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A numerical method for hypersingular integrals of the first kind

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Abstract

We derive an approximate solution for hypersingular integrals of the first kind. Chebyshev polynomials of the second kind are used to construct the interpolating polynomial. In turn, this polynomial approximates the crack opening displacement function of the density function. A collocation method is implemented, with the zeros of the Chebyshev polynomial of the first kind as the collocation points. As a result of these implementations, the whole integral equation is approximated by a system of algebraic equations which is mathematically tractable. The application and accuracy of the present method are illustrated with some relevant examples.

KEYWORDS:

Chebyshev polynomials, Collocation method, Gauss-Chebyshev quadrature, Hypersingular integrals of the first kind, Interpolation

1 | INTRODUCTION

A number of engineering disciplines often encounter integral equations of the form:

$$\mathop{=\!\!\!\!\!\!\int}_{-1}^{1} \frac{\phi(x)}{(x-\tau)^2}dx + \int_{-1}^{1} k(x,\tau)\phi(x)dx = \pi f(\tau), \quad -1 < \tau < 1, \tag{1}$$

which are more common in the study and modelling of aerodynamics [4], fracture mechanics [6], hydrodynamics [9], [12], [13], to mention just a few. The function $\phi(x)$ in Eq.(1) is unknown and is the function we seek to approximate. The density function $\phi(x)$ should satisfy $\phi(\pm 1) = 0$. The kernel $k(x,\tau)$ is square integrable on the domain $\{(x,\tau) \in [-1,1] \times [-1,1]\}$ and is known. The function $f(x)$ is smooth and is also known.

If we assume that $\phi(x)$ has the Hölder continuous derivative, then the first integral in Eq.(1) is understood in the Hadamard finite-part sense [8], [13], which is given by:

$$\oint_{-1}^{1} \frac{\phi(x)}{(x-\tau)^2} dx = lim_{\epsilon \to 0} \left\{ \int_{-1}^{x-\epsilon} \frac{\phi(x)}{(x-\tau)^2} dx + \int_{x+\epsilon}^{1} \frac{\phi(x)}{(x-\tau)^2} dx - \frac{2\phi(\tau)}{\epsilon} \right\}. \tag{2}$$

The problem of finding an approximate solution of integral equation (1) is not new, see for instance [5], [7], [11], [14], and [15] among others. In this paper we propose a method which does not feature in any of the mentioned works.
Following closely after this introduction are the preliminaries in Section 2, the description of the method in Section 3, Section 4 deals with convergence, illustrative examples are found in Section 5, while the conclusion is in Section 6.

2 | PRELIMINARIES

We give some well known but necessary properties of Chebyshev polynomials of both the first and second kind see for instance [3], [6], [10], and [14]. Let $\mathbb{P}_n$ be a space of polynomials of degree less than or equal to n.

2.1 | Chebyshev polynomials of the first kind

Chebyshev polynomials of the first kind of degree n are usually denoted by $T_n(x)$ and are given by

$$T_n(x) = cos(cos^{-1}(n(x))). \tag{3}$$

Using the convention $\theta = cos^{-1}(x),\ -1 \leq x \leq 1$, one obtains

$$T_n(cos(\theta)) = cos(n\theta). \tag{4}$$

The zeros of $T_n(x)$ are given by

$$x_i = cos\left(\frac{(2i-1)}{2n}\pi\right),\ i = 1, 2, \cdots, n. \tag{5}$$

2.2 | Chebyshev polynomials of the second kind

Chebyshev polynomials of the second kind of degree n are denoted by $U_n(x)$ and are given by

$$U_n(x) = \frac{sin((n+1)cos^{-1}(x))}{sin(cos^{-1}(x))}. \tag{6}$$

Similarly,

$$U_n(cos(\theta)) = \frac{sin((n+1)\theta)}{sin(\theta)}. \tag{7}$$

and

$$|U_n(x)| \leq n + 1. \tag{8}$$

The zeros of $U_n(x)$ are given by

$$x_i = cos\left(\frac{i}{n+1}\pi\right),\ i = 1, 2, \cdots, n. \tag{9}$$

2.3 | Some integrals involving $T_n(x)$ and $U_n(x)$

$$\fint_{-1}^{1} \frac{\sqrt{1-x^2}}{x-\tau} U_n(x)dx = -\pi T_{n+1}(\tau),\ -1<\tau<1,\ n\geq 0. \tag{10}$$

$$\fint_{-1}^{1} \frac{\sqrt{1-x^2}}{(x-\tau)^2} U_n(x)dx = -\pi (n+1) U_n(\tau),\ -1<\tau<1,\ n\geq 0. \tag{11}$$

2.4 | Interpolation error

Theorem 2.1
If $g(x)$ is a sufficiently smooth function defined on $[-1,1]$ and $g_n(x)$ is an interpolating polynomial of $g(x)$, then for each $x\in[-1,1]$, there exists $\xi\in(-1,1)$ such that

$$|g(x)-g_n(x)| \leq |\prod_{i=1}^{n}(x-x_i)| \frac{M_n}{(n+1)!} = E_n(g). \tag{12}$$

where $M_n = max\left\{g^{(n+1)}(\xi) | \xi\in(-1,1)\right\}$.

Remark. In the special case where $\{x_i\}_{i=0}^{n}$ are the zeros of $U_n(x)$, then Eq.(12) becomes

$$|g(x)-g_n(x)| \leq \frac{M_n}{2^n n!} = E_n(g). \tag{13}$$

3 | DESCRIPTION OF THE METHOD

We develop the method for two scenarios, namely

3.1 | Case 1: the kernel $k(x,\tau)=0$.

This reduces Eq.(1) into

$$\fint_{-1}^{1} \frac{\phi(x)}{(x-\tau)^2} dx = \pi f(\tau),\quad -1<\tau<1. \tag{14}$$

As mentioned earlier the density function should vanish at the end-points. We use a smooth unknown function $g(x)$ such that

$$\phi(x) = \sqrt{1-x^2} g(x). \tag{15}$$

In the field of fracture mechanics $g(x)$ models the crack opening displacement along the crack [11]. We then approximate $g(x)$ with an interpolating polynomial $g_n(x)$ defined by

$$g_n(x) = \sum_{i=1}^{n} \frac{g(x_i)U_n(x)}{(x-x_i)U_n'(x_i)} \approx g(x), \tag{16}$$

where x_i's are the zeros of $U_n(x)$. Consequently, we have the following approximation

$$\phi_n(x) = \sqrt{1-x^2} g_n(x) \approx \sqrt{1-x^2} g(x) = \phi(x). \tag{17}$$

If we use Eq.(16) in Eq.(15) and substitute into Eq.(14), we obtain the following approximate equation

$$\fint_{-1}^{1} \frac{\sqrt{1-x^2}\sum_{i=1}^{n} \frac{g(x_i)U_n(x)}{(x-x_i)U_n'(x_i)}}{(x-\tau)^2} dx = \pi f(\tau), \quad -1 < \tau < 1. \tag{18}$$

Upon rearrangement of Eq.(18), we obtain

$$\sum_{i=1}^{n} \frac{g(x_i)}{U_n'(x_i)} \fint_{-1}^{1} \frac{\sqrt{1-x^2}U_n(x)}{(x-x_i)(x-\tau)^2} dx = \pi f(\tau). \tag{19}$$

By applying the theory of partial fractions, we rewrite Eq.(19) as

$$\begin{aligned} \pi f(\tau) =& \sum_{i=1}^{n} \frac{g(x_i)}{U_n'(x_i)(x_i-\tau)^2} \int_{-1}^{1} \frac{\sqrt{1-x^2}U_n(x)}{x-x_i} dx \\ &- \sum_{i=1}^{n} \frac{g(x_i)}{U_n'(x_i)(x_i-\tau)^2} \fint_{-1}^{1} \frac{\sqrt{1-x^2}U_n(x)}{x-\tau} dx \,. \\ &- \sum_{i=1}^{n} \frac{g(x_i)(x_i-\tau)}{U_n'(x_i)(x_i-\tau)^2} \fint_{-1}^{1} \frac{\sqrt{1-x^2}U_n(x)}{(x-\tau)^2} dx \end{aligned} \tag{20}$$

Application of Eq.(10) and Eq.(11) to Eq.(20) gives

$$-\pi \sum_{i=1}^{n} \frac{g(x_i)}{U_n'(x_i)(x_i-\tau)^2} \left(T_{n+1}(x_i) - T_{n+1}(\tau) - (n+1)U_n(\tau)(x_i-\tau)\right) = \pi f(\tau). \tag{21}$$

Collocation of Eq.(21) at the zeros of $T_n(x)$, say x_j, translates the problem into finding the solution vector $g(x_i)$ of the following algebraic equation

$$-\pi \sum_{i=1}^{n} \frac{g(x_i)}{U_n'(x_i)(x_i-x_j)^2} \left(T_{n+1}(x_i) - T_{n+1}(x_j) - (n+1)U_n(x_j)(x_i-x_j)\right) = \pi f(x_j). \tag{22}$$

3.2 | Case 2: the kernel $k(x,\tau) \neq 0$.

Observe that

$$\frac{U_n(x)}{x-x_i} = 2^n \prod_{\substack{l=1 \\ i\neq l}}^{n} (x-x_l), \tag{23}$$

where x_i's and x_l's are the zeros of $U_n(x)$. The hypersingular part of Eq.(1) will be treated as in case 1. We shift our focus to the integral with the kernel $k(x,\tau)$. In view of Eq.(23), we rewrite the integral as

$$\begin{aligned} \int_{-1}^{1} k(x,\tau)\sqrt{1-x^2}g_n(x)dx =& \sum_{i=1}^{n} \frac{g(x_i)}{U_n'(x_i)} 2^n \int_{-1}^{1} \frac{k(x,\tau)(1-x^2)}{\sqrt{1-x^2}} \prod_{\substack{l=1 \\ i\neq l}}^{n} (x-x_l) dx \\ &\approx \sum_{i=1}^{n} \frac{g(x_i)}{U_n'(x_i)} \sum_{k=1}^{N} w_k G_i(t_k,\tau) \end{aligned} , \tag{24}$$

where

$$\begin{cases} w_k=\dfrac{\pi}{N}, \quad k=1,2,\cdots,N, \\ G_i(x,\tau)=2^n k(x,\tau)(1-x^2)\prod_{\substack{l=1\\ i\neq l}}^{n}(x-x_l), \\ t_k=cos\left(\dfrac{2k-1}{2N}\pi\right), \quad k=1,2,\cdots,N. \end{cases} \tag{25}$$

The approximation method in Eq.(24) is called the Gauss-Chebyshev Quadrature Rule and integrates polynomials $P(x)\in\mathbb{P}_{2N-1}$ exactly.

Combining Eq.(21) and Eq.(24), and in view of Eq.(1) we obtain our second algebraic equation, and that is

$$\begin{aligned} \pi f(x_j)=&-\pi\sum_{i=1}^{n}\frac{g(x_i)}{U_n'(x_i)(x_i-x_j)^2}\Big(T_{n+1}(x_i)-T_{n+1}(x_j)-(n+1)U_n(x_j)(x_i-x_j)\Big) \\ &+\sum_{i=1}^{n}\frac{g(x_i)}{U_n'(x_i)}\sum_{k=1}^{N}w_kG_i(t_k,x_j) \end{aligned}. \tag{26}$$

4 | CONVERGENCE ANALYSIS

In this section we show that the approximate solution $\phi_n(x)$ converges to the exact solution $\phi(x)$ in $L^2_\omega[-1,1]$ space.

Definition 4.1. Let $\omega(x)=\sqrt{1-x^2}$ and $L^2_\omega[-1,1]$ be a Hilbert space equipped with a real inner product

$$\langle u(x),v(x)\rangle=\int_{-1}^{1}\omega(x)u(x)v(x)dx \tag{27}$$

and a norm

$$\|u(x)\|_\omega=\left\{\int_{-1}^{1}\omega(x)|u(x)|^2dx\right\}^{\frac{1}{2}}. \tag{28}$$

Theorem 4.1
There exists a positive constant M_ω such that

$$\|\phi(x)-\phi_n(x)\|_\omega\leq M_\omega E_n(g). \tag{29}$$

Proof.

$$\begin{aligned} \|\phi(x)-\phi_n(x)\|_\omega=&\left\{\int_{-1}^{1}\omega(x)|\phi(x)-\phi_n(x)|^2dx\right\}^{\frac{1}{2}} \\ =&\left\{\int_{-1}^{1}\omega(x)|\omega(x)|^2|g(x)-g_n(x)|^2dx\right\}^{\frac{1}{2}} \\ \leq& E_n(g)\left\{\int_{-1}^{1}\omega(x)|\omega(x)|^2dx\right\}^{\frac{1}{2}} \\ =& M_\omega E_n(g) \end{aligned}. \tag{30}$$

where $M_\omega = \|\omega(x)\|_\omega$ and $E_n(g)$ is defined in Eq.(13). □

5 | ILLUSTRATIVE EXAMPLES

We consider two integral equations that are similar to Eq.(1) to demonstrate the application and suitability of the present method.

Example 5.1. [11]

$$\fint_{-1}^{1} \sqrt{1-x^2}\frac{g(x)}{(x-\tau)^2}dx = -\pi e^{\tau}. \tag{31}$$

This integral equation (31) is discussed in detail in [11] and it belongs to case 1. The function $g(x)$ is used to determine the stress intensity factor at the crack tips ± 1. Theoretical values of $g(\pm 1)$ are known see [11]. In Table 1 we compare our results, obtained using Eq.(22) of the present method, to these theoretical results. It is clear from the numerical results that our method

TABLE 1 Results for Example 5.1.

n	$g_n(1)$	$g(1)$	$g_n(-1)$	$g(-1)$
4	1.831223178	1.83122498	0.7009081793	0.70090677
6	1.831224982	1.83122498	0.7009067738	0.70090677

performs very well as it gives the exact solution for $n = 6$.

Example 5.2. [7]

$$\fint_{-1}^{1} \sqrt{1-x^2}\frac{g(x)}{(x-\tau)^2}dx + \int_{-1}^{1} \tau x\sqrt{1-x^2}g(x)dx = \pi\left(-8\tau^3 + \frac{17}{8}\tau - 1\right). \tag{32}$$

According to [7] the exact solution of Eq.(32) has $g(x) = 1 + 2x^3$. In Table 2 we compare the results, obtained using Eq.(26), to the exact solution. The numerical results are exact and this demonstrates the accuracy of our method. In Fig. 1 we plot the

TABLE 2 Results for Example 5.2, with $n = 4$ and $N = 4$.

x	$g_n(x)$	$g(x)$
-1.0000	−1.0000	−1.0000
-0.5000	0.7500	0.7500
0.0000	1.0000	1.0000
0.5000	1.2500	1.2500
1.0000	3.0000	3.0000

absolute error for Example 2. It is observed that the error remains at zero throughout the interval $[-1, 1]$. The exact solution $g(x)$ and the approximate solution $g_n(x)$ are plotted in Fig. 2.

6 | CONCLUSION

We have constructed a numerical method based on Chebyshev polynomials of the first and second kind. Using interpolation and collocation methods, the problem is translated into a linear system of algebraic equations. Convergence is proved and as *n* increases, the error decreases. Numerical illustrations demonstrate the accuracy and efficiency of the present method. Our results suggest that this method would, without a doubt, be very useful to researchers in the filed of fracture mechanics.

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