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RESEARCH ARTICLE

Modified subgradient extragradient method for approximating of the system of general equilibrium problem and fixed point problem

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Abstract

In this paper, we introduce the system of general equilibrium problem (SGEP) and new subgradient extragradient by using the concept of the set of solution of the modified variational inequality problem introduced by[3]. Then, we establish and prove weak and strong convergence theorem of the new subgradient extragradient algorithm for finding the set of the solutions of the SGEP under some suitable conditions of α_n and β_n with $\alpha_n + \beta_n \leq 1$. Moreover, we apply our main theorem to prove weak and strong convergence theorems for finding solutions of the generalized equilibrium problem, the system of equilibrium problem, the variational inequality problem and the general system of variational inequality problem. In the last section, we give three numerical examples to support our main result.

KEYWORDS:
equilibrium problem, subgradient extragradient algorithm, variational inequality problem, fixed point problem

1 | INTRODUCTION

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H.

The equilibrium problem is to find a point $x \in C$

$$F(x, y) \geq 0, \qquad \forall y \in C,$$

where $F : C \times C \to \mathbb{R}$ is bifunction. The set of all solutions of the equilibrium problem is denoted by $EP(F)$. Many problems in physic, optimization and economic are seeking some elements of $EP(F)$, see more detail in[1,8]. Over decades ago, there are many researches modified the equilibrium problems, see for instance[3,9].

For solving the equilibrium problems for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

$(A1)$ $F(x, x) = 0, \qquad \forall x \in C,$

$(A2)$ F *is monotone*, i.e., $F(x, y) + F(y, x) \leq 0, \qquad \forall x, y \in C,$

$(A3)$ $\forall x, y, z \in C,$

$$\lim_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x, y),$$

$(A4)$ $\forall x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Blum and Oettli[8] have proved the following lemma, which as a tool to solve equilibrium problems.

[0]**Abbreviations:** ANA, anti-nuclear antibodies; APC, antigen-presenting cells; IRF, interferon regulatory factor

Lemma 1. (See[8]) Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \qquad \forall x \in C.$$

Inspired and motivated by the concept of the $EP(F)$, we introduce *the system of general equilibrium problem (SGEP)*, which is to find $(x^*, y^*) \in C \times C$, such that

$$\begin{cases} F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - y^* + \lambda A y^*\rangle \geq 0, & \forall y \in C, \\ Q(y^*, x) + \frac{1}{r}\langle x - y^*, y^* - x^* + \beta B x^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1}$$

where $F, Q : C \times C \to \mathbb{R}$ are a bifunction and $A, B : C \to H$ are mappings, $\lambda, \beta, r > 0$ are three constants. In particular, if we put $F \equiv Q \equiv 0$ and $A \equiv B$, then the problem (1) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \beta A x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{2}$$

which is introduced by Verma[10], in 1999, and is called *the new system of variational inequalities problem*. Further, if we add up the requirement that $x^* = y^*$, then the problem (2) reduces to finding a point $x^* \in C$ such that

$$\langle A(x^*), x - x^*\rangle \geq 0, \qquad \forall x \in C, \tag{3}$$

which is introduced by Lions and Stampacchia[11], in 1964, and is called *the variational inequality problem (VIP)*. The set of all solutions of the variational inequality problem is denoted by $VI(C, A)$. Numerous problems in physic, game theory, finance, optimization and mechanics reduce to find an element of (3), see more detail in[11,12,13,14].

In 2013, Kangtunyakarn[3] modified the set of variational inequality as follows:

$$VI(C, aA + (1-a)B) = \{x \in C : \langle y - x, (aA + (1-a)B)x\rangle \geq 0, \ \ \forall y \in C, \ a \in (0,1)\}, \tag{4}$$

where $A, B : C \to H$ be two mappings. In particular, if we put $A \equiv B$, then the problem (3) is a special case of the problem (4) and he proved a strong convergence theorem for finding a common element of the set of fixed point problems of infinite family of strictly pseudo contractive mappings and the set of equilibrium problem and two set of variational inequality problems, which is related to (4) under suitable condition, see more detail in[3].

In 1953, Mann[15] introduced *Mann iteration* and is defined as follows:

$$\begin{cases} x_0 \in H \ \text{ arbitrary chosen}, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, & \forall n \geq 0, \end{cases} \tag{5}$$

where C is a nonempty closed convex subset of a normed space, $T : C \to C$ is a mapping and the sequence $\{\alpha_n\}$ is in the interval $(0, 1)$. If T is nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$) and some suitable condition of α_n then $\{x_n\}$ from algorithm (5) as only weakly converge to the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$). Thus, many mathematicians have been trying to modify Mann's iteration (5) and creat new iterative method to obtain their strong convergence theorem, see more detail in[16,17,18].

In 2000, Moudafi[16] introduced *the viscosity approximation method for nonexpansive mapping S* to prove $\{x_n\}$ converges strongly to $z = P_{F(S)} f(z)$ and the sequence $\{x_n\}$ generated by

$$\begin{cases} x_1 \in C \ \text{ arbitrary chosen}, \\ x_{n+1} = \frac{1}{1+\epsilon_n} S x_n + \frac{\epsilon_n}{1+\epsilon_n} f(x_n), & \forall n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\{\epsilon_n\} \subset (0, 1)$ satisfies certain conditions, $S : C \to C$ is a nonexpansive mapping and $f : C \to C$ is a contraction (i.e., there exists $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|, \forall x, y \in C$). Moreover, the viscosity approximation method for nonexpansive mapping S has been studied and developed in many reserchs, see previous studies in[20,19]. Notice that the sum of coefficients $\frac{1}{1+\epsilon_n}$ and $\frac{\epsilon_n}{1+\epsilon_n}$ in (6) is equal 1.

In 2017, Kanzow and Shehu[21] proved the strong convergence for a modified inexact Krasnoselskii-Mann iteration, which the sum of coefficients α_n, β_n and δ_n in (7) less than or equal 1 as follows:

$$x_{n+1} = \delta_n u + \alpha_n x_n + \beta_n T x_n + r_n, \qquad \forall n \geq 1, \tag{7}$$

where $x_1 \in H$, $u \in C$ denotes a fixed vector, r_n represents the residual, and the nonnegative real numbers $\alpha_n, \beta_n, \delta_n$ are chosen such that $\alpha_n+\beta_n+\delta_n \leq 1, n \geq 1$, and $T : H \to C$ is a nonexpansive mapping. Then the sequence $\{x_n\}$ generated by (7) strongly converges to a point in $F(T)$, which is the nearest point projection of u onto $F(T)$. Observe that this theorem more general than modified Halpern's iterative scheme.

In 1976, Korpelevich[22] proposed an algorithm for solving the VIP in Euclidean space, this method is called *the Extragradient Method* (see also[23]). In each iteration of her algorithm, in order to get the next iterate x_{k+1}, two orthogonal projections onto C are calculated, according to the following iterative step. Let $\{x_k\}$ and $\{y_k\}$ be the sequences generated by the following extragradient algorithm:

$$\begin{cases} y_k = P_C(x_k - \tau f(x_k)), \\ x_{k+1} = P_C(x_k - \tau f(y_k)), \end{cases}$$

where τ is some positive number and P_C denotes the Euclidean least distance projection onto C.

In 2011, Censor et al.[24] modified Korpelevich's method[22] by replacing the second projection onto the closed and convex subset C of Hilbert space with the one onto the subgradient half-space (T_k). This method is called the subgradient extragradient method. For the variational inequality, the subgradient extragradient is of the form

$$\begin{cases} x_0 \in H, \\ y_k = P_C(x_k - \tau f(x_k)), \\ T_k := \{w \in H | \langle (x_k - \tau f(x_k)) - y_k, w - y_k\rangle \leq 0\}, \\ x_{k+1} = P_{T_k}(x_k - \tau f(y_k)), \end{cases} \tag{8}$$

where $f : H \to H$ is Lipschitz continuous on C with constant $L > 0$ (i.e., $\|f(x) - f(y)\| \leq L\|x - y\|, \forall x, y \in C$) and $\tau \in (0, \frac{1}{L})$. Censor et al.[24] proved that the $\{x_n\}$ generated by (8) converges weakly to a solution of the variational inequality and used Lemma 2 for proof the strong convergence theorem of this iteration.

Lemma 2. (See[25]) Let H be a real Hilbert space and let D be a nonempty, closed and convex subset of H. Let the sequence $\{x_k\}_{k=0}^{\infty} \subset H$ be Fejér-monotone with respect to D, i.e., for every $u \in D$,

$$\|x_{k+1} - u\| \leq \|x_k - u\|, \qquad \forall k \geq 0.$$

Then $\{P_D(x_k)\}_{k=0}^{\infty}$ converges strongly to some $z \in D$.

Remark 1. Set T_k generated by the set of solution of VIP.

Inspired and motivated by problem (4), Censor et al.[24] and Kanzow and Shehu[21], we now present the new subgradient extragradient algorithm and the new iterative method for prove weak and strong convergence theorem of $\{x_n\}$ generated by the following algorithm:

Algorithm 1.1. Given $x_1 \in C$, let the sequence $\{x_n\}$ and $\{y_n\}$ be define by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) + \beta_n \varphi(x_n), \end{cases} \tag{9}$$

where $A, B, \overline{A}, \overline{B} : C \to H$ are $a, b, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively, $Q, F : C \times C \to \mathbb{R}$ are a bifunction satisfying A1)-A4), the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are in $[0,1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \geq 1$, $\eta = \min\{a, b\}$, $\lambda, \beta \in (0, 2\eta)$, $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$ and $a \in (0,1)$. Define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$, where $r, \beta, \lambda > 0$, for all $x \in C$, and T_r^F, T_r^Q define as same in Lemma 4 that is $T_r^F(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$ and $T_r^Q(x) = \{z \in C : Q(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$.

Moreover, we give a lemma that is more comprehensive than Lemma 2[25], which important tool for proof the strong convergence theorem of iteration $\{x_n\}$ generated by (9) show the next section.

The paper is therefore organized as follows: We first recall some basic definitions and we give a lemma, which is an important tool for proof weak and strong convergence of our main theorem in Sect. 2. We prove weak and strong convergence theorem of the new subgradient extragradient algorithm for finding the set of the solutions of the SGEP under some suitable conditions of α_n and β_n with $\alpha_n + \beta_n \leq 1$ in Sect. 3. An application, we apply our main theorem to prove weak and strong convergence theorems for finding solutions of the generalized equilibrium problem, the system of equilibrium problem, the variational inequality problem and the general system of variational inequality problem in Sect. 4. We give three numerical examples to support our main result in the last section.

2 | PRELIMINARIES

We write $x_k \rightharpoonup x$ to indicate that the sequence $\{x_k\}_{k=0}^{\infty}$ converges weakly to x and $x_k \to x$ to indicate that the sequence $\{x_k\}_{k=0}^{\infty}$ converges strongly to x. For each point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$. That is,

$$\|x - P_C(x)\| \leq \|x - y\|, \qquad \forall y \in C.$$

The mapping $P_C : H \to C$ is called *the metric projection of H onto C*. It is well known that P_C is a nonexpansive mapping of H onto C, i.e.,

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|, \qquad \forall x, y \in H.$$

The metric projection P_C is characterized[5] by the following two properties:

$$P_C(x) \in C$$

and

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0, \qquad \forall x \in H,\ y \in C. \tag{10}$$

and if C is a hyperplane, then (10) becomes an equality. It follows that

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \qquad \forall x \in H,\ y \in C.$$

Lemma 3. (See[4]) For a given $z \in H$ and $u \in C$,

$$u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0, \qquad \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C.

Lemma 4. (See[1]) Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$as follows:

$$T_z(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\},$$

for all $z \in H$. Then, the following hold:
(1) T_r is single-valued,
(2) T_r is firmly nonexpansive i.e.,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle, \quad \forall x, y \in H,$$

(3) $F(T_r) = EP(F)$,
(4) $EP(F)$ is closed and convex.

Lemma 5. (See[21]) Let X be a real inner product space. Then:
(a) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \qquad \forall x, y \in X.$
(b) $\|tx + sy\|^2 = t(t + s)\|x\|^2 + s(t + s)\|y\|^2 - st\|x - y\|^2, \qquad \forall x, y \in X, \forall s, t \in \mathbb{R}.$

Lemma 6. Let C be a nonempty closed convex subset of a real Hilbert spaces and let $F, Q : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $r > 0$, then the following equivalent.

(i) (x^*, y^*) is a solution of (1),

(ii) x^* is a fixed point of a mapping $\varphi : C \to C$ defined by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$ for all $\beta, \lambda > 0$ and $x \in C$, where $y^* = T_r^Q(I - \beta B)x^*$.

Proof. Let the following conditions hold. (i)⇒(ii) Let (x^*, y^*) be a solution of (1). For every $x, y \in C$, we obtain

$$F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - y^* + \lambda A y^*\rangle \geq 0,$$
$$Q(y^*, x) + \frac{1}{r}\langle x - y^*, y^* - x^* + \beta B x^*\rangle \geq 0.$$

From Lemma 4, we have
$$T_r^F(I-\lambda A)y^* = x^*, \tag{11}$$
and
$$T_r^Q(I-\beta B)x^* = y^*. \tag{12}$$
From (11) and (12), we have $x^* = T_r^F(I-\lambda A)T_r^Q(I-\beta B)x^* = \varphi(x^*)$.
Hence $x^* \in F(\varphi)$, where $y^* = T_r^Q(I-\beta B)x^*$.

(ii)⇒(i) Let $x^* \in F(\varphi)$ and $y^* = T_r^Q(I-\beta B)x^*$, we get
$$x^* = T_r^F(I-\lambda A)T_r^Q(I-\beta B)x^* = T_r^F(I-\lambda A)y^*. \tag{13}$$
From (13) and $y^* = T_r^Q(I-\beta B)x^*$, we have
$$\begin{aligned} F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - y^* + \lambda A y^*\rangle \geq 0, \qquad \forall y \in C,\\ Q(y^*, x) + \frac{1}{r}\langle x - y^*, y^* - x^* + \beta B x^*\rangle \geq 0, \qquad \forall x \in C. \end{aligned}$$
Then (x^*, y^*) is a solution of (1). □

Lemma 7. Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying
$$a_{n+1} \leq a_n + b_n, \qquad for\ all\ \ n \geq 0.$$

(i) If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

(ii) If $\sum_{n=0}^{\infty} b_n < \infty$ and $\{a_n\}_{n=0}^{\infty}$ has a subsequence converging to zero, then
$$\lim_{n\to\infty} a_n = 0.$$

Lemma 8. (See[3]) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α and β-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C, A) \bigcap VI(C, B) \neq \emptyset$. Then
$$VI(C, aA + (1-a)B) = VI(C, A) \bigcap VI(C, B), \qquad \forall a \in (0, 1).$$
Furthermore if $0 < \gamma < \min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1-a)B)$ is a nonexpansive mapping.

Remark 2. It is well known that $(aA + (1-a)B)$ is η-inverse strongly monotone, where $\eta = \min\{\alpha, \beta\}$.

Lemma 9. Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $\overline{A}, \overline{B} : C \to H$ be $\bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $x^* \in VI(C, \overline{A}) \cap VI(C, \overline{B})$, $\gamma \leq \bar{\eta} = min\{\bar{\alpha}, \bar{\beta}\}$ and $\bar{a} \in (0, 1)$, we have
$$\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \frac{\gamma}{\bar{\eta}})\|x_n - y_n\|^2 - (1 - \frac{\gamma}{\bar{\eta}})\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\|^2,$$
where sequence $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 1.1.

Proof. Let $x^* \in VI(C, \overline{A}) \cap VI(C, \overline{B})$.
By property of P_C, we have
$$\begin{aligned} \|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*\|^2 \leq& \|x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n - x^*\|^2 \\ &- \|x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\ =& \|x_n - x^*\|^2 - 2\gamma\langle x_n - x^*, (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\rangle + \gamma^2\|(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\|^2 \\ &- \Big(\|x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 + \gamma^2\|(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\|^2 \\ &- 2\gamma\langle x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n), (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\rangle\Big) \\ =& \|x_n - x^*\|^2 - \|x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\ &- 2\gamma\langle P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*, (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\rangle. \end{aligned} \tag{14}$$

From monotonicity of $(\bar{a}\overline{A} + (1-\bar{a})\overline{B})$, we have

$$\begin{aligned}
0 \leq & \langle (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n - (\bar{a}\overline{A} + (1-\bar{a})\overline{B})x^*, y_n - x^* \rangle \\
= & \langle (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - x^* \rangle - \langle (\bar{a}\overline{A} + (1-\bar{a})\overline{B})x^*, y_n - x^* \rangle \\
\leq & \langle (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - x^* \rangle \\
= & \langle (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \rangle + \langle (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^* \rangle .
\end{aligned}$$

It implies that

$$\langle (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, x^* - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \rangle \leq \langle (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \rangle . \quad (15)$$

From (14) and (15), we get

$$\begin{aligned}
\| P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^* \|^2 \leq & \| x_n - x^* \|^2 - \| x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \|^2 \\
& - 2\gamma \langle P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*, (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n \rangle \\
\leq & \| x_n - x^* \|^2 - \| x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \|^2 \\
& + 2\gamma \langle (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \rangle \\
= & \| x_n - x^* \|^2 - \| x_n - y_n \|^2 - \| y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \|^2 \\
& - 2\langle x_n - y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \rangle \\
& + 2\gamma \langle (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \rangle \\
= & \| x_n - x^* \|^2 - \| x_n - y_n \|^2 - \| y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \|^2 \\
& + 2\langle y_n - x_n + \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \rangle \\
= & \| x_n - x^* \|^2 - \| x_n - y_n \|^2 - \| y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \|^2 \\
& + 2\langle x_n - y_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n \rangle \\
= & \| x_n - x^* \|^2 - \| x_n - y_n \|^2 - \| y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \|^2 \\
& + 2\langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n \rangle \\
& + 2\langle \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n \rangle \\
\leq & \| x_n - x^* \|^2 - \| x_n - y_n \|^2 - \| y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \|^2 \\
& + 2\langle \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n \rangle \\
\leq & \| x_n - x^* \|^2 - \| x_n - y_n \|^2 - \| y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \|^2 \\
& + 2\gamma \| (\bar{a}\overline{A} + (1-\bar{a})\overline{B})x_n - (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n \| \| P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n \| \\
\leq & \| x_n - x^* \|^2 - \| x_n - y_n \|^2 - \| y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \|^2 \\
& + 2(\frac{\gamma}{\bar{\eta}}) \| x_n - y_n \| \| P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n \| \\
\leq & \| x_n - x^* \|^2 - \| x_n - y_n \|^2 - \| y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) \|^2 \\
& + \frac{\gamma}{\bar{\eta}}(\| x_n - y_n \|^2 + \| P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n \|^2) \\
= & \| x_n - x^* \|^2 - (1 - \frac{\gamma}{\bar{\eta}}) \| x_n - y_n \|^2 - (1 - \frac{\gamma}{\bar{\eta}}) \| P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n \|^2 .
\end{aligned}$$

□

Lemma 10. (See[4]) Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \iff u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Now, we present the following Lemma 11 for proving strong convergence theorem.

Lemma 11. Let H be a real Hilbert space and let S be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in S$,

$$\|x_{n+1} - u\| \leq \|x_n - u\| + b_n,$$

for every $n = 0, 1, 2, ...$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\{P_S x_n\}$ converges strongly to some $z \in S$.

Proof. Let $\epsilon > 0$. From Lemma 7, then $\lim_{n\to\infty} \|x_n - u\|$ exists and we have that $\{\|x_n - u\|\}$ is bounded, for all $u \in S$.
Put $u_n = P_S x_n$. We get

$$\begin{aligned}\|u_{n+1} - u_n\|^2 =& 2\|x_{n+1} - u_{n+1}\|^2 + 2\|x_{n+1} - u_n\|^2 - 4\|x_{n+1} - \frac{1}{2}(u_{n+1} + u_n)\|^2 \\ \leq& 2\|x_{n+1} - u_{n+1}\|^2 + 2\|x_{n+1} - u_n\|^2 - 4\|x_{n+1} - u_{n+1}\|^2 \\ =& 2\|x_{n+1} - u_n\|^2 - 2\|x_{n+1} - u_{n+1}\|^2 \\ \leq& 2\|x_n - u_n\|^2 - 2\|x_{n+1} - u_{n+1}\|^2 + 2b_n. \end{aligned} \tag{16}$$

It implies that

$$\|x_{n+1} - u_{n+1}\|^2 \leq \|x_n - u_n\|^2 + b_n.$$

From Lemma 7, then $\{\|x_n - u_n\|\}$ there exists.
By (16), we have $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$, then there exist $N \in \mathbb{N}$, such that $\|u_{n+1} - u_n\| \leq \frac{\epsilon}{2^n}$, for all $n \geq N$.
Thus, for every $p \in \mathbb{N}$, we have

$$\|u_{n+p} - u_n\| \leq \sum_{k=n}^{n+p-1} \|u_{k+1} - u_k\| \leq \epsilon \sum_{k=n}^{n+p-1} \frac{1}{2^k} \leq \epsilon(\frac{1}{2^{n-1}}) < \epsilon. \tag{17}$$

From (17), we have that $\{u_n\}$ is a cauchy sequence. Hence, $\{u_n\}$ converges strongly to some $z \in S$. □

3 | MAIN RESULTS

In this section, we prove weak and strong convergence of the new subgradient extragradient algorithm for finding the set of the solutions of the SGEP.

Theorem 1. Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $A, B, \overline{A}, \overline{B} : C \to H$ be $a, b, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $Q, F : C \times C \to \mathbb{R}$ be a bifunction satisfying A1)-A4). Define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$, where $r, \beta, \lambda > 0$ and for all $x \in C$, T_r^F, T_r^Q define as same in Lemma 4 that is $T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$ and $T_r^Q(x) = \{z \in C : Q(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi) \neq \emptyset$. For given $x_1 \in C$ and let the sequence $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B})y_n) + \beta_n \varphi(x_n), \end{cases} \tag{18}$$

where sequence $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$ and $\eta = \min\{a, b\}$, $\lambda, \beta \in (0, 2\eta)$, $\gamma \leq \bar{\eta} = min\{\bar{\alpha}, \bar{\beta}\}$, $\bar{a} \in (0, 1)$ satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$ for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_\xi(x_n).$$

Proof. First, we show that φ is a nonexpansive mapping for every $\lambda, \beta \in (0, 2\eta)$, where $\eta = \min\{a, b\}$. Let $x, y \in C$. Since A is a-inverse strongly monotone, we have

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 =&\|x-y-\lambda(Ax-Ay)\|^2\\ =&\|x-y\|^2-2\lambda\langle x-y, Ax-Ay\rangle+\lambda^2\|Ax-Ay\|^2\\ \leq&\|x-y\|^2-2\lambda a\|Ax-Ay\|^2+\lambda^2\|Ax-Ay\|^2\\ =&\|x-y\|^2-\lambda(2a-\lambda)\|Ax-Ay\|^2\\ \leq&\|x-y\|^2-\lambda(2\eta-\lambda)\|Ax-Ay\|^2\\ \leq&\|x-y\|^2. \end{aligned} \tag{19}$$

Thus $(I-\lambda A)$ is a nonexpansive mapping. By using the same method as (19), we have $(I-\beta B)$ is a nonexpansive mapping. Hence, $T_r^F(I-\lambda A)$ and $T_r^Q(I-\beta B)$ are nonexpansive mappings. It is easy to see that the mapping φ is a nonexpansive mapping.

Let $x^* \in VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi)$ and $\gamma \leq \bar{\eta} = min\{\bar{\alpha}, \bar{\beta}\}$. We divide the proof of this result into 3 steps.

Step 1. Show that $\{x_n\}_{n=0}^{\infty}$ is bounded.

From the definition of x_n, Lemma 5 and Lemma 9, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 =&\|\alpha_n P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)+\beta_n\varphi(x_n)-x^*\|^2\\ =&\|\alpha_n(P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*)+\beta_n(\varphi(x_n)-x^*)-(1-\alpha_n-\beta_n)x^*\|^2\\ \leq&\|\alpha_n(P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*)+\beta_n(\varphi(x_n)-x^*)\|^2-2(1-\alpha_n-\beta_n)\langle x^*, x_{n+1}-x^*\rangle\\ \leq&\|\alpha_n(P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*)+\beta_n(\varphi(x_n)-x^*)\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ =&\alpha_n(\alpha_n+\beta_n)\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*\|^2+\beta_n(\alpha_n+\beta_n)\|\varphi(x_n)-x^*\|^2\\ &-\alpha_n\beta_n\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-\varphi(x_n)\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ \leq&\alpha_n(\alpha_n+\beta_n)\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*\|^2+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2\\ &+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ \leq&\alpha_n(\alpha_n+\beta_n)\Big(\|x_n-x^*\|^2-(1-\frac{\gamma}{\bar{\eta}})\|x_n-y_n\|^2-(1-\frac{\gamma}{\bar{\eta}})\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\|^2\Big)\\ &+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ \leq&\alpha_n(\alpha_n+\beta_n)\|x_n-x^*\|^2+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ =&(\alpha_n+\beta_n)^2\|x_n-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ \leq&(\alpha_n+\beta_n)^2\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|x^*\|^2+(1-\alpha_n-\beta_n)\|x_{n+1}-x^*\|^2, \end{aligned}$$

which implies that

$$\|x_{n+1}-x^*\|^2 \leq \|x_n-x^*\|^2+\frac{1-\alpha_n-\beta_n}{\alpha_n+\beta_n}\|x^*\|^2, \tag{20}$$

there exists $M > 0$, such that

$$\|x_{n+1}-x^*\|^2 \leq \|x_n-x^*\|^2+(1-\alpha_n-\beta_n)M\|x^*\|^2. \tag{21}$$

By (21) and Lemma 7, then $\lim_{n\to\infty}\|x_n-x^*\|, \forall x^* \in \xi$ exists. So, we have the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded.

Step 2. Show that $\lim_{n\to\infty}\|\varphi(x_n)-x_n\| = 0$.

Let $W_n = x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n$.

From the definition of x_n, Lemma 5 and Lemma 9, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\quad=\|\alpha_n(P_{Q_n}W_n-x^*)+\beta_n(\varphi(x_n)-x^*)-(1-\alpha_n-\beta_n)x^*\|^2\\
&\quad\le\|\alpha_n(P_{Q_n}W_n-x^*)+\beta_n(\varphi(x_n)-x^*)\|^2-2(1-\alpha_n-\beta_n)\langle x^*,x_{n+1}-x^*\rangle\\
&\quad\le\|\alpha_n(P_{Q_n}W_n-x^*)+\beta_n(\varphi(x_n)-x^*)\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
&\quad\le\alpha_n(\alpha_n+\beta_n)\|P_{Q_n}W_n-x^*\|^2+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2-\alpha_n\beta_n\|P_{Q_n}W_n-\varphi(x_n)\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
&\quad\le\alpha_n(\alpha_n+\beta_n)\Big(\|x_n-x^*\|^2-(1-\frac{\gamma}{\bar\eta})\|x_n-y_n\|^2-(1-\frac{\gamma}{\bar\eta})\|P_{Q_n}W_n-y_n\|^2\Big)+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2\\
&\qquad-\alpha_n\beta_n\|P_{Q_n}W_n-\varphi(x_n)\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
&\quad=(\alpha_n+\beta_n)^2\|x_n-x^*\|^2-\alpha_n(\alpha_n+\beta_n)(1-\frac{\gamma}{\bar\eta})\Big(\|x_n-y_n\|^2+\|P_{Q_n}W_n-y_n\|^2\Big)\\
&\qquad-\alpha_n\beta_n\|P_{Q_n}W_n-\varphi(x_n)\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|,
\end{aligned}$$

which yields that

$$\begin{aligned}
&\alpha_n(\alpha_n+\beta_n)(1-\frac{\gamma}{\bar\eta})\Big(\|x_n-y_n\|^2+\|P_{Q_n}W_n-y_n\|^2\Big)+\alpha_n\beta_n\|P_{Q_n}W_n-\varphi(x_n)\|^2\\
&\qquad\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|.
\end{aligned}\tag{22}$$

From (22), $\lim_{n\to\infty}(\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2)=0$ and condition (ii), we have

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|P_{Q_n}W_n-y_n\|=\lim_{n\to\infty}\|P_{Q_n}W_n-\varphi(x_n)\|=0.\tag{23}$$

Since,

$$\|x_n-\varphi(x_n)\|\le\|x_n-y_n\|+\|y_n-P_{Q_n}W_n\|+\|P_{Q_n}W_n-\varphi(x_n)\|,$$

and (23), we get

$$\lim_{n\to\infty}\|\varphi(x_n)-x_n\|=0.\tag{24}$$

Step 3. Show that $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z\in\xi$ and $z=\lim_{n\to\infty}P_{\xi}(x_n)$.
Therefore it has at least one weak accumulation point. If $\bar x$ is a weak limit point of some subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$, then $x_{n_k}\rightharpoonup\bar x$ as $k\to\infty$.

Assume that $\bar x\ne\varphi(\bar x)$. By nonexpansiveness of φ, (24) and Opial's property, we have

$$\begin{aligned}
\liminf_{k\to\infty}\|x_{n_k}-\bar x\|&<\liminf_{k\to\infty}\|x_{n_k}-\varphi(\bar x)\|\\
&\le\liminf_{k\to\infty}(\|x_{n_k}-\varphi(x_{n_k})\|+\|\varphi(x_{n_k})-\varphi(\bar x)\|)\\
&\le\liminf_{k\to\infty}(\|x_{n_k}-\varphi(x_{n_k})\|+\|x_{n_k}-\bar x\|)\\
&\le\liminf_{k\to\infty}\|x_{n_k}-\bar x\|.
\end{aligned}$$

This is a contradiction, then we have

$$\bar x\in F(\varphi).\tag{25}$$

Assume that $\bar x\ne P_C(I-\gamma(\bar a\overline{A}+(1-\bar a)\overline{B}))\bar x$. By nonexpansiveness of $P_C(I-\gamma(\bar a\overline{A}+(1-\bar a)\overline{B}))$, (23) and Opial's property, we have

$$\begin{aligned}
\liminf_{k\to\infty}\|x_{n_k}-\bar x\|&<\liminf_{k\to\infty}\|x_{n_k}-P_C(I-\gamma(\bar a\overline{A}+(1-\bar a)\overline{B}))\bar x\|\\
&\le\liminf_{k\to\infty}(\|x_{n_k}-P_C(I-\gamma(\bar a\overline{A}+(1-\bar a)\overline{B}))x_{n_k}\|\\
&\quad+\|P_C(I-\gamma(\bar a\overline{A}+(1-\bar a)\overline{B}))x_{n_k}-P_C(I-\gamma(\bar a\overline{A}+(1-\bar a)\overline{B}))\bar x\|)\\
&\le\liminf_{k\to\infty}(\|x_{n_k}-P_C(I-\gamma(\bar a\overline{A}+(1-\bar a)\overline{B}))x_{n_k}\|+\|x_{n_k}-\bar x\|)\\
&\le\liminf_{k\to\infty}\|x_{n_k}-\bar x\|.
\end{aligned}$$

This is a contradiction, then we have

$$\bar x\in F(P_C(I-\gamma(\bar a\overline{A}+(1-\bar a)\overline{B}))).\tag{26}$$

By (26), Lemma 8 and Lemma 10, we get

$$\bar{x} \in VI(C, \overline{A}) \cap VI(C, \overline{B}). \tag{27}$$

From (25) and (27), we have

$$\bar{x} \in \xi.$$

In order to show that the entire sequence $\{x_n\}$ weakly converges to $\bar{x}$, assume $x_{n_k} \rightharpoonup \bar{x}'$ as $k \to \infty$, with $\bar{x}' \neq \bar{x}$ and $\bar{x}' \in VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi)$.
By the Opial condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \bar{x}\| &= \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| \\
&< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}'\| \\
&= \lim_{n\to\infty} \|x_n - \bar{x}'\| \\
&= \liminf_{k\to\infty} \|x_{n_k} - \bar{x}'\| \\
&< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| \\
&= \lim_{n\to\infty} \|x_n - \bar{x}\|,
\end{aligned}$$

and this is a contradiction, thus $\bar{x}' = \bar{x}$. This implies that the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to the same point $\bar{x} \in \xi$. Finally, if we take

$$u_n = P_{\xi} x_n, \tag{28}$$

then by (21) and Lemma 11, we see that $\{P_{\xi} x_n\}_{n=0}^{\infty}$ converges strongly to some $z \in \xi$. From (28) and Lemma 3, we get

$$\langle x_n - u_n, u_n - \bar{x} \rangle \geq 0, \qquad \forall \bar{x} \in \xi.$$

Take $n \to \infty$, we also have

$$\langle \bar{x} - z, z - \bar{x} \rangle \geq 0,$$

and hence $z = \bar{x}$. Therefore u_n converges strongly to $\bar{x} \in \xi$, this completes the proof. □

The following Corollary 1 is a special case of Theorem 1 if we put $\overline{A} \equiv \overline{B}$ in Theorem 1.

Corollary 1. Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $A, B, \overline{A} : C \to H$ be $a, b, \bar{\alpha}$-inverse strongly monotone, respectively. Let $Q, F : C \times C \to \mathbb{R}$ be a bifunction satisfying A1)-A4). Define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$, where $r, \beta, \lambda > 0$ and for all $x \in C$, T_r^F, T_r^Q define as same in Lemma 4 that is $T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$ and $T_r^Q(x) = \{z \in C : Q(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi) \neq \emptyset$. For given $x_1 \in C$ and let the sequence $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases}
y_n = P_C(x_n - \gamma \overline{A}(x_n)), \\
Q_n = \{z \in H : \langle x_n - \gamma \overline{A}(x_n) - y_n, y_n - z\rangle \geq 0\}, \\
x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma \overline{A}(y_n)) + \beta_n \varphi(x_n),
\end{cases} \tag{29}$$

where sequence $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$ and $\eta = \min\{a, b\}$, $\lambda, \beta \in (0, 2\eta)$, $\gamma \in (0, 2\bar{\alpha})$ satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$ for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

Proof. Putting $\overline{A} \equiv \overline{B}$ in Theorem 1, then we obtain the desired conclusion. □

4 | APPLICATION

4.1 | The generalized equilibrium and the system of equilibrium problems.

In this section, we obtain the following weak and strong convergence theorems for finding solutions of the generalized equilibrium and the system of equilibrium problems.

Put $A \equiv B \equiv 0$, in (1), the SGEP is reduced to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - y^*\rangle \geq 0, & \forall y \in C, \\ Q(y^*, x) + \frac{1}{r}\langle x - y^*, y^* - x^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{30}$$

(30) is called *the system of equilibrium problem*.

If $A \equiv B$, $F \equiv Q$, $r = 1$, $x^* = y^*$ and $\lambda = \beta = 1$, in (1), then the SGEP reduced to find $x^* \in C$ such that

$$F(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0, \qquad \forall y \in C, \tag{31}$$

where $A : C \to H$ is mapping, the problem (31) is called *the generalized equilibrium problem*. The set of solutions of (31) is denoted by $EP(F, A)$. The problem (30) and (31) covers various disciplines such as optimization problems, variational inequalities and the Nash equilibrium problem in noncooperative games, see literature in [1,6].

Theorem 2. Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $\overline{A}, \overline{B} : C \to H$ be $\bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $Q, F : C \times C \to \mathbb{R}$ be a bifunction satisfying A1)-A4). Define the mapping $\bar{\varphi} : C \to C$ by $\bar{\varphi}(x) = T_r^F(T_r^Q x)$, where $r > 0$ and for all $x \in C$, T_r^F, T_r^Q define as same in Lemma 4 that is $T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$ and $T_r^Q(x) = \{z \in C : Q(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\bar{\varphi}) \neq \emptyset$. For given $x_1 \in C$ and let the sequence $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B})y_n) + \beta_n \bar{\varphi}(x_n), \end{cases} \tag{32}$$

where sequence $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, $\bar{a} \in (0, 1)$ and $\gamma \leq \bar{\eta} = min\{\bar{\alpha}, \bar{\beta}\}$ satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$ for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_\xi(x_n).$$

Proof. If we put $A \equiv B \equiv 0$, in Theorem 1. The conclusion of Theorem 2 can be obtained from Theorem 1. □

Theorem 3. Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $A, \overline{A}, \overline{B} : C \to H$ be $a, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying A1)-A4). Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap EP(F, A) \neq \emptyset$. For given $x_1 \in C$ and let the sequence $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} F(u_n, y) + \langle Au, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B})y_n) + \beta_n u_n, \end{cases} \tag{33}$$

where sequence $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, $r > 0$, $\bar{a} \in (0, 1)$ and $\gamma \leq \bar{\eta} = min\{\bar{\alpha}, \bar{\beta}\}$ satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$ for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n)<\infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z=\lim_{n\to\infty} P_{\xi}(x_n).$$

Proof. If we put $A \equiv B$, $F \equiv Q$ and $\lambda=\beta$, in Theorem 1, then we obtain the desired conclusion. □

4.2 | The variational inequality and the general system of variational inequality problems

In this section, we obtain the following weak and strong convergence theorems for finding solutions of the variational inequality and the general system of variational inequality problems.

In 2008, Ceng et al.[7] introduced *the general system of variational inequalities problem (GSVIP)*, which is to find $(x^*,y^*)\in C\times C$ such that

$$\begin{aligned}&\langle \lambda Ay^*+x^*-y^*, x-x^*\rangle \geq 0, \qquad \forall x\in C,\\ &\langle \mu Bx^*+y^*-x^*, x-y^*\rangle \geq 0, \qquad \forall x\in C,\end{aligned} \tag{34}$$

where $A,B: C\to H$ are two mappings and $\lambda,\mu>0$ are two constants. Further, if we put $A\equiv B$ and $x^*=y^*$, then the problem (34) reduces to the variational inequality $VI(C,A)$.

Remark 3. Put $F\equiv Q\equiv 0$ in (1), we have (1) is reduced to GSVIP. So, (34) is a spacial case of SGEP.

Lemma 12. (See[7]) For given $x^*,y^*\in C$, (x^*,y^*) is a solution of problem (34) if and only if x^* is a fixed point of the mapping $G: C\to C$ defined by

$$G(x)=P_C\big(P_C(x-\mu Bx)-\lambda AP_C(x-\mu Bx)\big), \qquad \forall x\in C,$$

where $y^*=P_C(x^*-\mu Bx^*)$.

By use Theorem 1, we give a theorem involving to find the solution of the GSVIP as follows

Theorem 4. Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $A,B,\overline{A},\overline{B}: C\to H$ be $a,b,\bar{\alpha},\bar{\beta}$-inverse strongly monotone, respectively. Define the mapping $\varphi': C\to C$ by $\varphi'(x)=P_C(I-\lambda A)P_C(I-\beta B)x$ for all $\beta,\lambda>0$, $\forall x\in C$. Assume that $\xi=VI(C,\overline{A})\cap VI(C,\overline{B})\cap F(\varphi')\neq\emptyset$. For given $x_1\in C$ and let the sequence $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n=P_C(I-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B}))x_n,\\ Q_n=\{z\in H: \langle (I-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B}))x_n-y_n, y_n-z\rangle\geq 0\},\\ x_{n+1}=\alpha_nP_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)+\beta_n\varphi'(x_n),\end{cases} \tag{35}$$

where sequence $\{\alpha_n\},\{\beta_n\}\subset[0,1]$ with $\alpha_n+\beta_n\leq 1$, for all $n\in\mathbb{N}$, $\bar{a}\in(0,1)$, $\eta=\min\{a,b\}$, $\gamma\leq\bar{\eta}=min\{\bar{\alpha},\bar{\beta}\}$ and $\lambda,\beta\in(0,2\eta)$ satisfying the following conditions hold:

(i) $0<c\leq\beta_n\leq d<1$ for all $n\in\mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n)<\infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z\in\xi$ and furthermore,

$$z=\lim_{n\to\infty}P_{\xi}(x_n).$$

Proof. From Lemma 12 and putting $F\equiv Q\equiv 0$, in Theorem 1, then we obtain the desired conclusion. □

Next, we prove the fixed point problem, which uses our main theorem.

Remark 4. Let $T: C\to C$ be nonexpansive mapping with $F(T)\neq\emptyset$. Then $F(T)=VI(C,I-T)$.

Corollary 2. Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $T_i : C \to C$ be a nonexpansive mappings, for all $i = 1, 2, 3, 4$. Define the mapping $\varphi^* : C \to C$ by $\varphi^*(x) = P_C(I-\lambda(I-T_1))P_C(I-\beta(I-T_2))x$, where $\beta, \lambda > 0$ and for all $x \in C$. Assume that $\xi = \bigcap_{i=1}^{4} F(T_i) \neq \emptyset$. For given $x_1 \in C$ and let the sequence $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}(I - T_3) + (1 - \bar{a})(I - T_4)))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}(I - T_3) + (1 - \bar{a})(I - T_4)))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}(I - T_3) + (1 - \bar{a})(I - T_4))y_n) + \beta_n \varphi^*(x_n), \end{cases} \tag{36}$$

where sequence $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, $\bar{a} \in (0, 1)$ and $\lambda, \beta, \gamma \in (0, \frac{1}{2})$ satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$ for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_\xi(x_n).$$

Proof. The conclusion of Corollary 2 can be obtained from Theorem 4 and Remark 4. □

Corollary 3. Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $A, \overline{A}, \overline{B} : C \to H$ be $a, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap VI(C, A) \neq \emptyset$. For given $x_1 \in C$ and let the sequence $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B})y_n) + \beta_n P_C(I - \lambda A)x_n, \end{cases} \tag{37}$$

where sequence $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, $\lambda \in (0, 2a)$, $\bar{a} \in (0, 1)$ and $\gamma \leq \bar{\eta} = min\{\bar{\alpha}, \bar{\beta}\}$ satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$ for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_\xi(x_n).$$

Proof. If we put $A \equiv B$ and $\lambda = \beta$, in Theorem 4, then we obtain the desired conclusion. □

5 | NUMERICAL

In this section, we give the following example to support our main theorem.

Example 1. Let $\mathbb{R}$ be the set of real numbers, and let $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x} \cdot \mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2$, for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ and a usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Let $H = \mathbb{R}^2$, $C = [-100, 100] \times [-100, 100]$. Let $A, B, \overline{A}, \overline{B}$ be mappings from C to $\mathbb{R}^2$ defined by $A\mathbf{x} = (\frac{x_1}{2}, \frac{2x_2}{3})$, $B\mathbf{x} = (\frac{x_1}{3}, \frac{x_2}{4})$, $\overline{A}\mathbf{x} = (\frac{x_1}{3}, \frac{x_2}{3})$ and $\overline{B}\mathbf{x} = (\frac{x_1}{4}, \frac{x_2}{4})$, $\forall \mathbf{x} \in C$. Let the mapping $Q, F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(\mathbf{x}, \mathbf{y}) = \frac{-(x_1)^2 - (x_2)^2 + (y_1)^2 + (y_2)^2}{4}, \quad \forall \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \quad \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2,$$

and

$$Q(\mathbf{x},\mathbf{y}) = \frac{-(x_1)^2-(x_2)^2+(y_1)^2+(y_2)^2}{5}, \quad \forall \mathbf{x}=(x_1,x_2)\in\mathbb{R}^2, \quad \mathbf{y}=(y_1,y_2)\in\mathbb{R}^2.$$

Let $r, \beta, \lambda = 1$, the sequence $x^* = (x_1^*, x_2^*)$, $y^* = (y_1^*, y_2^*)$ and $y = (y_1, y_2)$

$$\begin{aligned}
0 \leq & F(x^*, y) + \langle y - x^*, x^* - (I-A)y^* \rangle \\
= & \frac{-(x_1)^2-(x_2)^2+(y_1)^2+(y_2)^2}{4} + \langle (y_1 - x_1^*, y_2 - x_2^*), (\frac{2x_1^*-y_1^*}{2}, \frac{3x_2^*-y_2^*}{3}) \rangle \\
= & \frac{-(x_1)^2-(x_2)^2+(y_1)^2+(y_2)^2}{4} + (y_1 - x_1^*)(\frac{2x_1^*-y_1^*}{2}) + (y_2 - x_2^*)(\frac{3x_2^*-y_2^*}{3}) \\
= & \left(\frac{3(y_1)^2 + (12x_1^* - 6y_1^*)y_1 - 15(x_1^*)^2 + 6x_1^* y_1^*}{12} \right) + \left(\frac{3(y_2)^2 + (12x_2^* - 4y_2^*)y_2 - 15(x_2^*)^2 + 4x_2^* y_2^*}{12} \right) \\
= & G_1(y_1) + G_2(y_2).
\end{aligned}$$

Let $G_1(y_1) = \left(\frac{3(y_1)^2 + (12x_1^* - 6y_1^*)y_1 - 15(x_1^*)^2 + 6x_1^* y_1^*}{12} \right)$ and $G_2(y_2) = \left(\frac{3(y_2)^2 + (12x_2^* - 4y_2^*)y_2 - 15(x_2^*)^2 + 4x_2^* y_2^*}{12} \right)$. $G_1(y_1)$ and $G_2(y_2)$ are quadratic functions with coefficients $a_1 = \frac{1}{4}$, $b_1 = x_1^* - \frac{y_1^*}{2}$, and $c_1 = \frac{-5(x_1^*)^2}{4} + \frac{x_1^* y_1^*}{2}$ of $G_1(y_1)$ and coefficients $a_2 = \frac{1}{4}$, $b_2 = x_2^* - \frac{y_2^*}{3}$, and $c_2 = \frac{-5(x_2^*)^2}{4} + \frac{x_2^* y_2^*}{3}$ of $G_2(y_2)$, respectively. Determine the discriminant Δ_1 of G_1 as follows:

$$\begin{aligned}
\Delta_1 = & b_1^2 - 4a_1 c_1 \\
= & (x_1^* - \frac{y_1^*}{2})^2 - 4\Big(\frac{1}{4}\Big)\Big(\frac{-5(x_1^*)^2}{4} + \frac{x_1^* y_1^*}{2}\Big) \\
= & (\frac{3x_1^* - y_1^*}{2})^2.
\end{aligned}$$

We know that $G_1(y_1) \geq 0$, for all $z \in \mathbb{R}$. If it has most one solution in $\mathbb{R}$, then $\Delta_1 \leq 0$. So, we obtain $x_1^* = \frac{y_1^*}{3}$. Next, we determine the discriminant Δ_2 of G_2 by using the same method as above, we obtain $x_2^* = \frac{2y_2^*}{9}$. That is $T_r^F(I - \lambda A)y^* = (\frac{y_1^*}{3}, \frac{2y_2^*}{9})$. After that, we find the solution of $y^* = (y_1^*, y_2^*)$ in this inequality $0 \leq Q(y^*, z) + \langle z - y^*, y^* - (I-B)x^* \rangle$. By using the same method as $T_r^F(I-\lambda A)y^*$, we obtain $T_r^Q(I - \beta B)x^* = (\frac{10x_1^*}{21}, \frac{15x_2^*}{28})$. That is $\varphi(x) = T_r^F(I-\lambda A)T_r^Q(I-\beta B)x = T_r^F(I-\lambda A)(\frac{10x_1}{21}, \frac{15x_2}{28}) = (\frac{10x_1}{63}, \frac{5x_2}{42})$.

Let $x_1 = (x_1^1, x_1^2)$ and $y_1 = (y_1^1, y_1^2) \in \mathbb{R}^2$. The sequences $\{x_n\}$ and $\{y_n\}$ are generated by (18), where $\eta, \bar{\eta}, \gamma = 1$, $\bar{a} = \frac{1}{2}$, $\alpha_n = \frac{1}{n} - \frac{1}{n^2}$ and $\beta_n = 1 - \frac{1}{n}$, for all $n \in \mathbb{N}$. From the definition of $A, B, \overline{A}, \overline{B}$ and φ, we have $VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi) = (0,0)$. From Theorem 1, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to $(0,0)$. For each $n \in \mathbb{N}$, we can rewrite (18) as follows:

$$\begin{aligned}
y_n = & P_C(I - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})))x_n, \\
Q_n = & \{z \in H : \langle (I - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})))x_n - y_n, y_n - z \rangle \geq 0\}, \\
x_{n+1} = & (\frac{1}{n} - \frac{1}{n^2})P_{Q_n}(x_n - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B}))y_n) + (1 - \frac{1}{n})\varphi(x_n),
\end{aligned}$$

where $P_C(x_1, x_2) = (\max\{\min\{x_1, 100\}, -100\}, \max\{\min\{x_2, 100\}, -100\})$.

The table 1 shows the values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = N = 20$.

TABLE 1 The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = N = 20$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(-10.0000,10.0000)	(-7.0833,7.0833)
2	(0.0000,0.0000)	(0.0000,0.0000)
3	(0.0000,0.0000)	(0.0000,0.0000)
⋮	⋮	⋮
10	(0.0000,0.0000)	(0.0000,0.0000)
⋮	⋮	⋮
18	(0.0000,0.0000)	(0.0000,0.0000)
19	(0.0000,0.0000)	(0.0000,0.0000)
20	(0.0000,0.0000)	(0.0000,0.0000)

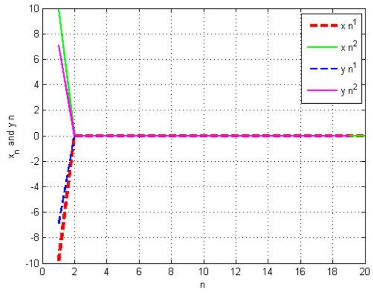

FIGURE 1 The convergence of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = N = 20$

Remark 5. If we choose $\overline{A} \equiv \overline{B}$ in Example 1, we can rewrite (29) as follows:

$$\begin{aligned} y_n =& P_C(x_n - \overline{A}(x_n)), \\ Q_n =& \{z \in H : \langle x_n - \overline{A}(x_n) - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} =& (\frac{1}{n} - \frac{1}{n^2})P_{Q_n}(x_n - \overline{A}(y_n)) + (1 - \frac{1}{n})\varphi(x_n), \end{aligned}$$

where $P_C(x_1, x_2) = (\max\{\min\{x_1, 100\}, -100\}, \max\{\min\{x_2, 100\}, -100\})$. From Corollary 1, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to $(0, 0)$.

The table 2 shows the values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = N = 20$.

TABLE 2 The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = N = 20$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(-10.0000,10.0000)	(-6.6667,6.6667)
2	(0.0000,0.0000)	(0.0000,0.0000)
3	(0.0000,0.0000)	(0.0000,0.0000)
⋮	⋮	⋮
10	(0.0000,0.0000)	(0.0000,0.0000)
⋮	⋮	⋮
18	(0.0000,0.0000)	(0.0000,0.0000)
19	(0.0000,0.0000)	(0.0000,0.0000)
20	(0.0000,0.0000)	(0.0000,0.0000)

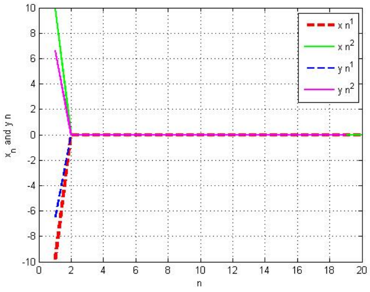

FIGURE 2 The convergence of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = N = 20$

Example 2. In this example, we use the same mappings in Example 1. Let the sequence $\{x_n\}$ and $\{y_n\}$ be generated by (18), where $\alpha_n = \frac{1}{n^2}$ and $\beta_n = 1 - \frac{1}{n^2}$ for all $n \in \mathbb{N}$. From the definition of $A, B, \overline{A}, \overline{B}$ and φ, we have $VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi) = (0, 0)$. We can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to $(0, 0)$. For each $n \in \mathbb{N}$, we can rewrite (18) as follows:

$$\begin{aligned} y_n =& P_C(I - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})))x_n, \\ Q_n =& \{z \in H : \langle (I - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} =& (\frac{1}{n^2})P_{Q_n}(x_n - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B}))y_n) + (1 - \frac{1}{n^2})\varphi(x_n), \end{aligned}$$

where $P_C(x_1, x_2) = (\max\{\min\{x_1, 100\}, -100\}, \max\{\min\{x_2, 100\}, -100\})$.

The table 3 shows the values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = N = 20$.

TABLE 3 The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = N = 20$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(-10.0000,10.0000)	(-7.0833,7.0833)
2	(-7.9340,7.9340)	(-5.6199,5.6199)
3	(-2.5182,2.2821)	(-1.7838,1.6165)
⋮	⋮	⋮
10	(0.0000,0.0000)	(0.0000,0.0000)
⋮	⋮	⋮
18	(0.0000,0.0000)	(0.0000,0.0000)
19	(0.0000,0.0000)	(0.0000,0.0000)
20	(0.0000,0.0000)	(0.0000,0.0000)

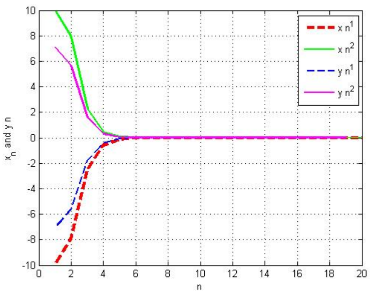

FIGURE 3 The convergence of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = N = 20$

Conclusion

1. Theorem 1 guarantees the convergence of $\{x_n\}$ and $\{y_n\}$ in Example 1.
2. Corollary 1 guarantees the convergence of $\{x_n\}$ and $\{y_n\}$ in Remark 5.
3. The convergence of $\{x_n\}$ and $\{y_n\}$ in an Example 1 is faster than the convergence of $\{x_n\}$ and $\{y_n\}$ in Example 2.

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Author contributions

The two authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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