DETERMINATION OF THE IMPULSIVE DIRAC SYSTEMS FROM A SET OF EIGENVALUES

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Abstract. In this work, we consider the inverse spectral problem for the impulsive Dirac systems on $(0,\pi)$ with the jump condition at the point $\frac{\pi}{2}$. We conclude that the matrix potential $Q(x)$ on the whole interval can be uniquely determined by a set of eigenvalues for two cases: (i) the matrix potential $Q(x)$ is given on $\left(0,\frac{(1+\alpha)\pi}{4}\right)$; (ii) the matrix potential $Q(x)$ is given on $\left(\frac{(1+\alpha)\pi}{4},\pi\right)$, where $0<\alpha<1$.

1. Introduction

Define $\rho(x)=\begin{cases}1, & x<\dfrac{\pi}{2}\\ \alpha, & x>\dfrac{\pi}{2}\end{cases}$ $(0<\alpha<1)$. Consider the following impulsive Dirac systems:

$$ly := By'(x)+Q(x)y(x)=\lambda\rho(x)y(x), \qquad x\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right), \tag{1.1}$$

with the boundary conditions

$$y_1(0)=0, \tag{1.2}$$
$$y_2(\pi)=0, \tag{1.3}$$

and the jump conditions

$$y\left(\frac{\pi}{2}+0\right)=Ay\left(\frac{\pi}{2}-0\right), \tag{1.4}$$

where

$$B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ Q(x)=\begin{pmatrix}p(x) & q(x)\\ q(x) & -p(x)\end{pmatrix},\ y(x)=\begin{pmatrix}y_1(x)\\ y_2(x)\end{pmatrix},$$

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$p(x)$ and $q(x)$ are real-valued functions in $L^2(0,\pi)$, λ is the spectral parameter, and $A = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \beta \in \mathbb{R}^+$. The problem (1.1)-(1.4), denoted by $L = L(p(x), q(x), \rho(x), \beta)$, is called a boundary value problem of the Dirac equations with the discontinuity conditions at $\frac{\pi}{2}$.

The boundary value problems with a discontinuous are related to discontinuous material characters of a intermediary. This kind of problem has been studied by many authors (see, e.g., [1, 9, 10]).

The inverse problem for the Dirac operator was completely solved by two spectra in [3, 4]. Mochizuki and Trooshin [8] studied the problem $L = L(p(x), q(x), 1, 1)$ with the separable boundary conditions. They gave the uniqueness theorem by a set of values of eigenfunctions in some internal point and spectrum. In [9], Ozkan and Amirov studied the boundary value problem $L = L(p(x), q(x), 1, \beta)$ and showed that the potential function can be uniquely determined by a set of values of eigenfunctions at some internal point and one spectrum. Amirov [1] gave representations of solutions of the Dirac equation, properties of spectral data and showed that the Dirac operator can be uniquely determined by the Weyl function on a finite interval $(0, \pi)$ for the problem $L = L(p(x), q(x), 1, \beta)$.

For the impulsive Dirac operator, Mamedov and Akcay [7] proved that the sequences of eigenvalues and normalizing numbers can uniquely determine the potential and they gave the theorem on the necessary and sufficient conditions for the solvability and a solution algorithm of the inverse problem for the boundary value problem $L = L(p(x), q(x), \rho(x), 1)$. In [10], Güldü studied the problem L and proved by Hochstadt and Lieberman's method [5] that if the potential function $p(x)$ is given on the interval $(\frac{\pi}{2}, \pi)$, then one spectrum can determine $p(x)$ on the whole interval.

In this paper, we consider the problem $L = L(p(x), q(x), \rho(x), \beta)$. It is shown two cases that (i) if the potential $p(x)$ and $q(x)$ are given on $(0, \frac{(1+\alpha)\pi}{4})$; (ii) if the potential $p(x)$ and $q(x)$ are given on $(\frac{(1+\alpha)\pi}{4}, \pi)$, respectively, then only a single spectrum is sufficient to determine $p(x)$, $q(x)$ on $(0, \pi)$, $\rho(x)$ and β.

2. Preliminaries

Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be the solutions of (1.1), satisfying the initial conditions

$$\varphi(0, \lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad \psi(\pi, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the jump condition (1.4), respectively. Denote $\sigma(x)=\int_0^x \rho(t)dt,\ \tau=\operatorname{Im}\lambda$.

From [6, 10], we can get that $\varphi(x,\lambda)$ has the following representation:

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^x K_1(x,t)\varphi_0(t,\lambda)dt,$$

where $\varphi_0(x,\lambda)=(\varphi_{01}(x,\lambda),\varphi_{02}(x,\lambda))^T$ satisfies the following forms:

$$\varphi_{01}(x,\lambda)=\begin{cases}\sin\lambda\sigma(x), & 0<x<\dfrac{\pi}{2},\\ A^+\cos\lambda\sigma(x)+A^-\cos\lambda(\pi-\sigma(x)), & \dfrac{\pi}{2}<x<\pi,\end{cases}\tag{2.1}$$

$$\varphi_{02}(x,\lambda)=\begin{cases}-\cos\lambda\sigma(x), & 0<x<\dfrac{\pi}{2},\\ A^+\sin\lambda\sigma(x)-A^-\sin\lambda(\pi-\sigma(x)), & \dfrac{\pi}{2}<x<\pi.\end{cases}\tag{2.2}$$

Similarly, we can compute that the following representation holds for $\psi(x,\lambda)$:

$$\psi(x,\lambda)=\psi_0(x,\lambda)+\int_x^\pi K_2(x,t)\psi_0(t,\lambda)dt,\tag{2.3}$$

where $\psi_0(x,\lambda)=(\psi_{01}(x,\lambda),\psi_{02}(x,\lambda))^T$ satisfies the following forms:

$$\psi_{01}(x,\lambda)=\begin{cases}\beta^+\sin\lambda(\sigma(\pi)-\sigma(x)) & \\ \qquad -\beta^-\sin\lambda(\sigma(\pi)+\sigma(x)-\pi), & 0<x<\dfrac{\pi}{2},\\ -\cos\lambda(\sigma(\pi)-\sigma(x)), & \dfrac{\pi}{2}<x<\pi,\end{cases}\tag{2.4}$$

$$\psi_{02}(x,\lambda)=\begin{cases}\beta^+\cos\lambda(\sigma(\pi)-\sigma(x)) & \\ \qquad +\beta^-\cos\lambda(\sigma(\pi)+\sigma(x)-\pi), & 0<x<\dfrac{\pi}{2},\\ \sin\lambda(\sigma(\pi)-\sigma(x)), & \dfrac{\pi}{2}<x<\pi,\end{cases}\tag{2.5}$$

where $A^\pm=\frac{1}{2}(\beta\pm\frac{1}{\beta})$, $\beta^\pm=\frac{1}{2}(\frac{1}{\beta}\pm\beta)$ and $K_n(x,t)=(K_{ijn}(x,t))_{i,j=1,2}$ $(n=1,2)$ with $K_{ijn}(x,t)$ are real-valued continuous functions for $i,j=1,2$.

Denote

$$\Delta(\lambda):=W[\varphi(x,\lambda),\psi(x,\lambda)]=\varphi_2(x,\lambda)\psi_1(x,\lambda)-\varphi_1(x,\lambda)\psi_2(x,\lambda).\tag{2.6}$$

The function $\Delta(\lambda)$ is called the characteristic function of L, which is entire in λ.

It follows from (2.3)-(2.5) and [10], we have

$$\Delta(\lambda) = \Delta_0(\lambda) + o(\exp|\tau|\sigma(\pi)), \tag{2.7}$$

where $\Delta_0(\lambda) = \beta^+ \sin\lambda\sigma(\pi) - \beta^- \sin\lambda(\sigma(\pi) - \pi)$.

Using the standard method in [2], or referring [10, 11], one can obtain the following Lemma.

Lemma 2.1. 1) *The problem L has an at most countable set of eigenvalues such that all of them are real and simple.*
2) *The eigenvalues denoted by $\{\lambda_n\}_{n\geq 0}$ can be represented by the following asymptotic formula for $n \to \infty$:*

$$\lambda_n = \frac{n\pi}{\sigma(\pi)}\Big(1 + O\Big(\frac{1}{n}\Big)\Big), \quad \lambda \in G_\varepsilon,$$

where $G_\varepsilon := \{\lambda : |\lambda - \lambda_n^0| \geq \varepsilon > 0, n \geq 0\}$.
3) $|\Delta(\lambda)| \geq C_\varepsilon \exp(|\tau|\sigma(\pi)) = C_\varepsilon \exp\left[\frac{(1+\alpha)\pi|\tau|}{2}\right]$ *for $\lambda \in G_\varepsilon$, where C_ε is some constant.*

3. Results

Together with the problem L we consider a boundary value problem $\tilde{L} = L(\tilde{p}(x), \tilde{q}(x), \tilde{\rho}(x), \tilde{\beta})$ of the same form but with the different coefficients $\tilde{p}(x)$, $\tilde{q}(x)$, $\tilde{\rho}(x)$ and $\tilde{\beta}$. We agree that if a certain symbol υ denotes an object related to L, then $\tilde{\upsilon}$ denote the analogous object related to $\tilde{L}$. In this paper the main results are as follows.

Theorem 3.1. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \geq 0$, $Q(x) = \tilde{Q}(x)$ on $\left(0, \frac{(1+\alpha)}{4}\pi\right)$, then $Q(x) = \tilde{Q}(x)$ almost everywhere on $(0, \pi)$.*

Theorem 3.2. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \geq 0$, $Q(x) = \tilde{Q}(x)$ on $\left(\frac{(1+\alpha)}{4}\pi, \pi\right)$, then $Q(x) = \tilde{Q}(x)$ almost everywhere on $(0, \pi)$.*

Before proving the results, we shall mention the following Lemma which will be needed later.

Lemma 3.3. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \geq 0$, then $\rho(x) = \tilde{\rho}(x)$ and $\beta = \tilde{\beta}$.*

Proof. It follows from Lemma 2 that $\alpha = \tilde{\alpha}$, that is $\rho(x) = \tilde{\rho}(x)$. We know that $\Delta(\lambda)$ and $\tilde{\Delta}(\lambda)$ are entire functions of λ of order 1. By the Hadamard's factorization theorem, the characteristic functions can be uniquely determined by the eigenvalues up to multiplicative constants. Similar to [11], since $\lambda_n = \tilde{\lambda}_n$ for all $n \geq 0$, we can get that

$\Delta(\lambda) = C\tilde{\Delta}(\lambda)$, where $C \neq 0$ is some constant. From (2.7), we have $\beta^+ = C\tilde{\beta}^+$ and $\beta^- = C\tilde{\beta}^-$. Thus,

$$\frac{1}{2}\Big(\beta \pm \frac{1}{\beta}\Big) = \frac{C}{2}\Big(\tilde{\beta} \pm \frac{1}{\tilde{\beta}}\Big). \tag{3.1}$$

Consequently, $\beta = C\tilde{\beta}$ and $\frac{1}{\beta} = C\frac{1}{\tilde{\beta}}$. In view of $\beta, \tilde{\beta} > 0$, we can obtain that $\beta = \tilde{\beta}$. □

Proof of Theorem 3.1. By virtue of Lemma 3.3, we know that $\rho(x) = \tilde{\rho}(x)$ and $\beta = \tilde{\beta}$. For convenience, denote $d = \frac{(1+\alpha)\pi}{4}$. Substituting $\lambda = \lambda_n$ into (2.6), we can get that for $n \geq 0$

$$\varphi_2(d, \lambda_n)\psi_1(d, \lambda_n) - \varphi_1(d, \lambda_n)\psi_2(d, \lambda_n) = 0.$$

If $\varphi_2(d, \lambda_n) \neq 0$, then

$$\frac{\varphi_1(d, \lambda_n)}{\varphi_2(d, \lambda_n)} = \frac{\psi_1(d, \lambda_n)}{\psi_2(d, \lambda_n)}, \quad n \geq 0. \tag{3.2}$$

The same relation holds for $\tilde{L}$:

$$\frac{\tilde{\varphi}_1(d, \lambda_n)}{\tilde{\varphi}_2(d, \lambda_n)} = \frac{\tilde{\psi}_1(d, \lambda_n)}{\tilde{\psi}_2(d, \lambda_n)}, \quad n \geq 0. \tag{3.3}$$

Since $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $(0, d)$, we can obtain that $\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda)$. That is, $\varphi_1(x, \lambda) = \tilde{\varphi}_1(x, \lambda)$ and $\varphi_2(x, \lambda) = \tilde{\varphi}_2(x, \lambda)$ for $x \in [0, d]$. Together 3.2 with 3.3, it yields

$$\psi_2(d, \lambda_n)\tilde{\psi}_1(d, \lambda_n) - \tilde{\psi}_2(d, \lambda_n)\psi_1(d, \lambda_n) = 0. \tag{3.4}$$

Note that $\varphi_2(d, \lambda_n) = 0$ implies $\psi_2(d, \lambda_n) = \tilde{\psi}_2(d, \lambda_n) = 0$, so this case also leads to 3.4.

Define

$$A(\lambda) = \psi_2(d, \lambda)\tilde{\psi}_1(d, \lambda) - \tilde{\psi}_2(d, \lambda)\psi_1(d, \lambda).$$

It is obvious that $A(\lambda)$ has zeros $\{\lambda_n\}_{n\geq 0}$. Next, we will show that $A(\lambda) \equiv 0$ in the whole complex plane.

From (2.3)-(2.5) and the similar representations for $\tilde{\psi}_1(x, \lambda)$ and $\tilde{\psi}_2(x, \lambda)$, we have

$$A(\lambda) = O(\exp 2|\tau|(\sigma(\pi) - \sigma(d))) = O(\exp |\tau|\sigma(\pi)), \quad |\lambda| \to \infty. \tag{3.5}$$

Define $G(\lambda) := \frac{A(\lambda)}{\Delta(\lambda)}$, which is entire in $\mathbb{C}$. It follows from (3.5) and 3) in Lemma 2.1 that

$$|G(\lambda)| \leq B_1, \quad \text{for } \lambda \in G_\varepsilon,$$

where B_1 is some a positive constant. Thus, by Liouville's theorem, we know that $G(\lambda)$ is constant. Furthermore, it follows from (2.3)-(2.5) and Riemann-Lebesque Lemma that for $\lambda \in \mathbb{R}$,

$$\lim_{\lambda \to \infty} G(\lambda) = 0,$$

which means $G(\lambda) = 0$. Thus, $A(\lambda) = 0$ for all λ in $\mathbb{C}$. Hence

$$\frac{\psi_2(d,\lambda)}{\psi_1(d,\lambda)} = \frac{\tilde{\psi}_2(d,\lambda)}{\tilde{\psi}_1(d,\lambda)}.$$

Note that $\frac{\psi_2(d,\lambda)}{\psi_1(d,\lambda)}$ is the Weyl function, defined in [1], of the boundary value problem for (1.1) on (d, π) with $y_1(d, \lambda) = 0$ and the jump condition (1.4). It has been proved in [1] that the Weyl function can uniquely determine the $p(x)$ and $q(x)$ on (d, π). Thus, we can get that $Q(x) = \tilde{Q}(x)$ a.e. on (d, π). This completes the proof. □

Proof of Theorem 3.2. By Theorem 3.1 and Lemma 3.3, we have $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$, $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on (d, π). So, $\psi(x, \lambda) = \tilde{\psi}(x, \lambda)$ on (d, π). From (3.2) and (3.3), we show that

$$\varphi_1(d, \lambda_n)\tilde{\varphi}_2(d, \lambda_n) - \tilde{\varphi}_1(d, \lambda_n)\varphi_2(d, \lambda_n) = 0.$$

From (2.1)-(2.2) and the similar representations for $\tilde{\varphi}_1(x, \lambda)$ and $\tilde{\varphi}_2(x, \lambda)$, we have

$$A_1(\lambda) = O(\exp 2|\tau|\sigma(d)) = O(\exp|\tau|\sigma(\pi)), \quad |\lambda| \to \infty. \tag{3.6}$$

Define $G_1(\lambda) := \frac{A_1(\lambda)}{\Delta(\lambda)}$, which is entire in $\mathbb{C}$. It follows from (3.6) and 3) in Lemma 2.1 that

$$|G_1(\lambda)| \le B_2, \quad \text{for } \lambda \in G_\varepsilon,$$

where B_2 is some a positive constant. Following the proof of Theorem 3.1, we have $A_1(\lambda) = 0$ for all λ in $\mathbb{C}$, so

$$\frac{\varphi_2(d,\lambda)}{\varphi_1(d,\lambda)} = \frac{\tilde{\varphi}_2(d,\lambda)}{\tilde{\varphi}_1(d,\lambda)}.$$

Note that $\frac{\varphi_2(d,\lambda)}{\varphi_1(d,\lambda)}$ is the Weyl function, defined in [1], of the boundary value problem for (1.1) on $(0, d)$ with $y_1(d, \lambda) = 0$ and the jump condition (1.4). It has been proved in [1] that the Weyl function can uniquely determine the $p(x)$ and $q(x)$ on $(0, d)$. Thus, we can get that $Q(x) = \tilde{Q}(x)$ a.e. on $(0, d)$. This completes the proof. □

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