s_k)_{k\geq 1}$ *and* $t = (t_k)_{k\geq 1}$ *in* $\mathbb{C}$ *satisfying* $|s_k| \leq |t_k|$, *for all* $k \geq 1$, *we have*
$$|s_k^\star| \leq |t_k|^\star, \quad \text{for every} \ \ k \geq 0. \tag{2.4}$$

$\diamondsuit$

Lemma 2.2 *For any fixed* x *and* y *in* $\mathbb{C}$, *the following properties hold.*

(i) *If we set* $t_k = yx^k$, *for all* $k \geq 1$, *then* $t_k^\star = yx^k(1+y)^{k-1}$, *for all* $k \geq 1$.

(ii) $\displaystyle\sum_{l=1}^{k} y^l \sum_{i_1+i_2+\ldots+i_l=k} 1 = y(1+y)^{k-1}$, *for every integer* $k \geq 1$.

(iii) $\sum_{i_1+i_2+\ldots+i_l=k} 1 = C_{k-1}^{l-1}$, *for every integers* $1 \leq l \leq k$ *and* $k \geq 1$.

(iv) *Let* $s = (s_k)_{k\geq 1}$ *be a sequence in* $\mathbb{C}$. *Suppose there exist* $\lambda > 0$ *and* $0 < q < \frac{1}{\lambda+1}$ *such that* $|s_k| \leq \lambda q^k$, *for every integer* $k \geq 1$, *then*
$$\sum_{k=1}^{\infty} |s_k| \leq \frac{\lambda q}{1-q}, \quad \sum_{k=0}^{\infty} |s_k^\star| \leq \frac{1-q}{1-q(1+\lambda)}.$$

$\diamondsuit$

Proof. (**i**) If $t_k = 0$, for all integer $k \geq 0$, it is clear that $t_k^\star = 0$, for all integer $k \geq 1$. Assume that x and y are non zero complex numbers and let $t_k = yx^k$, for all integer $k \geq 1$. By induction on the integer $k \geq 1$, let us show that $t_k^\star = yx^k(1+y)^{k-1}$ for all integer $k \geq 1$. For $k = 1$, the recurrence property is true since $t_1^\star = t_1 = yx$. Suppose that the recurrence property is valid to the order k ($k \geq 1$), and let's show that it remains valid to the order $k+1$. By the recurrence hypothesis and Eq. (2.2), we obtain

$$\begin{aligned}
t_{k+1}^\star &= \sum_{\nu=0}^{k} t_\nu^\star \, t_{k+1-\nu} \\
&= t_{k+1} + \sum_{\nu=1}^{k} yx^\nu(1+y)^{\nu-1} yx^{k+1-\nu} \\
&= yx^{k+1} + x^{k+1}y^2 \sum_{\nu=1}^{k} (1+y)^{\nu-1} \\
&= yx^{k+1} - x^{k+1}y^2 \frac{1-(1+y)^k}{y} \\
&= yx^{k+1}(1+y)^k.
\end{aligned}$$

(**ii**) Due to (**i**) and Lemma 2.1, we get

$$y(1+y)^{k-1} = \sum_{l=1}^{k} y^l \sum_{i_1+i_2+\ldots+i_l=k} 1, \quad \text{for every integer } k \geq 1 \quad \text{and every } y \ \text{ in } \mathbb{C}.$$

(**iii**) From (**ii**) and the binomial expansion formula, we have

$$\sum_{l=1}^{k} y^l \sum_{i_1+i_2+\ldots+i_l=k} 1 = \sum_{l=0}^{k-1} C_{k-1}^{l} y^{l+1} = \sum_{l=1}^{k} C_{k-1}^{l-1} y^l.$$

By identification, it follows that

$$\textstyle\sum_{i_1+i_2+\ldots+i_l=k} 1 = C_{k-1}^{l-1}, \quad \text{for every integers } 1 \le l \le k \;\text{ and } k \ge 1.$$

(**iv**) Let $(s_k)_{k\ge 1}$ be a sequence in $\mathbb{C}$ satisfying $|s_k| \le \lambda q^k$, for every integer $k \ge 1$, where $\lambda > 0$ and $0 < q < \frac{1}{1+\lambda}$. Clearly, $\sum_{k=1}^{\infty} |s_k| \le \lambda \sum_{k=1}^{\infty} q^k = \frac{\lambda q}{1-q} < +\infty$. Besides, $|s_k| \le |t_k|$, for every integer $k \ge 1$, where $t_k = \lambda q^k$, for every integer $k \ge 1$. By (i) and Remark 2.1, (Eq. (2.4)), it follows that $|s_k^\star| \le |t_k|^\star = \lambda q^k (1+\lambda)^{k-1}$, for every integer $k \ge 1$.

Thus,

$$\begin{aligned}
\sum_{k=0}^{\infty} |s_k^\star| &\le 1 + \sum_{k=1}^{\infty} |t_k|^\star \\
&\le 1 + \frac{\lambda}{1+\lambda} \sum_{k=1}^{\infty} (q(1+\lambda))^k \\
&\le 1 + \frac{\lambda q}{1 - q(1+\lambda)} \\
&\le \frac{1-q}{1-q(1+\lambda)}.
\end{aligned}$$

Q.E.D

Now, let's recall the Fubini's absolute convergence inversion criterion.

Lemma 2.3 *Let $(U_{n,l})_{n,l\ge 1}$ be a double sequence in $\mathbb{C}$ satisfying:*

(**i**) *For every fixed integer $n \ge 1$, the series $\sum_{l\ge 1} |U_{n,l}|$ converge.*

(**ii**) *The series $\sum_{n\ge 1} \sum_{l \ge 1} |U_{n,l}|$ converge.*

Then, we can interchange the order of summation in a doubly indexed infinite series:

$$\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} U_{n,l} = \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} U_{n,l}. \qquad \diamondsuit$$

Having obtained these results, we are ready to prove Eq. (2.1) when $(a_k)_k \equiv 1$.

Proposition 2.1 *For any sequence $s = (s_k)_{k\ge 1}$ in $\mathbb{C}$ such that $\tau(s) = \sum_{k=1}^{\infty} |s_k| < 1$ and $\sum_{k=0}^{\infty} |s_k^\star| < +\infty$, where $s^\star = (s_k^\star)_{k \ge 0}$ is given by (2.2), the following properties hold.*

(**i**) $\displaystyle\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} s_k \right)^n = \sum_{n=1}^{\infty} \sum_{l=1}^{n} \sum_{i_1+i_2+\ldots+i_l=n} s_{i_1} s_{i_2} \cdots s_{i_l}.$

(ii) *If* $\sum_{k=0}^{\infty} |s_k|^\star < +\infty$, *then we have*

(a) $$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} s_k\right)^n = \sum_{l=1}^{\infty}\sum_{k=l}^{\infty}\sum_{i_1+i_2+\ldots+i_l=k} s_{i_1}s_{i_2}\cdots s_{i_l}.$$

(b) $$\left(\sum_{k=1}^{\infty} s_k\right)^l = \sum_{k=l}^{\infty}\sum_{i_1+i_2+\ldots+i_l=k} s_{i_1}s_{i_2}\cdots s_{i_l},$$ *for every integer* $l \geq 1$.

(c) $$\left(\sum_{k=1}^{\infty} s_k z^k\right)^l = \sum_{k=l}^{\infty}\left(\sum_{i_1+i_2+\ldots+i_l=k} s_{i_1}s_{i_2}\cdots s_{i_l}\right) z^k,$$ *for* z *in* $\mathbb{C}$ *such that* $|z| < 1$ *and every integer* $l \geq 1$. $\diamondsuit$

Proof. (**i**) Setting

$$\mathcal{S} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} s_k\right)^n - \sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{i_1+i_2+\ldots+i_l=n} s_{i_1}s_{i_2}\cdots s_{i_l}.$$

By Eq. (2.3) and the fact that $s_0^\star = 1$, we can write

$$\mathcal{S} = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty} s_k\right)^n - \sum_{n=0}^{\infty} s_n^\star.$$

Putting $s_0 = 0$ and applying the Cauchy product formula of two absolutely convergent series, we obtain after taking (2.1) into account,

$$\begin{aligned}
\mathcal{S}\left(\sum_{k=1}^{\infty} s_k\right) &= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty} s_k\right)^{n+1} - \sum_{n=0}^{\infty} s_n^\star \sum_{k=0}^{\infty} s_k \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty} s_k\right)^{n} - 1 - \sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} s_\nu^\star\, s_{n-\nu}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty} s_k\right)^{n} - 1 - \sum_{n=1}^{\infty}\left(\sum_{\nu=0}^{n-1} s_\nu^\star\, s_{n-\nu}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty} s_k\right)^{n} - 1 - \sum_{n=1}^{\infty} s_n^\star \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty} s_k\right)^{n} - \sum_{n=0}^{\infty} s_n^\star = \mathcal{S}.
\end{aligned}$$

Hence, $\mathcal{S}\left(\sum_{k=1}^{\infty} s_k - 1\right) = 0$. Since $\left|\sum_{k=1}^{\infty} s_k\right| \leq \sum_{k=1}^{\infty} |s_k| < 1$, it comes that $\mathcal{S} = 0$. Accordingly, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} s_k\right)^n = \sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{i_1+i_2+\ldots+i_l=n} s_{i_1}s_{i_2}\cdots s_{i_l}.$$

(**ii**) Assume that $\sum_{k=0}^{\infty} |s_k|^\star < +\infty$. Due to (**i**), we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}s_k\right)^n=\sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{i_1+i_2+...+i_l=n}s_{i_1}s_{i_2}\cdots s_{i_l}=\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}U_{n,l},$$

where $U_{n,l}=\chi_{[|1,n|]}(l)\sum_{i_1+i_2+...+i_l=n}s_{i_1}s_{i_2}\cdots s_{i_l}$, for every $n,l\geq 1$ and $\chi_{[|1,n|]}$ is the characteristic function of $[|\ 1,n\ |]=\{1,2,...,n\}$.

For any fixed integer $n\geq 1$, we have

$$\sum_{l=1}^{\infty}|U_{n,l}|=\sum_{l=1}^{n}\left|\sum_{i_1+i_2+...+i_l=n}s_{i_1}s_{i_2}\cdots s_{i_l}\right|<+\infty.$$

Besides,

$$\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}|U_{n,l}|\leq\sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{i_1+i_2+...+i_l=n}|s_{i_1}||s_{i_2}|\cdots|s_{i_l}|\leq\sum_{n=1}^{\infty}|s_n|^{\star}<+\infty.$$

Using Lemma 2.3, it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}s_k\right)^n &= \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}U_{n,l}\\ &= \sum_{l=1}^{\infty}\sum_{n=1}^{\infty}U_{n,l}\\ &= \sum_{l=1}^{\infty}\sum_{n=l}^{\infty}\sum_{i_1+i_2+...+i_l=n}s_{i_1}s_{i_2}\cdots s_{i_l}.\end{aligned}$$

Hence, (**a**) holds.

Let z be a fixed complex number such that $|\ z\ |<1$. Setting $v_k=zs_k$, for all $k\geq 1$. Clearly $|v_k|\leq|s_k|$ for every $k\geq 1$. From Eq. (2.3), we have $|v_k|^{\star}\leq|s_k|^{\star},\ \forall\ k\geq 1$.

By the assumption, it comes that

$$\sum_{k=1}^{\infty}|v_k|\leq\sum_{k=1}^{\infty}|s_k|<1\qquad\text{and}\qquad\sum_{k=0}^{\infty}|v_k|^{\star}\leq\sum_{k=0}^{\infty}|s_k|^{\star}<+\infty.$$

So, while using (**a**), we obtain

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}v_k\right)^n=\sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{i_1+i_2+...+i_l=n}v_{i_1}v_{i_2}\cdots v_{i_l}=\sum_{l=1}^{\infty}\sum_{k=l}^{\infty}\sum_{i_1+i_2+...+i_l=k}v_{i_1}v_{i_2}\cdots v_{i_l}$$

i.e.,

$$\sum_{l=1}^{\infty}z^l\left(\sum_{k=1}^{\infty}s_k\right)^l=\sum_{l=1}^{\infty}z^l\left(\sum_{k=l}^{\infty}\sum_{i_1+i_2+...+i_l=k}s_{i_1}s_{i_2}\cdots s_{i_l}\right),$$

for every z in $\mathbb{C}$ with $|\ z\ |<1$.

By identification, we infer that

$$\left(\sum_{k=1}^{\infty} s_k\right)^l = \sum_{k=l}^{\infty} \sum_{i_1+i_2+\ldots+i_l=k} s_{i_1} s_{i_2} \cdots s_{i_l}, \quad \text{for every integer } \ l \geq 1.$$

Hence, (**b**) holds.

For a fixed z in $\mathbb{C}$, with $|z| < 1$, let $(w_k)_{k\geq 1}$ be the sequence in $\mathbf{C}$, given by $w_k = z^k s_k$, for every $k \geq 1$. Clearly, $|w_k| \leq |s_k|$ for all $k \geq 1$. By Eq. (2.3), we get $|w_k|^\star \leq |s_k|^\star, \ \forall\, k \geq 1$.

By the assumption, it comes that

$$\sum_{k=1}^{\infty} |w_k| \leq \sum_{k=1}^{\infty} |s_k| < 1 \qquad \text{and} \qquad \sum_{k=0}^{\infty} |w_k|^\star \leq \sum_{k=0}^{\infty} |s_k|^\star < +\infty.$$

Due to (**b**), it follows that

$$\left(\sum_{k=1}^{\infty} s_k z^k\right)^l = \sum_{k=l}^{\infty} \left(\sum_{i_1+i_2+\ldots+i_l=k} s_{i_1} s_{i_2} \cdots s_{i_l} \right) z^k,$$

for every $z \in \mathbb{C}$ with $|z| < 1$ and every integer $l \geq 1$.
Hence (**c**) holds. Q.E.D

More general, we have the following result.

Theorem 2.1 *Let $a = (a_k)_{k\geq 1}$ and $s = (s_k)_{k\geq 1}$ be two complex sequences such that $a = (a_k)_{k\geq 1}$ is bounded, $\tau(s) = \sum_{k=1}^{\infty} |s_k| < 1$ and $\sum_{k=0}^{\infty} |s_k|^\star < +\infty$, where $|s|^\star = (|s_k|^\star)_{k\geq 0}$ is defined as:*

$$\begin{cases} |s_0|^\star = 1, \\ |s_k|^\star = \displaystyle\sum_{l=1}^{k} \sum_{i_1+i_2+\ldots+i_l=k} |s_{i_1}||s_{i_2}| \cdots |s_{i_l}|, \quad k \geq 1. \end{cases}$$

Then, we have:

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} s_k\right)^n &= \sum_{n=1}^{\infty} a_n \sum_{l=n}^{\infty} \sum_{i_1+i_2+\ldots+i_n=l} s_{i_1} s_{i_2} \cdots s_{i_n} \\ &= \sum_{l=1}^{\infty} \sum_{n=1}^{l} a_n \sum_{i_1+i_2+\ldots+i_n=l} s_{i_1} s_{i_2} \cdots s_{i_n}. \end{aligned}$$

$\diamondsuit$

Proof. The first equality can be deduced from Proposition 2.1 (**b**). To prove the second equality, we write $\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} s_k\right)^n$ as

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} s_k\right)^n = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} V_{n,l},$$

with

$$V_{n,l} = a_n \ \chi_{[|n,+\infty|[}(l) \sum_{i_1+i_2+\ldots+i_n=l} s_{i_1} s_{i_2} \cdots s_{i_n},$$

for every integers $n, l \geq 1$ and where $\chi_{[|n,+\infty|[}$ is the characteristic function of $[| \ n, +\infty \ | \ [$. From the assumption and Proposition 2.1 (**b**), we have

$$\sum_{l=1}^{\infty} |V_{n,l}| \leq |a_n| \sum_{l=n}^{\infty} \sum_{i_1+i_2+\ldots+i_n=l} |s_{i_1}||s_{i_2}| \cdots |s_{i_n}| = |a_n| \left(\tau(s)\right)^n < +\infty,$$

for every integer $n \geq 1$.
Besides,

$$\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} |V_{n,l}| \leq \sum_{n=1}^{\infty} |a_n| \left(\tau(s)\right)^n.$$

Taking nto account the fact that $a = (a_n)_{n \geq 1}$ is a bounded sequence, i.e., there exists $M > 0$ such that $|a_n| \leq M$, for every $n \geq 1$ and since $\tau(s) < 1$, we obtain

$$\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} |V_{n,l}| \leq M \sum_{n=1}^{\infty} (\tau(s))^n = \frac{M\tau(s)}{1 - \tau(s)} < +\infty.$$

By virtue of Lemma 2.3, this implies

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} s_k \right)^n &= \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} V_{n,l} \\ &= \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} V_{n,l} \\ &= \sum_{l=1}^{\infty} \sum_{n=1}^{l} a_n \sum_{i_1+i_2+\ldots+i_n=l} s_{i_1} s_{i_2} \cdots s_{i_l}. \end{aligned}$$

Hence, the second equality holds. Q.E.D

Lemma 2.1, Proposition 2.1 and Theorem 2.1 remain true if we replace the complex sequence $(s_k)_{k \geq 1}$ by a sequence of bounded linear operators $(A_k)_{k \geq 1}$ on a Banach space X. To this end, let $(A_k^\star)_{k \geq 0}$ denote the transformed sequence defined as:

$$A_0^\star = I, \quad A_k^\star = \sum_{\nu=0}^{k-1} A_\nu^\star \ A_{k-\nu}, \ \text{ for every } \ k \geq 1 \tag{2.5}$$

and $(\|A_k\|^\star)_{k \geq 0}$ the sequence given by:

$$\|A_0\|^\star = 1, \quad \|A_k\|^\star = \sum_{\nu=0}^{k-1} \|A_\nu\|^\star \ \|A_{k-\nu}\|, \ \text{ for every } \ k \geq 1. \tag{2.6}$$

Then, we can deduce the following results:

Lemma 2.4 *Let $(A_k)_{k \geq 1}$ be a sequence of bounded linear operators on a Banach space X and $(A_k^\star)_{k \geq 0}$ the transformed sequence defined in Eq. (2.5). Then, we have*

$$A_k^\star = \sum_{l=1}^{k} \sum_{i_1+i_2+\ldots+i_l=k} s_{i_1} s_{i_2} s_{i_3} \ldots s_{i_l}, \ \text{ for every } \ k \geq 1.$$

(Here $1 \leq i_l \leq k, \quad \text{for every} \quad l = 1, 2, \cdots, k \quad \text{and} \quad k \geq 1$) $\diamondsuit$

Proof. The proof is similar to that of Lemma 2.1. Q.E.D

Proposition 2.2 *Let $(A_k)_{k\geq 1}$ be a sequence of bounded linear operators on a Banach space X and $(A_k^\star)_{k\geq 0}$ (resp., $(\|A_k\|^\star)_{k\geq 0}$) the transformed sequence defined in Eq. (2.5) (resp., Eq. (2.6)). If $\sum_{k=1}^{\infty} \|A_k\| < 1$ and $\sum_{k=0}^{\infty} \|A_k^\star\| < +\infty$, then the following assertions hold.*

(i) $$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} A_k \right)^n = \sum_{n=1}^{\infty} \sum_{l=1}^{n} \sum_{i_1+i_2+\ldots+i_l=n} A_{i_1} A_{i_2} \cdots A_{i_l}.$$

(ii) *If $\sum_{k=0}^{\infty} \|A_k\|^\star < +\infty$, then we have*

(a) $$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} A_k \right)^n = \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} \sum_{i_1+i_2+\ldots+i_l=k} A_{i_1} A_{i_2} \cdots A_{i_l}.$$

(b) $$\left(\sum_{k=1}^{\infty} A_k \right)^l = \sum_{k=l}^{\infty} \sum_{i_1+i_2+\ldots+i_l=k} A_{i_1} A_{i_2} \cdots A_{i_l},$$ *for every integer $l \geq 1$.*

(c) $$\left(\sum_{k=1}^{\infty} A_k z^k \right)^l = \sum_{k=l}^{\infty} \left(\sum_{i_1+i_2+\ldots+i_l=k} A_{i_1} A_{i_2} \cdots A_{i_l} \right) z^k,$$ *for z in $\mathbb{C}$ such that $|z| < 1$ and every integer $l \geq 1$.* $\diamondsuit$

Proof. (**i**) The proof can be sketched in a similar way to that in Proposition 2.1 (**i**) since the Cauchy product formula of two absolutely convergent series remains true for series with terms in a Banach algebra.

(**ii**) To prove (**a**), it suffices to apply the Fubini's absolute convergence inversion criterion for double sequences of a Banach space (see [20, *Théorème* 12]) to the sequence of operators $(A_{n,l})_{n,l\geq 1}$, where $A_{n,l} = \chi_{[|1,n|]}(l) \sum_{i_1+i_2+\ldots+i_l=n} A_{i_1} A_{i_2} \cdots A_{i_l}$, for every $n, l \geq 1$ and $\chi_{[|1,n|]}$ denotes the characteristic function of $[|\, 1, n \,|] = \{1, 2, \ldots, n\}$.
To show (**b**), we apply (**a**) to the sequence of operators $(B_k)_{k\geq 1}$, where $B_k = zA_k$, $\forall k \geq 1$, with z is a complex number such that $|z| < 1$.
Finally, the equality of (**c**) can be deduced from the one of (**b**) by considering the sequence of operators $(C_k)_{k\geq 1}$, with $C_k = z^k A_k$, $\forall k \geq 1$, and z is a complex number such that $|z| < 1$.

We close this section by the following result.

Theorem 2.2 *Let $(a_k)_{k\geq 1}$ be a bounded complex sequence and $(A_k)_{k\geq 1}$ a sequence of bounded linear operators on a Banach space X such that $\sum_{k=1}^{\infty} \|A_k\| < 1$ and $\sum_{k=0}^{\infty} \|A_k\|^\star < +\infty$, where $(\|A_k\|^\star)_{k\geq 0}$ is defined in Eq. (2.6)).*

Then, we have:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} A_k\right)^n &= \sum_{n=1}^{\infty} a_n \sum_{l=n}^{\infty} \sum_{i_1+i_2+\ldots+i_n=l} A_{i_1}A_{i_2}\cdots A_{i_n} \\ &= \sum_{l=1}^{\infty}\sum_{n=1}^{l} a_n \sum_{i_1+i_2+\ldots+i_n=l} A_{i_1}A_{i_2}\cdots A_{i_n}.\end{aligned}$$

$\diamondsuit$

Proof. The first equality follows immediately from Proposition 2.2. To prove the second equality, it suffices to apply the Fubini's absolute convergence inversion criterion for double sequences of a Banach space (see [20, *Théorème* 12]) to the sequence of operators $(B_{n,l})_{n,l\geq 1}$, where $B_{n,l} = a_n\ \chi_{[|n,+\infty|[}(l) \sum_{i_1+i_2+\ldots+i_n=l} A_{i_1}A_{i_2}\cdots A_{i_n}$, for every integers $n, l \geq 1$ and $\chi_{[|n,+\infty|[}$ is the characteristic function of $[|\ n, +\infty\ |\ [$. Q.E.D

3 Main results

Throughout this section, we will consider the following hypotheses:

$(H1)$ Let T_0 be a linear operator on a Banach space X with domain $\mathcal{D}(T_0)$ such that T_0 is closed and has isolated discrete eigenvalues.

$(H2)$ Let $T_1, T_2, T_3, \ldots$ be some linear operators on X having the same domain $\mathcal{D}$ and satisfying:

$$\mathcal{D}(T_0) \subset \mathcal{D} \text{ and there exist } a, b > 0 \text{ such that for every } k \geq 1$$
$$\|T_k\varphi\| \leq a\|\varphi\| + b\|T_0\varphi\|, \quad \text{for all } \varphi \in \mathcal{D}(T_0). \tag{3.1}$$

$(H3)$ Consider $(\xi_k)_{k\geq 0}$ a sequence of complex numbers verifying $\xi_0 = 1$ and $\tau(\xi) = \Sigma_{k=1}^{\infty}|\xi_k| < \infty$.

The first result of this section is formulated in the following theorem.

Theorem 3.1 *Assume that the assumptions $(H1)$- $(H3)$ hold. Then the series $\sum_{k\geq 0}\xi_k T_k\varphi$ converges for all $\varphi \in \mathcal{D}(T_0)$. If $T(\xi)\varphi$ denotes its limit, then we define a linear operator $T(\xi)$ with domain $\mathcal{D}(T_0)$. In addition, if $\tau(\xi) < \frac{1}{b}$ then $T(\xi)$ is closed.* $\diamondsuit$

Proof. Let $\varphi \in \mathcal{D}(T_0)$ and $n \in \mathbb{N}^\star$. Using Eq. (3.1) we get

$$\begin{aligned}\left\|\sum_{k=0}^{n}\xi_k T_k\varphi\right\| &\leq \sum_{k=0}^{n}|\xi_k|\|T_k\varphi\| \\ &\leq \|T_0\varphi\| + \sum_{k=1}^{n}|\xi_k|\|T_k\varphi\| \\ &\leq \|T_0\varphi\| + \sum_{k=1}^{n}|\xi_k|\left(a\|\varphi\| + b\|T_0\varphi\|\right) \\ &\leq \tau(\xi)a\|\varphi\| + (1+\tau(\xi)b)\,\|T_0\varphi\|.\end{aligned}$$

Hence, the series $\sum_{k\geq 0} \xi_k T_k \varphi$ is convergent. Setting $T(\xi)\varphi$ its limit, we define a linear operator $T(\xi)$ with domain $\mathcal{D}(T_0)$. Similarly, by Eq. (3.1), we obtain

$$\|(T(\xi) - T_0)\varphi\| \leq \tau(\xi)a\|\varphi\| + \tau(\xi)b\|T_0\varphi\|, \quad \text{for all} \;\; \varphi \in \mathcal{D}(T_0).$$

Since T_0 is a closed operator and $\tau(\xi) < \frac{1}{b}$, we deduce in view of [15, Théorème 1] that the operator $T(\xi)$ is also closed. Q.E.D

In particular, if we take $\xi_k = \varepsilon^k q^{k-1}$, $T_0 = A_0$ and $T_k = \frac{1}{q^{k-1}}\, A_k \quad \forall k \geq 1$, where $q > 0$, ε is a complex number and $(A_k)_{k\geq 1}$ are linear operators on X verifying Eq. (1.2), we regain the analytic perturbation $A(\varepsilon)$ (see Eq. (1.1)) considered by B. Sz. Nagy in [15]. More precisely, we have the following result.

Corollary 3.1 *Let $A_0, A_1, A_2, A_3, \ldots$ be linear operators on X such that A_0 is closed with domain $\mathcal{D}(A_0)$ and $A_1, A_2, A_3, \ldots$ are with the same domain $\mathcal{D} \supset \mathcal{D}(A_0)$ and verifying:*

$$\|A_k\varphi\| \leq q^{k-1}\left(a\|\varphi\| + b\|A_0\varphi\|\right), \quad \textit{for all} \;\; \varphi \in \mathcal{D}(A_0), \tag{3.2}$$

where a, b and q are strictly positive numbers. Then the series $\sum_{k\geq 0} \varepsilon^k A_k \varphi$ converges for all $\varphi \in \mathcal{D}(A_0)$ and for $|\varepsilon| < \frac{1}{q}$. If $A(\varepsilon)\varphi$ denotes its limit, then we define a linear operator $A(\varepsilon)$ with domain $\mathcal{D}(A_0)$. For $|\varepsilon| < \frac{1}{q+b}$, the operator $A(\varepsilon)$ is closed. $\diamondsuit$

Proof. We have $\tau(\xi) = \sum_{k=1}^{\infty} |\xi_k| = |\varepsilon| \sum_{k=0}^{\infty} (|\varepsilon| q)^k$. If $|\varepsilon| < \frac{1}{q}$, then $\tau(\xi) = \frac{|\varepsilon|}{1-|\varepsilon|q} < +\infty$. Hence, the series $\sum_{k\geq 0} \varepsilon^k A_k \varphi$ converges for all $|\varepsilon| < \frac{1}{q}$. Moreover, $\tau(\xi) < \frac{1}{b}$ if and only if $|\varepsilon| < \frac{1}{q+b}$. So, $A(\varepsilon)$ is closed for $|\varepsilon| < \frac{1}{q+b}$. Q.E.D

Remark 3.1 *Notice here that Corollary 3.1 was first cited in [15, Théorème 3]. Then, we can say that the perturbation $T(\xi)$ is more general than the one adressed in [14] − [15] where B. Sz. Nagy dealt with an analytic operator with one perturbation parameter ε.* $\diamondsuit$

Let $n \in \mathbb{N}^*$, λ_n the isolated eigenvalue number n of the operator T_0 with multiplicity one, $d_n = d(\lambda_n, \sigma(T_0) \setminus \{\lambda_n\})$: the distance between λ_n and $\sigma(T_0) \setminus \{\lambda_n\}$ and $\mathcal{C}_n = \mathcal{C}(\lambda_n, r_n)$: the closed circle with center λ_n and radii $r_n = \frac{d_n}{2}$. Since $(T_0 - zI)^{-1}$ is a regular analytic function of $z \in \rho(T_0)$, $\|(T_0 - zI)^{-1}\|$ is a continuous function. So, we denote by:

$$M_n := \max_{z_n \in \mathcal{C}_n} \|(T_0 - z_n I)^{-1}\|,$$

$$N_n := \max_{z_n \in \mathcal{C}_n} \|T_0(T_0 - z_n I)^{-1}\| = \max_{z_n \in \mathcal{C}_n} \|I + z_n(T_0 - z_n I)^{-1}\|$$

and

$$\alpha_n := aM_n + bN_n.$$

Proposition 3.1 *Assume that the assumptions $(H1)$-$(H3)$ hold. If $\tau(\xi) < \frac{1}{\alpha_n}$, then the resolvent of $T(\xi)$ at $z_n \in \mathcal{C}_n$ is well defined. Let $R_{z_n}(\xi) := (T(\xi) - z_n I)^{-1}$, then for $\tau(\xi) < \frac{1}{\alpha_n}$, the following assertions hold:*

(i) the operator $R_{z_n}(\xi)$ can be developed into a series

$$R_{z_n}(\xi) := R_{z_n,0}(\xi) + R_{z_n,1}(\xi) + R_{z_n,2}(\xi) + \cdots + R_{z_n,l}(\xi) + \cdots ,$$

where

$$\begin{array}{rcl} R_{z_n,0}(\xi) & := & R_{z_n} := (T_0 - z_n I)^{-1} \\ R_{z_n,l}(\xi) & := & (-1)^l \displaystyle\sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} \xi_{i_1}\xi_{i_2}\cdots\xi_{i_l}\ R_{z_n}\ T_{i_1}R_{z_n}\ T_{i_2}R_{z_n}\cdots T_{i_l}R_{z_n}, \quad \forall\ l \geq 1 \end{array} \tag{3.3}$$

(ii) we have

$$\|R_{z_n,l}(\xi)\| \leq M_n(\alpha_n \tau(\xi))^l, \quad \textit{for every} \ \ l \geq 0. \qquad \diamondsuit$$

Proof. (i) Let $z_n \in \mathcal{C}_n$. Using Eq. (3.1), we infer that for all $g \in X \setminus \{0\}$ and for all $k \geq 1$,

$$\begin{aligned} \|T_k(T_0 - z_n I)^{-1} g\| &\leq a\|(T_0 - z_n I)^{-1} g\| + b\|T_0(T_0 - z_n I)^{-1} g\| \\ &\leq \left(a\|(T_0 - z_n I)^{-1}\| + b\|T_0(T_0 - z_n I)^{-1}\|\right)\|g\| \ . \end{aligned}$$

So,

$$\|T_k(T_0 - z_n I)^{-1}\| \leq \alpha_n, \quad \text{for all} \ \ k \geq 1. \tag{3.4}$$

We claim that $\mathcal{C}_n \subset \rho(T(\xi))$. Indeed, for $z_n \in \mathcal{C}_n$ we have

$$\begin{array}{rcl} T(\xi) - z_n I & = & T_0 + \xi_1 T_1 + \xi_2 T_2 + \ldots - z_n I \\ & = & \left(I + \xi_1 T_1 (T_0 - z_n I)^{-1} + \xi_2 T_2 (T_0 - z_n I)^{-1} + \ldots\right)(T_0 - z_n I) \\ & = & \left(I + \displaystyle\sum_{k=1}^{\infty} \xi_k T_k R_{z_n}\right)(T_0 - z_n I) \\ & = & (I + S)\,(T_0 - z_n I), \end{array} \tag{3.5}$$

where $S = \sum_{k=1}^{\infty} \xi_k T_k R_{z_n}$. In view of Eq. (3.4), we have if $\tau(\xi) < \frac{1}{\alpha_n}$ then

$$\|S\| \leq \textstyle\sum_{k=1}^{\infty} \|\xi_k T_k R_{z_n}\| \leq \alpha_n \tau(\xi) < 1.$$

Hence, $I + S$ is invertible with bounded inverse. Since $z_n \in \mathcal{C}_n \subset \rho(T_0)$, then Eq. (3.5) implies that $T(\xi) - z_n I$ is also invertible with bounded inverse. Hence $z_n \in \rho\left(T(\xi)\right)$, which ends the proof of the claim.

Now Eq. (3.5) yields

$$\begin{aligned} R_{z_n}(\xi) &= R_{z_n}(I+S)^{-1} \\ &= R_{z_n} \sum_{\nu=0}^{\infty} (-S)^{\nu} \\ &= R_{z_n} \sum_{\nu=0}^{\infty} \left(\sum_{k=1}^{\infty} -\xi_k T_k R_{z_n} \right)^{\nu} \\ &= R_{z_n} + R_{z_n} \sum_{\nu=1}^{\infty} \left(\sum_{k=1}^{\infty} -\xi_k T_k R_{z_n} \right)^{\nu}. \end{aligned}$$

Since $\sum_{k=1}^{\infty} \| -\xi_k T_k R_{z_n} \| < 1$, then by Proposition 2.2 (**ii**) (**a**) we get

$$\begin{aligned} R_{z_n}(\xi) &= R_{z_n} + R_{z_n} \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} (-\xi_{i_1} T_{i_1} R_{z_n})\ (-\xi_{i_2} T_{i_2} R_{z_n}) \cdots (-\xi_{i_l} T_{i_l} R_{z_n}) \\ &= R_{z_n} + R_{z_n} \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} (-1)^l\ \xi_{i_1} \xi_{i_2} \cdots \xi_{i_l}\ T_{i_1} R_{z_n}\ T_{i_2} R_{z_n}\ \cdots T_{i_l} R_{z_n} \\ &= \sum_{l=0}^{\infty} R_{z_n,l}(\xi), \end{aligned}$$

where

$$\begin{cases} R_{z_n,0}(\xi) := R_{z_n} := (T_0 - z_n I)^{-1} \\ R_{z_n,l}(\xi) := (-1)^l \displaystyle\sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_l}\ R_{z_n}\ T_{i_1} R_{z_n}\ T_{i_2} R_{z_n}\ \cdots T_{i_l} R_{z_n}, \quad \forall\ l \geq 1. \end{cases}$$

(*ii*) By the fact that $\|R_{z_n}\| \leq M_n$ and $\|T_k R_{z_n}\| \leq \alpha_n$ for every $k \geq 1$ (see Eq. (3.4)) and using Proposition 2.1 (**ii**) (**b**), the following estimations hold

$$\begin{aligned} \|R_{z_n,l}(\xi)\| &\leq M_n \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} \alpha_n^l |\xi_{i_1}|\ |\xi_{i_2}| \cdots |\xi_{i_l}| \\ &\leq M_n \alpha_n^l \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} |\xi_{i_1}|\ |\xi_{i_2}| \cdots |\xi_{i_l}| \\ &\leq M_n \alpha_n^l (\tau(\xi))^l, \end{aligned}$$

for every $l \geq 1$. This implies that

$$\|R_{z_n,l}(\xi)\| \leq M_n (\alpha_n \tau(\xi))^l, \quad \text{for every} \quad l \geq 0.$$ Q.E.D

Consider the operator $A(\varepsilon) := A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \cdots + \varepsilon^k A_k + \cdots$ defined in Corollary 3.1. Analogously, let $\tilde{\lambda_n}$ be the isolated eigenvalue number n of the operator A_0 with multiplicity one, $\tilde{d_n} = d(\tilde{\lambda_n}, \sigma(A_0) \setminus \{\tilde{\lambda_n})$: the distance between $\tilde{\lambda_n}$ and $\sigma(A_0) \setminus \{\tilde{\lambda_n}\}$

and $\tilde{\mathcal{C}_n} = \mathcal{C}(\tilde{\lambda_n}, \tilde{r_n})$: the closed circle with center $\tilde{\lambda_n}$ and radii $\tilde{r_n} = \frac{\tilde{d_n}}{2}$. Denoting by $\tilde{M_n}$, $\tilde{N_n}$ and $\tilde{\alpha_n}$ the following numbers

$$\begin{cases} \tilde{M_n} := \max\limits_{z_n \in \tilde{\mathcal{C}_n}} \|(A_0 - z_n I)^{-1}\|, \\ \tilde{N_n} := \max\limits_{z_n \in \tilde{\mathcal{C}_n}} \|A_0 (A_0 - z_n I)^{-1}\| = \max\limits_{z_n \in \tilde{\mathcal{C}_n}} \|I + z_n (A_0 - z_n I)^{-1}\|, \\ \tilde{\alpha_n} := a\tilde{M_n} + b\tilde{N_n}, \end{cases}$$

we can see the following result.

Corollary 3.2 *Suppose that the assumptions of Corollary 3.1 hold. Let $\tilde{R}_{z_n}(\varepsilon) := (A(\varepsilon) - z_n I)^{-1}$ denote the resolvent of $A(\varepsilon)$ at $z_n \in \tilde{\mathcal{C}_n}$. If $|\varepsilon| < \frac{1}{q + \tilde{\alpha_n}}$, then:*

(i) the operator $\tilde{R}_{z_n}(\varepsilon)$ can be developed into an entire series

$$\tilde{R}_{z_n}(\varepsilon) := \tilde{R}_{z_n,0} + \varepsilon \tilde{R}_{z_n,1} + \varepsilon^2 \tilde{R}_{z_n,2} + \cdots + \varepsilon^l \tilde{R}_{z_n,l} + \cdots ,$$

where

$$\begin{aligned} \tilde{R}_{z_n,0} &:= \tilde{R}_{z_n} := (A_0 - z_n I)^{-1} \\ \tilde{R}_{z_n,l} &:= \sum_{l=l}^{k} (-1)^l \sum_{i_1+i_2+\cdots+i_l=k} \tilde{R}_{z_n}\ A_{i_1} \tilde{R}_{z_n}\ A_{i_2} \tilde{R}_{z_n} \cdots A_{i_l} \tilde{R}_{z_n}, \quad \forall\ l \geq 1, \end{aligned} \tag{3.6}$$

(ii) we have

$$\|\tilde{R}_{z_n,l}\| \leq \tilde{M_n} \tilde{\alpha_n} (q + \tilde{\alpha_n})^{l-1}, \quad \text{for every} \ \ l \geq 1. \qquad \diamondsuit$$

Proof. (i) Let $\xi_k = \varepsilon^k q^{k-1}$, $T_0 = A_0$ and $T_k = \frac{1}{q^{k-1}}\ A_k$, $\forall k \geq 1$. In view of Proposition 3.1, we obtain

$$\begin{aligned} &\tilde{R}_{z_n}(\xi) \\ &= \tilde{R}_{z_n} + \sum_{l=1}^{\infty} (-1)^l \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_l}\ \tilde{R}_{z_n}\ T_{i_1} \tilde{R}_{z_n}\ T_{i_2} \tilde{R}_{z_n} \cdots T_{i_l} \tilde{R}_{z_n} \\ &= \tilde{R}_{z_n} + \sum_{l=1}^{\infty} (-1)^l \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} \varepsilon^{i_1} q^{i_1-1}\ \varepsilon^{i_2} q^{i_2-1} \cdots \varepsilon^{i_l} q^{i_l-1}\ \tilde{R}_{z_n} \frac{A_{i_1} \tilde{R}_{z_n}}{q^{i_1-1}} \frac{A_{i_2} \tilde{R}_{z_n}}{q^{i_2-1}} \cdots \frac{A_{i_l} \tilde{R}_{z_n}}{q^{i_l-1}} \\ &= \tilde{R}_{z_n} + \sum_{l=1}^{\infty} (-1)^l \sum_{k=l}^{\infty} \varepsilon^k \sum_{i_1+i_2+\cdots+i_l=k} \tilde{R}_{z_n}\ A_{i_1} \tilde{R}_{z_n}\ A_{i_2} \tilde{R}_{z_n} \cdots A_{i_l} \tilde{R}_{z_n} \\ &= \tilde{R}_{z_n} + \sum_{k=1}^{\infty} \varepsilon^k \sum_{l=1}^{k} (-1)^l \sum_{i_1+i_2+\cdots+i_l=k} \tilde{R}_{z_n}\ A_{i_1} \tilde{R}_{z_n}\ A_{i_2} \tilde{R}_{z_n} \cdots A_{i_l} \tilde{R}_{z_n} \\ &= \sum_{k=0}^{\infty} \varepsilon^k \tilde{R}_{z_n,k}\ , \end{aligned}$$

where

$$\begin{cases} \tilde{R}_{z_n,0} := \tilde{R}_{z_n} := (A_0 - z_n I)^{-1} \\ \tilde{R}_{z_n,k} := \sum_{l=1}^{k} (-1)^l \sum_{i_1+i_2+\cdots+i_l=k} \tilde{R}_{z_n}\ A_{i_1} \tilde{R}_{z_n}\ A_{i_2} \tilde{R}_{z_n} \cdots A_{i_l} \tilde{R}_{z_n}, \quad \forall\ k \geq 1. \end{cases}$$

(ii) In view of Eq. (3.2), we get $\|A_k R_{z_n}\| \leq q^{k-1}\tilde{\alpha_n}$. So, by Lemma 2.2 (**ii**) we have for every $k \geq 1$

$$
\begin{aligned}
\|\tilde{R}_{z_n,k}\| &\leq \sum_{l=1}^{k} \sum_{i_1+i_2+\cdots+i_l=k} \tilde{M}_n \left(q^{i_1-1}\tilde{\alpha_n}\right) \left(q^{i_2-1}\tilde{\alpha_n}\right) \cdots \left(q^{i_1-1}\tilde{\alpha_n}\right) \\
&\leq \tilde{M}_n\, q^k \sum_{l=1}^{k} \sum_{i_1+i_2+\cdots+i_l=k} \left(\frac{\tilde{\alpha_n}}{q}\right)^l \\
&\leq \tilde{M}_n\, q^k \sum_{l=1}^{k} \left(\frac{\tilde{\alpha_n}}{q}\right)^l \sum_{i_1+i_2+\cdots+i_l=k} 1 \\
&\leq \tilde{M}_n\, q^k\, \frac{\tilde{\alpha_n}}{q} \left(1+\frac{\tilde{\alpha_n}}{q}\right)^{k-1} \\
&\leq \tilde{M}_n \tilde{\alpha_n} \left(q+\tilde{\alpha_n}\right)^{k-1}.
\end{aligned}
$$

Q.E.D

Remarks 3.1 (i) *Notice that Corollary 3.2 improves* [15, p. 133]. *Indeed, in* [15], *B. Sz. Nagy has proved that the resolvent* $\tilde{R}_{z_n}(\varepsilon)$ *of* $A(\varepsilon)$ *at* $z_n \in \tilde{\mathcal{C}}_n$ *can be developed into an entire series without giving the explicit expression of the coefficients of this series. Besides, he has given the estimation of these coefficients by comparing them to those of an other series. The main novelty here, is that we give the explicit expression of the coefficients* $(\tilde{R}_{z_n,k})_{k\geq 1}$, *which allows us to estimate them.*

(ii) *Proposition 3.1 extends* [15, p. 133] *since in our considerations, we deal with a non-analytic perturbation including more than one parameter, whereas in* [15], *B. Sz. Nagy has dealt with an analytic operator with one perturbation parameter* ε. $\diamondsuit$

Having obtained these results, we are now ready to investigate under the spectral properties of the perturbed operator $T(\xi)$. In that line our first result asserts:

Theorem 3.2 *Assume that hypotheses* $(H1)$-$(H3)$ *hold. Let* φ_n *(respectively,* φ_n^**) be an eigenvector of* T_0 *(respectively,* T_0^**: the adjoint of* T_0*) associated to the eigenvalue* λ_n *(respectively,* $\overline{\lambda_n}$*) such that* $\|\varphi_n\| = \|\varphi_n^*\| = \omega_n$ *and* $\varphi_n^*(\varphi_n) = 1$. *Then, the following assertions hlod.*

(i) *If* $\tau(\xi) < \frac{1}{\alpha_n(1+r_n M_n)}$, *then* $T(\xi)$ *has a unique point* $\lambda_n(\xi)$ *of its spectrum in the neighborhood of* λ_n *and this point is also with multiplicity one,*

(ii) *If* $\tau(\xi) < \frac{1}{\alpha_n(1+\omega_n^2 r_n M_n)}$, *then the eigenvalue* $\lambda_n(\xi)$ *can be developed into a series*

$$\lambda_n(\xi) := \lambda_n + \lambda_{n,1}(\xi) + \lambda_{n,2}(\xi) + \cdots + \lambda_{n,i}(\xi) + \cdots,$$

where

$$\lambda_{n,i}(\xi) = \sum_{l=0}^{i-1} b_{n,l}(\xi)\, c_{n,i-l}(\xi) \quad \textit{for all} \quad i \geq 1,$$

with

$$\begin{aligned}
b_{n,0}(\xi) &= 1, \\
b_{n,l}(\xi) &= \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} \beta_{i_1}\ \beta_{i_2}\ \cdots \beta_{i_l}, \quad \forall l \geq 1 \\
\beta_k &= -\varphi_n^* \left(P_{n,k}(\xi)\varphi_n\right), \quad \forall\ k \geq 1, \\
P_{n,k}(\xi) &= \frac{-1}{2i\pi} \int_{\mathcal{C}_n} R_{z_n,k}(\xi)\ dz_n, \quad \forall\ \ k \geq 1, \\
c_{n,k}(\xi) &= \varphi_n^* \left(B_{n,k}(\xi)\varphi_n\right), \quad \forall\ k \geq 1, \\
B_{n,k}(\xi) &= \frac{-1}{2i\pi} \int_{\mathcal{C}_n} (z_n - \lambda_n) R_{z_n,k}(\xi)\ dz_n, \quad \forall\ \ k \geq 1,
\end{aligned} \tag{3.7}$$

and $(R_{z_n,k}(\xi))_{k\geq 1}$ *is defined in Eq.* (3.3),

(*iii*) *We have*

$$|\lambda_{n,i}(\xi)| \leq \frac{\omega_n^2 r_n^2 M_n \left(1 - \alpha_n \tau(\xi)\right)}{\alpha_n \tau(\xi) - 1 + \omega_n^2 r_n M_n} \ \left(\left(\frac{\omega_n^2 r_n M_n \alpha_n \tau(\xi)}{1 - \alpha_n \tau(\xi)} \right)^i - (\alpha_n \tau(\xi))^i \right), \quad \forall\ i \geq 1.$$

$\diamondsuit$

Proof. (*i*) Let $P_n(\xi)$ (resp., P_n) be the spectral projection of $T(\xi)$ (resp., T_0) corresponding to the eigenvalue $\lambda_n(\xi)$ (resp., λ_n) and $\mathcal{R}_n(\xi)$ (resp., $\mathcal{R}_n$) the eigenspace for $\lambda_n(\xi)$ (resp., λ_n). In view of Proposition 3.1, we have

$$\begin{aligned}
P_n(\xi) &= \frac{-1}{2\pi i} \int_{\mathcal{C}_n} R_{z_n}(\xi)\ dz_n \\
&= \frac{-1}{2\pi i} \int_{\mathcal{C}_n} \left(R_{z_n} + \sum_{k=1}^{\infty} R_{z_n,k}(\xi) \right)\ dz_n \\
&= P_n + \sum_{k=1}^{\infty} P_{n,k}(\xi),
\end{aligned}$$

where

$$P_{n,k}(\xi) = \frac{-1}{2\pi i} \int_{\mathcal{C}_n} R_{z_n,k}(\xi)\ dz_n, \quad \forall\ k \geq 1.$$

Since

$$\|P_{n,k}(\xi)\| \leq \frac{1}{2\pi} \int_{\mathcal{C}_n} \|R_{z_n,k}(\xi)\|\ dz_n \leq r_n M_n (\alpha_n \tau(\xi))^k,$$

then, for $\tau(\xi) < \frac{1}{\alpha_n(1+r_n M_n)}$ we have

$$\begin{aligned}
\|P_n(\xi) - P_n\| &= \left\| \sum_{k=1}^{\infty} P_{n,k}(\xi) \right\| \\
&\leq \sum_{k=1}^{\infty} \|P_{n,k}(\xi)\| \\
&\leq \sum_{k=1}^{\infty} r_n M_n (\alpha_n \tau(\xi))^k \\
&\leq \frac{r_n M_n \alpha_n \tau(\xi)}{1 - \alpha_n \tau(\xi)} \\
&< 1.
\end{aligned}$$

Consequently, $\dim \mathcal{R}_n(\xi) = \dim \mathcal{R}_n = 1$. So, $T(\xi)$ has a unique eigenvalue $\lambda_n(\xi)$ in $\mathcal{C}_n$, for $\tau(\xi) < \frac{1}{\alpha_n(1+r_n M_n)}$. Moreover, $\lambda_n(\xi)$ is with multiplicity one.

(ii) Let

$$\varphi_n(\xi) = \frac{P_n(\xi)\varphi_n}{[\varphi_n^*(P_n(\xi)\varphi_n)]^{\frac{1}{2}}}$$

be an eigenvector of $T(\xi)$ associated to the eigenvalue $\lambda_n(\xi)$. We have

$$\begin{aligned}
\lambda_n(\xi) - \lambda_n &= (\lambda_n(\xi) - \lambda_n) \, \frac{\varphi_n^*(\varphi_n(\xi))}{\varphi_n^*(\varphi_n(\xi))} \\
&= \frac{\varphi_n^*((\lambda_n(\xi) - \lambda_n)\varphi_n(\xi))}{\varphi_n^*(\varphi_n(\xi))} \\
&= \frac{\varphi_n^*(T(\xi)\varphi_n(\xi) - \lambda_n \varphi_n(\xi))}{\varphi_n^*(\varphi_n(\xi))} \\
&= \frac{\varphi_n^*((T(\xi) - \lambda_n I)P_n(\xi)\varphi_n)}{\varphi_n^*(P_n(\xi)\varphi_n)}
\end{aligned} \tag{3.8}$$

Since $\varphi_n^*(P_n\varphi_n) = \varphi_n^*(\varphi_n) = 1$, then Eq. (3.8) implies that

$$\lambda_n(\xi) - \lambda_n = \frac{\varphi_n^*((T(\xi) - \lambda_n I)P_n(\xi)\varphi_n)}{1 + \varphi_n^*((P_n(\xi) - P_n)\varphi_n)}. \tag{3.9}$$

Or, for $\tau(\xi) < \frac{1}{\alpha_n(1+r_n M_n \omega_n^2)}$ we have

$$\begin{aligned}
\|\varphi_n^*((P_n(\xi) - P_n)\varphi_n)\| &\leq \|\varphi_n^*\| \, \|(P_n(\xi) - P_n)\varphi_n\| \\
&\leq \|P_n(\xi) - P_n\| \, \|\varphi_n\| \, \|\varphi_n^*\| \\
&\leq \frac{r_n M_n \alpha_n \tau(\xi)}{1 - \alpha_n \tau(\xi)} \, \omega_n^2 \\
&< 1.
\end{aligned}$$

So,

$$\frac{1}{1 + \varphi_n^*((P_n(\xi) - P_n)\varphi_n)} = \sum_{\nu=0}^{\infty} (-1)^\nu \left[\varphi_n^*((P_n(\xi) - P_n)\varphi_n)\right]^\nu. \tag{3.10}$$

On the other hand, we have

$$
\begin{aligned}
(T(\xi)-\lambda_n I)\,P_n(\xi) &= T(\xi)P_n(\xi)-\lambda_n P_n(\xi)\\
&= \frac{-1}{2\pi i}\int_{\mathcal{C}_n} z_n R_{z_n}(\xi)\;dz_n-\lambda_n\;\left(\frac{-1}{2\pi i}\right)\int_{\mathcal{C}_n} R_{z_n}(\xi)\;dz_n\\
&= \frac{-1}{2\pi i}\int_{\mathcal{C}_n}(z_n-\lambda_n)R_{z_n}(\xi)\;dz_n\\
&= \frac{-1}{2\pi i}\int_{\mathcal{C}_n}(z_n-\lambda_n)\left(R_{z_n}+\sum_{k=1}^{\infty}R_{z_n,k}(\xi)\right)dz_n\\
&= \frac{-1}{2\pi i}\int_{\mathcal{C}_n}(z_n-\lambda_n)R_{z_n}\;dz_n+\sum_{k=1}^{\infty}\left(\frac{-1}{2\pi i}\int_{\mathcal{C}_n}(z_n-\lambda_n)R_{z_n,k}(\xi)\;dz_n\right)\\
&= \sum_{k=1}^{\infty}B_{n,k}(\xi), \qquad (3.11)
\end{aligned}
$$

where

$$B_{n,k}(\xi)=\frac{-1}{2\pi i}\int_{\mathcal{C}_n}(z_n-\lambda_n)R_{z_n,k}(\xi)\;dz_n.$$

Consequently, Eqs (3.9), (3.10) and (3.11) imply that

$$
\begin{aligned}
\lambda_n(\xi)-\lambda_n &= \sum_{\nu=0}^{\infty}(-1)^{\nu}\left[\varphi_n^*\left((P_n(\xi)-P_n)\varphi_n\right)\right]^{\nu}\quad\varphi_n^*\left(\sum_{l=1}^{\infty}B_{n,l}(\xi)\varphi_n\right)\\
&= \sum_{\nu=0}^{\infty}(-1)^{\nu}\left(\sum_{k=1}^{\infty}\varphi_n^*\left(P_{n,k}(\xi)\varphi_n\right)\right)^{\nu}\quad\sum_{l=1}^{\infty}\varphi_n^*\left(B_{n,l}(\xi)\varphi_n\right)\\
&= \sum_{\nu=0}^{\infty}\left(\sum_{k=1}^{\infty}-\varphi_n^*\left(P_{n,k}(\xi)\varphi_n\right)\right)^{\nu}\quad\sum_{l=1}^{\infty}\varphi_n^*\left(B_{n,l}(\xi)\varphi_n\right) \qquad (3.12)
\end{aligned}
$$

Denoting by $\beta_k=-\varphi_n^*\left(P_{n,k}(\xi)\varphi_n\right),\forall\, k\geq 1$, we can easily check that if $\tau(\xi)<\frac{1}{\alpha_n(1+r_nM_n\omega_n^2)}$, then $\sum_{k=1}^{\infty}|\beta_k|<1$ and $\sum_{k=0}^{\infty}\tilde{\beta}_k^{\star}<+\infty$, where $\tilde{\beta}_k^{\star}=\sum_{l=1}^{k}\sum_{i_1+i_2+\cdots+i_l=k}|\beta_{i_1}|\;|\beta_{i_2}|\;\cdots|\beta_{i_l}|$. Hence, in view of Proposition 2.1 (**ii**)(**b**), we get

$$\sum_{\nu=0}^{\infty}\left(\sum_{k=1}^{\infty}-\varphi_n^*\left(P_{n,k}(\xi)\varphi_n\right)\right)^{\nu}=\sum_{l=0}^{\infty}b_{n,l}(\xi),$$

where $b_{n,0}(\xi)=1$, $b_{n,l}(\xi)=\sum_{k=l}^{\infty}\sum_{i_1+i_2+\cdots+i_l=k}\beta_{i_1}\;\beta_{i_2}\;\cdots\beta_{i_l},\;\;\forall l\geq 1$. So, using Eq. (3.12) we get

$$\lambda_n(\xi)-\lambda_n=\sum_{l=0}^{\infty}b_{n,l}(\xi)\;\sum_{k=1}^{\infty}c_{n,k}(\xi),$$

where $c_{n,k}(\xi)=\varphi_n^*\left(B_{n,k}(\xi)\varphi_n\right),\;\;\forall\, k\geq 1$. By the Cauchy product of two series, we get

$$\lambda_n(\xi)-\lambda_n=\sum_{i=1}^{\infty}\lambda_{n,i}(\xi),\;\text{ where }\lambda_{n,i}(\xi)=\sum_{l=0}^{i-1}b_{n,l}(\xi)\;c_{n,i-l}(\xi).$$

(iii) In order to estimate the coefficients $(|\lambda_{n,i}(\xi)|)_{i\geq 1}$, we shall first estimate $(|b_{n,l}(\xi)|)_{l\geq 0}$ and $(|c_{n,k}(\xi)|)_{k\geq 1}$. Indeed, in view of Proposition 2.1 (**ii**)(**b**), we have for $l \geq 1$

$$|b_{n,l}(\xi)| \leq \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} |\beta_{i_1}|\,|\beta_{i_2}|\,\cdots|\beta_{i_l}| \leq \left(\sum_{k=1}^{\infty}|\beta_k|\right)^l.$$

Since

$$|\beta_k| \leq \|\varphi_n^*\|\|P_{n,k}(\xi)\|\,\|\varphi_n\| \leq \omega_n^2 M_n r_n (\alpha_n \tau(\xi))^k, \quad \forall k \geq 1,$$

we obtain

$$|b_{n,l}(\xi)| \leq \left(\frac{\omega_n^2 M_n r_n \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)}\right)^l, \quad \forall l \geq 1.$$

As $b_{n,0}(\xi) = 1$, we get

$$|b_{n,l}(\xi)| \leq \left(\frac{\omega_n^2 M_n r_n \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)}\right)^l, \quad \forall l \geq 0. \tag{3.13}$$

On the other hand, we have for $l \geq 1$

$$|c_{n,l}(\xi)| \leq \|\varphi_n^*\|\|B_{n,l}(\xi)\|\,\|\varphi_n\| \leq \omega_n^2 \frac{1}{2\pi}\left\|\int_{\mathcal{C}_n}(z_n-\lambda_n)R_{z_n,k}(\xi)dz_n\right\| \leq \omega_n^2 r_n^2 M_n(\alpha_n\tau(\xi))^l. \tag{3.14}$$

Consequently, combining Eqs (3.13) and (3.14), we get

$$\begin{aligned}|\lambda_{n,i}(\xi)| &\leq \sum_{l=0}^{i-1}|b_{n,l}(\xi)|\,|c_{n,i-l}(\xi)|\\ &\leq \omega_n^2 r_n^2 M_n \sum_{l=0}^{i-1}\left(\frac{\omega_n^2 M_n r_n\alpha_n\tau(\xi)}{1-\alpha_n\tau(\xi)}\right)^l\,(\alpha_n\tau(\xi))^{i-l}.\end{aligned}$$

Regarding that $M_n \geq \frac{1}{r_n}$ and $\omega_n^2 \geq 1$, we obtain $1-\omega_n^2 M_n r_n \leq 0$. So, $\tau(\xi) \neq \frac{1}{\alpha_n}(1-\omega_n^2 M_n r_n)$. Hence, $\frac{\omega_n^2 M_n r_n \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)} \neq \alpha_n\tau(\xi)$ and then we get

$$\begin{aligned}|\lambda_{n,i}(\xi)| &\leq \omega_n^2 r_n^2 M_n \alpha_n \tau(\xi)\,\frac{\left(\frac{\omega_n^2 r_n M_n\alpha_n\tau(\xi)}{1-\alpha_n\tau(\xi)}\right)^i-(\alpha_n\tau(\xi))^i}{\frac{\omega_n^2 r_n M_n\alpha_n\tau(\xi)}{1-\alpha_n\tau(\xi)}-\alpha_n\tau(\xi)}\\ &\leq \frac{\omega_n^2 r_n^2 M_n\,(1-\alpha_n\tau(\xi))}{\alpha_n\tau(\xi)-1+\omega_n^2 r_n M_n}\left(\left(\frac{\omega_n^2 r_n M_n\alpha_n\tau(\xi)}{1-\alpha_n\tau(\xi)}\right)^i-(\alpha_n\tau(\xi))^i\right).\end{aligned}$$

Q.E.D

We have the analogue of Theorem 3.2 if we consider the analytic operator

$$A(\varepsilon) := A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots + \varepsilon^k A_k + \cdots.$$

Corollary 3.3 *Suppose that the assumptions of Corollary 3.1 are satisfied. Let $\tilde{\varphi}_n$ (respectively, $\tilde{\varphi}_n^*$) be an eigenvector of A_0 (respectively, A_0^*: the adjoint of A_0) associated to the eigenvalue $\tilde{\lambda}_n$ (respectively, $\overline{\tilde{\lambda}_n}$) such that $\|\tilde{\varphi}_n\| = \|\tilde{\varphi}_n^*\| = \tilde{\omega}_n$ and $\tilde{\varphi}_n^*(\tilde{\varphi}_n) = 1$. Then:*

(i) for $|\varepsilon| < \frac{1}{q+\tilde{\alpha}_n+\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n}$, $A(\varepsilon)$ has a unique point $\tilde{\lambda}_n(\varepsilon)$ of its spectrum in the neighborhood of $\tilde{\lambda}_n$ and this point is also with multiplicity one,

(ii) if $|\varepsilon| < \frac{1}{q+\tilde{\alpha}_n+\tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n}$, then $\tilde{\lambda}_n(\varepsilon)$ can be developed into an entire series

$$\tilde{\lambda}_n(\varepsilon) := \tilde{\lambda}_n + \varepsilon\tilde{\lambda}_{n,1} + \varepsilon^2\tilde{\lambda}_{n,2} + \cdots + \varepsilon^k\tilde{\lambda}_{n,k} + \cdots ,$$

where

$$\tilde{\lambda}_{n,k} = \sum_{i=0}^{k-1} \tilde{b}_{n,i}\ \tilde{c}_{n,k-i}., \quad \textit{for all} \quad k \geq 1,$$

with

$$\begin{array}{rcl}
\tilde{b}_{n,k} &=& \displaystyle\sum_{l=0}^{k}(-1)^l \sum_{i_1+i_2+\cdots+i_l=k} \tilde{\beta}_{i_1}\ \tilde{\beta}_{i_2}\ \cdots\tilde{\beta}_{i_l}, \quad \forall \quad k\geq 1,\\
\tilde{\beta}_k &=& \tilde{\varphi}_n^*\left(\tilde{P}_{n,k}\tilde{\varphi}_n\right), \quad \forall \quad k\geq 1,\\
\tilde{P}_{n,k} &=& \displaystyle\frac{-1}{2i\pi}\int_{\tilde{\mathcal{C}}_n}\tilde{R}_{z_n,k}\ dz_n, \quad \forall \quad k\geq 1,\\
\tilde{c}_{n,k} &=& \tilde{\varphi}_n^*\left(\tilde{B}_{n,k}\tilde{\varphi}_n\right), \quad \forall \quad k\geq 1,\\
\tilde{B}_{n,k} &=& \displaystyle\frac{-1}{2i\pi}\int_{\tilde{\mathcal{C}}_n}(z_n-\tilde{\lambda}_n)\tilde{R}_{z_n,k}\ dz_n, \quad \forall \quad k\geq 1,
\end{array} \tag{3.15}$$

and $\left(\tilde{R}_{z_n,k}\right)_{k\geq 1}$ is defined in Eq. (3.6).

(iii) we have

$$|\tilde{\lambda}_{n,k}| \leq \tilde{\omega}_n^2\tilde{r}_n^2\tilde{M}_n\tilde{\alpha}_n\left((q+\tilde{\alpha}_n+\tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n)^{k-1} - (q+\tilde{\alpha}_n)^{k-1}\right), \quad \textit{for all} \quad k\geq 1. \tag{3.16}$$

$\diamondsuit$

Proof. (i) The result is an immediate consequence of Theorem 3.2, it suffices to take $T_0 = A_0$, $T_k = \frac{1}{q^{k-1}}\ A_k$ and $\xi_k = \varepsilon^k q^{k-1}$, $\forall k \geq 1$. In this case, we have

$$\tau(\xi) = \tfrac{|\varepsilon|}{1-|\varepsilon|q} < \tfrac{1}{\tilde{\alpha}_n(1+\tilde{r}_n\tilde{M}_n)} \quad \text{if and only if} \quad |\varepsilon| < \tfrac{1}{q+\tilde{\alpha}_n+\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n}.$$

(ii) Let $\tilde{P}_n(\varepsilon)$ (resp., $\tilde{P}_n$) be the spectral projection of $A(\varepsilon)$ (resp., A_0) corresponding to the eigenvalue $\tilde{\lambda}_n(\varepsilon)$. Consider $\tilde{\varphi}_n(\varepsilon) = \frac{\tilde{P}_n(\varepsilon)\tilde{\varphi}_n}{[\tilde{\varphi}_n^*(\tilde{P}_n(\varepsilon)\tilde{\varphi}_n)]^{\frac{1}{2}}}$ an eigenvector of $A(\varepsilon)$ associated to the eigenvalue $\tilde{\lambda}_n(\varepsilon)$. Making the same reasoning as the one in the proof of Theorem 3.2 (ii), we obtain

$$\tilde{\lambda}_n(\varepsilon) - \tilde{\lambda}_n = \frac{\tilde{\varphi}_n^*\left((A(\varepsilon)-\tilde{\lambda}_nI)\ \tilde{P}_n(\varepsilon)\tilde{\varphi}_n\right)}{1+\tilde{\varphi}_n^*((\tilde{P}_n(\varepsilon)-\tilde{P}_n)\tilde{\varphi}_n)}. \tag{3.17}$$

If $|\varepsilon| < \frac{1}{q+\tilde{\alpha}_n+\tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n}$, we have $\|\tilde{\varphi}_n^*((\tilde{P}_n(\varepsilon)-\tilde{P}_n)\tilde{\varphi}_n)\| < 1$. So,

$$\begin{aligned}\frac{1}{1+\tilde{\varphi}_n^*((\tilde{P}_n(\varepsilon)-\tilde{P}_n)\tilde{\varphi}_n)} &= \sum_{\nu=0}^{\infty}(-1)^\nu\left[\tilde{\varphi}_n^*\left((\tilde{P}_n(\varepsilon)-\tilde{P}_n)\tilde{\varphi}_n\right)\right]^\nu \\ &= \sum_{\nu=0}^{\infty}\left(\sum_{k=1}^{\infty}-\varepsilon^k\tilde{\varphi}_n^*(\tilde{P}_{n,k}\varphi_n)\right)^\nu.\end{aligned}$$

Since $\sum_{k=1}^{\infty}\left|\varepsilon^k\tilde{\varphi}_n^*\left(\tilde{P}_{n,k}\tilde{\varphi}_n\right)\right| < 1$, for $|\varepsilon| < \frac{1}{q+\tilde{\alpha}_n+\tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n}$, we get in view of Proposition 2.1 (**i**),

$$\sum_{\nu=0}^{\infty}\left(\sum_{k=1}^{\infty}-\varepsilon^k\tilde{\varphi}_n^*\left(\tilde{P}_{n,k}\tilde{\varphi}_n\right)\right)^\nu = \sum_{k=0}^{\infty}\varepsilon^k\tilde{b}_{n,k}, \tag{3.18}$$

where $\tilde{b}_{n,0}=1$ and $\tilde{b}_{n,k}=\sum_{l=1}^{k}(-1)^l\sum_{i_1+i_2+\cdots+i_l=k}\beta_{i_1}\ \beta_{i_2}\ \cdots\beta_{i_l}$ and $\beta_k=\tilde{\varphi}_n^*\left(\tilde{P}_{n,k}\tilde{\varphi}_n\right)$, $\forall\ k\geq 1$.

On the other hand, in view of Corollary 3.2 and proceeding as in Theorem 3.2, we get

$$\left(A(\varepsilon)-\tilde{\lambda}_nI\right)\tilde{P}_n(\varepsilon)=\sum_{k=1}^{\infty}\varepsilon^k\tilde{B}_{n,k},\ \text{where } \tilde{B}_{n,k}=\frac{-1}{2\pi i}\int_{\tilde{\mathcal{C}}_n}(z_n-\tilde{\lambda}_n)\tilde{R}_{z_n,k}\,dz_n,\quad \forall k\geq 1.$$

So we deduce that,

$$\tilde{\varphi}_n^*\left(\left(A(\varepsilon)-\tilde{\lambda}_nI\right)\tilde{P}_n(\varepsilon)\tilde{\varphi}_n\right)=\sum_{k=1}^{\infty}\varepsilon^k\tilde{c}_{n,k},\ \text{where } \tilde{c}_{n,k}=\tilde{\varphi}_n^*\left(\tilde{B}_{n,k}\tilde{\varphi}_n\right),\quad \forall k\geq 1. \tag{3.19}$$

Consequently, Eqs (3.17), (3.18) and (3.19) yield

$$\tilde{\lambda}_n(\varepsilon)-\tilde{\lambda}_n=\sum_{k=0}^{\infty}\ \varepsilon^k\tilde{b}_{n,k}\sum_{k=1}^{\infty}\varepsilon^k\tilde{c}_{n,k}=\sum_{k=1}^{\infty}\varepsilon^k\tilde{\lambda}_{n,k},\ \text{where } \tilde{\lambda}_{n,k}=\sum_{i=0}^{k-1}\tilde{b}_{n,i}\ \tilde{c}_{n,k-i}.$$

(iii) We have

$$|\tilde{b}_{n,k}|\leq\sum_{l=1}^{k}\sum_{i_1+i_2+\cdots+i_l=k}|\beta_{i_1}|\ |\beta_{i_2}|\ \cdots|\beta_{i_l}|\,.$$

Or,

$$|\beta_k|=\left|\tilde{\varphi}_n^*\left(\tilde{P}_{n,k}\tilde{\varphi}_n\right)\right|\leq\tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n(q+\tilde{\alpha}_n)^{k-1}.$$

Hence,

$$\begin{aligned}|\tilde{b}_{n,k}| &\leq \tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n\sum_{l=1}^{k}\sum_{i_1+i_2+\cdots+i_l=k}(q+\tilde{\alpha}_n)^{i_1-1}(q+\tilde{\alpha}_n)^{i_2-1}\cdots(q+\tilde{\alpha}_n)^{i_l-1} \\ &\leq \tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n\sum_{l=1}^{k}\sum_{i_1+i_2+\cdots+i_l=k}(q+\tilde{\alpha}_n)^{k-l} \\ &\leq \tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n(q+\tilde{\alpha}_n)^k\sum_{l=1}^{k}\frac{1}{(q+\tilde{\alpha}_n)^l}\sum_{i_1+i_2+\cdots+i_l=k}1.\end{aligned}$$

Using Lemma 2.2 (**ii**), we deduce that

$$\left|\tilde{b}_{n,k}\right| \leq \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n (1+q+\tilde{\alpha}_n)^{k-1}. \tag{3.20}$$

On the other hand, we have

$$|\tilde{c}_{n,k}| \leq \left|\tilde{\varphi}_n^* \left(\tilde{B}_{n,k}\tilde{\varphi}_n\right)\right| \leq \tilde{\omega}_n^2 \tilde{r}_n^2 \tilde{M}_n \tilde{\alpha}_n (q+\tilde{\alpha}_n)^{k-1}. \tag{3.21}$$

Consequently, Eqs (3.20) and (3.21) imply

$$\begin{aligned}
|\tilde{\lambda}_{n,k}| &\leq \sum_{i=1}^{k-1} |\tilde{b}_{n,i}|\ |\tilde{c}_{n,k-i}| \\
&\leq \tilde{\omega}_n^4 \tilde{r}_n^3 \tilde{M}_n^2 \tilde{\alpha}_n^2 \sum_{i=1}^{k-1} \left(q+\tilde{\alpha}_n+\tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n\right)^{i-1} (q+\tilde{\alpha}_n)^{k-1-i} \\
&\leq \tilde{\omega}_n^4 \tilde{r}_n^3 \tilde{M}_n^2 \tilde{\alpha}_n^2 \ \frac{\left(q+\tilde{\alpha}_n+\tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n\right)^{k-1} - (q+\tilde{\alpha}_n)^{k-1}}{\tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n} \\
&\leq \tilde{\omega}_n^2 \tilde{r}_n^2 \tilde{M}_n \tilde{\alpha}_n \left(\left(q+\tilde{\alpha}_n+\tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n\right)^{k-1} - (q+\tilde{\alpha}_n)^{k-1}\right).
\end{aligned}$$

Q.E.D

Remarks 3.2 (*i*) *As in* [*15*, p. 136], *we prove that the eigenvector* $\tilde{\lambda}_n(\varepsilon)$ *of* $A(\varepsilon)$ *can be developed into an entire series. The main contribution here is that we give the explicit expression of the coefficients* $(\tilde{\lambda}_{n,i})_{i\geq 1}$*, which enables us to estimate them; whereas in* [*15*], *the author has given the estimation of these coefficients by comparing them to those of an other series. This approach can't be applied if we consider the non-analytic operator* $T(\xi)$.

(*ii*) *Its is clear here that Theorem 3.2 is more general than* [*15*, p. 136] *since we deal with a non-analytic perturbation including more than one parameter.* $\diamondsuit$

Now, setting $\varphi_n(\xi)$ an eigenvector of $T(\xi)$ associated to the eigenvalue $\lambda_n(\xi)$, we prove the analogue of Theorem 3.2 for $\varphi_n(\xi)$.

Theorem 3.3 *Assume that the hypotheses* $(H1)$*-*$(H3)$ *hold. Let* φ_n *(respectively,* φ_n^**) be an eigenvector of* T_0 *(respectively,* T_0^**: the adjoint of* T_0*) associated to the eigenvalue* λ_n *(respectively,* $\overline{\lambda_n}$*) such that* $\|\varphi_n\| = \|\varphi_n^*\|$ *and* $\varphi_n^*(\varphi_n) = 1$*. Then, for* $\tau(\xi) < \frac{1}{\alpha_n(1+r_n M_n \omega_n^2)}$ *we have:*

(*i*) *the eigenvector* $\varphi_n(\xi)$ *can be developed into a series*

$$\varphi_n(\xi) := \varphi_n + \varphi_{n,1}(\xi) + \varphi_{n,2}(\xi) + \cdots + \varphi_{n,i}(\xi) + \cdots,$$

where

$$\begin{cases} \varphi_{n,i}(\xi) := \sum_{k=1}^{i} d_{n,k}(\xi) P_{n,i-k}(\xi)\varphi_n, \quad \forall\ i \geq 1 \\ d_{n,0}(\xi) = 1, \\ d_{n,l}(\xi) = \sum_{k=l}^{\infty} C^l_{-\frac{1}{2}} \sum_{i_1+i_2+\cdots+i_l=k} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_l}, \quad \forall \quad l \geq 1, \\ \\ \gamma_l = \varphi_n^*(P_{n,l}(\xi)\ \varphi_n), \quad \forall \quad l \geq 1 \end{cases}$$

and $(P_{n,k}(\xi))_{k\geq 1}$ *is defined in Eq.* (3.7).

(ii) the coefficients $(\varphi_{n,i}(\xi))_{i\geq 1}$ *satisfy*

$$\|\varphi_{n,i}(\xi)\| \leq \frac{\omega_n^3 r_n^2 M_n^2}{\omega_n^2 r_n M_n - 1 + \alpha_n \tau(\xi)} \left[\left(\frac{\omega_n^2 r_n M_n \alpha_n \tau(\xi)}{1 - \alpha_n \tau(\xi)} \right)^i - (\alpha_n \tau(\xi))^i \right].$$

$\diamondsuit$

Proof. (i) We have

$$\begin{aligned} \varphi_n(\xi) &= \frac{P_n(\xi)\varphi_n}{[\varphi_n^*(P_n(\xi)\varphi_n)]^{\frac{1}{2}}} \\ &= [\varphi_n^*(P_n(\xi)\varphi_n)]^{-\frac{1}{2}} P_n(\xi)\varphi_n \\ &= (1 + \varphi_n^*((P_n(\xi) - P_n)\varphi_n)))^{-\frac{1}{2}} \sum_{l=0}^{\infty} P_{n,l}(\xi)\varphi_n, \end{aligned}$$

where $(P_{n,k}(\xi))_{k\geq 1}$ is defined in Eq. (3.7). Since for $\tau(\xi) < \frac{1}{\alpha_n(1+r_n M_n \omega_n^2)}$, we have $\|\varphi_n^*((P_n(\xi) - P_n)\varphi_n)\| < 1$ then

$$(1 + \varphi_n^*((P_n(\xi) - P_n)\varphi_n)))^{-\frac{1}{2}} = \sum_{\nu=0}^{\infty} C^{\nu}_{-\frac{1}{2}} \left[\varphi_n^* \left((P_n(\xi) - P_n)\varphi_n\right)\right]^{\nu}.$$

So,

$$\begin{aligned} \varphi_n(\xi) &= \sum_{\nu=0}^{\infty} C^{\nu}_{-\frac{1}{2}} \left(\varphi_n^* \left((P_n(\xi) - P_n)\varphi_n\right)\right)^{\nu} \sum_{l=0}^{\infty} P_{n,l}(\xi)\varphi_n \\ &= \sum_{\nu=0}^{\infty} C^{\nu}_{-\frac{1}{2}} \left(\varphi_n^* \left(\sum_{k=1}^{\infty} P_{n,k}(\xi)\varphi_n \right) \right)^{\nu} \sum_{l=0}^{\infty} P_{n,l}(\xi)\varphi_n. \end{aligned} \tag{3.22}$$

Setting $\gamma_k = \varphi_n^*\left(P_{n,k}(\xi)\varphi_n\right),\ \forall\ k \geq 1$, we can easily check that if $\tau(\xi) < \frac{1}{\alpha_n(1+r_n M_n \omega_n^2)}$, then $\sum_{k=1}^{\infty}|\gamma_k| < 1$ and $\sum_{k=0}^{\infty} \tilde{\gamma}_k^{\star} < +\infty$, where $\tilde{\gamma}_k^{\star} = \sum_{l=1}^{k} \sum_{i_1+i_2+\cdots+i_l=k} |\gamma_{i_1}|\ |\gamma_{i_2}|\ \cdots |\gamma_{i_l}|$. Consequently, in view of Theorem 2.1 (**i**), we get

$$\sum_{\nu=0}^{\infty} C^{\nu}_{-\frac{1}{2}} \left(\sum_{k=1}^{\infty} \varphi_n^*(P_{n,k}(\xi)\varphi_n) \right)^{\nu} = \sum_{l=0}^{\infty} d_{n,l}(\xi),$$

where

$$\begin{cases} d_{n,0}(\xi) = 1, \\ d_{n,l}(\xi) = C^l_{-\frac{1}{2}} \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_l}, \quad \forall \quad l \geq 1. \end{cases}$$

By the Cauchy product of the two series we get

$$\varphi_n(\xi) = \sum_{l=0}^{\infty} d_{n,l}(\xi) \sum_{l=0}^{\infty} P_{n,l}(\xi)\varphi_n = \sum_{i=0}^{\infty} \varphi_{n,i}(\xi),$$

where

$$\begin{cases} \varphi_{n,0}(\xi) := \varphi_n, \\ \varphi_{n,i}(\xi) := \sum_{k=1}^{i} d_{n,k}(\xi) P_{n,i-k}(\xi)\varphi_n, \quad \forall\ i \geq 1. \end{cases}$$

(ii) Since $\left| C^{\nu}_{-\frac{1}{2}} \right| < \frac{1}{2}, \ \ \forall\ \nu \geq 1$, then for every $l \geq 1$ we have

$$|d_{n,l}(\xi)| \leq \frac{1}{2} \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} |\gamma_{i_1}|\ |\gamma_{i_2}| \cdots |\gamma_{i_l}|.$$

Denoting that for all $l \geq 1$

$$|\gamma_l| = |\varphi_n^*(P_{n,l}(\xi)\ \varphi_n)| \leq \omega_n^2 r_n M_n \left(\alpha_n \tau(\xi)\right)^l,$$

we get in view of Proposition 2.1 (**ii**)(**b**),

$$\begin{aligned} |d_{n,l}(\xi)| \ &\leq\ \frac{1}{2} \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} \left(\omega_n^2 r_n M_n\right)^l \left(\alpha_n \tau(\xi)\right)^k \\ &\leq\ \frac{1}{2} \left(\omega_n^2 r_n M_n\right)^l \sum_{k=l}^{\infty} \sum_{i_1+i_2+\cdots+i_l=k} \left(\alpha_n \tau(\xi)\right)^{i_1+i_2+\cdots+i_l} \\ &\leq\ \frac{1}{2} \left(\omega_n^2 r_n M_n\right)^l \left(\sum_{k=1}^{\infty} \left(\alpha_n \tau(\xi)\right)^k \right)^l \\ &\leq\ \frac{1}{2} \left(\omega_n^2 r_n M_n\right)^l \left(\frac{\alpha_n \tau(\xi)}{1 - \alpha_n \tau(\xi)} \right)^l. \end{aligned}$$

Hence,

$$|d_{n,l}(\xi)| \leq \frac{1}{2} \left(\frac{\omega_n^2 r_n M_n \alpha_n \tau(\xi)}{1 - \alpha_n \tau(\xi)} \right)^l, \quad \text{for all} \ \ l \geq 0. \tag{3.23}$$

On the other hand, we have

$$\|P_{n,i-k}(\xi)\varphi_n\| \leq \omega_n r_n M_n \left(\alpha_n \tau(\xi)\right)^{i-k}, \quad \forall i \geq 1 \text{ and } \forall k \geq 0. \tag{3.24}$$

Consequently, Eqs (3.23) and (3.24) imply that for all $i \geq 1$

$$\begin{aligned}
\|\varphi_{n,i}(\xi)\| &\leq \sum_{k=1}^{i} |d_{n,k}(\xi)| \|P_{n,i-k}(\xi)\varphi_n\| \\
&\leq \sum_{k=1}^{i} \left(\frac{\omega_n^2 r_n M_n \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)} \right)^k \omega_n r_n M_n \left(\alpha_n\tau(\xi)\right)^{i-k} \\
&\leq \omega_n r_n M_n \sum_{k=1}^{i} \left(\frac{\omega_n^2 r_n M_n \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)} \right)^k \left(\alpha_n\tau(\xi)\right)^{i-k} \\
&\leq \frac{\omega_n^3 r_n^2 M_n^2 \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)} \sum_{k=1}^{i} \left(\frac{\omega_n^2 r_n M_n \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)} \right)^{k-1} \left(\alpha_n\tau(\xi)\right)^{i-k}. \qquad (3.25)
\end{aligned}$$

Now, making the same reasoning as the one in the proof of Theorem 3.2, we have $\frac{\omega_n^2 r_n M_n \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)} \neq \alpha_n\tau(\xi)$. So, Eq. (3.25) yields for every $i \geq 1$

$$\begin{aligned}
\|\varphi_{n,i}(\xi)\| &\leq \frac{\omega_n^3 r_n^2 M_n^2 \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)} \frac{\left(\frac{\omega_n^2 r_n M_n \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)}\right)^i - \left(\alpha_n\tau(\xi)\right)^i}{\frac{\omega_n^2 r_n M_n \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)} - \alpha_n\tau(\xi)} \\
&\leq \frac{\omega_n^3 r_n^2 M_n^2}{\alpha_n\tau(\xi) - 1 + \omega_n^2 r_n M_n} \left(\left(\frac{\omega_n^2 r_n M_n \alpha_n \tau(\xi)}{1-\alpha_n\tau(\xi)} \right)^i - \left(\alpha_n\tau(\xi)\right)^i \right).
\end{aligned}$$

Q.E.D

Let us consider the analytic perturbation $A(\varepsilon)$ (see Eq. (1.1)) and let $\tilde{\varphi}_n(\varepsilon)$ be an eigenvector of $A(\varepsilon)$ associated to the eigenvalue $\tilde{\lambda}_n(\varepsilon)$. Then, we can see the following result.

Corollary 3.4 *Suppose that the assumptions of Corollary 3.1 are satisfied. Let $\tilde{\varphi}_n$ (respectively, $\tilde{\varphi}_n^*$) be an eigenvector of A_0 (respectively, A_0^*: the adjoint of A_0) associated to the eigenvalue $\tilde{\lambda}_n$ (respectively, $\overline{\tilde{\lambda}_n}$) such that $\|\tilde{\varphi}_n\| = \|\tilde{\varphi}_n^*\|$ and $\tilde{\varphi}_n^*(\tilde{\varphi}_n) = 1$. Then, for $|\varepsilon| < \frac{1}{q+\tilde{\alpha}_n+\tilde{\omega}_n^2\tilde{r}_n\tilde{M}_n\tilde{\alpha}_n}$, we have:*

(i) the eigenvector $\tilde{\varphi}_n(\varepsilon)$ can be developed into an entire series

$$\tilde{\varphi}_n(\varepsilon) := \tilde{\varphi}_n + \varepsilon\tilde{\varphi}_{n,1} + \varepsilon^2\tilde{\varphi}_{n,2} + \cdots + \varepsilon^k\tilde{\varphi}_{n,k} + \cdots,$$

where

$$\begin{cases}
\tilde{\varphi}_{n,i} := \displaystyle\sum_{k=1}^{i} \tilde{d}_{n,k}\ \tilde{P}_{n,i-k}\tilde{\varphi}_n, \quad \forall\ i \geq 1, \\
\tilde{d}_{n,k} = C^k_{-\frac{1}{2}} \displaystyle\sum_{l=1}^{k} \sum_{i_1+i_2+\cdots+i_l=k} \tilde{\gamma}_{i_1}\tilde{\gamma}_{i_2}\cdots\tilde{\gamma}_{i_l}, \quad \forall\ \ k \geq 1, \\
\tilde{\gamma}_k = \tilde{\varphi}_n^*(\tilde{P}_{n,k}\ \tilde{\varphi}_n), \quad \forall\ \ k \geq 1
\end{cases}$$

and $\left(\tilde{P}_{n,k}\right)_{k\geq 1}$ *is defined in Eq.* (3.15).

(*ii*) *the coefficients* $(\tilde{\varphi}_{n,i})_{i\geq 1}$ *satisfy*

$$\|\tilde{\varphi}_{n,i}\| \leq \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n \left[\frac{\left(q + \tilde{\alpha}_n + \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n\right)^i - (q + \tilde{\alpha}_n)^i}{q + \tilde{\alpha}_n} \right]. \tag{3.26}$$

$\diamondsuit$

Proof. (*i*) We have

$$\tilde{\varphi}_n(\varepsilon) = \frac{\tilde{P}_n(\varepsilon)\tilde{\varphi}_n}{[\tilde{\varphi}_n^*(\tilde{P}_n(\varepsilon)\tilde{\varphi}_n)]^{\frac{1}{2}}}.$$

Making the same reasoning as the one in the proof of Theorem 3.3 and using Theorem 2.1, we get

$$\left(1 + \tilde{\varphi}_n^*((\tilde{P}_n(\varepsilon) - \tilde{P}_n)\tilde{\varphi}_n)\right)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \varepsilon^k \tilde{d}_{n,k},$$

where

$$\begin{cases} \tilde{d}_{n,0} = 1, \\ \tilde{d}_{n,k} = \displaystyle\sum_{l=1}^{k} C_{-\frac{1}{2}}^{l} \sum_{i_1+i_2+\cdots+i_l=k} \tilde{\gamma}_{i_1}\tilde{\gamma}_{i_2}\cdots\tilde{\gamma}_{i_l}, \quad \forall \ \ k \geq 1, \\ \tilde{\gamma}_k = \tilde{\varphi}_n^*(\tilde{P}_{n,k}\ \tilde{\varphi}_n), \quad \forall \ \ k \geq 1, \\ \tilde{P}_{n,k} \ = \dfrac{-1}{2i\pi} \displaystyle\int_{\tilde{\mathcal{C}}_n} \tilde{R}_{z_n,k}\ dz_n, \quad \forall \ \ k \geq 1. \end{cases}$$

So, we have

$$\tilde{\varphi}_n(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \tilde{d}_{n,k} \sum_{k=0}^{\infty} \varepsilon^k \tilde{P}_{n,k}\tilde{\varphi}_n = \sum_{i=0}^{\infty} \varepsilon^i \tilde{\varphi}_{n,i},$$

where

$$\begin{cases} \tilde{\varphi}_{n,0} := \tilde{\varphi}_n, \\ \tilde{\varphi}_{n,i} := \displaystyle\sum_{k=1}^{i} \tilde{d}_{n,k}\ \tilde{P}_{n,i-k}\tilde{\varphi}_n, \quad \forall\ i \geq 1. \end{cases}$$

(*ii*) We have

$$\left|\tilde{d}_{n,k}\right| = \sum_{l=1}^{k} \sum_{i_1+i_2+\cdots+i_l=k} |\tilde{\gamma}_{i_1}|\ |\tilde{\gamma}_{i_2}| \cdots |\tilde{\gamma}_{i_l}|, \forall \ \ k \geq 1.$$

Since

$$|\tilde{\gamma}_k| \leq \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n (q + \tilde{\alpha}_n)^{k-1}, \quad \forall \ \ k \geq 1,$$

then using Lemma 2.2 (**ii**)(**b**) we deduce that for all $k \geq 1$

$$\begin{aligned}
|d_{n,k}| &\leq \sum_{l=1}^{k} \left(\tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n\right)^l \sum_{i_1+i_2+\cdots+i_l=k} (q+\tilde{\alpha}_n)^{k-l} \\
&\leq (q+\tilde{\alpha}_n)^k \sum_{l=1}^{k} \left(\frac{\tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n}{q+\tilde{\alpha}_n}\right)^l \sum_{i_1+i_2+\cdots+i_l=k} 1 \\
&\leq (q+\tilde{\alpha}_n)^k \frac{\tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n}{q+\tilde{\alpha}_n} \left(1 + \frac{\tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n}{q+\tilde{\alpha}_n}\right)^{k-1} \\
&\leq \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n \left(q+\tilde{\alpha}_n + \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n\right)^{k-1}. \qquad (3.27)
\end{aligned}$$

Noting that

$$\left\|\tilde{P}_{n,k}\right\| \leq \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n (q+\tilde{\alpha}_n)^{k-1}, \quad \forall k \geq 1,$$

we get in view of Eq. (3.27) that for all $i \geq 1$,

$$\begin{aligned}
\|\tilde{\varphi}_{n,i}\| &\leq \omega_n^2 r_n^2 M_n^2 \alpha_n^2 \sum_{k=1}^{i} \left(q+\tilde{\alpha}_n + \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n\right)^{k-1} (q+\tilde{\alpha}_n)^{i-k-1} \\
&\leq \frac{\omega_n^2 r_n^2 M_n^2 \alpha_n^2}{q+\tilde{\alpha}_n} \sum_{k=1}^{i} \left(q+\tilde{\alpha}_n + \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n\right)^{k-1} (q+\tilde{\alpha}_n)^{i-k} \\
&\leq \frac{\omega_n^2 r_n^2 M_n^2 \alpha_n^2}{q+\tilde{\alpha}_n} \frac{\left(q+\tilde{\alpha}_n + \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n\right)^{i} - (q+\tilde{\alpha}_n)^i}{\tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n} \\
&\leq \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n \left[\frac{\left(q+\tilde{\alpha}_n + \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n\right)^{i} - (q+\tilde{\alpha}_n)^i}{q+\tilde{\alpha}_n}\right].
\end{aligned}$$

Q.E.D

Remarks 3.3 (*i*) *In Corollary 3.4, we give some supplements to* [*15, p.* 136]. *Indeed, in* [*15*] *has proved that the eigenvector $\tilde{\lambda}_n(\varepsilon)$ of $A(\varepsilon)$ can be developed into an entire series of ε and he has given the estimation of the coefficients $(\tilde{\varphi}_{n,i})_{i\geq 1}$, by comparing them to those of an other series. In our considerations, we estimate the coefficients $(\tilde{\varphi}_{n,i})_{i\geq 1}$ by means of their expression. The obtained estimations are more precise than the one given by B. Sz. Nagy since Eq. (3.26) implies that*

$$\begin{aligned}
\|\varphi_{n,i}\| &\leq \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n \frac{\left(q+\tilde{\alpha}_n + \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n\right)^{i}}{q+\tilde{\alpha}_n} \\
&\leq \tilde{r}_n \tilde{M}_n \left(q+\tilde{\alpha}_n + \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n\right)^{i} \\
&\leq \tilde{\omega}_n \tilde{r}_n \tilde{M}_n \left(q+\tilde{\alpha}_n + \tilde{\omega}_n^2 \tilde{r}_n \tilde{M}_n \tilde{\alpha}_n\right)^{i}.
\end{aligned}$$

(ii) Theorem 3.3 extends [15, p. 136] to a new type of perturbed operator depending on many parameters. This new situation is much wider in the scope of applications. $\diamondsuit$

4 Application to a Gribov operator

In this section, we consider the Gribov operator $H_{\lambda",\lambda',\mu,\lambda}$ defined in Eq. (1.7).
Let S, H_0 and H_1 be the operators defined by:

$$\begin{cases} S : \mathcal{D}(S) \subset E \longrightarrow E \\ \qquad\qquad \varphi \longrightarrow S\varphi(z) = A^{*2}A^2\varphi(z) \\ \mathcal{D}(S) = \{\varphi \in E \text{ such that } S\varphi \in E\}, \end{cases}$$

$$\begin{cases} H_0 : \mathcal{D}(H_0) \subset E \longrightarrow E \\ \qquad\qquad \varphi \longrightarrow H_0\varphi(z) = A^*A\varphi(z) \\ \mathcal{D}(H_0) = \{\varphi \in E \text{ such that } H_0\varphi \in E\} \end{cases}$$

and

$$\begin{cases} H_1 : \mathcal{D}(H_1) \subset E \longrightarrow E \\ \qquad\qquad \varphi \longrightarrow H_1\varphi(z) = A^*(A + A^*)A\varphi(z) \\ \mathcal{D}(H_1) = \{\varphi \in E \text{ such that } H_1\varphi \in E\}. \end{cases}$$

Remark 4.1 *Due to [10, Lemme 3 p. 112], H_0 is a self-adjoint operator with compact resolvent. Moreover, $\{e_n(z) = \frac{z^n}{\sqrt{n!}}\}_1^\infty$ is a system of eigenvectors associated to the eigenvalues $\{n\}$. So, the spectral decomposition of H_0 is given by:*

$$H_0 = \sum_{n=1}^{\infty} n\langle ., e_n\rangle e_n. \qquad \diamondsuit$$

Let $G = H_0^3$. So, G is defined by

$$\begin{cases} G : \mathcal{D}(G) \subset E \longrightarrow E \\ \qquad\qquad \varphi \longrightarrow G\varphi = \displaystyle\sum_{n=1}^{\infty} n^3\langle \varphi, e_n\rangle e_n \\ \mathcal{D}(G) = \left\{\varphi \in E \text{ such that } \displaystyle\sum_{n=1}^{\infty} n^6|\langle \varphi, e_n\rangle|^2 < \infty\right\}. \end{cases}$$

The expression of $H_{\lambda",\lambda',\mu,\lambda}$ becomes then:

$$H_{\lambda",\lambda',\mu,\lambda} := \lambda" G + \lambda' S + \mu H_0 + i\lambda H_1.$$

Regarding [10], we can see the following result:

Proposition 4.1 *We have the following assertions:*
(i) G is a closed linear operator with dense domain.
(ii) The resolvent set of G is not empty. In fact, $0 \in \rho(G)$.
(iii) G is a self-adjoint operator with compact resolvent.
(iv) The eigenvalues of G are simple and isolated. $\diamondsuit$

Let $T_0 := \lambda" G$, $\mathcal{D} = \mathcal{D}(S) \cap \mathcal{D}(H_0) \cap \mathcal{D}(H_1)$ and T_1 (respectively, T_2 and T_3) the restriction of $\lambda" S$ (respectively, $\frac{\lambda"}{c} H_0$ and $\frac{\lambda"}{(1+2\sqrt{2})c} H_1$) to $\mathcal{D}$, where $c = \left\| \left(H_0^{-1}\right)^3 \right\|^{\frac{1}{6}}$. Hence, the operators $(T_k)_{1 \leq k \leq 3}$ have the same domain $\mathcal{D}$ and we have $\mathcal{D}(T_0) \subset \mathcal{D}$.
Moreover, the operator $H_{\lambda",\lambda',\mu,\lambda}$ can be written as:

$$H_{\lambda",\lambda',\mu,\lambda} := T_0 + \xi_1 T_1 + \xi_2 T_2 + \xi_3 T_3,$$

where $\xi_1 = \frac{\lambda'}{\lambda"}$, $\quad \xi_2 = \frac{c\mu}{\lambda"}$ and $\quad \xi_3 = \frac{(1+2\sqrt{2})c\lambda}{\lambda"}$.

The first result of this section is formulated in the following proposition.

Proposition 4.2 *The operators $(T_k)_{1 \leq k \leq 3}$ satisfy the following estimation*

$$\|T_k \varphi\| \leq a\|\varphi\| + b\|T_0 \varphi\|, \quad \textit{for all} \ \ \varphi \in \mathcal{D}(T_0),$$

where $a = \frac{\lambda"}{3}$ *and* $b = \frac{2}{3}$. $\diamondsuit$

Proof. Due to [4, Lemma 4.1, (ii)], we have

$$\|S\varphi\| \leq \|G\varphi\|^{\frac{2}{3}} \, \|\varphi\|^{\frac{1}{3}}, \ \text{ for all } \varphi \in \mathcal{D}(G).$$

Using Young's inequality, we obtain

$$\|S\varphi\| \leq \frac{1}{3} \, \|\varphi\| + \frac{2}{3} \, \|G\varphi\|, \ \text{ for all } \varphi \in \mathcal{D}(G).$$

Hence,

$$\|T_1\varphi\| \leq \frac{\lambda"}{3} \, \|\varphi\| + \frac{2}{3} \, \|T_0\varphi\|, \quad \text{for all } \ \varphi \in \mathcal{D}(T_0).$$

Moreover, we have

$$\|H_0\varphi\| = \left(\sum_{n=1}^{\infty} n^2 \left|\langle \varphi, e_n \rangle\right|^2\right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{\infty} n^3 \left|\langle \varphi, e_n \rangle\right|^2\right)^{\frac{1}{2}} \leq \left\| H_0^{\frac{3}{2}} \varphi \right\|, \quad \text{for all } \ \varphi \in \mathcal{D}\left(H_0^{\frac{3}{2}}\right). \tag{4.1}$$

Taking into account Cauchy Schwartz's inequality, we get

$$\left\| H_0^{\frac{3}{2}} \varphi \right\|^2 = \left\langle H_0^{\frac{3}{2}} \varphi, H_0^{\frac{3}{2}} \varphi \right\rangle = \left\langle H_0^3 \varphi, \varphi \right\rangle \leq \|H_0^3 \varphi\| \|\varphi\|, \quad \text{for all } \ \varphi \in \mathcal{D}\left(H_0^3\right).$$

So, Eq. (4.1) together with the fact that $\mathcal{D}\left(H_0^3\right) \subset \mathcal{D}\left(H_0^{\frac{3}{2}}\right)$ yield

$$\|H_0\varphi\| \leq \|H_0^3\varphi\|^{\frac{1}{2}}\ \|\varphi\|^{\frac{1}{2}}, \quad \text{for all } \varphi \in \mathcal{D}\left(H_0^3\right). \tag{4.2}$$

On the other hand, since $0 \in \rho(H_0)$, we get for $\varphi \in \mathcal{D}\left(H_0^3\right)$

$$\begin{aligned} \left\|H_0^3\varphi\right\|^{\frac{1}{2}}\ \|\varphi\|^{\frac{1}{2}} &= \left\|\left(H_0^{-1}\right)^3 H_0^3\varphi\right\|^{\frac{1}{6}} \left\|H_0^3\varphi\right\|^{\frac{1}{2}}\ \|\varphi\|^{\frac{1}{3}} \\ &\leq \left\|\left(H_0^{-1}\right)^3\right\|^{\frac{1}{6}} \left\|H_0^3\varphi\right\|^{\frac{2}{3}}\ \|\varphi\|^{\frac{1}{3}}. \end{aligned} \tag{4.3}$$

Consequently, Eqs (4.2) and (4.3) imply that

$$\|H_0\varphi\| \leq c\left\|H_0^3\varphi\right\|^{\frac{2}{3}}\ \|\varphi\|^{\frac{1}{3}}, \quad \text{for all } \varphi \in \mathcal{D}\left(H_0^3\right),$$

where $c = \left\|\left(H_0^{-1}\right)^3\right\|^{\frac{1}{6}}$. Now, using Young's inequality, we obtain

$$\|H_0\varphi\| \leq \frac{c}{3}\|\varphi\| + \frac{2}{3}\ c\ \|G\varphi\|, \quad \text{for all } \varphi \in \mathcal{D}\left(G\right).$$

Hence,

$$\|T_2\varphi\| \leq \frac{\lambda"}{3}\ \|\varphi\| + \frac{2}{3}\ \|T_0\varphi\|, \quad \text{for all } \ \varphi \in \mathcal{D}(T_0).$$

Now, let us prove the third inequality. In view of [7, Proposition 6.3], we have

$$\|H_1\varphi\| \leq \left(1 + 2\sqrt{2}\right)\ \|H_0^{\frac{3}{2}}\|, \quad \text{for all } \varphi \in \mathcal{D}\left(H_0^3\right).$$

Making the same reasoning as above and using Young's inequality, we infer that for $\varphi \in \mathcal{D}\left(H_0^3\right)$

$$\begin{aligned} \|H_1\varphi\| &\leq \left(1 + 2\sqrt{2}\right)\ c\ \|H_0^3\varphi\|^{\frac{2}{3}}\|\varphi\|^{\frac{1}{3}} \\ &\leq \frac{1}{3}\left(1 + 2\sqrt{2}\right)c\|\varphi\| + \frac{2}{3}\left(1 + 2\sqrt{2}\right)c\|H_0^3\varphi\|. \end{aligned}$$

Consequently, we have

$$\|T_3\varphi\| \leq \frac{\lambda"}{3}\ \|\varphi\| + \frac{2}{3}\ \|T_0\varphi\|, \quad \text{for all } \ \varphi \in \mathcal{D}(T_0).$$

Q.E.D

Proposition 4.3 *If $\tau(\xi) < \frac{3}{2}$, then the operator $H_{\lambda",\lambda',\mu,\lambda}$ is closed.* ◊

Proof. The result is an immediate consequence of Theorem 3.1, Propositions 4.1 and 4.2. Q.E.D

Now, we are in position to state the objective of this section.

Theorem 4.1 *(i) For $\tau(\xi)$ enough small, the operator $H_{\lambda",\lambda',\mu,\lambda}$ has a unique point $\lambda_n(\xi)$ of its spectrum in the neighborhood of $\lambda_n = n^3$ and this point is also with multiplicity one. Moreover, $\lambda_n(\xi)$ can be developed into a series*

$$\lambda_n(\xi) := n^3 + \lambda_{n,1}(\xi) + \lambda_{n,2}(\xi) + ... + \lambda_{n,i}(\xi) + ...$$

(ii) Let φ_n be an eigenvector of T_0 associated to the eigenvalue λ_n. Then, setting $\varphi_n(\xi)$ an eigenvector of $H_{\lambda",\lambda',\mu,\lambda}$ associated to the eigenvalue $\lambda_n(\xi)$, we have for ξ enough small $\varphi_n(\xi)$ can be developed into a series

$$\varphi_n(\xi) := e_n + \varphi_{n,1}(\xi) + \varphi_{n,2}(\xi) + ... + \varphi_{n,i}(\xi) + ... \qquad \diamondsuit$$

Proof. The result follows from Theorem 3.3, Propositions 4.1, 4.2 and 4.3. Q.E.D

References

[1] M. Aimar, A. Intissar and A. Jeribi, *On an unconditionl basis of generalized eigenvectors of the nonself-adjoint Gribov operator in Bargmann space.* J. Math. Anal. Appl., 231, 588-602, (1999).

[2] V. Bargmann, *On a Hilbert space of analytic functions and associated integral transform.* Comm. Pure Appl. Math., 14, 187-214, (1961).

[3] N. Ben Ali and A. Jeribi, *On the Riesz basis of a family of analytic operators in the sens of Kato and application to the problem of radiation of a vibrating structure in a light fluid.* J. Math. Anal. Appl., 320, 78-94, (2006).

[4] S. Charfi, A. Damergi and A. Jeribi, *On a Riesz basis of finite-dimensional invariant subspaces and application to Gribov operator in Bargmann space.* Linear and Multilinear Algebra, 61 (11), 1577-1591, (2013).

[5] H. Ellouze, I. Feki and A. Jeribi, *On a Riesz basis of exponentials related to the eigenvalues of an analytic operator and application to a non-selfadjoint problem deduced from a perturbation method for sound radiation.* Journal of Mathematical Physics, 54, (2013).

[6] I. Feki, A. Jeribi and R. Sfaxi, *On an unconditionl basis of generalized eigenvectors of an analytic operator and application to a problem of radiation of a vibrating structure in a light fluid.* J. Math. Anal. Appl., 375, 261-269, (2011).

[7] I. Feki, A. Jeribi and R. Sfaxi, *On a Schauder basis related to the eigenvectors of a family of non-selfadjoint analytic operators and applications.* Anal. Math. Phys., 3 (4), 311-331 (2013).

[8] I. Feki, A. Jeribi and R. Sfaxi, *On a Riesz basis of eigenvectors of a nonself-adjoint analytic operator and applications.* Linear and Multilinear Algebra, 62, 1049-1068 (2014).

[9] V.N. Gribov, *A Reggeon diagram technique.* Sov. Phys. JETP, 26 (2), 414-423, (1968).

[10] A. Intissar, *Analyse fonctionnelle et théorie spectrale pour les opérateurs compacts non-autoadjoints.* Editions CEPADUES, (1997).

[11] A. Jeribi and A. Intissar, *On a Riesz basis of generalized eigenvectors of the nonself-adjoint problem deduced from a perturbation method for sound radiation by a vibrating plate in a light fluid.* J. Math. Anal. Appl., 292, 1-16, (2004).

[12] A. Jeribi, *Spectral Theory and Applications of Linear Operators and Block Operator Matrices.* Springer-Verlag, New-York, (2015).

[13] T. Kato, *On the convergence of the perturbation method.* I. Progress of Theoretical Physics, 4, 514-523, (1949).

[14] B. Sz. Nagy, *Perturbations des transformations autoadjointes dans l'espace de Hilbert.* Comment. Math. Helv., 19, 347-366, (1947).

[15] B. Sz. Nagy, *Perturbations des transformations linéaires fermées.* Acta Sci. Math. Szeged., 14, 125-137, (1951).

[16] L. Rayleigh, *The theory of sound.* Vol. 1, (1927).

[17] F. Rellich, *Storungstheorie der Spektralzerlegung.* I. Mitt. Math. Ann., 113, 600-619, (1937).

[18] E. Schrödinger, *Quantisierung als Eigenwertproblem.* Ann. Physik, 80, 437-490, (1926).

[19] E. Schrödinger, *Collected papers on wave mechanics.* London and Glasgow, (1928).

[20] P. Vollat, *Familles sommables dans un espace vectoriel normé, Applications aux séries dans un espace vectoriel normé.* http://pavollat.free.fr/, (2009).