Strong coupled fixed point analysis in fuzzy metric spaces and an application to Urysohn integral equations

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Abstract: The aim of this paper is to establish some strong coupled fixed point theorems via a new concept of cyclic contractive type mappings in the context of fuzzy metric spaces. Moreover, we ensure the existence of a common solution of the two Urysohn type integral equations:

$$\xi(l) = \int_a^b K_1(l, s, \xi(s))ds + h_1(l),$$
$$\xi(l) = \int_a^b K_2(l, s, \xi(s))ds + h_2(l),$$

where $l \in [a, b] \subset \mathbb{R}$, $\xi, h_1, h_2 \in C([a, b], \mathbb{R})$ and $K_1, K_2 : [a, b]^2 \times \mathbb{R} \to \mathbb{R}$.

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1 Introduction

By using the notion of a fuzzy metric space in the sense of Kramosil et al. [12], George and Veeramani proved in [6] a celebrated fuzzy version of the Banach contraction principle. Furthermore, some fixed point results in the context of fuzzy metric spaces can be found in ([1, 7–10, 14, 15, 17, 18] the references are therein).

In 2009, Lakshmikantham et al. [13], reintroduced the concept of a coupled fixed point which has many applications in partial differential equations and boundary value problems. Later on, Zhu and Xiao in [21], gave some counterexamples for the help of contraction conditions in fuzzy metric spaces and proved a coupled fixed point theorem under some hypotheses of fuzzy metric and t-norm.

Kirk et al. [11], proved some fixed point results for a special type of mapping known as a cyclic contractive type mapping. Choudhury et al. [4], obtained a cyclic coupled Kannan type contraction result in a complete metric space for a strong coupled fixed point. Some more related results can be found in [5, 16, 20].

In this paper, we define a cyclic coupled Kannan type fuzzy contraction and a cyclic coupled Chatterjea type fuzzy contraction in fuzzy metric spaces. We prove some generalized strong coupled fixed point results with illustrative examples. Moreover, we give an integral type application that is, the two Urysohn type integral equations to prove the existence result for common solution in fuzzy metric spaces.

2 Preliminaries

Definition 2.1 ([19]). An operation $* : [0,1]^2 \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

(1) $*$ is commutative, associative and is continuous,

(2) $\forall\ a \in [0,1]$, and $1 * a = a$,

(3) $\forall\ a, b, c, d \in [0,1]$ and $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$.

The basic t-norms; minimum, product and Lukasiewicz continuous t-norms are defined as (see [19]):

$$a * b = \min\{a, b\}, \quad a * b = ab, \quad \text{and} \quad a * b = \max\{a + b - 1, 0\}.$$

Definition 2.2 ([6]). A three-tuple $(X, M_\delta, *)$ is said to be a fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm and M_δ is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(1) $M_\delta(\xi, \eta, t) > 0$ and $M_\delta(\xi, \eta, t) = 1 \Leftrightarrow \xi = \eta$,

(2) $M_\delta(\xi, \eta, t) = M_\delta(\eta, \xi, t)$,

(3) $M_\delta(\xi, \eta, t + s) \geq M_\delta(\xi, \omega, t) * M_\delta(\omega, \eta, s)$,

(4) $M_\delta(\xi, \eta, \cdot) : (0, \infty) \to [0,1]$ is continuous,

$\forall\ \xi, \eta, \omega \in X$ and $s, t > 0$.

Definition 2.3 ([6]). Let (ξ_n) be a sequence in a fuzzy metric space $(X, M_\delta, *)$ and a point $\xi \in X$. Then

(i) (ξ_n) is said to converge to ξ, if $\lim_{n\to\infty} M_\delta(\xi_n, \xi, t) = 1$, $\forall\ t > 0$.

(ii) (ξ_n) is said to be a Cauchy sequence, if for $r \in (0,1)$ and $\exists\ n_1 \in \mathbb{N}$ such that $M_\delta(\xi_m, \xi_n, t) > 1 - r$, $\forall\ t > 0$ and $m, n \geq n_1$.

(iii) $(X, M_\delta, *)$ is said to be complete iff every Cauchy sequence is convergent in X.

Definition 2.4. A sequence $(\xi_n)_{n\geq 1}$ in a fuzzy metric space $(X, M_\delta, *)$ is said to be fuzzy contractive if $\exists\ 0 < a < 1$ such that

$$\frac{1}{M_\delta(\xi_n, \xi_{n+1}, t)} - 1 \leq a\left(\frac{1}{M_\delta(\xi_{n-1}, \xi_n, t)} - 1\right)$$

for $t > 0$.

Definition 2.5 ([2]). A fuzzy metric M_δ is triangular in a fuzzy metric space $(X, M_\delta, *)$, if

$$\frac{1}{M_\delta(\xi, \eta, t)} - 1 \leq \left(\frac{1}{M_\delta(\xi, \omega, t)} - 1\right) + \left(\frac{1}{M_\delta(\omega, \eta, t)} - 1\right),$$

for all $\xi, \eta, \omega \in X$ and $t > 0$.

Definition 2.6. A mapping $G : X \to X$ is said to be fuzzy contractive in a fuzzy metric space $(X, M_\delta, *)$, if there exists $a \in (0, 1)$ such that

$$\frac{1}{M_\delta(G\xi, G\eta, t)} - 1 \leq a\left(\frac{1}{M_\delta(\xi, \eta, t)} - 1\right), \tag{2.1}$$

for all $\xi, \eta \in X$ and $t > 0$. a is called the contraction constant of G.

Definition 2.7 ([4]). Let A and B be two nonempty subsets of a set X. We say that a function $G : X \times X \to X$ is a cyclic map with respect to A and B if $G(\xi, \eta) \in A$ when $\xi \in B$ and $\eta \in A$, and $G(\xi, \eta) \in B$ when $\xi \in A$ and $\eta \in B$.

Definition 2.8 ([3,4]). Let X be a nonempty set. A pair $(\xi, \eta) \in X \times X$ is called a coupled fixed point of the mapping $G : X \times X \to X$ if $G(\xi, \eta) = \xi$ and $G(\eta, \xi) = \eta$. It is known as a strong coupled fixed point if $\xi = \eta$, that is, $G(\xi, \xi) = \xi$.

Definition 2.9 ([4]). Let A and B be two nonempty subsets of a metric space (X, δ). Then a mapping $G : X \times X \to X$ is called a cyclic coupled Kannan type contraction with respect to A and B, if G is cyclic w.r.t A and B, and satisfying for some $a \in (0, 1/2)$,

$$\delta(G(\xi, \eta), G(u, v)) \leq a(\delta(\xi, G(\xi, \eta)) + \delta(u, G(u, v))), \tag{2.2}$$

where $\xi, v \in A$ and $\eta, u \in B$.

Theorem 2.10 ([4]). *Let A and B be two nonempty closed subsets of a complete metric space (X, δ) and $G : X \times X \to X$ be a cyclic coupled Kannan type contraction w.r.t A and B such that $A \cap B \neq \emptyset$. Then G has a strong coupled fixed point in $A \cap B$.*

3 Main results

In this section, we define cyclic coupled Kannan type fuzzy contractions and cyclic coupled Chatterjea type fuzzy contractions in fuzzy metric spaces. Moreover, we prove some strong coupled fixed point theorems for cyclic coupled contractive type mappings in fuzzy metric spaces.

Definition 3.1. Let A and B be two nonempty subsets of a fuzzy metric space $(X, M_\delta, *)$ and $G : X \times X \to X$ be a mapping. Such G is called a cyclic coupled Kannan type fuzzy contraction w.r.t A and B, if G is cyclic w.r.t A and B, and for some $a \in (0, 1/2)$,

$$\frac{1}{M_\delta(G(\xi,\eta), G(u,v), t)} - 1 \leq a\left(\frac{1}{M_\delta(\xi, G(\xi,\eta), t)} - 1 + \frac{1}{M_\delta(u, G(u,v), t)} - 1\right), \tag{3.1}$$

where $\xi, v \in A,\ \eta, u \in B$ and $t > 0$.

Definition 3.2. Let A and B be two nonempty subsets of a fuzzy metric space $(X, M_\delta, *)$, and a mapping $G : X \times X \to X$ is called a cyclic coupled Chatterjea type fuzzy contraction w.r.t A and B, if G is cyclic w.r.t A and B, and for some $a \in (0, 1/2)$,

$$\frac{1}{M_\delta(G(\xi,\eta), G(u,v), t)} - 1 \leq a\left(\frac{1}{M_\delta(u, G(\xi,\eta), t)} - 1 + \frac{1}{M_\delta(\xi, G(u,v), t)} - 1\right), \tag{3.2}$$

where $\xi, v \in A,\ \eta, u \in B$ and $t > 0$.

Moreover, we combine (3.1) and (3.2) for a more generalized cyclic coupled type fuzzy contraction in fuzzy metric spaces $(X, M_\delta, *)$ and we shall prove a related strong coupled fixed point theorem. A mapping $G : X \times X \to X$ is said a generalized cyclic coupled type fuzzy contraction w.r.t A and B, if G is cyclic w.r.t A and B, and for some $a, b \in (0, 1/2)$, it satisfies

$$\begin{aligned}\frac{1}{M_\delta(G(\xi,\eta), G(u,v), t)} - 1 \leq{}& a\left(\frac{1}{M_\delta(\xi, G(\xi,\eta), t)} - 1 + \frac{1}{M_\delta(u, G(u,v), t)} - 1\right) \\ &+ b\left(\frac{1}{M_\delta(u, G(\xi,\eta), t)} - 1 + \frac{1}{M_\delta(\xi, G(u,v), t)} - 1\right),\end{aligned} \tag{3.3}$$

where $\xi, v \in A,\ \eta, u \in B$ and $t > 0$.

Let us prove our first main result.

Theorem 3.3. *Let A and B be two nonempty closed subsets of a complete fuzzy metric space $(X, M_\delta, *)$, in which M_δ is triangular and $G : X \times X \to X$ is a generalize cyclic coupled type fuzzy contraction mapping w.r.t A and B. Let G satisfy (3.3) with $2(a + b) < 1$. Then $A \cap B \neq \emptyset$ and G has a strong coupled fixed point in $A \cap B$.*

Proof. Let $\xi_0 \in A$ and $\eta_0 \in B$ be two arbitrary elements and let (ξ_n) and (η_n) be two sequences in A and B, respectively such that

$$\xi_{n+1} = G(\eta_n, \xi_n) \quad \text{and} \quad \eta_{n+1} = G(\xi_n, \eta_n), \quad \text{for all } n \geq 0. \tag{3.4}$$

Then by using (3.3), we have

$$\begin{aligned}
&\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1=\frac{1}{M_\delta(G(\eta_n,\xi_n),G(\xi_{n+1},\eta_{n+1}),t)}-1\\
&\le a\left(\frac{1}{M_\delta(\eta_n,G(\eta_n,\xi_n),t)}-1+\frac{1}{M_\delta(\xi_{n+1},G(\xi_{n+1},\eta_{n+1}),t)}-1\right)\\
&+b\left(\frac{1}{M_\delta(\xi_{n+1},G(\eta_n,\xi_n),t)}-1+\frac{1}{M_\delta(\eta_n,G(\xi_{n+1},\eta_{n+1}),t)}-1\right)\\
&=a\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1+\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right)+b\left(\frac{1}{M_\delta(\eta_n,\eta_{n+2},t)}-1\right)\\
&\le a\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1+\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right)+b\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1+\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right).
\end{aligned}$$

After simplification, we can get

$$\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\le h\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right),\qquad \text{where } 0<h=\frac{a+b}{1-(a+b)}<1, \tag{3.5}$$

for $t>0$. Similarly, for $t>0$,

$$\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\le h\left(\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right),\qquad \text{where } 0<h=\frac{a+b}{1-(a+b)}<1. \tag{3.6}$$

Now, by adding (3.5) and (3.6), we have

$$\left(\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right)+\left(\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\right)\le h\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1+\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right). \tag{3.7}$$

Again, in view of (3.3), we have

$$\begin{aligned}
&\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1=\frac{1}{M_\delta(G(\xi_{n-1},\eta_{n-1}),G(\eta_n,\xi_n),t)}-1\\
&\le a\left(\frac{1}{M_\delta(\xi_{n-1},G(\xi_{n-1},\eta_{n-1}),t)}-1+\frac{1}{M_\delta(\eta_n,G(\eta_n,\xi_n),t)}-1\right)\\
&+b\left(\frac{1}{M_\delta(\eta_n,G(\xi_{n-1},\eta_{n-1}),t)}-1+\frac{1}{M_\delta(\xi_{n-1},G(\eta_n,\xi_n),t)}-1\right)\\
&=a\left(\frac{1}{M_\delta(\xi_{n-1},\eta_n,t)}-1+\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right)+b\left(\frac{1}{M_\delta(\xi_{n-1},\xi_{n+1},t)}-1\right)\\
&\le a\left(\frac{1}{M_\delta(\xi_{n-1},\eta_n,t)}-1+\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right)+b\left(\frac{1}{M_\delta(\xi_{n-1},\eta_n,t)}-1+\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right).
\end{aligned}$$

After simplification, we can get

$$\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\le h\left(\frac{1}{M_\delta(\xi_{n-1},\eta_n,t)}-1\right),\qquad \text{where } 0<h=\frac{a+b}{1-(a+b)}<1, \tag{3.8}$$

for $t>0$. Similarly,

$$\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\le h\left(\frac{1}{M_\delta(\eta_{n-1},\xi_n,t)}-1\right),\qquad \text{where } 0<h=\frac{a+b}{1-(a+b)}<1. \tag{3.9}$$

By adding (3.8) and (3.9),

$$\left(\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right)+\left(\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\right)\le h^2\left(\frac{1}{M_\delta(\xi_{n-1},\eta_n,t)}-1+\frac{1}{M_\delta(\eta_{n-1},\xi_n,t)}-1\right).$$

Continuing this process, one obtains

$$\left(\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right)+\left(\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\right)\le h^{n+1}\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right), \tag{3.10}$$

for all $n\ge 0$. Now,

$$\begin{aligned}
&\frac{1}{M_\delta(\xi_{m+1},\eta_{m+1},t)}-1=\frac{1}{M_\delta(G(\eta_m,\xi_m),G(\xi_m,\eta_m),t)}-1\\
&\le a\left(\frac{1}{M_\delta(\eta_m,\xi_{m+1},t)}-1+\frac{1}{M_\delta(\xi_m,\eta_{m+1},t)}-1\right)+b\left(\frac{1}{M_\delta(\xi_m,\xi_{m+1},t)}-1+\frac{1}{M_\delta(\eta_m,\eta_{m+1},t)}-1\right)\\
&\le a\left(\frac{1}{M_\delta(\eta_m,\xi_{m+1},t)}-1+\frac{1}{M_\delta(\xi_m,\eta_{m+1},t)}-1\right)\\
&+b\left(\frac{1}{M_\delta(\xi_m,\eta_{m+1},t)}-1+\frac{1}{M_\delta(\eta_{m+1},\xi_{m+1},t)}-1+\frac{1}{M_\delta(\eta_m,\xi_{m+1},t)}-1+\frac{1}{M_\delta(\xi_{m+1},\eta_{m+1},t)}-1\right)\\
&\le (a+b)\left(\frac{1}{M_\delta(\eta_m,\xi_{m+1},t)}-1+\frac{1}{M_\delta(\xi_m,\eta_{m+1},t)}-1\right)+2b\left(\frac{1}{M_\delta(\eta_{m+1},\xi_{m+1},t)}-1\right).
\end{aligned}$$

Clearly, we have

$$\frac{1}{M_\delta(\xi_{m+1},\eta_{m+1},t)}-1\le\lambda\left(\frac{1}{M_\delta(\eta_m,\xi_{m+1},t)}-1+\frac{1}{M_\delta(\xi_m,\eta_{m+1},t)}-1\right),$$

where $\lambda=(a+b)/(1-2b)$ and by using (3.10), for $t>0$, we have that

$$\frac{1}{M_\delta(\xi_{m+1},\eta_{m+1},t)}-1\le\lambda h^m\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M(\eta_0,\xi_1,t)}-1\right),\quad\text{for } m\ge 0. \tag{3.11}$$

Since M_δ is triangular, and by (3.10) and (3.11), we have

$$\begin{aligned}
&\left(\frac{1}{M_\delta(\xi_n,\xi_{n+1},t)}-1\right)+\left(\frac{1}{M_\delta(\eta_n,\eta_{n+1},t)}-1\right)\\
&\le\left(\frac{1}{M_\delta(\xi_n,\eta_n,t)}-1+\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right)+\left(\frac{1}{M_\delta(\eta_n,\xi_n,t)}-1+\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right)\\
&=\left(\frac{1}{M_\delta(\xi_n,\eta_n,t)}-1+\frac{1}{M_\delta(\eta_n,\xi_n,t)}-1\right)+\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1+\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right)\\
&\le 2\lambda h^{n-1}\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right)+h^n\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M(\eta_0,\xi_1,t)}-1\right)\\
&=\left(1+\frac{2\lambda}{h}\right)h^n\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right),\quad\text{for } n\ge 0.
\end{aligned} \tag{3.12}$$

Without loss of generality, we may assume that $m \geq n$.

$$\begin{aligned}\frac{1}{M_\delta(\xi_n,\xi_m,t)}-1 &\leq \sum_{k=n}^{m-1}\left(\frac{1}{M_\delta(\xi_k,\xi_{k+1},t)}-1\right)\\ &\leq \sum_{k=n}^{m-1}\left(1+\frac{2\lambda}{h}\right)h^k\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right)\\ &\leq \left(1+\frac{2\lambda}{h}\right)\frac{h^n}{1-h}\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right)\\ &\to 0, \quad \text{as } n\to\infty.\end{aligned}$$

This shows that (ξ_n) is a Cauchy sequence and hence it is convergent in X. Since A is a closed subset of X, one writes

$$\xi_n \to \xi \in A, \quad \text{as } n \to \infty. \tag{3.13}$$

Similarly,

$$\eta_n \to \eta \in B, \quad \text{as } n \to \infty. \tag{3.14}$$

Hence, from (3.13) and (3.14),

$$\lim_{n\to\infty} M_\delta(\xi_n,\eta_n,t) = M_\delta(\xi,\eta,t), \quad \text{for} \ \ t>0.$$

Since M_δ is triangular, by (3.10) and (3.12), we get

$$\begin{aligned}\frac{1}{M_\delta(\xi_n,\eta_n,t)}-1 &\leq \left(\frac{1}{M_\delta(\xi_n,\xi_{n+1},t)}-1\right)+\left(\frac{1}{M_\delta(\xi_{n+1},\eta_n,t)}-1\right)\\ &\leq \left(\frac{h+2\lambda}{h}+1\right)h^n\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right)\\ &\to 0, \quad \text{as } n\to\infty.\end{aligned}$$

Therefore, $M_\delta(\xi,\eta,t)=1$. This implies that $\xi=\eta\in A\cap B$.

Now, we show that ξ is a strong coupled fixed point of G. Since M_δ is triangular, one has

$$\frac{1}{M_\delta(\xi,G(\xi,\eta),t)}-1 \leq \left(\frac{1}{M_\delta(\xi,\xi_{n+1},t)}-1\right)+\left(\frac{1}{M_\delta(\xi_{n+1},G(\xi,\eta),t)}-1\right), \quad \text{for } t>0. \tag{3.15}$$

Then, by view of (3.3), (3.13) and (3.14), for $t>0$, we have

$$\begin{aligned}\frac{1}{M_\delta(\xi_{n+1},G(\xi,\eta),t)}-1 &= \frac{1}{M_\delta(G(\eta_n,\xi_n),G(\xi,\eta),t)}-1\\ &\leq a\left(\frac{1}{M_\delta(\eta_n,G(\eta_n,\xi_n),t)}-1+\frac{1}{M_\delta(\xi,G(\xi,\eta),t)}-1\right)\\ &\quad+b\left(\frac{1}{M_\delta(\xi,G(\eta_n,\xi_n),t)}-1+\frac{1}{M_\delta(\eta_n,G(\xi,\eta),t)}-1\right)\\ &\leq a\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1+\frac{1}{M_\delta(\xi,G(\xi,\eta),t)}-1\right)\\ &\quad+b\left(\frac{1}{M_\delta(\xi,\xi_{n+1},t)}-1+\frac{1}{M_\delta(\eta_n,\xi,t)}-1+\frac{1}{M_\delta(\xi,G(\xi,\eta),t)}-1\right)\\ &\to\ (a+b)\left(\frac{1}{M_\delta(\xi,G(\xi,\eta),t)}-1\right), \quad \text{as} \ \ n\to\infty.\end{aligned}$$

Then

$$\limsup_{n\to\infty}\left(\frac{1}{M_\delta(\xi_{n+1},G(\xi,\eta),t)}-1\right)\le(a+b)\left(\frac{1}{M_\delta(\xi,G(\xi,\eta),t)}-1\right),\quad \text{for}\ \ t>0.$$

Hence, we put this together with (3.15),

$$\frac{1}{M_\delta(\xi,G(\xi,\eta),t)}-1\le(a+b)\left(\frac{1}{M_\delta(\xi,G(\xi,\eta),t)}-1\right),\quad \text{for}\ \ t>0.$$

Since $a+b<1$, $M_\delta(\xi,G(\xi,\eta),t)=1\Rightarrow G(\xi,\eta)=\xi=\eta$. That is, ξ is a strong coupled fixed point of G.

Corollary 3.4. *Let A and B be two nonempty closed subsets of a complete fuzzy metric space $(X,M_\delta,*)$, in which M_δ is triangular. A mapping $G:X\times X\to X$ is a cyclic coupled Kannan type fuzzy contraction w.r.t A and B (it satisfies (3.1)). Then $A\cap B\neq\emptyset$ and G has a strong coupled fixed point in $A\cap B$.*

Corollary 3.5. *Let A and B be two nonempty closed subsets of a complete fuzzy metric space $(X,M_\delta,*)$ in which M_δ is triangular. A mapping $G:X\times X\to X$ is a cyclic coupled Chatterjea type fuzzy contraction w.r.t A and B (it satisfies (3.2)). Then $A\cap B\neq\emptyset$ and G has a strong coupled fixed point in $A\cap B$.*

Remark 3.6. In a special case, Theorem 3.3, Corollary 3.4 and including Theorem 2.10 (i.e., [4, Theorem 5]) are all as same results.

To support our main result, we present the following example.

Example 3.7. Let $X=\mathbb{R}$, $*$ be a continuous t-norm and $M_\delta:X^2\times(0,\infty)\to[0,1]$ be defined as

$$M_\delta(\xi,\eta,t)=\frac{t}{t+\delta(\xi,\eta)},$$

where $\delta(\xi,\eta)=|\xi-\eta|$, for all $\xi,\eta\in X$ and $t>0$. Let $A=[0,1]$ and $B=[0,1/2]$. Then A and B are two nonempty closed subset of X and $\delta(A,B)=0$.

Let $G:X\times X\to X$ be defined as

$$G(\xi,\eta)=\frac{\xi+2\eta}{10}.$$

Then easily it can be verified that G is a cyclic mapping with respect to A and B, for all $\xi,v\in A$ and

$\eta, u \in B$. Now, we have

$$\begin{aligned}
\frac{1}{M_\delta(G(\xi,\eta),G(u,v),t)} - 1 &= \frac{1}{t}\ \delta(G(\xi,\eta),G(u,v)) \\
&= \frac{1}{t}\left|\frac{\xi+2\eta}{10} - \frac{u+2v}{10}\right| \\
&= \frac{1}{t}\left|\frac{\xi-u}{10} + \frac{\eta-v}{5}\right| \\
&\le \frac{9}{20t}\left|\frac{9(\xi+u)}{10} - \frac{2(\eta+v)}{10}\right| \\
&= \frac{1}{4t}\left|\frac{9(\xi+u)}{10} - \frac{2(\eta+v)}{10}\right| + \frac{1}{5t}\left|\frac{9(\xi+u)}{10} - \frac{2(\eta+v)}{10}\right| \\
&= \frac{1}{4t}\left(\left|\xi - \frac{\xi+2\eta}{10} + u - \frac{u+2v}{10}\right|\right) + \frac{1}{5t}\left(\left|u - \frac{\xi+2\eta}{10} + \xi - \frac{u+2v}{10}\right|\right) \\
&\le \frac{1}{4t}\left(\left|\xi - \frac{\xi+2\eta}{10}\right| + \left|u - \frac{u+2v}{10}\right|\right) + \frac{1}{5t}\left(\left|u - \frac{\xi+2\eta}{10}\right| + \left|\xi - \frac{u+2v}{10}\right|\right) \\
&= \frac{1}{4t}\left(\delta(\xi,G(\xi,\eta)) + \delta(u,G(u,v))\right) + \frac{1}{5t}\left(\delta(u,G(\xi,\eta)) + \delta(\xi,G(u,v))\right) \\
&= \frac{1}{4}\left(\frac{1}{M_\delta(\xi,G(\xi,\eta),t)} - 1 + \frac{1}{M_\delta(u,G(u,v),t)} - 1\right) \\
&+ \frac{1}{5}\left(\frac{1}{M_\delta(u,G(\xi,\eta),t)} - 1 + \frac{1}{M_\delta(\xi,G(u,v),t)} - 1\right).
\end{aligned}$$

Hence (3.3) is satisfied. Thus, all the conditions of Theorem 3.3 are satisfied with $a = 1/4$ and $b = 1/5$ (for $t > 0$). Here, 0 is the strong coupled fixed point of G, that is, $G(0,0) = 0$.

Theorem 3.8. *Let A and B be two nonempty closed subsets of a complete fuzzy metric space $(X, M_\delta, *)$ in which M_δ is triangular and $G : X \times X \to X$ be a cyclic coupled contractive type mapping w.r.t A and B so that for some $a \in [0,1)$,*

$$\begin{aligned}
&\frac{1}{M_\delta(G(\xi,\eta),G(u,v),t)} - 1 \\
&\le a\left(\frac{1}{min\{M_\delta(\xi,G(\xi,\eta),t), M_\delta(u,G(u,v),t), M_\delta(u,G(\xi,\eta)), M_\delta(\xi,G(u,v),t)\}} - 1\right),
\end{aligned} \tag{3.16}$$

where $\xi, v \in A$, $\eta, u \in B$ and $t > 0$. Then $A \cap B \neq \emptyset$ and G has a strong coupled fixed point in $A \cap B$.

Proof. Let $\xi_0 \in A$ and $\eta_0 \in B$ be two arbitrary elements. Let (ξ_n) and (η_n) be two sequences in A and B, respectively, such that

$$\xi_{n+1} = G(\eta_n, \xi_n) \quad \text{and} \quad \eta_{n+1} = G(\xi_n, \eta_n), \quad \text{for all } n \ge 0. \tag{3.17}$$

Now, we have to show that (ξ_n) is a Cauchy sequence. First, we prove that

$$\frac{1}{M_\delta(\xi_{n+1}, \eta_{n+2}, t)} - 1 \le \frac{a}{1-a}\left(\frac{1}{M_\delta(\eta_n, \xi_{n+1}, t)} - 1\right), \quad \text{for } t > 0. \tag{3.18}$$

Then by using (3.16), we have

$$\begin{aligned}
&\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1=\frac{1}{M_\delta(G(\eta_n,\xi_n),G(\xi_{n+1},\eta_{n+1}),t)}-1\\
&\leq a\left(\frac{1}{\min\{M_\delta(\eta_n,G(\eta_n,\xi_n),t),M_\delta(\xi_{n+1},G(\xi_{n+1},\eta_{n+1},t),M_\delta(\xi_{n+1},G(\eta_n,\xi_n),t),M_\delta(\eta_n,G(\xi_{n+1},\eta_{n+1}),t)\}}-1\right)\\
&=a\left(\frac{1}{\min\{M_\delta(\eta_n,\xi_{n+1},t),M_\delta(\xi_{n+1},\eta_{n+2},t),M_\delta(\xi_{n+1},\xi_{n+1},t),M_\delta(\eta_n,\eta_{n+2},t)\}}-1\right)\\
&=a\left(\frac{1}{\min\{M_\delta(\eta_n,\xi_{n+1},t),M_\delta(\xi_{n+1},\eta_{n+2},t),1,M_\delta(\eta_n,\eta_{n+2},t)\}}-1\right). \qquad (3.19)
\end{aligned}$$

Then, we may have the following four cases:

(i) If $M_\delta(\eta_n,\xi_{n+1},t)$ is minimum, then $\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right)$ will be the maximum in (3.19), such that

$$\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\leq a\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right)\leq\frac{a}{1-a}\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right)\quad\text{for } t>0.$$

(3.18) holds as $a<a/(1-a)$, since $a\in[0,1)$.

(ii) If $M_\delta(\xi_{n+1},\eta_{n+2},t)$ is minimum, then $\left(\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right)$ will be the maximum in (3.19), such that

$$\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\leq a\left(\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right),\ \text{this implies } M_\delta(\xi_{n+1},\eta_{n+2},t)=1,\quad\text{for } t>0.$$

In this case, immediately (3.18) follows, since $a\in[0,1)$.

(iii) If 1 is minimum, then the right-hand side of (3.19) will be equal to zero, such that

$$\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\leq a.0,\ \text{this implies that } M_\delta(\xi_{n+1},\eta_{n+2},t)=1,\quad\text{for } t>0.$$

Hence (3.18) holds.

(iv) If $M_\delta(\eta_n,\eta_{n+2},t)$ is minimum, then $\left(\frac{1}{M_\delta(\eta_n,\eta_{n+2},t)}-1\right)$ will be the maximum in (3.19). Since M_δ triangular,

$$\begin{aligned}
\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1&\leq a\left(\frac{1}{M_\delta(\eta_n,\eta_{n+2},t)}-1\right)\\
&\leq a\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1+\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right)\\
&\leq\frac{a}{1-a}\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right).
\end{aligned}$$

It follows that (3.18) holds, for $t>0$.

From all cases for $n\geq 0$ and $t>0$, we have that

$$\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\leq\kappa\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right),\quad\text{where } \kappa=\frac{a}{1-a}<1. \qquad (3.20)$$

Similarly, we prove that

$$\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\leq\frac{a}{1-a}\left(\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right),\quad\text{for } t>0. \tag{3.21}$$

Then, again by using (3.16), we have

$$\begin{aligned}
&\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1=\frac{1}{M_\delta(G(\xi_n,\eta_n),G(\eta_{n+1},\xi_{n+1}),t)}-1\\
&\leq a\left(\frac{1}{\min\{M_\delta(\xi_n,G(\xi_n,\eta_n),t),M_\delta(\eta_{n+1},G(\eta_{n+1},\xi_{n+1}),t),M_\delta(\eta_{n+1},G(\xi_n,\eta_n),t),M_\delta(\xi_n,G(\eta_{n+1},\xi_{n+1}),t)\}}-1\right)\\
&=a\left(\frac{1}{\min\{M_\delta(\xi_n,\eta_{n+1},t),M_\delta(\eta_{n+1},\xi_{n+2},t),M_\delta(\eta_{n+1},\eta_{n+1},t),M_\delta(\xi_n,\xi_{n+2},t)\}}-1\right)\\
&=a\left(\frac{1}{\min\{M_\delta(\xi_n,\eta_{n+1},t),M_\delta(\eta_{n+1},\xi_{n+2},t),1,M_\delta(\xi_n,\xi_{n+2},t)\}}-1\right).
\end{aligned}\tag{3.22}$$

Then again, we may have the following four cases:

(i) If $M_\delta(\xi_n,\eta_{n+1},t)$ is minimum, then $\left(\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right)$ will be the maximum in (3.22), such that

$$\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\leq a\left(\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right)\leq\frac{a}{1-a}\left(\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right).$$

That is, (3.21) holds as $a<a/(1-a)$, since $a\in[0,1)$.

(ii) If $M_\delta(\eta_{n+1},\xi_{n+2},t)$ is minimum, then $\left(\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\right)$ will be the maximum in (3.22), such that

$$\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\leq a\left(\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\right),\ \text{this implies } M_\delta(\eta_{n+1},\xi_{n+2},t)=1,\quad\text{for } t>0.$$

In this case, immediately (3.21) follows, since $a\in[0,1)$.

(iii) If 1 is minimum, then the right-hand side of (3.22) will be equal to zero, such that

$$\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\leq a.0,\ \text{this implies that } M_\delta(\eta_{n+1},\xi_{n+2},t)=1,\quad\text{for } t>0.$$

Hence (3.21) holds.

(iv) If $M_\delta(\xi_n,\xi_{n+2},t)$ is minimum, then $\left(\frac{1}{M_\delta(\xi_n,\xi_{n+2},t)}-1\right)$ will be the maximum in (3.22). Since M_δ triangular,

$$\begin{aligned}
\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1&\leq a\left(\frac{1}{M_\delta(\xi_n,\xi_{n+2},t)}-1\right)\\
&\leq a\left(\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1+\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\right)\\
&\leq\frac{a}{1-a}\left(\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right),
\end{aligned}$$

it follows that (3.21) holds, for $t>0$.

Hence, from all all cases,

$$\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\le\kappa\left(\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right),\quad\text{where }\kappa=\frac{a}{1-a}<1. \tag{3.23}$$

Now, by adding (3.20) and (3.23), we have that

$$\left(\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right)+\left(\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\right)\le\kappa\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1+\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right). \tag{3.24}$$

Again, by (3.16) and similarly we can get the following

$$\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\le\kappa\left(\frac{1}{M_\delta(\xi_{n-1},\eta_n,t)}-1\right),\quad\text{where }\kappa=\frac{a}{1-a}<1. \tag{3.25}$$

Also,

$$\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\le\kappa\left(\frac{1}{M_\delta(\eta_{n-1},\xi_n,t)}-1\right),\quad\text{where }\kappa=\frac{a}{1-a}<1. \tag{3.26}$$

By adding (3.25) and (3.26),

$$\left(\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right)+\left(\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\right)\le\kappa^2\left(\frac{1}{M_\delta(\xi_{n-1},\eta_n,t)}-1+\frac{1}{M_\delta(\eta_{n-1},\xi_n,t)}-1\right).$$

Continuing this process,

$$\left(\frac{1}{M_\delta(\xi_{n+1},\eta_{n+2},t)}-1\right)+\left(\frac{1}{M_\delta(\eta_{n+1},\xi_{n+2},t)}-1\right)\le\kappa^{n+1}\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M(\eta_0,\xi_1,t)}-1\right). \tag{3.27}$$

Hence (3.27) holds for all $n\ge 0$. Consider,

$$\begin{aligned}&\frac{1}{M_\delta(\xi_{m+1},\eta_{m+1},t)}-1=\frac{1}{M_\delta(G(\eta_m,\xi_m),G(\xi_m,\eta_m),t)}-1\\&\le a\left(\frac{1}{\min\{M_\delta(\eta_m,G(\eta_m,\xi_m),t),M_\delta(\xi_m,G(\xi_m,\eta_m),t),M_\delta(\xi_m,G(\eta_m,\xi_m),t),M_\delta(\eta_m,G(\xi_m,\eta_m),t)\}}-1\right)\\&\le a\left(\frac{1}{\min\{M_\delta(\eta_m,\xi_{m+1},t),M_\delta(\xi_m,\eta_{m+1},t),M_\delta(\xi_m,\xi_{m+1},t),M_\delta(\eta_m,\eta_{m+1},t)\}}-1\right).\end{aligned} \tag{3.28}$$

Then, again we may have the following four cases:

(a) If $M_\delta(\eta_m,\xi_{m+1},t)$ is minimum, then $\left(\frac{1}{M_\delta(\eta_m,\xi_{m+1},t)}-1\right)$ will be the maximum in (3.28), such that

$$\frac{1}{M_\delta(\xi_{m+1},\eta_{m+1},t)}-1\le a\left(\frac{1}{M_\delta(\eta_m,\xi_{m+1},t)}-1\right)\le\kappa\left(\frac{1}{M_\delta(\eta_m,\xi_{m+1},t)}-1\right)\quad\text{for }t>0,$$

where $a<\kappa=a/(1-a)<1$, since $a\in[0,1)$.

(b) If $M_\delta(\xi_m,\eta_{m+1},t)$ is minimum, then $\left(\frac{1}{M_\delta(\xi_m,\eta_{m+1},t)}-1\right)$ will be the maximum in (3.28), such that

$$\frac{1}{M_\delta(\xi_{m+1},\eta_{m+1},t)}-1\le a\left(\frac{1}{M_\delta(\xi_m,\eta_{m+1},t)}-1\right)\le\kappa\left(\frac{1}{M_\delta(\xi_m,\eta_{m+1},t)}-1\right),\quad\text{for }t>0,$$

where $a<\kappa=a/(1-a)<1$, since $a\in[0,1)$.

(c) If $M_\delta(\xi_m, \xi_{m+1}, t)$ is minimum, then $\left(\frac{1}{M_\delta(\xi_m,\xi_{m+1},t)} - 1\right)$ will be the maximum in (3.28). Since M_δ triangular,

$$\begin{aligned}\frac{1}{M_\delta(\xi_{m+1}, \eta_{m+1}, t)} - 1 &\leq a\left(\frac{1}{M_\delta(\xi_m, \xi_{m+1}, t)} - 1\right) \\ &\leq a\left(\frac{1}{M_\delta(\xi_m, \eta_{m+1}, t)} - 1 + \frac{1}{M_\delta(\eta_{m+1}, \xi_{m+1}, t)} - 1\right) \\ &\leq \kappa\left(\frac{1}{M_\delta(\xi_m, \eta_{m+1}, t)} - 1\right),\end{aligned}$$

where $a < \kappa = a/(1-a) < 1$ and $t > 0$.

(d) If $M_\delta(\eta_m, \eta_{m+1}, t)$ is minimum, then $\left(\frac{1}{M_\delta(\eta_m,\eta_{m+1},t)} - 1\right)$ will be the maximum in (3.28). Since M_δ triangular,

$$\begin{aligned}\frac{1}{M_\delta(\xi_{m+1}, \eta_{m+1}, t)} - 1 &\leq a\left(\frac{1}{M_\delta(\eta_m, \eta_{m+1}, t)} - 1\right) \\ &\leq a\left(\frac{1}{M_\delta(\eta_m, \xi_{m+1}, t)} - 1 + \frac{1}{M_\delta(\xi_{m+1}, \eta_{m+1}, t)} - 1\right) \\ &\leq \kappa\left(\frac{1}{M_\delta(\eta_m, \xi_{m+1}, t)} - 1\right),\end{aligned}$$

where $a < \kappa = a/(1-a) < 1$ and $t > 0$.

Hence, from (a) and (d), we have

$$\frac{1}{M_\delta(\xi_{m+1}, \eta_{m+1}, t)} - 1 \leq \kappa\left(\frac{1}{M_\delta(\eta_m, \xi_{m+1}, t)} - 1\right), \quad \text{for } t > 0. \tag{3.29}$$

Using (b) and (c), we have

$$\frac{1}{M_\delta(\xi_{m+1}, \eta_{m+1}, t)} - 1 \leq \kappa\left(\frac{1}{M_\delta(\xi_m, \eta_{m+1}, t)} - 1\right), \quad \text{for } t > 0. \tag{3.30}$$

By adding (3.29) and (3.30), we have

$$\frac{1}{M_\delta(\xi_{m+1}, \eta_{m+1}, t)} - 1 \leq \beta\left(\frac{1}{M_\delta(\xi_m, \eta_{m+1}, t)} - 1 + \frac{1}{M_\delta(\eta_m, \xi_{m+1}, t)} - 1\right),$$

where $\beta = \kappa/2$ and using (3.27), we have

$$\frac{1}{M_\delta(\xi_{m+1}, \eta_{m+1}, t)} - 1 \leq \beta\kappa^m\left(\frac{1}{M_\delta(\xi_0, \eta_1, t)} - 1 + \frac{1}{M_\delta(\eta_0, \xi_1, t)} - 1\right), \quad \text{for } m \geq 0. \tag{3.31}$$

Since M_δ is triangular, and by (3.27) and (3.31), we have

$$\begin{aligned}
&\left(\frac{1}{M_\delta(\xi_n,\xi_{n+1},t)}-1\right)+\left(\frac{1}{M_\delta(\eta_n,\eta_{n+1},t)}-1\right)\\
&\le\left(\frac{1}{M_\delta(\xi_n,\eta_n,t)}-1+\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1\right)+\left(\frac{1}{M_\delta(\eta_n,\xi_n,t)}-1+\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right)\\
&=\left(\frac{1}{M_\delta(\xi_n,\eta_n,t)}-1+\frac{1}{M_\delta(\eta_n,\xi_n,t)}-1\right)+\left(\frac{1}{M_\delta(\eta_n,\xi_{n+1},t)}-1+\frac{1}{M_\delta(\xi_n,\eta_{n+1},t)}-1\right)\\
&\le 2\beta\kappa^{n-1}\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right)+\kappa^n\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right)\\
&=\left(1+\frac{2\beta}{\kappa}\right)\kappa^n\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right),\quad \text{for } n\ge 0. \qquad (3.32)
\end{aligned}$$

Without loss of generality, we may assume that $m>n$, we have

$$\begin{aligned}
\frac{1}{M_\delta(\xi_n,\xi_m,t)}-1 &\le \sum_{k=n}^{m-1}\left(\frac{1}{M_\delta(\xi_k,\xi_{k+1},t)}-1\right)\\
&\le \sum_{k=n}^{m-1}\left(1+\frac{2\beta}{\kappa}\right)\kappa^k\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right)\\
&\le \left(1+\frac{2\beta}{\kappa}\right)\frac{\kappa^n}{1-\kappa}\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right)\\
&\to 0,\quad \text{as } n\to\infty.
\end{aligned}$$

This shows that (ξ_n) is a Cauchy sequence and hence it is convergent in X. Since A is a closed subsets of X,

$$\xi_n\to\xi\in A,\quad \text{as } n\to\infty. \qquad (3.33)$$

Similarly,

$$\eta_n\to\eta\in B,\quad \text{as } n\to\infty. \qquad (3.34)$$

Hence, from (3.33) and (3.34), we have

$$\lim_{n\to\infty} M_\delta(\xi_n,\eta_n,t)=M_\delta(\xi,\eta,t),\quad \text{for } \ t>0.$$

Since M_δ is triangular, by (3.27) and (3.32), we have

$$\begin{aligned}
\frac{1}{M_\delta(\xi_n,\eta_n,t)}-1 &\le \left(\frac{1}{M_\delta(\xi_n,\xi_{n+1},t)}-1\right)+\left(\frac{1}{M_\delta(\xi_{n+1},\eta_n,t)}-1\right)\\
&\le \left(\frac{\kappa+2\beta}{\kappa}+1\right)\kappa^n\left(\frac{1}{M_\delta(\xi_0,\eta_1,t)}-1+\frac{1}{M_\delta(\eta_0,\xi_1,t)}-1\right)\\
&\to 0,\quad \text{as } n\to\infty.
\end{aligned}$$

Therefore, $M_\delta(\xi,\eta,t)=1$and this implies that $\xi=\eta\in A\cap B$.

Now, we show that ξ is a strong coupled fixed point of G. Since M_δ is triangular, we have

$$\frac{1}{M_\delta(\xi,G(\xi,\eta),t)}-1\le\left(\frac{1}{M_\delta(\xi,\xi_{n+1},t)}-1\right)+\left(\frac{1}{M_\delta(\xi_{n+1},G(\xi,\eta),t)}-1\right),\quad \text{for } t>0. \qquad (3.35)$$

Then, by (3.16), (3.33) and (3.34), we have

$$
\begin{aligned}
&\frac{1}{M_\delta(\xi_{n+1}, G(\xi,\eta), t)} - 1 = \frac{1}{M_\delta(G(\eta_n,\xi_n), G(\xi,\eta), t)} - 1 \\
&\le a\left(\frac{1}{\min\{M_\delta(\eta_n, G(\eta_n,\xi_n), t), M_\delta(\xi, G(\xi,\eta), t), M_\delta(\xi, G(\eta_n,\xi_n), t), M_\delta(\eta_n, G(\xi,\eta), t)\}} - 1\right) \\
&\le a\left(\frac{1}{\min\{M_\delta(\eta_n, \xi_{n+1}, t), M_\delta(\xi, G(\xi,\eta), t), M_\delta(\xi, \xi_{n+1}, t), M_\delta(\eta_n, G(\xi,\eta), t)\}} - 1\right) \\
&\to a\left(\frac{1}{\min\{1, M_\delta(\xi, G(\xi,\eta), t)\}} - 1\right), \quad \text{as } n \to \infty.
\end{aligned}
$$

If 1 is the minimum of $\{1, M_\delta(\xi, G(\xi,\eta), t)\}$, then directly from (3.35) we can get that $M_\delta(\xi, G(\xi,\eta), t) = 1$, as $n \to \infty$. This implies that $G(\xi,\eta) = \xi = \eta$. Secondly, if $M_\delta(\xi, G(\xi,\eta), t)$ is the minimum of $\{1, M_\delta(\xi, G(\xi,\eta), t)\}$, then

$$
\limsup_{n\to\infty}\left(\frac{1}{M_\delta(\xi_{n+1}, G(\xi,\eta), t)} - 1\right) \le a\left(\frac{1}{M_\delta(\xi, G(\xi,\eta), t)} - 1\right), \quad \text{for } t > 0.
$$

Now, from (3.35), we have

$$
\begin{aligned}
\frac{1}{M_\delta(\xi, G(\xi,\eta), t)} - 1 &\le a\left(\frac{1}{M_\delta(\xi, G(\xi,\eta), t)} - 1\right) \\
(1-a)\left(\frac{1}{M_\delta(\xi, G(\xi,\eta), t)} - 1\right) &\le 0, \quad \text{for } t > 0,
\end{aligned}
$$

which is a contradiction. Hence $M_\delta(\xi, G(\xi,\eta), t) = 1$ implies that $G(\xi,\eta) = \xi = \eta$. Hence, ξ is a strong coupled fixed point of G.

Corollary 3.9. *Let A and B be two nonempty closed subsets of a complete fuzzy metric space $(X, M_\delta, *)$ in which M_δ is triangular. Let $G : X \times X \to X$ be a cyclic coupled contractive type mapping w.r.t A and B: for some $a \in [0,1)$,*

$$
\frac{1}{M_\delta(G(\xi,\eta), G(u,v), t)} - 1 \le a\left(\frac{1}{min\{M_\delta(\xi, G(\xi,\eta), t), M_\delta(u, G(u,v), t)\}} - 1\right), \tag{3.36}
$$

where $\xi, v \in A$, $\eta, u \in B$ and $t > 0$. Then $A \cap B \neq \emptyset$ and G has a strong coupled fixed in $A \cap B$.

Corollary 3.10. *Let A and B be two nonempty closed subsets of a complete fuzzy metric space $(X, M_\delta, *)$ in which M_δ is triangular and $G : X \times X \to X$ is a cyclic coupled contractive type mapping w.r.t A and B: for some $a \in [0,1)$,*

$$
\frac{1}{M_\delta(G(\xi,\eta), G(u,v), t)} - 1 \le a\left(\frac{1}{min\{M_\delta(u, G(\xi,\eta)), M_\delta(\xi, G(u,v), t)\}} - 1\right), \tag{3.37}
$$

where $\xi, v \in A$, $\eta, u \in B$ and $t > 0$. Then $A \cap B \neq \emptyset$ and G has a strong coupled fixed point in $A \cap B$.

Example 3.11. As from Example 3.7, and in view of (3.16), we have that

$$
\begin{aligned}
\frac{1}{M_\delta(G(\xi,\eta),G(u,v),t)}-1 &= \frac{1}{t}\ \delta(G(\xi,\eta),G(u,v)) \\
&= \frac{1}{t}\left|\frac{\xi+2\eta}{10}-\frac{u+2v}{10}\right| \\
&= \frac{1}{t}\left|\frac{\xi-u}{10}+\frac{2(\eta-v)}{10}\right| \\
&\le \frac{4}{50t}\left(\max\left\{|9\xi-2\eta|,|9u-2v|,|10u-\xi-2\eta|,|10\xi-u-2v|\right\}\right) \\
&= \frac{4}{5t}\left(\max\left\{\left|\frac{9\xi-2\eta}{10}\right|,\left|\frac{9u-2v}{10}\right|,\left|\frac{10u-\xi-2\eta}{10}\right|,\left|\frac{10\xi-u-2v}{10}\right|\right\}\right) \\
&= \frac{4}{5t}\left(\max\left\{\left|\xi-\frac{\xi+2\eta}{10}\right|,\left|u-\frac{u+2v}{10}\right|,\left|u-\frac{\xi+2\eta}{10}\right|,\left|\xi-\frac{u+2v}{10}\right|\right\}\right) \\
&= \frac{4}{5t}\left(\max\left\{\delta(\xi,G(\xi,\eta)),\delta(u,G(u,v)),\delta(u,G(\xi,\eta)),\delta(\xi,G(u,v))\right\}\right) \\
&\le \frac{4}{5t}\left(\frac{1}{\min\{M_\delta(\xi,G(\xi,\eta),t),M_\delta(u,G(u,v),t),M_\delta(u,G(\xi,\eta)),M_\delta(\xi,G(u,v),t)\}}-1\right).
\end{aligned}
$$

Hence all the conditions of Theorem 3.3 are satisfied with $a=4/5$ for $t>0$, and 0 is the strong coupled fixed point of G, that is, $G(0,0)=0$.

4 An application on Urysohn integral equations

In this section, we prove an existence result of a common solution for two Urysohn type integral equations. Let $X=C([a,b],\mathbb{R})$ and we define the supremum norm on X as

$$\|\xi\| = \sup_{r\in[a,b]}|\xi(r)| \quad \text{where } \xi\in C([a,b],\mathbb{R}),$$

Let δ be the induced metric defined as

$$\delta(\xi,\eta) = \sup_{r\in[a,b]}|\xi(r)-\eta(r)| \quad \text{where } \xi,\eta\in C([a,b],\mathbb{R}).$$

The binary $*$ is defined by $a*b=ab$, for all $a,b\in[a,b]$. A fuzzy metric $M_\delta: X\times X\times(0,\infty)\to[0,1]$ is given by

$$M_\delta(\xi,\eta,t)=\frac{t}{t+\delta(\xi,\eta)}, \qquad \text{where } \delta(\xi,\eta)=\|\xi-\eta\|, \tag{4.1}$$

for $t>0$ and $\xi,\eta\in([a,b],\mathbb{R})$. This M_δ is called a standard fuzzy metric induced by δ. Then easily one can show that M_δ is triangular and $(X,M_\delta,*)$ is a complete fuzzy metric space.

Now, we are in the position to give a common solution for two Urysohn type integral equations.

Theorem 4.1. *The two Urysohn type integral equations are*

$$\xi(l)=\int_a^b K_1(l,s,\xi(s))ds+h_1(l), \tag{4.2}$$

$$\xi(l)=\int_a^b K_2(l,s,\xi(s))ds+h_2(l), \tag{4.3}$$

where $l \in [a,b] \subset \mathbb{R}$ *and* $\xi, h_1, h_2 \in X$.
Assume that $K_1, K_2 : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ *are such that* $A_{(\xi,\eta)}, B_{(u,v)} \in X$ *for* $\xi, v \in A$ *and* $\eta, u \in B$ *with* $A, B \subseteq X$ *and*

$$A_{(\xi,\eta)}(l) = \int_a^b K_1(l,s,(\xi,\eta)(s))ds, \quad B_{(u,v)}(l) = \int_a^b K_2(l,s,(u,v)(s))ds, \quad \forall\ l \in [a,b].$$

If there exists $\lambda \in (0,1)$ *such that*

$$\|(A_{(\xi,\eta)} + h_1) - (B_{(u,v)} + h_2)\| \le \lambda N((\xi,\eta),(u,v)),$$

where

$$N((\xi,\eta),(u,v)) = \ max \left\{ \begin{array}{c} \|A_{(\xi,\eta)} + h_1 - \xi\| + \|B_{(u,v)} + h_2 - u\|, \\ \|A_{(\xi,\eta)} + h_1 - u\| + \|B_{(u,v)} + h_2 - \xi\| \end{array} \right\}.$$

Then the two Urysohn type integral equations (4.2) *and* (4.3) *have a unique common solution.*

Proof. Define a mapping $G : X \times X \to X$ as

$$G(\xi,\eta) = A_{(\xi,\eta)} + h_1 \quad \text{and} \quad G(u,v) = B_{(u,v)} + h_2.$$

If

$$N((\xi,\eta),(u,v)) = \|A_{(\xi,\eta)} + h_1 - \xi\| + \|B_{(u,v)} + h_2 - u\|,$$

then

$$\|G(\xi,\eta) - G(u,v)\| \le \lambda(\|G(\xi,\eta) - \xi\| + \|G(u,v) - u\|),$$

for all $\xi, v \in A$ and $\eta, u \in B$ with $A, B \subseteq X$. By Theorem 3.3 with $\lambda = a$ and $b = 0$ in Theorem 3.3, the two Urysohn type integral equations (4.2) and (4.3) have a unique common solution.

Again, If

$$N((\xi,\eta),(u,v)) = \|A_{(\xi,\eta)} + h_1 - u\| + \|B_{(u,v)} + h_2 - \xi\|,$$

then

$$\|G(\xi,\eta) - G(u,v)\| \le \lambda(\|G(\xi,\eta) - u\| + \|G(u,v) - \xi\|),$$

for all $\xi, v \in A$ and $\eta, u \in B$, with $A, B \subseteq X$. By Theorem 3.3 with $\lambda = b$ and $a = 0$ in Theorem 3.3, the two Urysohn type integral equations (4.2) and (4.3) have a unique common solution.

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